An eikonal-inspired approach to gravitational scattering and waveforms

Gravitation et Cosmologie ($\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$), Institut d'Astrophysique de Paris

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Based on:

- 2306.16488: Report on the gravitational eikonal

  In collaboration with: Paolo Di Vecchia, Rodolfo Russo, Gabriele Veneziano

- 2303.07006, 2312.14710: One-loop $2 \to 3$ amplitude

  In collaboration with: Alessandro Georgoudis, Ingrid Vazquez-Holm

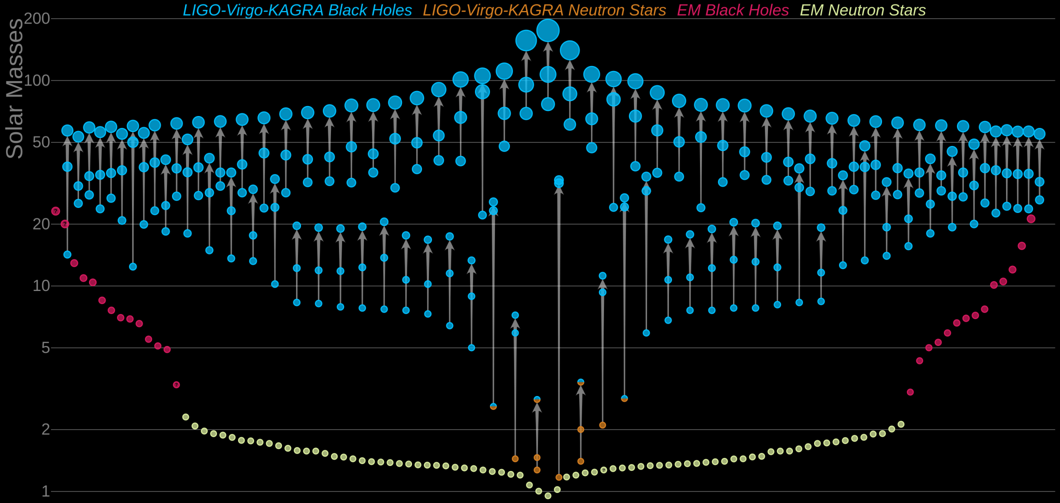
- 2312.07452, 2402.06361: Analysis of the NLO waveform

  In collaboration with: Alessandro Georgoudis, Rodolfo Russo

Outline

Outline

Masses in the Stellar Graveyard



LIGO-Virgo-KAGRA | Aaron Geller | Northwestern

Analytical Approximation Methods

- Post-Newtonian (PN): expansion "for small G and small v"

$$\frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \ll 1\,.$$

- Post-Minkowskian (PM): expansion "for small G"

$$\frac{Gm}{rc^2} \ll 1\,, \qquad \text{generic } \frac{v^2}{c^2}\,.$$

- Self-Force: expansion in the near-probe limit $m_2 \ll m_1$ or

$$m = m_1 + m_2\,, \qquad \nu = \frac{m_1 m_2}{m^2} \ll 1\,.$$

General Relativity from Scattering Amplitudes

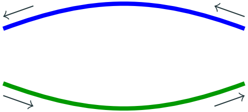
Key Idea: Extract the PM gravitational dynamics from scattering amplitudes.

- Weak-coupling expansion $\leftrightarrow$ PM expansion

Weak-coupling:	$\mathcal{A}_0 = \mathcal{O}(G)$	$\mathcal{A}_1 = \mathcal{O}(G^2)$	$\mathcal{A}_2 = \mathcal{O}(G^3)$	$\mathcal{A}_3 = \mathcal{O}(G^4)$
PM:	1PM	2PM	3PM	4PM
				State of the art
				...

- Lorentz invariance $\leftrightarrow$ generic velocities
- Study scattering events, then export to bound trajectories (V_{eff}, analytic continuation...)

Recent Progress on "Point Particles"



$\mathcal{A}_0^{(4)} = \mathcal{O}(G)$

[Geissler '59]

$\mathcal{A}_1^{(4)} = \mathcal{O}(G^2)$

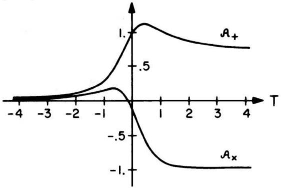
[Westpfahl '85]
[Cheung, Rothstein, Solon '18]
[Collado, Di Vecchia, Russo '19]

$\mathcal{A}_2^{(4)} = \mathcal{O}(G^3)$

[Bern et al. '19]
[Di Vecchia, CH, Russo, Veneziano '20, '21]
[Damgaard et al. 21]
[Brandhuber et al. 21]

$\mathcal{A}_3^{(4)} = \mathcal{O}(G^4)$

[Bern et al. '21]
[Dlapa et al. '21,'22]
[Damgaard et al. '23]



$\mathcal{A}_0^{\mu\nu} = \mathcal{O}(G^{\frac{3}{2}})$

[Kovacs, Thorne '78]
[Goldberger, Ridgway '16]
[Luna, Nicholson, O'Connell, White '17]
[Jakobsen, Mogull, Plefka, Steinhoff '21]
[Mougiakakos, Riva, Vernizzi '21]
[De Angelis, Gonzo, Novichkov '23]
[Brandhuber et al. '23]
[Aoude, Haddad, CH, Helset '23]

$\mathcal{A}_1^{\mu\nu} = \mathcal{O}(G^{\frac{5}{2}})$

[Brandhuber et al. '23]
[Herderschee, Roiban, Teng '23]
[Elkhidir, O'Connell, Sergola, Vazquez-Holm '23]
[Georgoudis, CH, Vazquez-Holm '23]
[Caron-Huot, Giroux, Hannesdottir, Mizera '23]
[Georgoudis, CH, Russo '23, '24]
[Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, Teng '24]

Ref. [Bini, Damour, Geralico '23] reports **mismatches** with MPM-PN formalism (?)

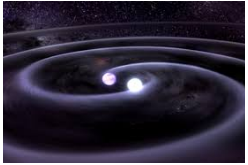
Dissipative Observables

- Radiated energy-momentum P^μ carried away by gravitational waves

	Point-particle	Tides	Spins
$\mathcal{O}(G^3)$ L.O.	[Herrmann, Parra-Martinez, Ruf, Zeng '21]	[Mougiakakos, Riva, Vernizzi '22]	[Riva, Vernizzi, Wong '22]
$\mathcal{O}(G^4)$ N.L.O.	[Dlapa, Kälin, Liu, Neef, Porto '22]	[Jakobsen, Mogull, Plefka, Sauer '23]	[Jakobsen, Mogull, Plefka, Sauer '23]

- Angular momentum loss $J^{\mu\nu}$, angular momentum + mass dipole (rotation + boost charge) carried away by the gravitational field

	Point-particle	Tides	Spins
$\mathcal{O}(G^2)$ L.O.	[Damour '20; Gralla, Lobo '21]	-	[Alessio, Di Vecchia '22]
$\mathcal{O}(G^3)$ N.L.O.	[Manohar, Ridgway, Shen '22] [Di Vecchia, CH, Russo, Veneziano '22]	[CH '22]	[CH '23]



Outline

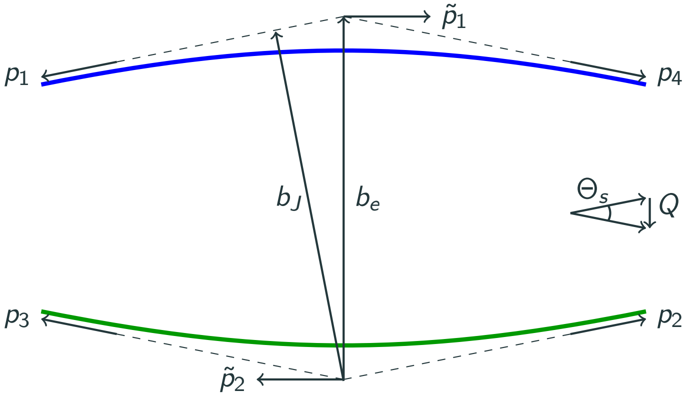
Kinematics of Classical Post-Minkowskian (PM) Scattering

$$\tilde{p}_1^\mu = m_1 \tilde{u}_1^\mu = \frac{1}{2}(p_4^\mu - p_1^\mu)$$

$$\tilde{p}_2^\mu = m_2 \tilde{u}_2^\mu = \frac{1}{2}(p_3^\mu - p_2^\mu)$$

$$Q^\mu = p_1^\mu + p_4^\mu = -p_2^\mu - p_3^\mu$$

$$b_e^\mu = b_J^\mu - \left(\frac{\check{v}_1^\mu}{2m_1} - \frac{\check{v}_2^\mu}{2m_2}\right) Qb$$

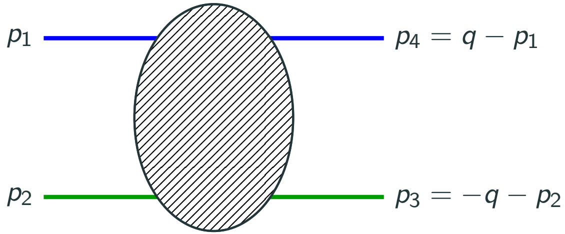


In this way, $v_1 \cdot b_J = v_2 \cdot b_J = 0$ and $\tilde{u}_1 \cdot b_e = \tilde{u}_2 \cdot b_e = 0$. Classical PM regime:

$$\frac{Gm^2}{\hbar} \underset{CL}{\gg} 1\,, \qquad \frac{Gm}{b} \underset{\mathrm{PM}}{\ll} 1\,, \qquad \boxed{\frac{\hbar}{m} \ll Gm \ll b} \qquad \sigma = \frac{1}{\sqrt{1-v^2}} \geq 1 \text{ (generic)}.$$

Kinematics of the Elastic $2 \to 2$ Amplitude, $\mathcal{A}^{(4)}$

$$\bar{p}_1^\mu = \frac{1}{2}(p_4^\mu - p_1^\mu)$$
$$\bar{p}_2^\mu = \frac{1}{2}(p_3^\mu - p_2^\mu)$$
$$q^\mu = p_1^\mu + p_4^\mu = -p_2^\mu - p_3^\mu$$



Defining velocities by $p_1^\mu = -m_1 v_1^\mu$, $p_2^\mu = -m_2 v_2^\mu$

$$\sigma = -v_1 \cdot v_2\,.$$

Dual velocities: $v_1^\mu = \sigma \check{v}_2^\mu + \check{v}_1^\mu$, $v_2^\mu = \sigma \check{v}_1^\mu + \check{v}_2^\mu$ obey $\check{v}_i \cdot v_j = -\delta_{ij}$.

