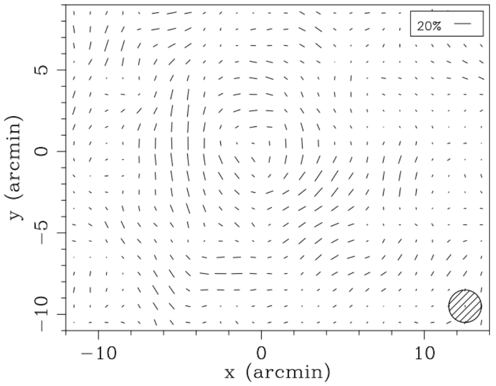
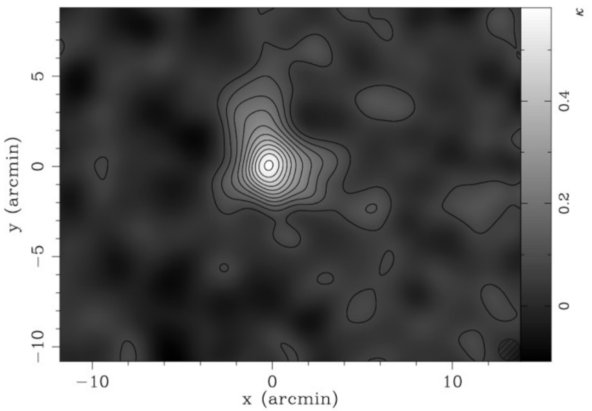
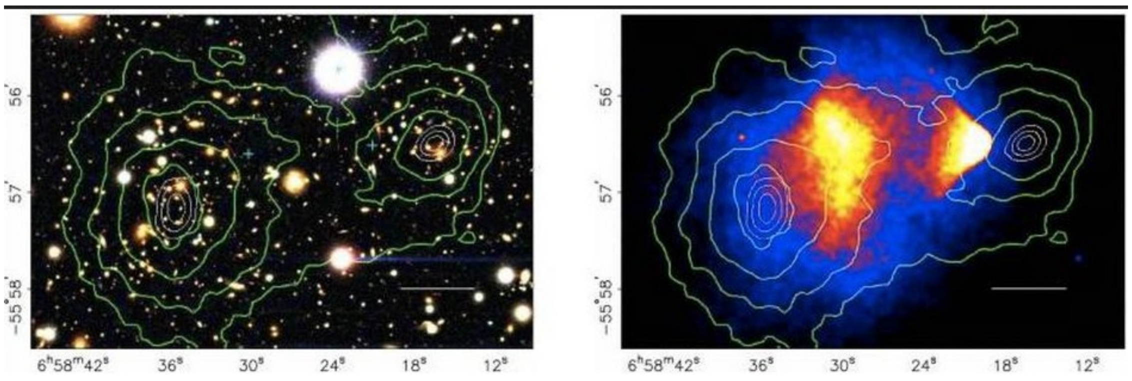
Cluster of galaxies



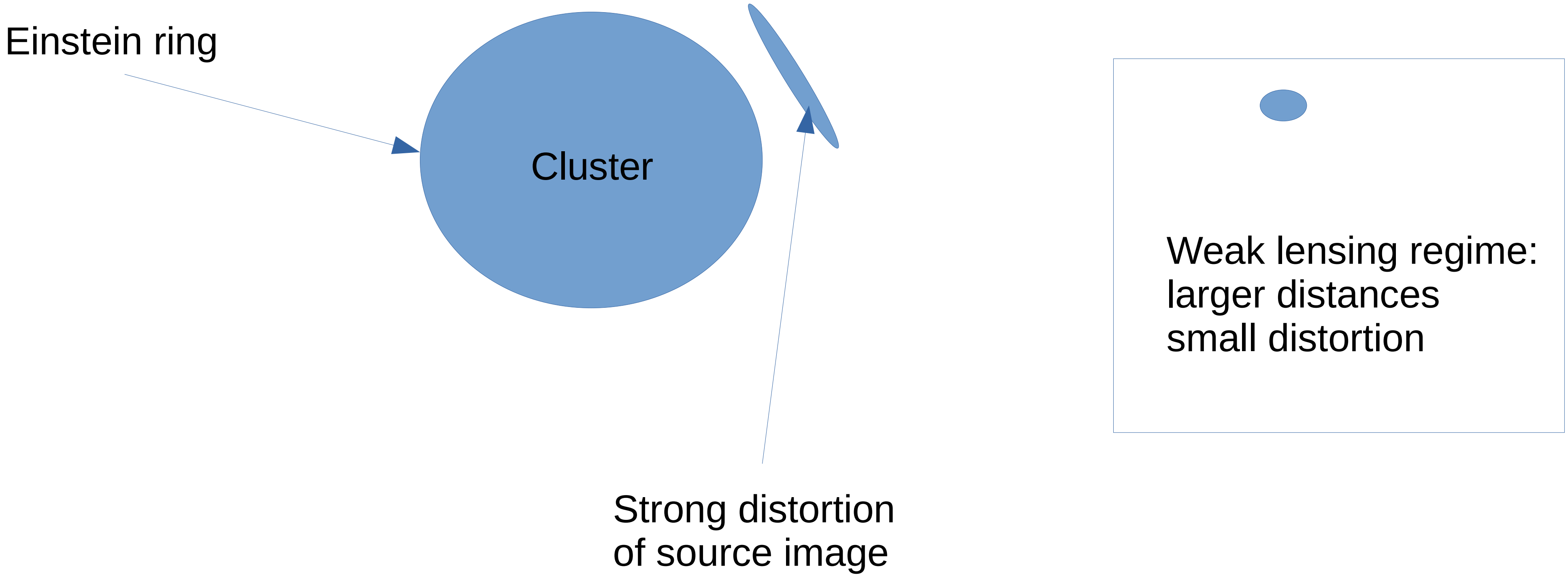
The weak lensing method

Basic ideas
Shift-Jacobian
Shear-moments relation
From shear to convergence
Some applications
Practical problems
Strong-lensing/weak-lensing combination

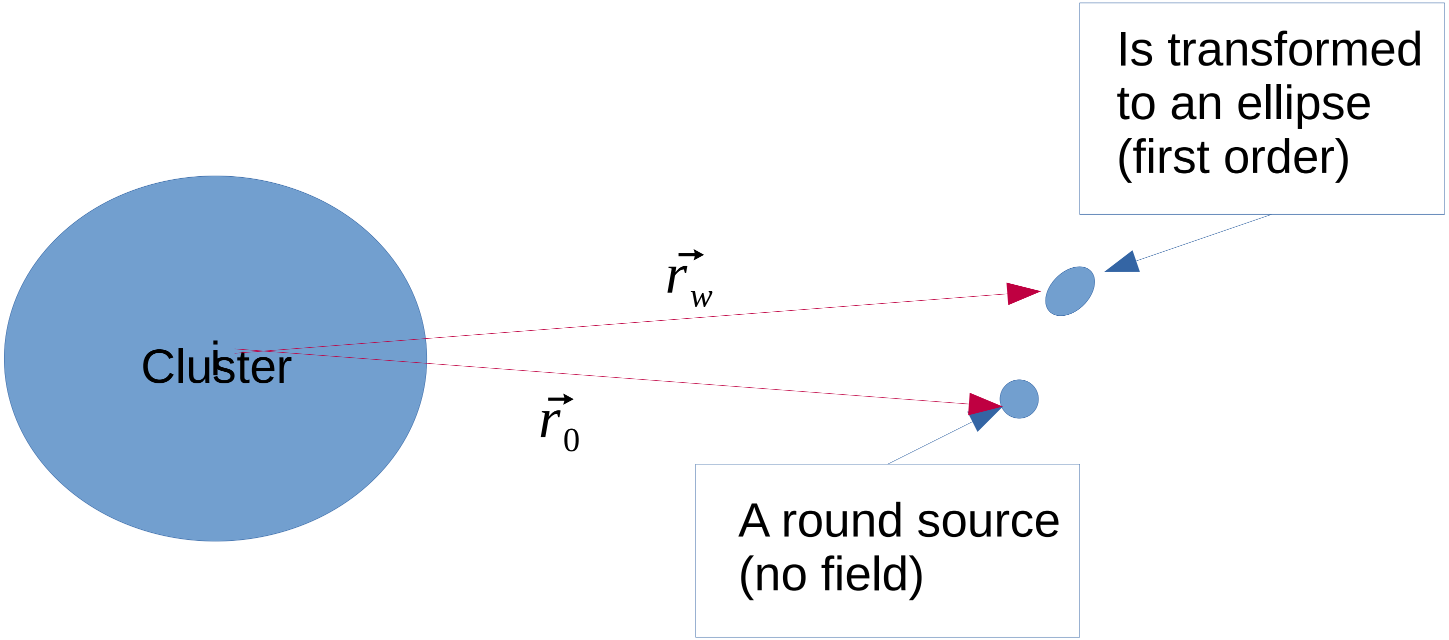




The weak lensing regime



The weak lensing regime



Additionally the source position is shifted: $\vec{r}_0 \neq \vec{r}_w$

The weak lensing regime

The lens equation: $\vec{r}_S = \vec{r} - \vec{\nabla}\phi$

In the weak-field regime we make a local approximation of the lens equation near the source position $\vec{r}_0$