The Elastic Eikonal

- From q to b: Fourier transform $[q \sim \mathcal{O}(\frac{\hbar}{b})]$

$$\tilde{\mathcal{A}}(b) = \frac{1}{4Ep} \int \frac{d^{D-2}q}{(2\pi)^{D-2}}\, e^{ib\cdot q} \mathcal{A}(s,q)\,, \qquad \boxed{1 + i\tilde{\mathcal{A}}(b) = e^{2i\delta(b)}}$$

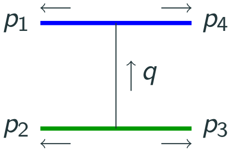
with $2\delta = 2\delta_0 + 2\delta_1 + 2\delta_2 + \cdots \sim \frac{Gm^2}{\hbar}\left(\log b + \frac{Gm}{b} + \left(\frac{Gm}{b}\right)^2 + \cdots\right)$

- From b to Q: stationary-phase approximation $[Q \sim \mathcal{O}(p \cdot \frac{Gm}{b})]$

$$\int d^{D-2}b\, e^{-ib\cdot Q} e^{i2\delta(b)} \implies Q_\mu = \frac{\partial\, \mathrm{Re}\, 2\delta}{\partial b^\mu}$$

Example: the 1PM Eikonal in General Relativity

- Tree-level amplitude in $D = 4 - 2\epsilon$ dimensions



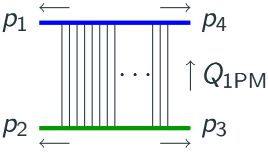
$$\mathcal{A}_0(s,q) = \frac{32\pi G m_1^2 m_2^2 (\sigma^2 - \frac{1}{2-2\epsilon})}{q^2} + \cdots$$

$$\tilde{\mathcal{A}}_0(s,b) = \frac{4Gm_1 m_2(\sigma^2 - \frac{1}{2-2\epsilon})}{2\sqrt{\sigma^2-1}} \frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}} .$$

- Matching to the eikonal exponentiation [Kabat, Ortiz '92; Bjerrum-Bohr et al. '18]

$$e^{2i\delta_0} \underset{\text{“small } G\text{”}}{\longrightarrow} 1 + i\tilde{\mathcal{A}}_0 \implies 2\delta_0 = \tilde{\mathcal{A}}_0 .$$

- From $2\delta_0$, we obtain the leading-order deflection



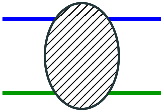
$$Q_{1\text{PM}} = -\frac{\partial 2\delta_0}{\partial b} = \frac{4Gm_1 m_2\left(\sigma^2 - \frac{1}{2}\right)}{b\sqrt{\sigma^2-1}}$$

$$\Theta_{1\text{PM}} = \frac{4GE\left(\sigma^2 - \frac{1}{2}\right)}{b(\sigma^2-1)} .$$

Elastic $2 \to 2$ Amplitude up to One Loop

[Cheung, Rothstein, Solon '18; Collado, Di Vecchia Russo '19]

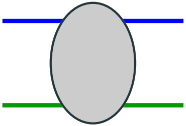
$\mathcal{A}^{(4)} =$



$= \mathcal{A}_0^{(4)} + \mathcal{A}_1^{(4)} + \cdots$

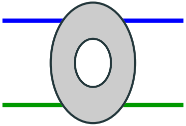
with

$\mathcal{A}_0^{(4)} =$

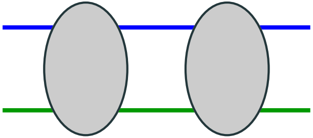


$$= \frac{32\pi G m_1^2 m_2^2 \left(\sigma^2 - \frac{1}{2-2\epsilon}\right)}{q^2} + \mathcal{O}(q^0)$$

$\mathcal{A}_1^{(4)} =$



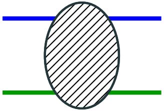
$= \mathrm{Re}\,\mathcal{A}_1^{(4)} + \frac{i}{2}$



$$\mathrm{Re}\,\mathcal{A}_1^{(4)} = 2\pi G m_1^2 m_2^2 (m_1 + m_2) \frac{3\pi(5\sigma^2 - 1)}{q} + \mathcal{O}(\log(q))$$

Impulse from the Eikonal Phase up to One Loop

[Collado, Di Vecchia, Russo '19]

$1 + i\,\mathrm{FT}$  $\sim e^{2i\delta}\,,\qquad 2\delta = 2\delta_0 + 2\delta_1 + \cdots \qquad Q_\mu = \frac{\partial 2\delta}{\partial b_e^\mu}$

- Tree level: $i\tilde{\mathcal{A}}_0 = 2i\delta_0$, so

$$2\delta_0 = \tilde{\mathcal{A}}_0^{(4)} = \frac{2Gm^2\nu(\sigma^2 - \frac{1}{2-2\epsilon})}{\sqrt{\sigma^2-1}}\frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}}\,,\qquad Q^\mu_{1\mathrm{PM}} = -\frac{4Gm^2\nu(\sigma^2-\frac{1}{2})}{b\sqrt{\sigma^2-1}}\frac{b_e^\mu}{b}\,.$$

- One loop: By the exponentiation $i\tilde{\mathcal{A}}_1 - \frac{1}{2!}(2i\delta_0)^2 = i\,\mathrm{Re}\,\tilde{\mathcal{A}}_1 = 2i\delta_1$, so

$$2\delta_1 = \mathrm{Re}\,\tilde{\mathcal{A}}_1^{(4)} = \frac{3\pi G^2 m^3\nu\left(5\sigma^2-1\right)}{4b\sqrt{\sigma^2-1}}\,,\qquad Q^\mu_{2\mathrm{PM}} = -\frac{3\pi G^2m^3\nu\left(5\sigma^2-1\right)}{4b^2\sqrt{\sigma^2-1}}\frac{b_e^\mu}{b}\,.$$

The 3PM Eikonal in General Relativity [Di Vecchia, CH, Russo, Veneziano '20, '21]

[Related work at 3PM: Bern al. '19; Damour '20; Herrmann et al. '21, Bjerrum-Bohr et al. '21; Brandhuber et al. '21]

- Eikonal phase:

$$\mathrm{Re}\,2\delta_2 = \frac{4G^3 m_1^2 m_2^2}{b^2}\left[\frac{s\left(12\sigma^4-10\sigma^2+1\right)}{2m_1 m_2\left(\sigma^2-1\right)^{\frac{3}{2}}}-\frac{\sigma\left(14\sigma^2+25\right)}{3\sqrt{\sigma^2-1}}-\frac{4\sigma^4-12\sigma^2-3}{\sigma^2-1}\,\mathrm{arccosh}\sigma\right]$$

$$+\,\mathrm{Re}\,2\delta_2^{\mathrm{RR}}$$

with

$$\mathrm{Re}\,2\delta_2^{\mathrm{RR}} = \frac{G}{2}Q_{\mathrm{1PM}}^2\mathcal{I}(\sigma)\,,\quad \mathcal{I}(\sigma)\equiv\frac{8-5\sigma^2}{3(\sigma^2-1)}+\frac{\sigma\left(2\sigma^2-3\right)}{(\sigma^2-1)^{3/2}}\,\mathrm{arccosh}\,\sigma\,.$$

- Infrared divergent exponential suppression:

$$\mathrm{Im}\,2\delta_2 = \frac{1}{\pi}\left[-\frac{1}{\epsilon}+\log(\sigma^2-1)\right]\mathrm{Re}\,2\delta_2^{\mathrm{RR}}+\cdots$$

Smoothness and Universality of $\mathrm{Re}\, 2\delta_2$ at High Energy [Di Vecchia, CH, Russo, Veneziano '20, '21]

At high energy, as $\sigma \to \infty$ and $s \sim 2m_1 m_2 \sigma$, i.e. in the massless limit:

- The *complete* eikonal phase is smooth, although the conservative and radiation-reaction parts separately diverge like $\log \sigma$
- Its expression is the same in $\mathcal{N} = 8$ supergravity and in GR,

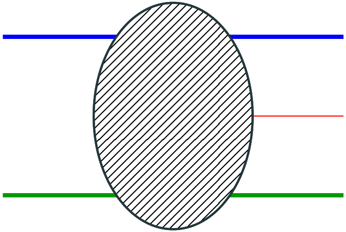
$$\mathrm{Re}\, 2\delta_2 \sim Gs\, \frac{\Theta_s^2}{4}\,, \qquad \Theta_s \sim \frac{4G\sqrt{s}}{b}$$

in agreement with [Amati, Ciafaloni, Veneziano '90].

Outline

Kinematics of the $2 \to 3$ Amplitude

$$\bar{p}_1^\mu = \frac{1}{2}(p_4^\mu - p_1^\mu)$$
$$\bar{p}_2^\mu = \frac{1}{2}(p_3^\mu - p_2^\mu)$$
$$q_1^\mu = p_1^\mu + p_4^\mu$$
$$q_2^\mu = p_2^\mu + p_3^\mu$$
$$0 = q_1^\mu + q_2^\mu + k^\mu$$



More invariants, besides q_1^2, q_2^2, also

$$\sigma = -v_1 \cdot v_2\,, \qquad \omega_1 = -v_1 \cdot k\,, \qquad \omega_2 = -v_2 \cdot k\,.$$

We denote by E, ω the total energy and the graviton frequency in the CoM frame,

$$E = \sqrt{-(p_1 + p_2)^2}\,, \qquad \omega = \frac{1}{E}(p_1 + p_2) \cdot k = \frac{1}{E}(m_1\omega_1 + m_2\omega_2)\,, \qquad \alpha_{1,2} = \frac{\omega_{1,2}}{\omega}\,.$$

Inelastic Final State [Di Vecchia, CH, Russo, Veneziano 2210.12118]

[cf. Kosower, Maybee, O'Connell '18; Damgaard, Planté, Vanhove '21; Cristofoli et al. '21]

Eikonal Exponentiation of Graviton Exchanges + Coherent Radiation:

$$e^{2i\hat{\delta}(b_1,b_2)} = e^{i\,\mathrm{Re}\,2\delta(b)}e^{i\int_k\left[\tilde{W}(k)a^\dagger(k)+\tilde{W}^*(k)a(k)\right]}\,.$$

- Final state, schematically:

$$|\mathrm{out}\rangle = e^{2i\hat{\delta}(b_1,b_2)}|\mathrm{in}\rangle$$

- Unitarity:

$$\langle\mathrm{out}|\mathrm{out}\rangle = \langle\mathrm{in}|\mathrm{in}\rangle = 1$$

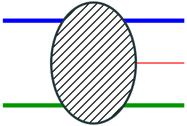
- Consistency with the elastic exponentiation: by the BCH formula,

$$\langle\mathrm{in}|\mathrm{out}\rangle = e^{i\,\mathrm{Re}\,2\delta(b)}e^{-\,\mathrm{Im}\,2\delta(b)} = e^{2i\delta(b)}$$

because, by unitarity, $\mathrm{Im}\,2\delta_2 = \frac{1}{2}\int_k \tilde{\mathcal{A}}_0^{(5)}\tilde{\mathcal{A}}_0^{(5)*}$.

$2 \to 3$ Amplitude up to One Loop

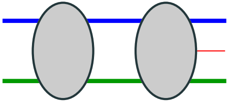
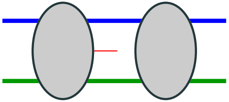
[Brandhuber et al. '23; Herderschee, Roiban, Teng 23; Elkhidir, O'Connell, Sergola, Vazquez-Holm '23] [Georgoudis, CH, Vazquez-Holm '23]

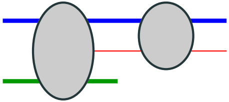
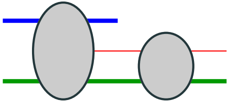
$\mathcal{A} =$  $= \mathcal{A}_0 + \mathcal{A}_1 + \cdots$

with $\mathcal{A}_0$ the tree-level amplitude, and

$$\mathcal{A}_1 = \mathcal{B}_1 + \frac{i}{2}(s + s') + \frac{i}{2}(c_1 + c_2)\,.$$

where $\mathcal{B}_1 = \mathrm{Re}\,\mathcal{A}_1$ and the unitarity cuts can be depicted as follows,

$s =$  $s' =$ 

$c_1 =$  $c_2 =$ 

Eikonal Waveform Kernel up to One Loop

In the eikonal approach, the asymptotic metric fluctuation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ sourced by the scattering (the waveform) is expressed formally as

$$h_{\mu\nu}(x) = \sqrt{32\pi G}\,\langle \text{out}|\hat{H}_{\mu\nu}(x)|\text{out}\rangle \sim \frac{4G}{\kappa r}\int_0^\infty e^{-i\omega U}\,\tilde{W}_{\mu\nu}(\omega n)\,\frac{d\omega}{2\pi} + (\text{c.c.})$$

where $\kappa = \sqrt{8\pi G}$, r is the distance from the observer and U the retarded time.

- Tree level:

$$W_0 = \mathcal{A}_0$$

- One loop:

$$W_1 = \mathcal{B}_1 + \frac{i}{2}(c_1 + c_2)\,.$$

To obtain this simple form for W_1, it is crucial to employ the "eikonal" or "average" variables $\tilde{u}_1$, $\tilde{u}_2$, b_e (and not v_1, v_2, b_J) [Georgoudis, CH, Russo '23]

[See also Caron-Huot, Giroux, Hannesdottir, Mizera '23; Bini, Damour, Geralico '23; Aoude, Haddad, CH, Helset '23]

Real Part of the Kernel

[See e.g. Herderschee, Roiban, Teng '23]

- Tree level: $\mathcal{A}_0$ is a relatively simple rational function
- One loop: We isolate the even and odd parts of $\mathcal{B}_1$ under $\omega_{1,2} \mapsto -\omega_{1,2}$,

$$\mathcal{B}_1 = \mathcal{B}_{1O} + \mathcal{B}_{1E} \,,$$

and $\mathcal{B}_{1O}$ is fixed in terms of the tree-level amplitude,

$$\mathcal{B}_{1O} = \left[1 - \frac{\sigma\left(\sigma^2 - \frac{3}{2}\right)}{(\sigma^2-1)^{3/2}}\right] \pi G E \omega \, \mathcal{A}_0$$

while

$$\mathcal{B}_{1E} = \left[\frac{A^R_{\omega_1}}{\omega_1^2 (q_2^2 + \omega_1^2)^{7/2}} + \frac{A^R_{q_1}}{\omega_2^2 q_1}\right] \frac{m_1^3 m_2^2}{q_2^2 \mathcal{Q}_1^4} + (1 \leftrightarrow 2) \,.$$

- Here, A^R_X are polynomials and $\mathcal{Q}_1$ denotes the spurious pole

$$\mathcal{Q}_1 = (q_1^2 - q_2^2)^2 - 4 q_1^2 \omega_1^2$$

Imaginary Part of the Kernel

[See e.g. Herderschee, Roiban, Teng '23]

The imaginary part is determined by the rescattering or Compton cuts, for instance

$$\frac{i}{2}\,c_1 = iGm_1\omega_1\left(-\frac{1}{\epsilon}+\log\frac{\omega_1^2}{\mu_{\text{IR}}^2}\right)[\mathcal{A}_0]_{D=4} + im_1^3m_2^2\mathcal{M}^{m_1^3m_2^2}\,,$$

$$\begin{aligned}\mathcal{M}^{m_1^3m_2^2} = &\frac{A^I_{\text{rat}}}{q_1^2q_2^2(\sigma^2-1)\omega_1\omega_2^2(q_2^2+\omega_1^2)^3\mathcal{Q}_1^3\mathcal{P}\mathcal{Q}}\\ &+\frac{A^I_{\omega_1}}{q_2^2\omega_1^2(q_2^2+\omega_1^2)^3\mathcal{P}\mathcal{Q}_1^4}\,\text{arcsinh}\,\frac{\omega_1}{q_2}+\frac{A^I_{q_1}}{q_2^2\omega_1(\sigma^2-1)\mathcal{P}^2\mathcal{Q}^2}\,\frac{\text{arccosh}\,\sigma}{\sqrt{\sigma^2-1}}\\ &+\frac{A^I_{\omega_1\omega_2}}{\omega_1\omega_2^2\mathcal{P}^2\mathcal{Q}^2}\,\log\frac{\omega_1^2}{\omega_2^2}+\frac{A^I_{q_1q_2}}{q_1^2q_2^2\mathcal{Q}_1^4\mathcal{P}\mathcal{Q}^2}\,\log\frac{q_1^2}{q_2^2}\end{aligned}$$

with

$$\mathcal{P} = -\omega_1^2+2\omega_1\omega_2\sigma-\omega_2^2\,,\qquad \mathcal{Q} = (q_1^2)^2\omega_1^2-2q_1^2q_2^2\omega_1\omega_2\sigma+(q_2^2)^2\omega_2^2\,.$$

Infrared Divergences Revisited

[Georgoudis, CH, Russo '24]

- Infrared divergences exponentiate in momentum space,

$$W = e^{-\frac{i}{\epsilon} GE\omega} \left[\mathcal{A}_0 + \mathcal{B}_1 + \frac{i}{2} (c_1 + c_2)^{\text{reg}} \right] = e^{-\frac{i}{\epsilon} GE\omega} W^{\text{reg}} ,$$

where

$$\frac{i}{2} c_1^{\text{reg}} = \frac{i}{2} c_1 + \frac{i}{\epsilon} Gm_1\omega_1 \mathcal{A}_0$$

- This also modifies the finite part by $\frac{i}{\epsilon} Gm_1\omega_1$ times the $\mathcal{O}(\epsilon)$ part of $\mathcal{A}_0$.
- After this step, the divergence can be canceled by redefining the origin of retarded time, arriving at the following well defined expression

$$h_{\mu\nu}(x) \sim \frac{4G}{\kappa r} \int_0^\infty e^{-i\omega U} \tilde{W}^{\text{reg}}_{\mu\nu}(\omega n) \frac{d\omega}{2\pi} + (\text{c.c.}) ,$$