Local coordinates in the source plane: $\vec{r}_S = \vec{r}_0 + \delta\vec{r}_S$

Local coordinates in the lens plane: $\vec{r} = \vec{r}_0 + \delta\vec{r}$

Whe will expand the potential near the source center $\vec{r}_0$

$\phi \simeq \phi(\vec{r}_0 + \delta\vec{r})$ to order 2 with: $\vec{r}_S = \vec{r}_0 + \delta\vec{r}_S$

$$\vec{r}_S = \vec{r} - \vec{\nabla}\phi$$

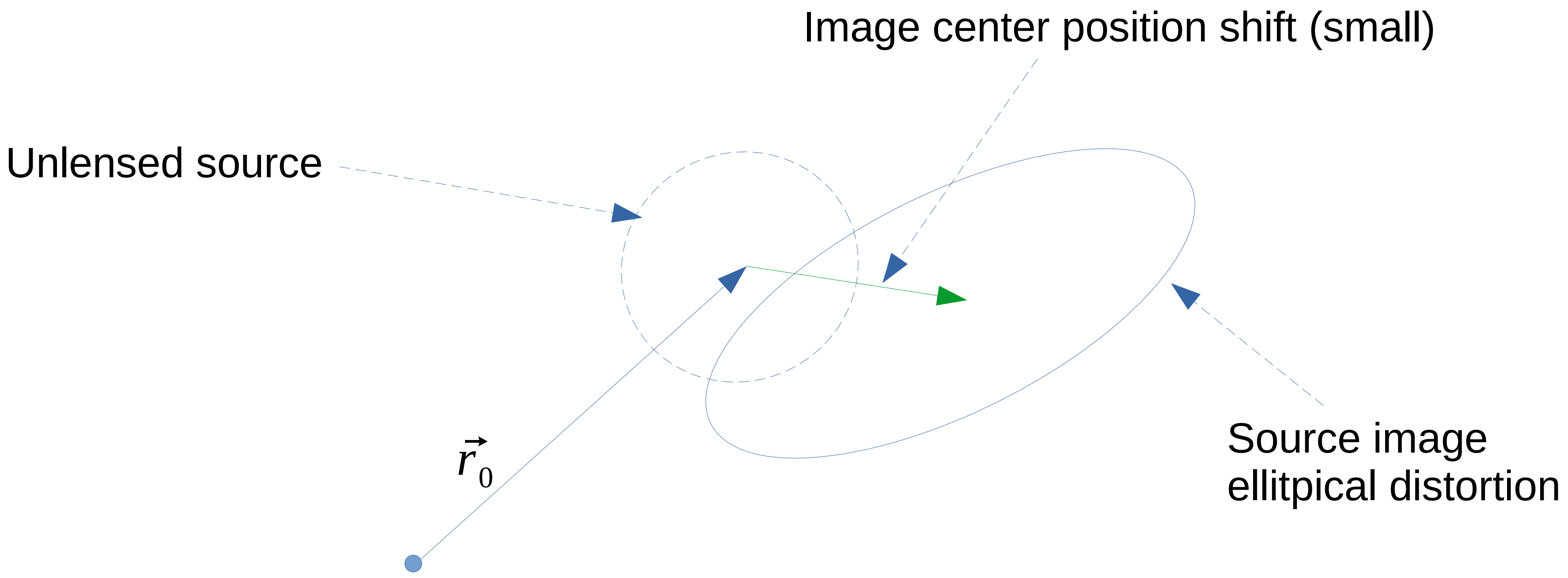


Image center position shift (small)
Unlensed source
$\vec{r}_0$
Source image
ellitpical distortion

First order effect: global shift

$$\phi(\vec{r}_0+\delta\vec{r})=\phi_0+\phi_1\,dx+\phi_2\,dy$$

$$\phi_i=\left[\frac{\delta\phi}{\delta x_i}\right]_{[\vec{r}=\vec{r}_0]}$$

Going back to the lens equation:

$$\begin{cases} x_S=x-\phi_{,x}=-\phi_1+x_0+dx \\ y_S=y-\phi_{,y}=-\phi_2+y_0+dy \end{cases}$$

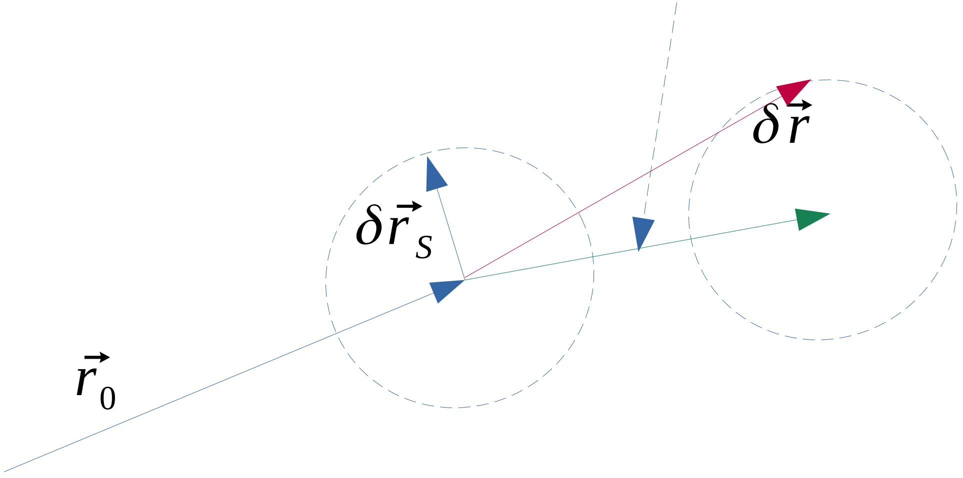
$$\vec{r}=\vec{r}_0+\delta\vec{r}$$

$$\vec{\nabla}\phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

$$\vec{r}_S=-\vec{\nabla}\phi+\vec{r}_0+\delta\vec{r}$$

$$\vec{r}_S = -\vec{\nabla}\phi + \vec{r}_0 + \delta\vec{r} \quad with \quad \vec{r}_S = \vec{r}_0 + \delta\vec{r}_S$$

$$\delta\vec{r}_S = \underbrace{-\vec{\nabla}\phi + \delta\vec{r}}_{\text{Shift}}$$



The shift affect the distribution on objects around the lens

We expand to order 2

$$\phi(\vec{r}_0+\delta\vec{r})=\phi_0+\vec{\nabla}\phi.\delta\vec{r}+\frac{1}{2}\phi_{11}dx^2+\phi_{12}dx\,dy+\frac{1}{2}\phi_{22}dy^2 \quad \text{with:} \quad \phi_{ij}=\left[\frac{\delta^2\phi}{\delta x_i\,\delta x_j}\right]_{\vec{r}=\vec{r}_0}$$

Going back to the lens equation:

$$\begin{cases} x_S=x-\phi_{,x}=-\phi_1+x_w+dx-\phi_{11}dx-\phi_{12}dy \\ y_S=y-\phi_{,y}=-\phi_2+y_w+dy-\phi_{22}dy-\phi_{12}dx \end{cases}$$

$$M=\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{12} & \phi_{22} \end{pmatrix}$$

↓

$$\vec{r}_S=-\vec{\nabla}\phi+\vec{r}_0+\delta\vec{r}-M\,\delta\vec{r}$$

$$\vec{r}_S = -\vec{\nabla}\phi + \vec{r}_0 + \delta\vec{r} - M\,\delta\vec{r} \quad \textit{with} \quad \vec{r}_S = \vec{r}_0 + \delta\vec{r}_S$$

$$\delta\vec{r}_S = -\vec{\nabla}\phi + \delta\vec{r} - M\,\delta\vec{r}$$

We introduce the centered (shift free) coordinate $\delta\vec{u} = \delta\vec{r} - \vec{q}_0$

here $\vec{q}_0$ is the shift at order 2

$$\delta\vec{r}_S = \underbrace{-\vec{\nabla}\phi + \vec{q}_0 - M\,\vec{q}_0}_{} + \delta\vec{u} - M\,\delta\vec{u}$$

No shift must be = 0

Introducing $J = \begin{pmatrix} 1-\phi_{11} & -\phi_{12} \\ -\phi_{12} & 1-\phi_{22} \end{pmatrix}$

$$\delta\vec{r}_S = -\vec{\nabla}\phi + \vec{q}_0 - M\vec{q}_0 + \delta\vec{u} - M\,\delta\vec{u} = -\vec{\nabla}\phi + J\vec{q}_0 + J\,\delta\vec{u}$$