Regions in the Soft Limit

[Georgoudis, CH, Russo '23]

Letting $k^\mu = \omega\, n^\mu$, we target non-analytic pieces as $\omega \to 0$, i.e. $\boxed{\omega \ll b^{-1}}$

$$\tilde{W} = \tilde{W}^{[\omega^{-1}]} + \tilde{W}^{[\log\omega]} + \tilde{W}^{[\omega^0]} + \tilde{W}^{[\omega(\log\omega)^2]} + \tilde{W}^{[\omega\log\omega]} + \dots$$

- Region 1: $\boxed{\omega \ll q_\perp \sim b^{-1}}$ The amplitude simplifies and FT become elementary,

$$\int \frac{d^{2-2\epsilon}q_\perp}{(2\pi)^{2-2\epsilon}}\,(q_\perp^2)^\nu\, e^{ib\cdot q_\perp} = \frac{4^\nu}{\pi^{1-\epsilon}}\frac{\Gamma(1+\nu-\epsilon)}{\Gamma(-\nu)(b^2)^{1+\nu-\epsilon}}$$

- Region 2: $\boxed{\omega \sim q_\perp \ll b^{-1}}$ FT turn into ordinary integrals. At tree level,

$$I_{i_1 i_2} = \int \frac{d^{2-2\epsilon}q_\perp}{(2\pi)^{2-2\epsilon}}\frac{1}{\left(q_\perp^2 + \frac{\omega^2\alpha_2^2}{\sigma^2-1}\right)^{i_1}\left((q_\perp - n_\perp)^2 + \frac{\omega^2\alpha_1^2}{\sigma^2-1}\right)^{i_2}}$$

$$I_{10} = \frac{\Gamma(\epsilon)}{(4\pi)^{1-\epsilon}}\left(\frac{\alpha_2^2\omega^2}{\sigma^2-1}\right)^{-\epsilon} \qquad I_{01} = \frac{\Gamma(\epsilon)}{(4\pi)^{1-\epsilon}}\left(\frac{\alpha_1^2\omega^2}{\sigma^2-1}\right)^{-\epsilon} \qquad I_{11} = \frac{\sqrt{\sigma^2-1}}{4\pi\alpha_1\alpha_2\omega^2}\operatorname{arccosh}\sigma$$

Universal Terms

- Leading $1/\omega$ soft term (memory effect in time domain) [matches Weinberg '64; Sahoo, Sen '18; '21]

$$\tilde{W}^{[\omega^{-1}]} = \frac{i\kappa Q}{b\omega\tilde{\alpha}_1^2\tilde{\alpha}_2^2}(\tilde{\alpha}_1\tilde{u}_2\cdot\varepsilon - \tilde{\alpha}_2\tilde{u}_1\cdot\varepsilon)(2\tilde{\alpha}_1\tilde{\alpha}_2 b_e\cdot\varepsilon + b_e\cdot n(\tilde{\alpha}_1\tilde{u}_2\cdot\varepsilon + \tilde{\alpha}_2\tilde{u}_1\cdot\varepsilon))$$

- Subleading $\log\omega$ soft term [matches Sahoo, Sen '18; '21]

$$\tilde{W}^{[\log\omega]} = \kappa\frac{2Gm_1m_2\sigma(2\sigma^2-3)}{\tilde{\alpha}_1\tilde{\alpha}_2(\sigma^2-1)^{3/2}}(\tilde{\alpha}_1\tilde{u}_2\cdot\varepsilon - \tilde{\alpha}_2\tilde{u}_1\cdot\varepsilon)^2\log\left(\frac{\omega b\, e^{\gamma}}{2\sqrt{\sigma^2-1}}\right) + 2iGE\omega\,\tilde{W}_0^{[\omega^{-1}]}\log\omega$$

- Sub-subleading $\omega(\log\omega)^2$ soft term [matches Sahoo, Sen '18; '21]

$$\tilde{W}^{[\omega(\log\omega)^2]} = 2iGE\omega\tilde{W}_0^{[\log\omega]}\log\omega$$

Key simplification:

By dimensional analysis, for the log terms, we can focus on one region (the easier one)!

- For later, we need the complete ω^0 piece of the tree-level result,

$$\tilde{W}_0^{[\omega^0]} = \kappa(\tilde{\alpha}_1 \tilde{u}_2 \cdot \varepsilon - \tilde{\alpha}_2 \tilde{u}_1 \cdot \varepsilon)^2 \left[\frac{G m_1 m_2 \sigma (2\sigma^2 - 3)}{\tilde{\alpha}_1 \tilde{\alpha}_2 (\sigma^2 - 1)^{3/2}} \log(\tilde{\alpha}_1 \tilde{\alpha}_2) - \frac{2 G m_1 m_2 (2\sigma^2 - 1)}{\tilde{\mathcal{P}} \sqrt{\sigma^2 - 1}} \right]$$
$$+ \frac{4 G m_1 m_2}{\tilde{\mathcal{P}}} \left[\frac{(\tilde{\alpha}_1 \tilde{u}_2 \cdot \varepsilon - \tilde{\alpha}_2 \tilde{u}_1 \cdot \varepsilon)^2}{\tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\mathcal{P}}} \left(g_3 \operatorname{arccosh} \sigma + g_2 \log \frac{\tilde{\alpha}_1}{\tilde{\alpha}_2} \right) \right.$$
$$\left. + \frac{2\sigma^2 - 1}{2 b^2 \tilde{\alpha}_1^2 \sqrt{\sigma^2 - 1}} g_1 \right] + i b_2 \cdot n\, \omega \tilde{W}_0^{[\omega^{-1}]} \,.$$

- For this one, both regions are needed!

- Tree-level $\omega \log \omega$ piece [matches Ghosh, Sahoo '21]

$$\tilde{W}_0^{[\omega \log \omega]} = \kappa \frac{2iGm_1 m_2 \sigma(2\sigma^2 - 3)}{\tilde{\alpha}_1 \tilde{\alpha}_2 (\sigma^2 - 1)^{3/2}} (\tilde{\alpha}_1 \, \tilde{u}_2 \cdot \varepsilon - \tilde{\alpha}_2 \, \tilde{u}_1 \cdot \varepsilon)$$
$$\times \left[\tilde{\alpha}_1 \tilde{\alpha}_2 \, b_e \cdot \varepsilon + \tilde{\alpha}_2 (b_1 \cdot n)(\tilde{u}_1 \cdot \varepsilon) - \tilde{\alpha}_1 (b_2 \cdot n)(\tilde{u}_2 \cdot \varepsilon)\right] \omega \log \omega$$

- Non-universal one-loop $\omega \log \omega$ piece. $\mathcal{B}_{1O}$ contributes in the obvious way, while $\mathcal{B}_{1E}$ does not contribute. Finally,

$$\frac{i}{2}(\tilde{c}_1 + \tilde{c}_2)^{[\omega \log \omega]} = iGE \left[-\frac{1}{\epsilon} + \log \frac{\alpha_1 \alpha_2}{\mu_{\rm IR}^2}\right] \omega \tilde{W}_0^{[\log \omega]} + 2iGE\omega \log \omega \, \tilde{W}_0^{[\omega^0]} + i\tilde{\mathcal{M}}_1^{[\omega \log \omega]}$$

with

$$i\tilde{\mathcal{M}}_1^{[\omega \log \omega]} = i\kappa\omega \log \omega \, G^2 \, m_1^2 m_2 \frac{2\sigma(\alpha_1 \, u_2 \cdot \varepsilon - \alpha_2 \, u_1 \cdot \varepsilon)^2}{(\sigma^2 - 1)^{3/2} \tilde{\mathcal{P}}}$$
$$\times \left[\frac{2\sigma^2 - 3}{\tilde{\mathcal{P}}} \left(f_3 \frac{\operatorname{arccosh} \sigma}{(\sigma^2 - 1)^{3/2}} + f_2 \frac{1}{\alpha_2} \log \frac{\alpha_1}{\alpha_2}\right) - \frac{f_1}{\alpha_2(\sigma^2 - 1)}\right] + (1 \leftrightarrow 2) \, .$$

Comparison with Predictions from MPM Formalism

[Georgoudis, CH, Russo '23]

- The result for the $\omega \log \omega$ term was given explicitly in the PN expansion using the *Multipolar post-Minkowskian* (MPM) formalism in [Bini, Damour, Geralico '23], where a **mismatch** was found when comparing with the amplitude-based result starting at 2.5PN ($\sim 1/c^5$)
- We find that **agreement is restored** after performing the following supertranslation [Veneziano, Vilkovisky '22]

$$U \mapsto U - T(n)\,, \qquad T(n) = 2G(m_1\alpha_1 \log \alpha_1 + m_2\alpha_2 \log \alpha_2)$$

or more precisely

$$\delta_T h_{AB} = -T(n)\,\partial_U h_{AB} + r\,[2D_A D_B - \gamma_{AB}\Delta]\,T(n)$$

where only the first term on the RHS (the non-static one) matters.
Here, $n^\mu = (1, \hat{n})$, $e_A^\mu = \partial_A n^\mu$, $h_{AB} = r^2 e_A^\mu e_B^\nu h_{\mu\nu}$, $\gamma_{AB} = e_A \cdot e_B$, D_A is the associated covariant derivative, $\Delta = D_A D^A$.