Image distortion ($J\,\delta\vec{u}$)

No shift → $\vec{q}_0 = J^{-1}\vec{\nabla}\phi \simeq \vec{\nabla}\phi$

Finally in the centered coordinates: → $\delta\vec{r}_S = J\,\delta\vec{u}$

$$\delta\vec{r}_S = J\,\delta\vec{u}$$

$$J = \begin{pmatrix} 1-\phi_{11} & -\phi_{12} \\ -\phi_{12} & 1-\phi_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_S}{\partial x} & \frac{\partial y_S}{\partial x} \\ \frac{\partial x_S}{\partial y} & \frac{\partial y_S}{\partial y} \end{pmatrix}$$

$$x_s = x - \frac{\partial \phi}{\partial x}$$

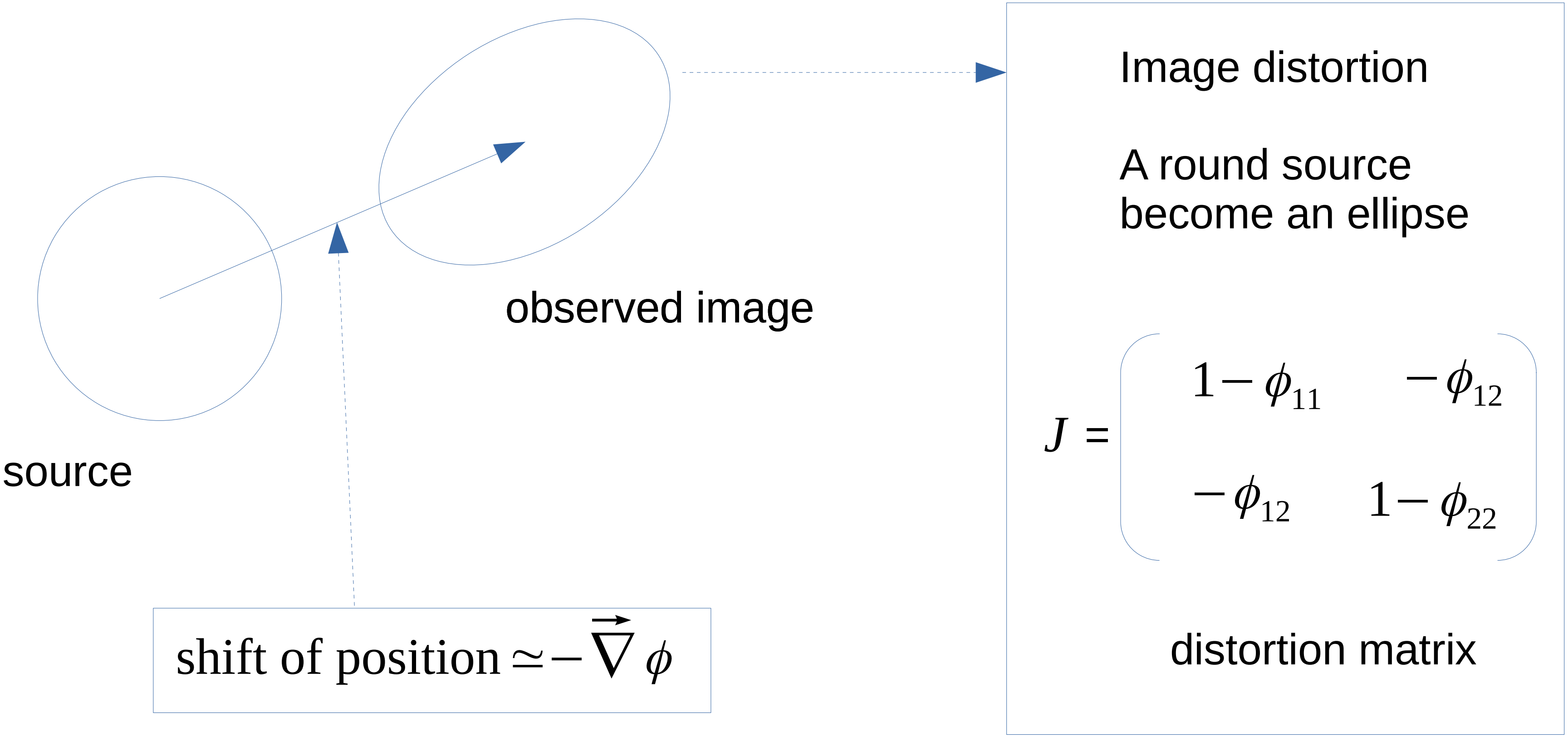
$$y_s = y - \frac{\partial \phi}{\partial y}$$

$$\phi_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}$$

$$(x_1 \equiv x \quad \text{and} \quad x_2 \equiv y)$$

The effect of the second term is a distortion of the source
An initially round source is transformed to an elliptical one

J is the Jacobian matrix


source
observed image
shift of position $\simeq -\vec{\nabla}\phi$
Image distortion
A round source
become an ellipse
$J = \begin{pmatrix} 1-\phi_{11} & -\phi_{12} \\ -\phi_{12} & 1-\phi_{22} \end{pmatrix}$
distortion matrix

Re-writing the Jacobian by introducing the convergence κ

$$J = \begin{pmatrix} 1-\phi_{11} & -\phi_{12} \\ -\phi_{12} & 1-\phi_{22} \end{pmatrix} = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix} = (1-\kappa) \begin{pmatrix} 1-g_1 & -g_2 \\ -g_2 & 1+g_1 \end{pmatrix}$$

$$\kappa = \frac{1}{2}(\phi_{11}+\phi_{22}) \qquad \gamma_1 = \frac{1}{2}(\phi_{11}-\phi_{22}) \qquad \gamma_2 = \phi_{12}$$

$$g_i = \frac{\gamma_i}{(1-\kappa)}$$ (reduced shear)

What is observable

$$J = \begin{pmatrix} 1-\phi_{11} & -\phi_{12} \\ -\phi_{12} & 1-\phi_{22} \end{pmatrix} = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix} = (1-\kappa) \begin{pmatrix} 1-g_1 & -g_2 \\ -g_2 & 1+g_1 \end{pmatrix}$$

We don’t observe the effect of convergence
It represents an absolute unknown scale

We observe only the reduced shear

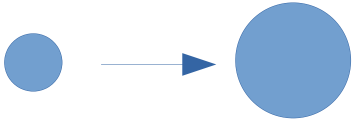
$$g_i = \frac{\gamma_i}{(1-\kappa)} \simeq \gamma_i$$

Reduced shear and shear equivalent in the weak lensing regime

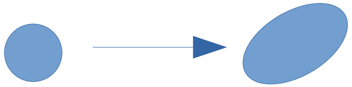
Interpretation

$$J = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$$

Effect of convergence



Effect of shear



What we observe and measure

The second order moments

$$Q_{ij} = \int \Sigma(\vec{r})\, x_i\, x_j\, d^2 x$$

And the associated matrix

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{pmatrix}$$

Transforming the second order moment matrix

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{pmatrix}$$

Must look like the observable part of J

$$J_o = \begin{pmatrix} 1-g_1 & g_2 \\ g_2 & 1+g_1 \end{pmatrix}$$

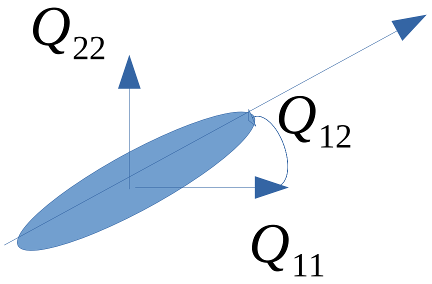
$$Q = Q_0 \begin{pmatrix} 1-\alpha & \beta \\ \beta & 1+\alpha \end{pmatrix}$$

$$\alpha = \frac{Q_{22} - Q_{11}}{Q_{11} + Q_{22}}$$

$$\beta = \frac{2\,Q_{21}}{Q_{11} + Q_{22}}$$

$$(\alpha, \beta) \quad \textit{are directly related with} \quad (g_1, g_2)$$

The second order moments are associated with an equivalent ellitpical contour and thus a quadratic form



As a consequence it is useful to represent the effect of shear in terms of quadratic forms and their associated matrix

How does the shear transformation affect the ellitpicities of galaxies ?

$$\alpha = \frac{Q_{22} - Q_{11}}{Q_{11} + Q_{22}} \qquad \beta = \frac{2 Q_{21}}{Q_{11} + Q_{22}}$$

Let say we have some initial value for these 2 parameters
Then we apply a shear transformation

X *represents the coordinate system* $\qquad Y \equiv J_o X \qquad J_o = \begin{pmatrix} 1 - g_1 & g_2 \\ g_2 & 1 + g_1 \end{pmatrix}$

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

The effect of the transformation is to transform the elliptical contour represented by the second order moments into another elliptical contour

An elliptical contour is represented by the quadratic form associated with the moments:

$$q = X^T Q X$$

Introducing $Y = J_0 X$ Q transforms to: $q_s = X^T J_o^T Q J_o X$

Thus $Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{pmatrix}$ Transforms to: $Q_s = J_o^T Q J_o$

With $$J_o = \begin{pmatrix} 1-g_1 & g_2 \\ g_2 & 1+g_1 \end{pmatrix}$$

To first order in g_i and for a circular contour $Q_s = J_o^T J_o$

$$\alpha = \frac{Q_{22}-Q_{11}}{Q_{11}+Q_{22}} = g_1 \qquad \beta = \frac{2Q_{21}}{Q_{11}+Q_{22}} = g_2$$

For a general non-circular contours $Q_s = J_o^T Q \, J_o$

Applying a shear transformation leads to:

$\alpha_s = \alpha + \mu \, g_1$ $\beta_s = \beta + \mu \, g_2$ $With: \;\; \mu \simeq 1$

$\mu \simeq 1 - \langle e \rangle^2$ $With: \;\; \langle e \rangle \simeq 0.25$ ➔ $\mu \simeq 1$

Kaiser & Squires (1993)

By averaging on a distribution of randomly oriented sources

$$\langle \alpha \rangle = \left\langle \frac{Q_{22} - Q_{11}}{Q_{11} + Q_{22}} \right\rangle = 0 \qquad \langle \beta \rangle = \left\langle \frac{2\, Q_{12}}{Q_{11} + Q_{22}} \right\rangle = 0$$

We have: $\langle \alpha_s \rangle \simeq g_1$ $\langle \beta_s \rangle \simeq g_2$

In practice some complications may occur in the statistics of orientations and ellipticities

Using complex ellipticity and shear

Defining the complex ellipticity:

Bartelmann & Schneider (2001)

$$\epsilon = \frac{Q_{11} - Q_{22} + 2 I Q_{12}}{Q_{11} + Q_{22} + 2\sqrt{Q_{11} Q_{22} - Q_{12}^2}}$$

$$\epsilon^s = \frac{\epsilon + g}{1 + g^* \epsilon} \simeq \epsilon + g$$

$$g = g_1 + I g_2 \simeq \gamma_1 + I \gamma_2$$

In the weak lensing regime

g^* : Complex conjugate

On average the statistical mean of the complex ellipticity for unlensed sources should be zero

Once the shear is known and a shear map is obtained

The simplest approach is to fit a model for the potential

This model must reproduce the shear map through a least-square minimization

Making a general model free map
Estimating the convergence from the shear

$$\gamma_1 = \frac{1}{2}(\phi_{11} - \phi_{22}) \quad ; \quad \gamma_2 = \phi_{12} \quad ; \quad \gamma = \gamma_1 + I\,\gamma_2$$

With: $$\phi = \frac{1}{\pi} \int \kappa(\vec{u}) \log(|\vec{r} - \vec{u}|)\, d^2\vec{u}$$

Then: $$\gamma(\vec{r}) = \frac{1}{\pi} \int \kappa(\vec{u})\, \chi(\vec{r} - \vec{u})\, d^2\vec{u}$$ and $$\chi(\vec{r}) = \frac{x^2 - y^2 - 2\,Ixy}{|r|^4}$$

Estimating the convergence from the shear

The integral: $$\gamma(\vec{r}) = \frac{1}{\pi}\int \kappa(\vec{u})\,\chi(\vec{r}-\vec{u})\,d^2\vec{u}$$

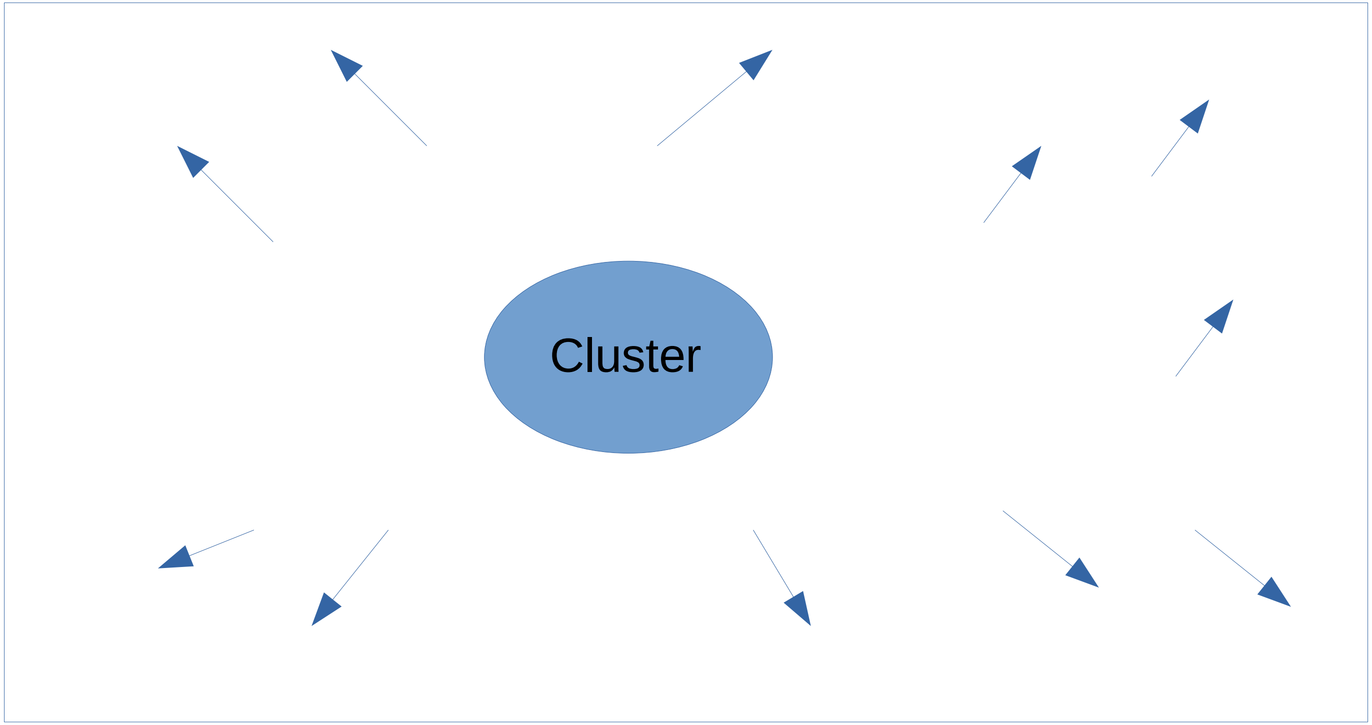
Has an inversion formula (see Kaiser & Squires 1993)

$$\kappa(\vec{r}) - \kappa_0 = \frac{1}{\pi}\int \gamma(\vec{u})\,\chi^*(\vec{r}-\vec{u})\,d^2\vec{u}$$

Thus basically the convergence is obtained by convolving the shear with a kernel

First we build a shear map

The moments: Q_{ij} are estimated from the data by using background galaxies



The shear is estimated from the moments: $\epsilon = \frac{Q_{11} - Q_{22} + 2IQ_{12}}{Q_{11} + Q_{22} + 2\sqrt{Q_{11}Q_{22} - Q_{12}^2}} \simeq \epsilon_0 + g$

This map is convolved with the kernel

$$\kappa(\vec{r}) - \kappa_0 = \frac{1}{\pi} \int \gamma(\vec{u})\, \chi^*(\vec{r} - \vec{u})\, d^2\vec{u}$$

The convergence (projected surface density) is obtained

There are many approach to reconstruct the convergence from the shear

Not necessary by using the inversion formula we just presented

Since the shear is related to the convergence by a convolution
Fourier methods are natural

But other means like for instance maximum entropy may be also used
To recover the convergence

$$\gamma(\vec{r}) = \frac{1}{\pi} \int \kappa(\vec{u}) \chi(\vec{r} - \vec{u}) d^2 \vec{u} \quad ; \quad \chi(\vec{r}) = \frac{x^2 - y^2 - 2Ixy}{|r|^4}$$

Note: note all sources around the cluster are at the same distance

We must have spectroscopy and redshift data

1) to eliminate foreground objects (galaxies closer to us than the lens)

2) to estimate the distances background sources (galaxies behind the lens)

When no spectroscopy is available photometric redshifts are used instead to estimate
The redshifts

$$\vec{\beta} = \vec{\theta} - \frac{D_{LS}}{D_L}\vec{\alpha} \quad \rightarrow \quad \vec{r}_S = \vec{r} - \frac{D_{LS}}{D_S D_L}\vec{\hat{\alpha}} = \theta - \vec{\nabla}\phi$$