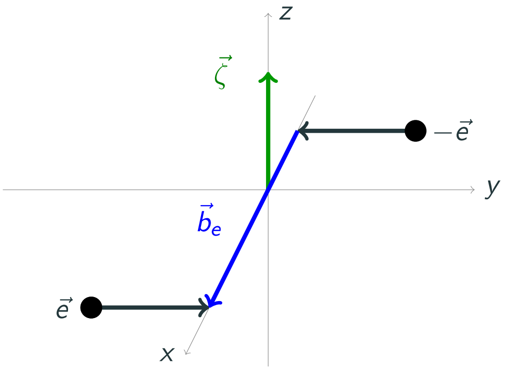
The PN Limit

- We consider the PN expansion

$$p_\infty = \sqrt{\sigma^2 - 1} = \mathcal{O}(\lambda)\,, \qquad \omega = \mathcal{O}(\lambda) \quad \text{as } \lambda \to 0$$

Reference vectors in the CoM frame:

$$t^\alpha = (1, 0, 0, 0)$$
$$b_e^\alpha = (0, b, 0, 0)$$
$$e^\alpha = (0, 0, 1, 0)$$
$$\zeta^\mu = (0, 0, 0, 1)$$



Multipolar Decomposition

[Blanchet, Damour '85; Blanchet '13]

- We define the dimensionless frequency

$$u = \frac{\omega b}{p_\infty}\,,$$

which does not scale in the PN limit.

- It is convenient to express the waveform in terms of the radiative multipoles, i.e. of symmetric trace-free (STF) tensors $\mathrm{U}_L(u)$, $\mathrm{V}_L(u)$,

$$h_{ij}^{\mathrm{TT}} = \frac{4G}{r}\sum_{\ell=2}^{\infty}\frac{1}{\ell!}\left[n_{L-2}\,\mathrm{U}_{ijL-2}(u) - \frac{2\ell}{\ell+1}\,n_{cL-2}\epsilon_{cd(i}\,\mathrm{V}_{j)dL-2}(u)\right]^{\mathrm{TT}}$$

- **We computed all building blocks of the kernel to NNNLO** in the small λ limit and extracted the associated multipoles.

Taking the Limit: Simplifications and Challenges

[Georgoudis, CH, Russo '24]

- The amplitude dramatically simplifies and FT become trivial,

$$\int \frac{d^2 q_\perp}{(2\pi)^2} \left(1 + \frac{p_\infty^2 q_\perp^2}{\omega^2}\right)^\nu e^{ib\cdot q_\perp} = \frac{2^\nu}{\pi b^2} \frac{K_{1+\nu}(u)}{\Gamma(-\nu)\, u^{\nu-1}}$$

  No need to consider different regions.
- The dependence on p_∞ in the relativistic invariants σ, ω_1, ω_2, q_1^2, q_2^2 enters via square roots and is rather **intricate**.
  This can be solved by expanding the elementary variables separately.
- The **spurious poles** induce large inverse powers of λ, which requires to expand the numerators to very high order, especially for $c_1 + c_2$.
  This can be solved by making an ansatz for the expansion and fixing it by numerical evaluation of the limit.

See also [Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, Teng '24]

Different Scaling of the Building Blocks

[Georgoudis, CH, Russo '24]

- Tree level: $\mathcal{A}_0 \longrightarrow G\lambda^{-1}\left(1+\lambda+\lambda^2+\lambda^3+\lambda^4+\lambda^5+\lambda^6+\cdots\right)$
- We further break down $\mathcal{B}_{1O} = \mathcal{B}_{1O}^{(i)} + \mathcal{B}_{1O}^{(h)}$,

$$\mathcal{B}_{1O}^{(h)} = -\frac{\sigma(\sigma^2-\frac{3}{2})}{(\sigma^2-1)^{3/2}}\,\pi G E\omega\mathcal{A}_0\,, \qquad \mathcal{B}_{1O}^{(i)} = \pi G E\omega\mathcal{A}_0$$

so that

$$\mathcal{B}_{1O}^{(h)} \longrightarrow G^2\lambda^{-3}\left(1+\lambda+\lambda^2+\lambda^3+\lambda^4+\lambda^5+\lambda^6+\cdots\right)$$
$$\mathcal{B}_{1O}^{(i)} \longrightarrow G^2\lambda^{0}\left(1+\lambda+\lambda^2+\lambda^3+\cdots\right)$$

and

$$\mathcal{B}_{1E} \longrightarrow G^2\lambda^{-1}\left(1+\lambda+\lambda^2+\lambda^3+\lambda^4+\cdots\right)$$

- C^{reg} defined by "pulling out" the tail $\log\omega$ as $\frac{i}{2}(c_1+c_2)^{\text{reg}} = 2iGE\omega\log\frac{\omega}{\mu_{\text{IR}}}\mathcal{A}_0 + C^{\text{reg}}$ behaves as follows,

$$C^{\text{reg}} \longrightarrow G^2\lambda^{0}\left(1+\lambda+\lambda^2+\lambda^3+\cdots\right)$$

Different Scaling of the Building Blocks

[Georgoudis, CH, Russo '23; '24]

Let us show the scaling in detail for the quadrupole, $\mathrm{U}_{ij} = \mathrm{U}_2$:

	0PN	0.5PN	1PN	1.5PN	2PN	2.5PN	3PN
$\mathcal{A}_0 \longrightarrow \mathrm{U}_2 \sim$	$G\lambda^{-1}(1$	$+$	λ^2	$+$	λ^4	$+$	λ^6
$\mathcal{B}_{1O}^{(i)} \longrightarrow \mathrm{U}_2 \sim$	$G^2\lambda^{-3}(1$	$+$	λ^2	$+$	λ^4	$+$	λ^6
$\mathcal{B}_{1O}^{(h)} \longrightarrow \mathrm{U}_2 \sim$				$G^2\lambda^0(1$	$+$	λ^2	$+\cdots$
$\mathcal{B}_{1E} \longrightarrow \mathrm{U}_2 \sim$			$G^2\lambda^1(1$	$+$	λ^2	$+$	λ^4
$C^{\mathrm{reg}} \longrightarrow \mathrm{U}_2 \sim$				$G^2\lambda^0(1$	$+$	λ^2	$+\cdots$

We have calculated the leading contributions to $\mathrm{U}_{2,3,4,5}$, $\mathrm{V}_{2,3,4}$ and the next-to-leading corrections to $\mathrm{U}_{2,3}$, V_2

Examples

[Georgoudis, CH, Russo '24]

Newtonian quadrupole at tree level,

$$\mathrm{U}_{11}^{\mathrm{LO}} = -\frac{4Gm^2\nu}{3p_\infty}(K_0(u) + 3uK_1(u))\,,$$

$$\mathrm{U}_{12}^{\mathrm{LO}} = -\frac{4iGm^2\nu}{p_\infty}(uK_0(u) + K_1(u))\,,$$

$$\mathrm{U}_{22}^{\mathrm{LO}} = \frac{4Gm^2\nu}{3p_\infty}(2K_0(u) + 3uK_1(u))\,,$$

$$\mathrm{U}_{33}^{\mathrm{LO}} = -\frac{4Gm^2\nu K_0(u)}{3p_\infty}$$

1PN quadrupole correction due to $\mathcal{B}_{1E}$,

$$\mathrm{U}_{E11} = -\mathrm{U}_{E22} = -\frac{6\pi G^2 m^3\nu}{bp_\infty}(1+u)\,e^{-u}\,,$$

$$\mathrm{U}_{E12} = -\frac{6i\pi G^2 m^3\nu}{bp_\infty}\left(\frac{1}{u} + 1 + u\right)e^{-u}\,,$$

while e.g. one component at 2PN is

$$\mathrm{U}_{E33}^{\mathrm{NLO}} = -\frac{\pi G^2 m^3 \nu p_\infty}{b}(2\nu - 5)(u+1)\,e^{-u}\,.$$

Emitted Energy-Momentum

[Georgoudis, CH, Russo '24]

Using the multipoles obtained in this way, we get

$$
\begin{aligned}
&E_{\rm rad}/(m\nu^2)\\
&= \frac{G^3 m^3}{b^3}\,\pi p_\infty \left[\frac{37}{15} + \left(\frac{1357}{840} - \frac{37\nu}{30}\right) p_\infty^2\right]\\
&+ \frac{G^4 m^4}{b^4 p_\infty}\left[\frac{1568}{45} + \left(\frac{18608}{525} - \frac{1136}{45}\nu\right) p_\infty^2\right]\\
&+ \frac{G^4 m^4}{b^4} p_\infty^2 \left[\frac{3136}{45} + \left(\frac{1216}{105} - \frac{2272}{45}\nu\right) p_\infty^2\right]\\
&+ \cdots
\end{aligned}
$$

$$
\begin{aligned}
&P^\mu_{\rm rad}/(m\nu^2\sqrt{1-4\nu})\\
&= \frac{G^3 m^3}{b^3}\,\pi \left[-\frac{37}{30} p_\infty^2 + \left(\frac{37}{60}\nu - \frac{839}{1680}\right) p_\infty^4\right] e^\mu\\
&+ \frac{G^4 m^4}{b^4}\left[-\frac{64}{3} + \left(\frac{32}{3}\nu - \frac{1664}{175}\right) p_\infty^2\right] e^\mu\\
&+ \frac{G^4 m^4}{b^4} p_\infty^3 \left[\left(\frac{1491}{400} - \frac{26757}{5600} p_\infty^2\right)\pi \frac{b_e^\mu}{b}\right.\\
&+ \left.\left(-\frac{128}{3} + \left(\frac{64}{3}\nu - \frac{192}{75}\right) p_\infty^2\right) e^\mu\right]\\
&+ \cdots
\end{aligned}
$$

The component along b_e^μ of $P^\mu_{\rm rad}$ is sensitive to $C^{\rm reg}$ and to the ϵ/ϵ!