Thus ϕ must be re-scaled as a function of D_S

Example of shear maps and cluster surface

Density reconstruction form the litterature

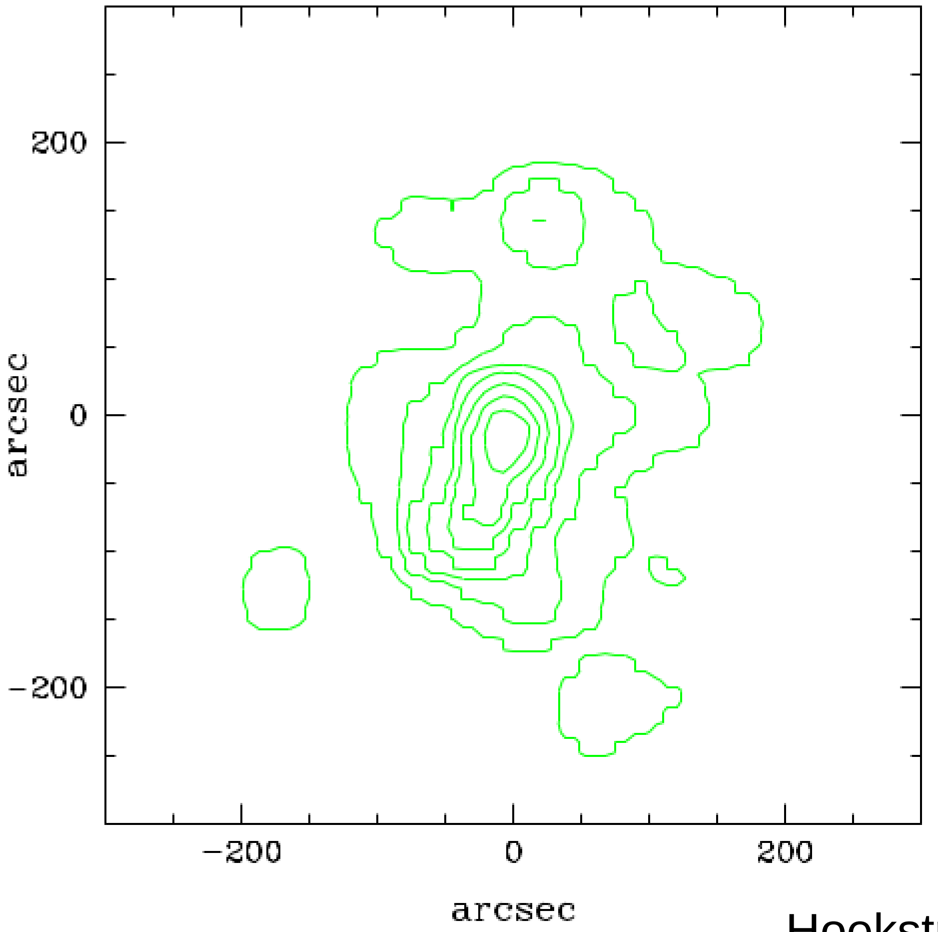
Reconstructions for 3 clusters of galaxies

Cl1358+62 (1998)

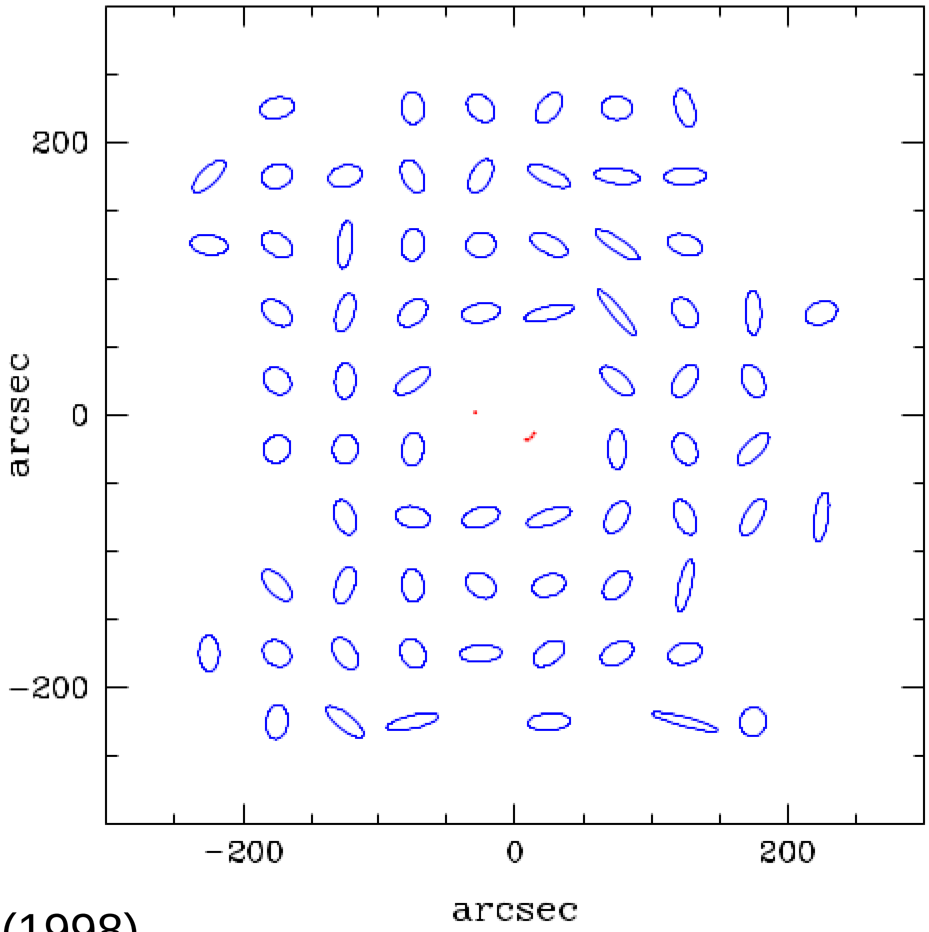
MS1054-03 (2000)

Abell 1689 (2007)

Light distribution in the cluster Cl1358+62

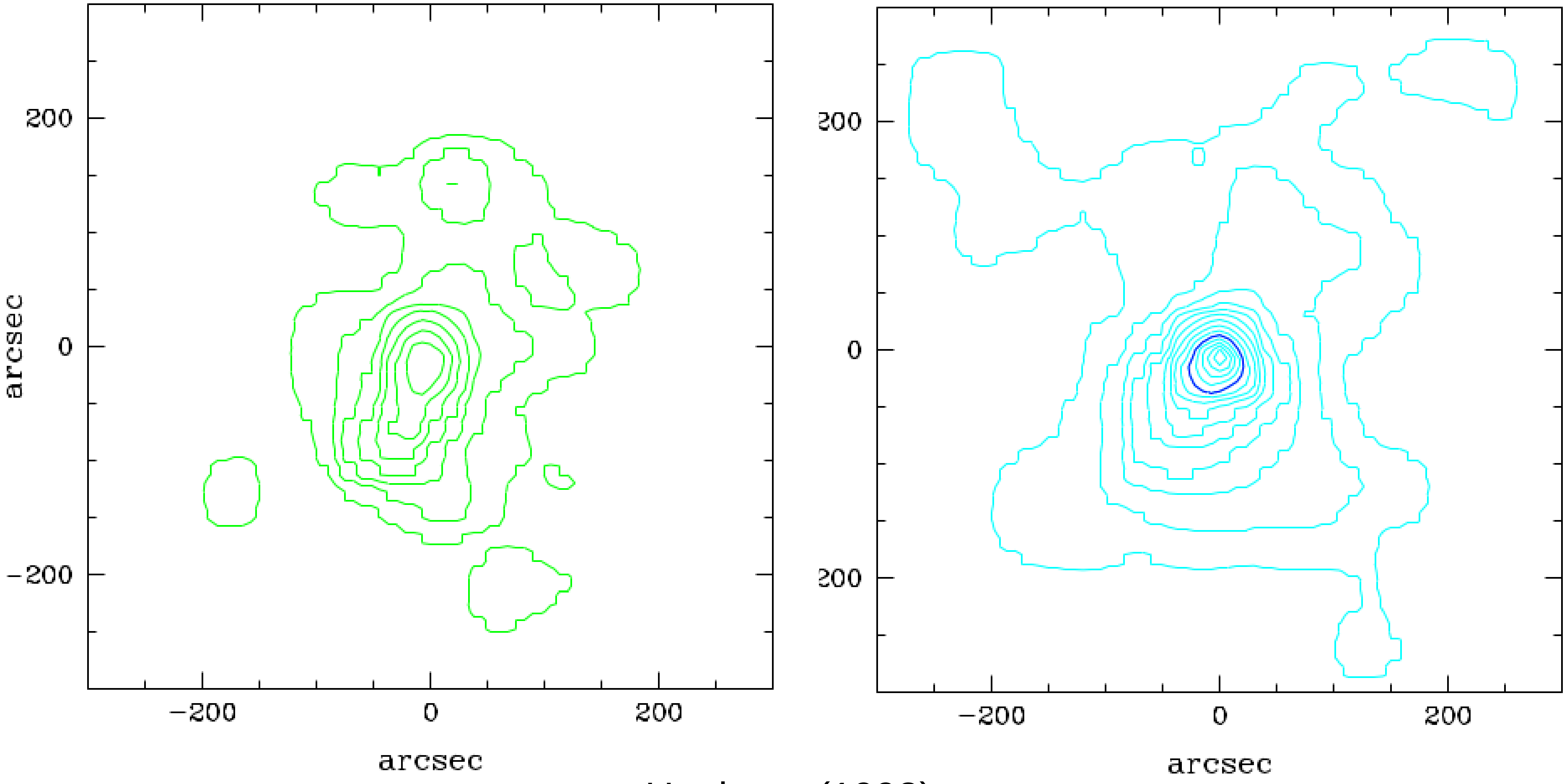


Shear map (equivalent elliptical distortion)



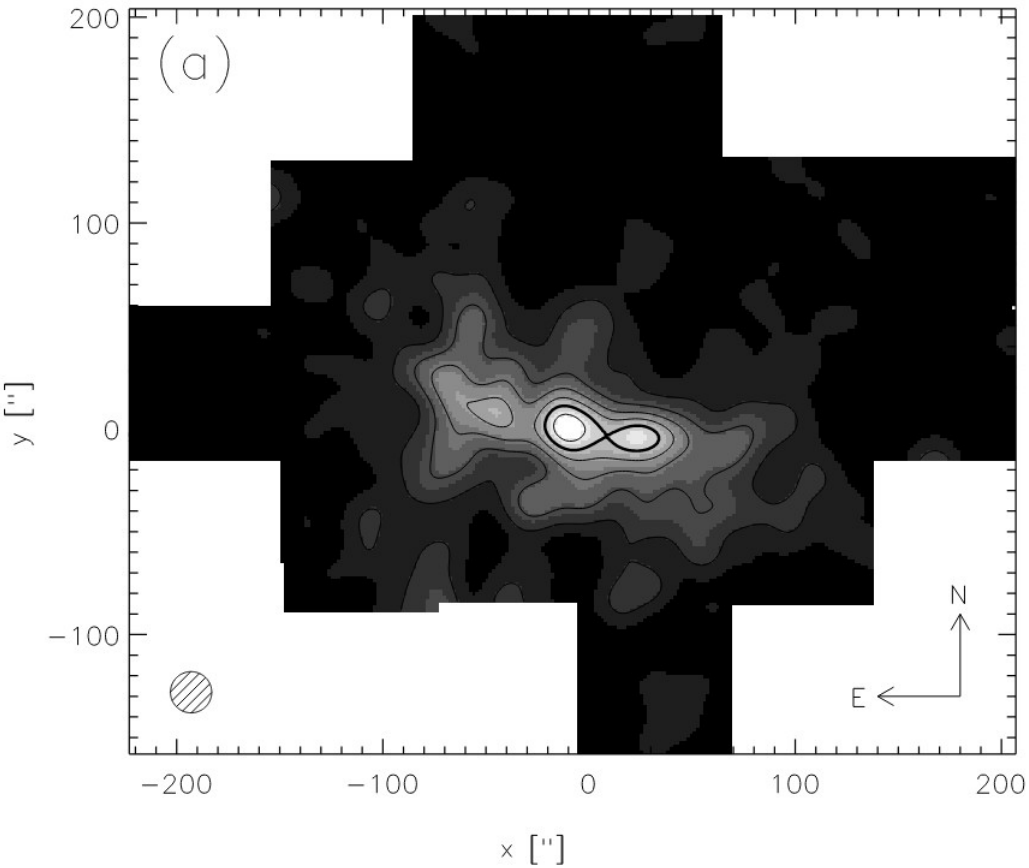
Hoekstra (1998)

Light distribution in the cluster Cl1358+62 Surface density reconstruction

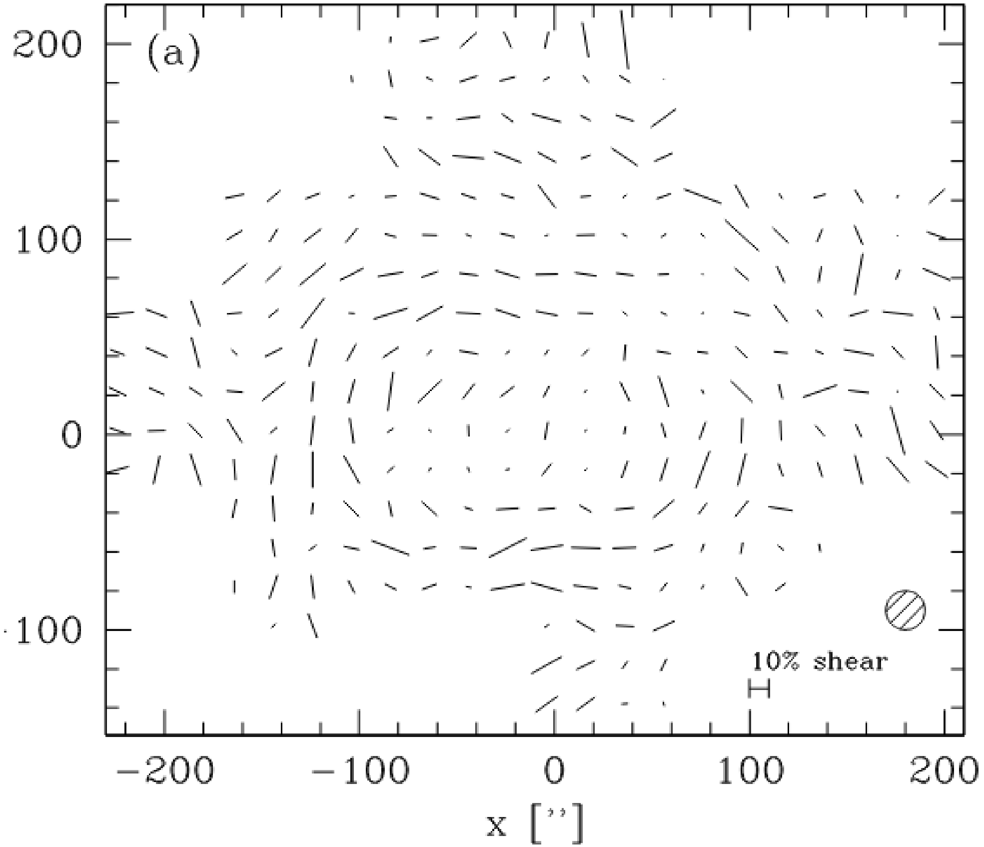


Hoekstra (1998)

Light distribution in MS1054-03

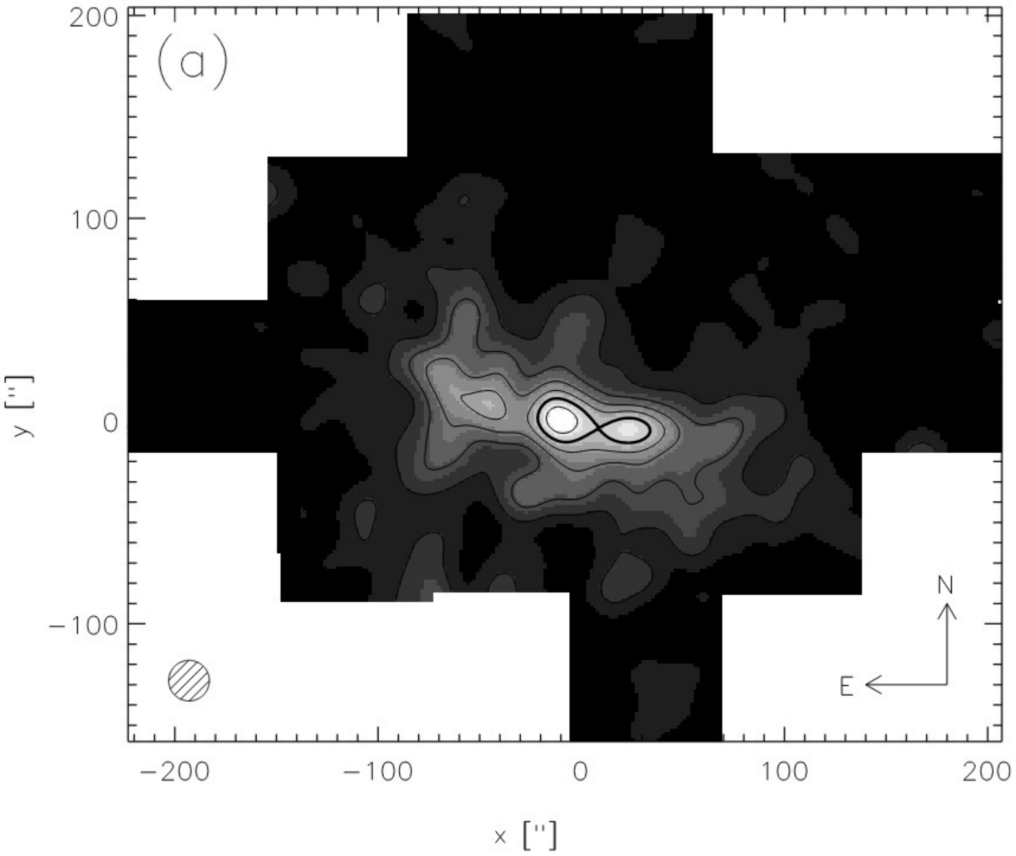


Shear map (vector representation)