Perfect agreement with [Bini, Damour, Geralico '21; '22; Dlapa, Kälin, Liu, Neef, Porto '22]

Comparison with MPM Formalism

[Georgoudis, CH, Russo '24]

- Integer PN terms arise from various corrections to the trajectories.
- Half-odd PN: **Tail formula**

$$\mathrm{U}_L^{\text{tail}} = \frac{2GE}{c^3} i\omega \mathrm{U}_L^{\text{tree}} \left(\log \frac{\omega}{\mu_{\text{IR}}} - \kappa_\ell - \frac{i\pi}{2} \right)$$

(similarly for $\mathrm{V}_L(u)$ with π_ℓ)

- Half-odd PN: **Nonlinear** effects, e.g.

$$U_{ij}^{QQ} = \frac{G}{c^5} \left[\frac{1}{7} I^{(5)}_{a\langle i} I_{j\rangle a} - \frac{5}{7} I^{(4)}_{a\langle i} I^{(1)}_{j\rangle a} - \frac{2}{7} I^{(3)}_{a\langle i} I^{(2)}_{j\rangle a} \right]$$

- Half-odd PN: **Radiation-reaction**

$$x^\mu_{RR} = \frac{8G^2 m^2 p_\infty \nu}{5b^2 r} \left(b^2 e^\mu - (r + p_\infty t) b_e^\mu \right) \qquad U_{ij}^{RR} = 2m\nu \frac{d^2}{dt^2} \left(x_{\langle i} x^{RR}_{j\rangle} \right)$$

We checked that C^{reg} completely agrees with the MPM prediction given by tail+nonlinear+radiation-reaction up to and including 2.5PN.

See also [Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, Teng '24]

Outline

Radiated Energy-Momentum

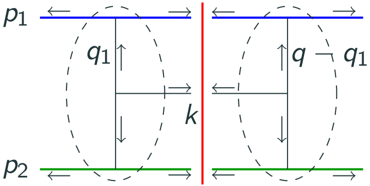
[Kosower, Maybee, O'Connell '18; Herrmann, Parra-Martinez, Ruf, Zeng '21] [Di Vecchia, CH, Russo, Veneziano '22]

- $\langle \text{out}|\hat{P}^\alpha|\text{out}\rangle = \boldsymbol{P}^\alpha$
- In terms of the waveform

$$\boldsymbol{P}^\alpha = \int_k k^\alpha \tilde{A}^{\mu\nu}(k) \left(\eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma} \right) \tilde{A}^{*\rho\sigma}(k) \equiv \int_k k^\alpha \tilde{A}\tilde{A}^* \,.$$

- Recast as the FT of a cut in momentum-space (reverse unitarity)

$$\boldsymbol{P}^\alpha = \text{FT} \int d(\text{LIPS}) k^\alpha$$



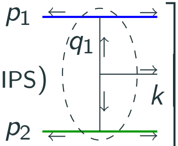
Same integrals appearing in the $2 \to 2$ amplitude!

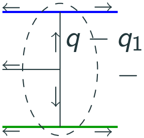
- $\langle \text{out}|\hat{J}^{\alpha\beta}|\text{out}\rangle = \boldsymbol{J}^{\alpha\beta} + \mathcal{J}^{\alpha\beta}$ where [Manohar, Ridgway, Shen '22] [Di Vecchia, CH, Russo '22]

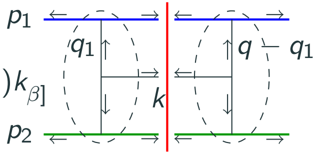
$$\boldsymbol{J}_{\alpha\beta} = \boldsymbol{J}^{(o)}_{\alpha\beta} + \boldsymbol{J}^{(s)}_{\alpha\beta}\,, \qquad i\boldsymbol{J}^{(o)}_{\alpha\beta} = \int_k k_{[\alpha} \frac{\partial \tilde{\mathcal{A}}^{(5)}}{\partial k^{\beta]}} \tilde{\mathcal{A}}^{(5)*}\,, \qquad \boldsymbol{J}^{(s)}_{\alpha\beta} = 2i \int_k \tilde{\mathcal{A}}^{(5)\mu}_{[\alpha} \tilde{\mathcal{A}}^{(5)*}_{\beta]\mu} \ .$$

- Reverse unitarity: $q_{\|2} = -u_2 \cdot q$ [Di Vecchia, CH, Russo, Veneziano '22]

$$i\boldsymbol{J}^{(o)}_{\alpha\beta} = \text{FT}\int k_{[\alpha} \frac{\partial}{\partial k^{\beta]}} \Bigg[d(\text{LIPS}) \ \ldots \Bigg] \ldots - u_{2[\alpha}\, \text{FT} \frac{\partial}{\partial q_{\|2}} \int d(\text{LIPS})\, k_{\beta]} \ \ldots$$







- $\boldsymbol{P}^{\alpha}$, $\boldsymbol{J}^{\alpha\beta}$ transform as follows under translations $x^{\mu} \mapsto x^{\mu} + a^{\mu}$,

$$\boldsymbol{P}^{\alpha} \mapsto \boldsymbol{P}^{\alpha}\,, \qquad \boldsymbol{J}^{\alpha\beta} \mapsto \boldsymbol{J}^{\alpha\beta} + a^{[\alpha}\boldsymbol{P}^{\beta]}\,.$$

Angular Momentum Balance

- Convenient functions: $\mathcal{C}\sqrt{\sigma^2-1} = -\mathcal{E}_+ + \sigma\mathcal{E}_-$, $\mathcal{F} = \pm\mathcal{E}_\pm \mp \frac{1}{2}\,\mathcal{E}$.
- Radiated angular momentum [Manohar, Ridgway, Shen '21]

$$\boldsymbol{J}^{\alpha\beta} = \frac{G^3 m_1^2 m_2^2}{b^3}\,\mathcal{F}\left(b^{[\alpha}\check{u}_1^{\beta]} - b^{[\alpha}\check{u}_2^{\beta]}\right).$$

- Radiative changes of mechanical angular momentum [Di Vecchia, CH, Russo, Veneziano '22]

$$\Delta\boldsymbol{L}_1^{\alpha\beta} = \frac{G^3 m_1^2 m_2^2}{b^3}\left[+\frac{\mathcal{E}_+ b^{[\alpha}u_1^{\beta]}}{\sigma-1} - \frac{1}{2}\,\mathcal{E}\,b^{[\alpha}\check{u}_2^{\beta]}\right]$$

$$\Delta\boldsymbol{L}_2^{\alpha\beta} = \frac{G^3 m_1^2 m_2^2}{b^3}\left[-\frac{\mathcal{E}_+ b^{[\alpha}u_2^{\beta]}}{\sigma-1} + \frac{1}{2}\,\mathcal{E}\,b^{[\alpha}\check{u}_1^{\beta]}\right]$$

$$\boxed{\boldsymbol{J}^{\alpha\beta} + \Delta\boldsymbol{L}_1^{\alpha\beta} + \Delta\boldsymbol{L}_2^{\alpha\beta} = 0}$$

$$\frac{\mathcal{E}}{\pi} = f_1 + f_2\log\frac{\sigma+1}{2} + f_3\frac{\sigma\,\mathrm{arccosh}\,\sigma}{2\sqrt{\sigma^2-1}}\,, \qquad \frac{\mathcal{C}}{\pi} = g_1 + g_2\log\frac{\sigma+1}{2} + g_3\frac{\sigma\,\mathrm{arccosh}\,\sigma}{2\sqrt{\sigma^2-1}}$$

Static Modes in the Eikonal Operator

[Di Vecchia, CH, Russo '22]

Operator dressing:

$$S_{s.r.} = e^{\int_k \left[f^{\mu\nu}(k) a^{\dagger}_{\mu\nu}(k) - f^{*\mu\nu}(k) a_{\mu\nu}(k) \right]} .$$

- $f^{\mu\nu}(k) = F^{\mu\nu}_{TT}(k)$ [Weinberg '64,'65]

$$F^{\mu\nu}(k) = \sum_n \frac{\sqrt{8\pi G}\, p_n^\mu p_n^\nu}{p_n \cdot k - i0} ,$$

  and $\int_k = \int \frac{d^D k}{(2\pi)^D} 2\pi\delta(k^2)\theta(k^0)\theta(\Lambda - k^0)$, with Λ a cutoff.

- Key identification: $p_1 + p_4 = Q = -p_2 - p_3$ with $Q_\mu = \frac{\partial \operatorname{Re} 2\delta}{\partial b^\mu}$.
- $e^{2i\hat{\delta}(b_1,b_2)} \mapsto S_{s.r.} e^{2i\hat{\delta}(b_1,b_2)}$ includes static or "Coulombic" modes

Angular Momentum of the Static Gravitational Field $\mathcal{J}_{\alpha\beta}$

[Di Vecchia, CH, Russo '22] [see also: Veneziano, Vilkovisky '22; Javadinezhad, Porrati '22; Riva, Vernizzi, Wong '23].

Angular momentum/mass dipole loss due to static modes:

$$\mathcal{J}^{\alpha\beta} = \frac{G}{2}\sum_{n,m}\left[\left(\sigma_{nm}^2 - \tfrac{1}{2}\right)\frac{\frac{\sigma_{nm}\,\mathrm{arccosh}\,\sigma_{nm}}{\sqrt{\sigma_{nm}^2-1}} - 1}{\sigma_{nm}^2 - 1} - \frac{2\sigma_{nm}\,\mathrm{arccosh}\,\sigma_{nm}}{\sqrt{\sigma_{nm}^2-1}}\right](\eta_n - \eta_m)\,p_n^{[\alpha}p_m^{\beta]}\,.$$

- Here $-\eta_n\eta_m p_n \cdot p_m = m_n m_m \sigma_{nm}$ with $\eta_n = +1$ ($\eta_n = -1$) if $n \in \mathrm{out}$ ($n \in \mathrm{in}$)
- Matches [Damour '20; Manohar, Ridgway, Shen '22; Bini, Damour '22] up to $\mathcal{O}(G^3)$ upon expanding

$$\mathcal{J}^{\alpha\beta} = -\frac{G}{2}(p_1 - p_2)^{[\alpha}Q^{\beta]}\mathcal{I}(\sigma) + \mathcal{O}(G^4)\,, \qquad Q^\mu = Q^\mu_{1\mathrm{PM}} + Q^\mu_{2\mathrm{PM}} + \mathcal{O}(G^3)$$

- **Easy** to include tidal [CH '22] and spin [Alessio, Di Vecchia '22] [CH '23] effects, via Q^α.

Radiated Angular Momentum Due to Linear Tidal Effects [CH '22]

- Pseudo stress-energy tensor with linear **tidal effects** $t^{\mu\nu}_{E,B}$ given in [Mougiakakos, Riva, Vernizzi '22],

$$\mathcal{A}^{\mu\nu}_{E,B} = (8\pi G)^{3/2} 4 m_1^2 m_2^2 \, t^{\mu\nu}_{E,B} / (q_1^2 q_2^2) \,.$$

- Wilson coefficients

$$c_{E_i^2} = \tfrac{1}{6} k_i^{(2)} R_i^5 / G \,, \qquad c_{B_i^2} = \tfrac{1}{32} j_i^{(2)} R_i^5 / G$$

with radii $R_i = G m_i / K_i$, and k_i, j_i the Love numbers. Compactness $K_i \simeq 0, 1$.

- Using $\mathcal{A}^{\mu\nu}_{E,B}$, I reproduce $\boldsymbol{P}^{\mu}_{\rm tid}$ of [Mougiakakos, Riva, Vernizzi '22] and obtain the **new result**

$$\boldsymbol{J}^{\alpha\beta}_{\rm tid} = \frac{15\pi G^3 m_1^2 m_2^2}{64\, b^7} \sum_{X=E,B} \frac{c_{X_1^2}}{m_1} \left(\mathcal{C}^X b^{[\alpha} u_1^{\beta]} + \mathcal{D}^X u_2^{[\alpha} b^{\beta]} \right) + (1 \leftrightarrow 2)$$

which is an exact function of σ, with $\mathcal{C}^X$, $\mathcal{D}^X$ involving $\log \frac{\sigma+1}{2}$ and $\frac{\operatorname{arccosh}\sigma}{\sqrt{\sigma^2-1}}$.

- Later fully confirmed by and independent worldline QFT calculation in [Jakobsen, Mogullo, Plefka, Sauer '24]

Radiated Angular Momentum Due to Spin-Orbit Effects [CH '23]

- Pseudo stress-energy tensor with linear (spin-orbit) and quadratic (spin-spin) dependence on the **classical spins** given in [Riva, Vernizzi, Wong '22],

$$\mathcal{A}^{\mu\nu}_{s_{1,2}} = (8\pi G)^{3/2} 4 m_1^2 m_2^2\, t^{\mu\nu}_{s_{1,2}}/(q_1^2 q_2^2)\,.$$

- Spin tensors $s_i^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} u_{i\rho} s_{i\sigma}$ related to the mass-rescaled spin vectors s_i^μ $(i = 1, 2)$

$$|s_i| < G m_i \ll b\,, \qquad s_i^\mu u_{i\mu} = 0 \text{ (SSC)}$$

- Using $\mathcal{A}^{\mu\nu}_{s_{1,2}}$, I confirm the linear part of $\boldsymbol{P}^\mu_{s_{1,2}}$ by [Riva, Vernizzi, Wong '22] and obtain

$$\boldsymbol{J}^{\alpha\beta}_{s_1} = \frac{\pi G^3 m_1^2 m_2^2}{b^5}\left[\left(c_{u_1 b} u_1^{[\alpha} b^{\beta]} + c_{u_2 b} u_2^{[\alpha} b^{\beta]}\right) u_2^\rho b^\sigma + \left(c_{u_1\zeta} u_1^{[\alpha} \zeta^{\beta]} + c_{u_2\zeta} u_2^{[\alpha} \zeta^{\beta]}\right) u_2^\rho \zeta^\sigma + c_{b\zeta} b^{[\alpha} \zeta^{\beta]} b^\rho \zeta^\sigma\right] s_{1\rho\sigma}$$

which is an exact function of σ, with c_{vw} involving $\log\frac{\sigma+1}{2}$ and $\frac{\operatorname{arccosh}\sigma}{\sqrt{\sigma^2-1}}$.

- Note the dependence on $\zeta_\mu = -\frac{1}{b p_\infty}\,\epsilon_{\mu\alpha\beta\gamma} u_1^\alpha u_2^\beta b^\gamma$.

Geometry of $2 \to 2$ Scattering with Spin

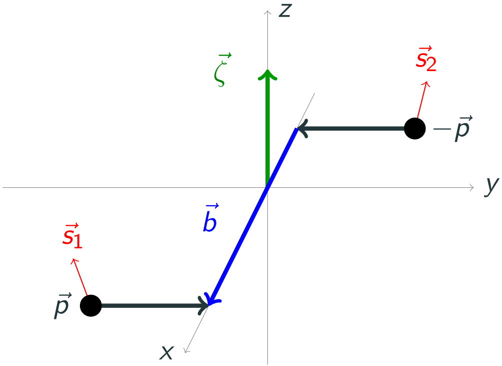
Reference vectors in the CoM frame:

$$
\begin{aligned}
t^\alpha &= (1,0,0,0)\\
b^\alpha &= (0,b,0,0)\\
p^\alpha &= (0,0,p,0)\\
\zeta^\mu &= (0,0,0,1)
\end{aligned}
$$

where

$$
t^\alpha = -\frac{p_1^\alpha + p_2^\alpha}{E}, \qquad p^\alpha = +m_1[u_1^\alpha + t^\alpha(t\cdot u_1)]
$$
$$
= -m_2[u_2^\alpha + t^\alpha(t\cdot u_2)]
$$

are the time direction and the spatial momentum in that frame.



Recoil [CH '23]

- Balance law $J^{\mu\nu} = -\Delta J^{\mu\nu}_{\text{mech}}$
- Total **angular momentum vector** $(J_{\text{mech}})_\mu = -\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma} t^\alpha J^{\beta\gamma}_{\text{mech}}$ with respect to the time-direction in the center-of-mass frame t^α.
- Then, $J^{*\mu} = -\Delta J^\mu_{\text{mech}}$ with

$$J_\mu = -\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma} t^\alpha J^{\beta\gamma}\,, \qquad R_\mu = -\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}(-\Delta t^\alpha) J^{\beta\gamma}\,.$$

- The recoil term is crucial in order to compare $J^{*\mu}$ to its tensor analog $J^{\mu\nu}$,

$$J^{*\mu} = \zeta^\mu \left[\frac{1}{bp}(b_\alpha J^{\alpha\beta} p_\beta) - (\zeta \cdot s_1)\frac{p \cdot \boldsymbol{P}}{E}\right] + \frac{b^\mu}{b}\left[\frac{1}{p}(p_\alpha J^{\alpha\beta}\zeta_\beta) - \frac{(b \cdot s_1)}{b}\frac{p \cdot \boldsymbol{P}}{E}\right]$$
$$+ \frac{p^\mu}{bp}(\zeta_\alpha J^{\alpha\beta} b_\beta) + t^\mu \left[pb\frac{\zeta \cdot \boldsymbol{P}}{E} - \frac{m_1}{p}(s_1 \cdot e)\frac{p \cdot \boldsymbol{P}}{E}\right]$$

  where p^μ is the spatial momentum in the center-of-mass frame.
- Full agreement with the PN results [Cho, Kälin, Porto '21] [Bini, Geralico, Rettegno '23] and with the full PM result [Jakobsen, Mogull, Plefka, Sauer '23]

Analytic Continuation to the Bound Case

[Saketh, Vines, Steinhoff, Buonanno '21; Cho, Kälin, Porto '21] [CH '23]

- The results discussed so far hold for the **scattering kinematics**, in which the total center-of-mass energy is

$$E = \sqrt{m_1^2 + 2m_1 m_2\,\sigma + m_2^2} \geq m_1 + m_2\,, \qquad \sigma \geq 1\,.$$

- To analytically continue $J(L = pb, a_1, \sigma)$ to the **bound-state kinematics**, $\sigma < 1$, one can sum the two branch choices $\sqrt{\sigma^2 - 1} \to \pm i\sqrt{1 - \sigma^2}$

$$J^{\text{bound}}(L, a_1, \sigma) = J(L, a_1, \sigma)_+ + J(L, a_1, \sigma)_-$$

- The $\mathcal{O}(G^3)$ result $J^{\mathcal{O}(G^3)}(L, a_1, \sigma)$ is an analytic function of σ for $\mathrm{Re}\,\sigma > -1$, so

$$J^{\mathcal{O}(G^3)\text{bound}}(L, a_1, \sigma) = 2J^{\mathcal{O}(G^3)}(L, a_1, \sigma)\,.$$

Summary and Outlook

- The **eikonal approach** provides a flexible and conceptually transparent framework to calculate scattering observables, including the **impulse**, the **waveform** and the emitted **energy and angular momentum**.
- The comparison with the **MPM-PN** results is interesting both technically and conceptually. There is full agreement up to and including 2.5PN once the amplitudes and the MPM results are written in the same **BMS** frame

For the future:

- Is the choice of BMS frame relevant in other comparisons (PN versus NR, PN versus NRGR-EFT)? Is it relevant for bound orbits?
- Analytic results beyond soft/PN limit?
- When does the naive eikonal exponentiation break down? (If it does)
- Analytic continuation? [Adamo, Gonzo, Ilderton '24]
- NNLO waveform?