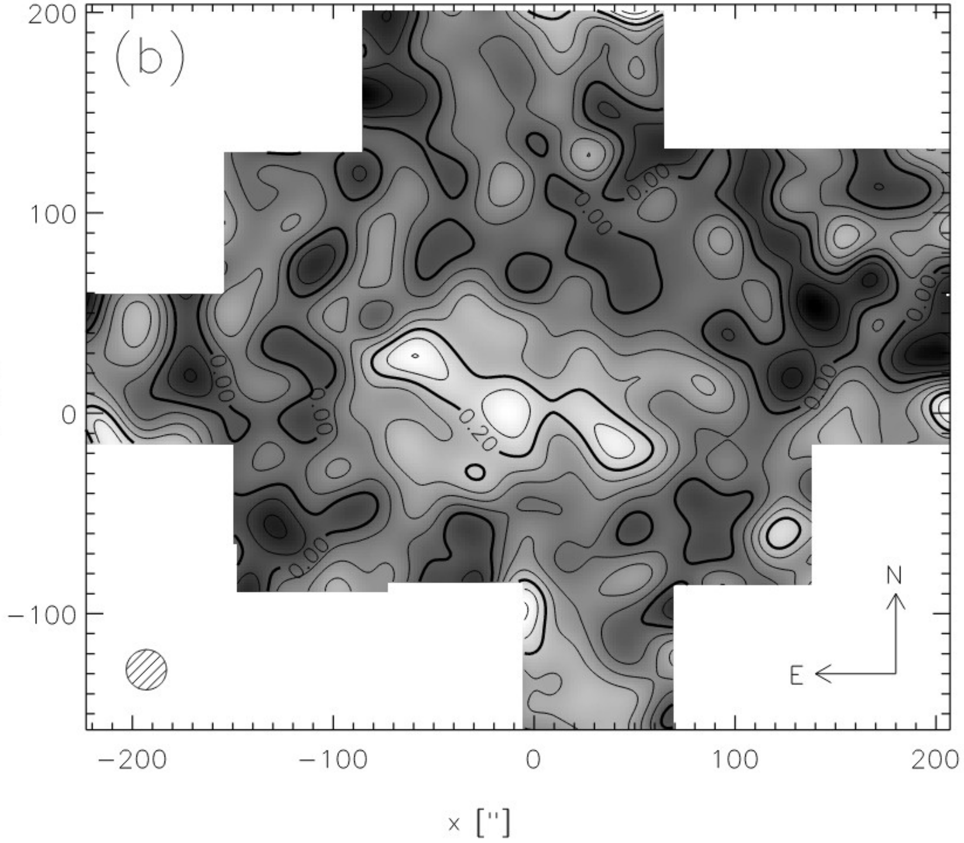


Hoekstra (2000)

Light distribution in MS1054-03



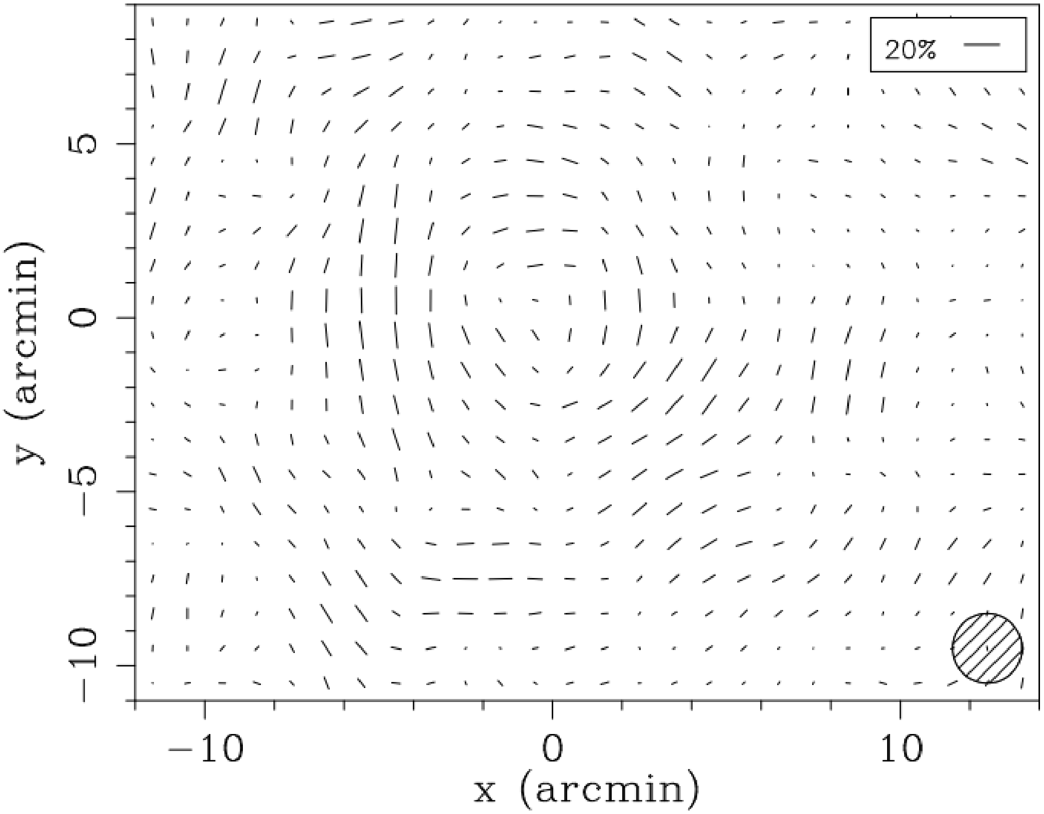
Surface brightness reconstruction



Hoekstra (2000)

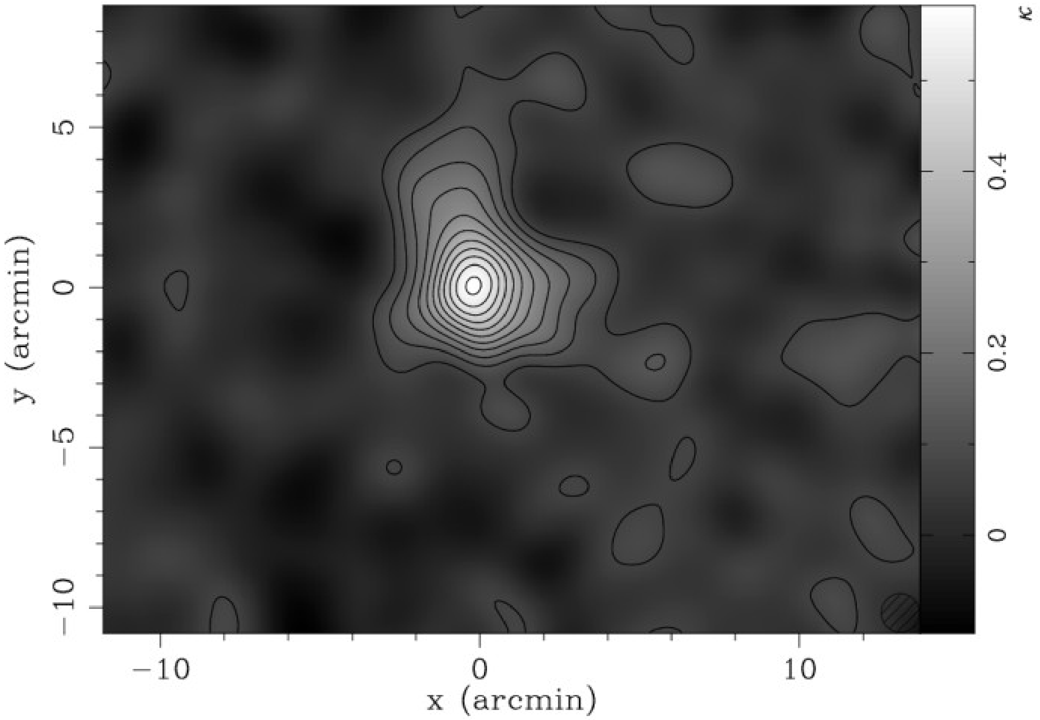
Light distribution in the cluster Abell 1689