ADDITIONAL MATERIAL

The Initial State

- We model the initial state by $|\text{in}\rangle = |1\rangle \otimes |2\rangle$, with

$$|1\rangle = \int_{-p_1} \varphi_1(-p_1)\, e^{ib_1\cdot p_1} | -p_1\rangle$$
$$|2\rangle = \int_{-p_2} \varphi_2(-p_2)\, e^{ib_2\cdot p_2} | -p_2\rangle$$

and $\int_{-p_i} = \int 2\pi\delta(p_i^2 + m_i^2)\theta(-p_i^0)\frac{d^D p_i}{(2\pi)^D}$ the LIPS measure.

- Wavepackets $\varphi_i(-p_i)$ peaked around the classical incoming momenta.
- Impact parameter $b^\mu = b_1^\mu - b_2^\mu$ lies in the transverse plane $b \cdot p_1 = 0 = p_2 \cdot b$.

Elastic and Inelastic Fourier Transforms

- Elastic Fourier transform:

$$\begin{aligned}\mathrm{FT}\,\mathcal{A}^{(4)} &= \int \frac{d^D q}{(2\pi)^D}\, 2\pi\delta(2m_1 v_1\cdot q)\, 2\pi\delta(2m_2 v_2\cdot q) e^{ib\cdot q}\mathcal{A}^{(4)}(q)\\ &= \frac{1}{4Ep}\int \frac{d^{D-2}q}{(2\pi)^{D-2}}\, e^{ib\cdot q}\mathcal{A}(s,q) = \tilde{\mathcal{A}}^{(4)}\,.\end{aligned}$$

- Inelastic Fourier transform:

$$\begin{aligned}\mathrm{FT}\,\mathcal{A}^{(5)} &= \int \frac{d^D q_1}{(2\pi)^D}\frac{d^D q_2}{(2\pi)^D}\,(2\pi)^D\delta^{(D)}(q_1+q_2+k)\\ &\times 2\pi\delta(2m_1 v_1\cdot q_1) 2\pi\delta(2m_2 v_2\cdot q_2) e^{ib_1\cdot q_1 + ib_2\cdot q_2}\mathcal{A}^{(5)}(q_1,q_2,k)\\ &= \tilde{\mathcal{A}}^{(5)}(k)\,.\end{aligned}$$

N-Operator, T-Operator and Unitarity

- N-operator: $S = e^{iN}$

$$N = -i\log(1+iT) = T - \frac{i}{2}T^2 + \cdots$$

up to one loop.

- Unitarity: $S^\dagger S = 1$,

$$\frac{1}{2}(T - T^\dagger) = +\frac{i}{2}T^\dagger T$$

- We shall denote by $\mathcal{B}$ the N-matrix elements, just like $\mathcal{A}$ denotes the the usual amplitudes (T-matrix elements). Then, by unitarity,

$$\mathcal{B}_0 = \mathcal{A}_0\,, \qquad \mathcal{B}_1 = \mathrm{Re}\,\mathcal{A}_1\,.$$

Waveform KMOC Kernel up to One Loop

In the KMOC approach [Kosower, Maybee, O'Connell '18; Cristofoli, Gonzo, Kosower, O'Connell '21], the asymptotic metric fluctuation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ sourced by the scattering (the waveform) is expressed as the FT of a momentum space kernel: formally,

$$h_{\mu\nu}(x) \sim \frac{4G}{\kappa r} \int_0^\infty e^{-i\omega U} \tilde{W}_{\mu\nu}(\omega n) \frac{d\omega}{2\pi} + (\text{c.c.})$$

where $\kappa = \sqrt{8\pi G}$, r is the distance from the observer and U the retarded time.

- Tree level:

$$W_0 = \mathcal{A}_0$$

- One loop:

$$W_1 = \mathcal{B}_1 + \frac{i}{2}(s - s') + \frac{i}{2}(c_1 + c_2) \,.$$

The difference of two-massive-particle cuts appears [Caron-Huot, Giroux, Hannesdottir, Mizera '23],

$$i\, s_- = \frac{i}{2}(s - s')$$

$i\tilde{s}_-$ as a Taylor Expansion

[Georgoudis, CH, Russo '23]

- In any CoM translation frame,

$$b_1^\mu = E_2 b_J^\mu / E \,, \qquad b_2^\mu = -E_1 b_J^\mu / E \,,$$

much like for the NLO impulse we can show that, in $D = 4 - 2\epsilon$,

$$\boxed{i\,\tilde{s}_- = Q\,\bar{\partial}\tilde{W}_0^{\mu\nu} + i\omega\,(\delta U)\tilde{W}_0^{\mu\nu}}$$

with [fully confirmed by Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, Teng '24 in the PN limit]

$$\delta U = GE\left[\mu_{\text{IR}}^{2\epsilon}\,\frac{\sigma\left(\sigma^2 - \frac{3-4\epsilon}{2-2\epsilon}\right)}{(\sigma^2-1)^{3/2}}\,\frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}} + \frac{(m_1+m_2\sigma)(m_2+m_1\sigma)}{m_1^2+2m_1m_2\sigma+m_2^2}\,\frac{2\sigma^2-1}{(\sigma^2-1)^{3/2}}\right]$$

- It can be reabsorbed by the familiar rotation from v_1, v_2, b_J to $\tilde{u}_1$, $\tilde{u}_2$, b_e plus a retarded time translation $U \mapsto U + \delta U$.
- Working with the latter set of variables, we can focus on

$$W^{\text{eik}} = \mathcal{A}_0 + \mathcal{B}_1 + \frac{i}{2}(c_1 + c_2)\,.$$

Radiative Functions (Point Particles)

$$\frac{\mathcal{E}}{\pi} = f_1 + f_2 \log\frac{\sigma+1}{2} + f_3 \frac{\sigma \operatorname{arccosh}\sigma}{2\sqrt{\sigma^2-1}}\,, \qquad \frac{\mathcal{C}}{\pi} = g_1 + g_2 \log\frac{\sigma+1}{2} + g_3 \frac{\sigma \operatorname{arccosh}\sigma}{2\sqrt{\sigma^2-1}}$$

$$f_1 = [210\sigma^6 - 552\sigma^5 + 339\sigma^4 - 912\sigma^3 + 3148\sigma^2 - 3336\sigma + 1151]/[48(\sigma^2-1)^{3/2}]$$

$$f_2 = [-35\sigma^4 + 60\sigma^3 - 150\sigma^2 + 76\sigma - 5]/(8\sqrt{\sigma^2-1})$$

$$f_3 = [(2\sigma^2-3)\left(35\sigma^4 - 30\sigma^2 + 11\right)]/[8\left(\sigma^2-1\right)^{3/2}]$$

$$g_1 = [105\sigma^7 - 411\sigma^6 + 240\sigma^5 + 537\sigma^4 - 683\sigma^3 + 111\sigma^2 + 386\sigma - 237]/[24(\sigma^2-1)^2]$$

$$g_2 = [35\sigma^5 - 90\sigma^4 - 70\sigma^3 + 16\sigma^2 + 155\sigma - 62]/[4(\sigma^2-1)]$$

$$g_3 = -[(2\sigma^2-3)\left(35\sigma^5 - 60\sigma^4 - 70\sigma^3 + 72\sigma^2 + 19\sigma - 12\right)]/[4\left(\sigma^2-1\right)^2]$$

Digression: From the Deflection Angle to the Precession Angle

We introduce the effective potential $V(r)$

$$p^2 = p_r^2 + \frac{J^2}{r^2} + V(r)\,, \qquad V(r) = -\left(\frac{G}{r} f_1 + \frac{G^2}{r^2} f_2 + \frac{G^3}{r^3} f_3 + \cdots\right)$$

to extract information about the bound system as well.

- Matching to the **conservative** PM deflection angle, one can fix f_1, f_2, f_3. E.g. in GR, [Bern et al. '19, Damour '20]

$$f_1 = 4m_1^2 m_2^2(\sigma^2 - \tfrac{1}{2})/E\,, \qquad f_2 = \frac{3}{2}(m_1 + m_2)m_1^2 m_2^2(5\sigma^2 - 1)/E\,,$$

- Analytically continuing to $\sigma < 1$ (**bound case**) and working in the Post-Newtonian limit $v_\infty = \sqrt{1-\sigma^2} \to 0$ for fixed $\alpha \equiv G m_1 m_2/(J v_\infty)$ matches the corresponding orders in [Blanchet '13]

$$\Delta\Phi = -2\pi + 2J\int_{r_-}^{r_+} \frac{dr}{r^2\sqrt{p^2 - \frac{J^2}{r^2} - V(r)}} = 3v_\infty^2\alpha^2 - \frac{3}{4}v_\infty^4\alpha^2\left[2\nu - 5 + 5\alpha^2(2\nu - 7)\right]$$