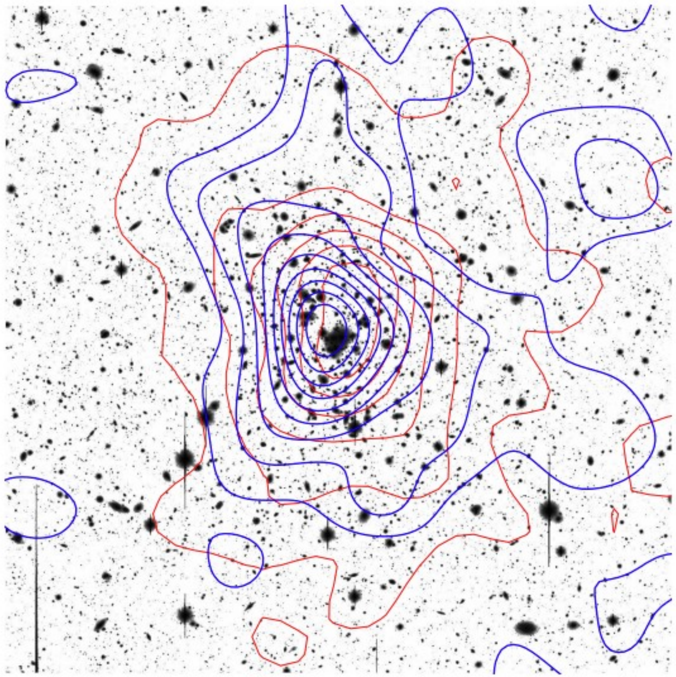
Shear Map (vector representation)



Oguri etal. (2007) – obtained with the Subaru telescope

Surface density reconstruction
Oguri etal. (2007)

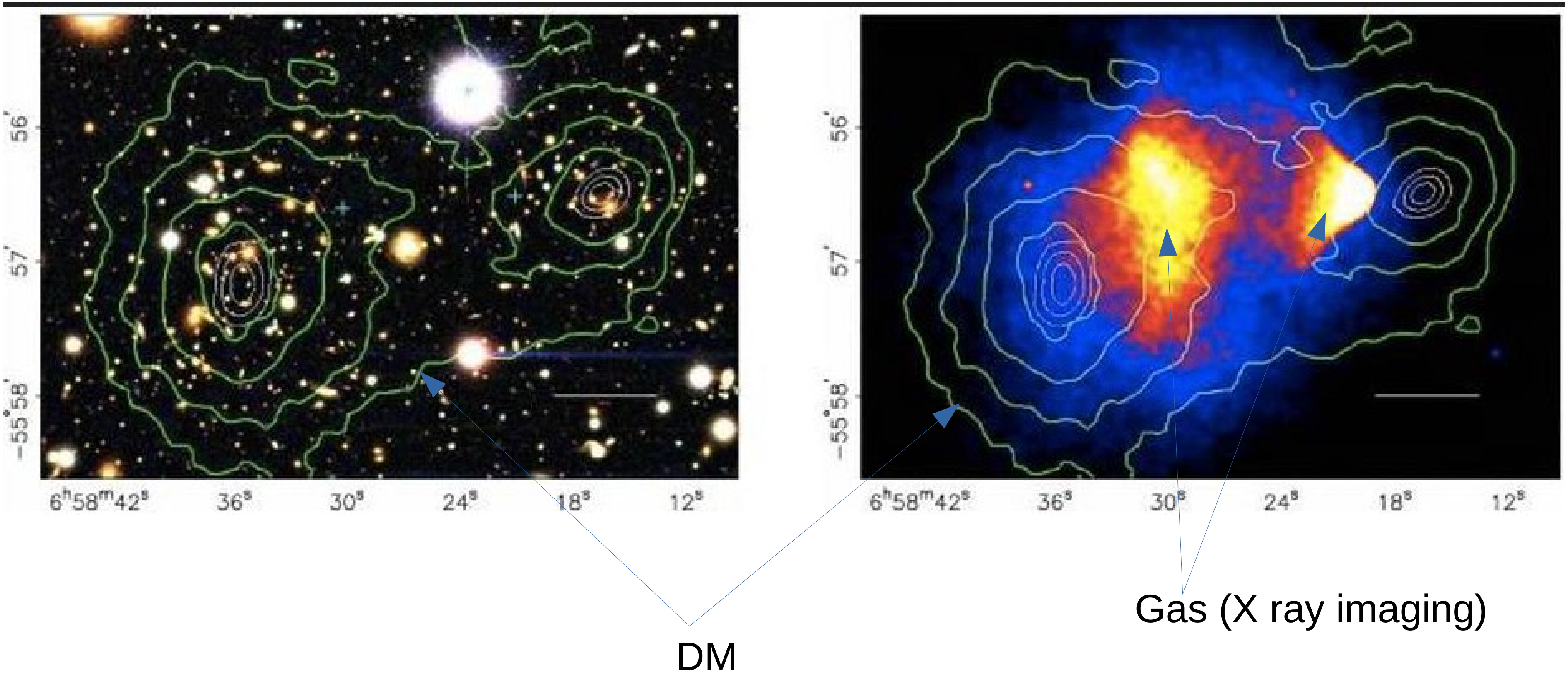




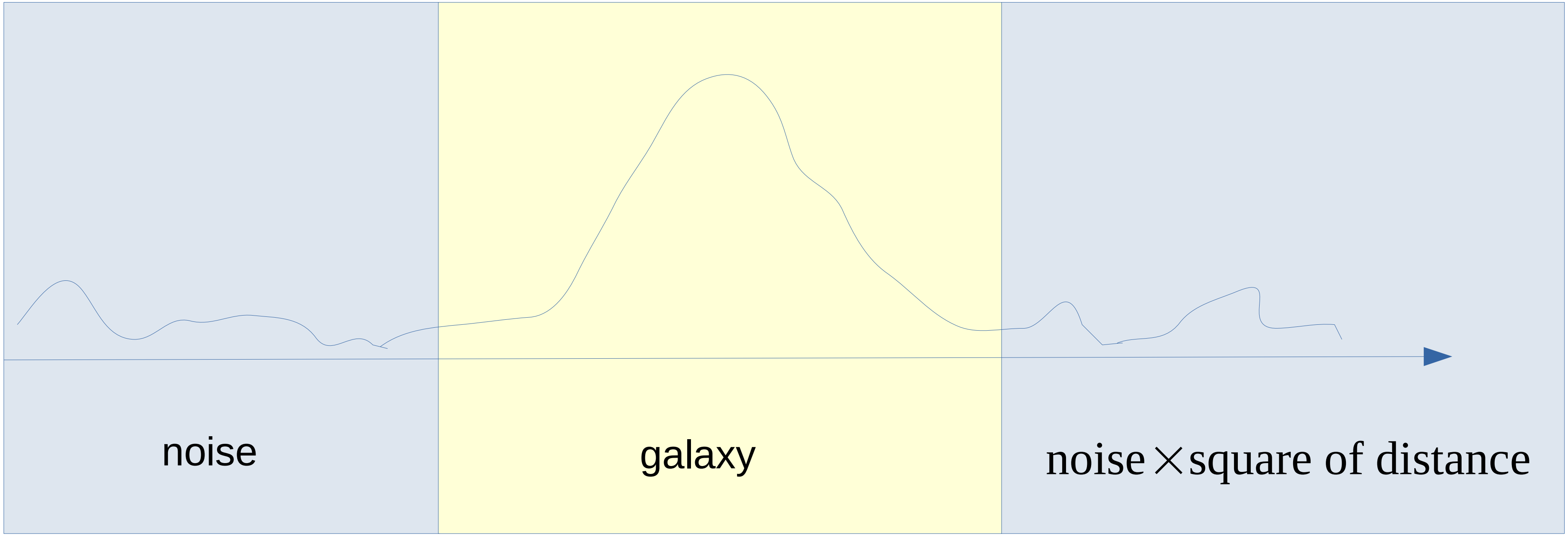
Comparison of contours
Light: red
Weak lensing: blue

Important results from weak lensing: the bullet cluster

The distribution of baryons and DM are different



Practical problems with the estimation of moments



$$Q_{ij} = \int \Sigma(\vec{r})\, x_i x_j\, d^2x \quad \rightarrow \quad \text{is quickly dominated by noise out of the galaxy}$$

Generic problem: the integral does not converge...

Practical problems with the estimation of moments

Generic problem: the integral does not converge…
→ solution use some weight function

Second problem: in practice the data are convolved with the PSF

Solution: actual moments are the sum of the galaxy moments +PSF moments

Problem: the PSF may not have converging moments

Solution to the problem of PSF non converging moments:

We estimate the associated quadratic form in another way

For instance fit some generic quadratic function to the data

$$F\left(a_0 x^2 + a_1 x y + a_2 y^2\right)$$

Method: convolve the generic function with the PSF
and then fit the parameters of the quadratic form

In practice we may use Gaussians (F is an exponential)
or any other functions

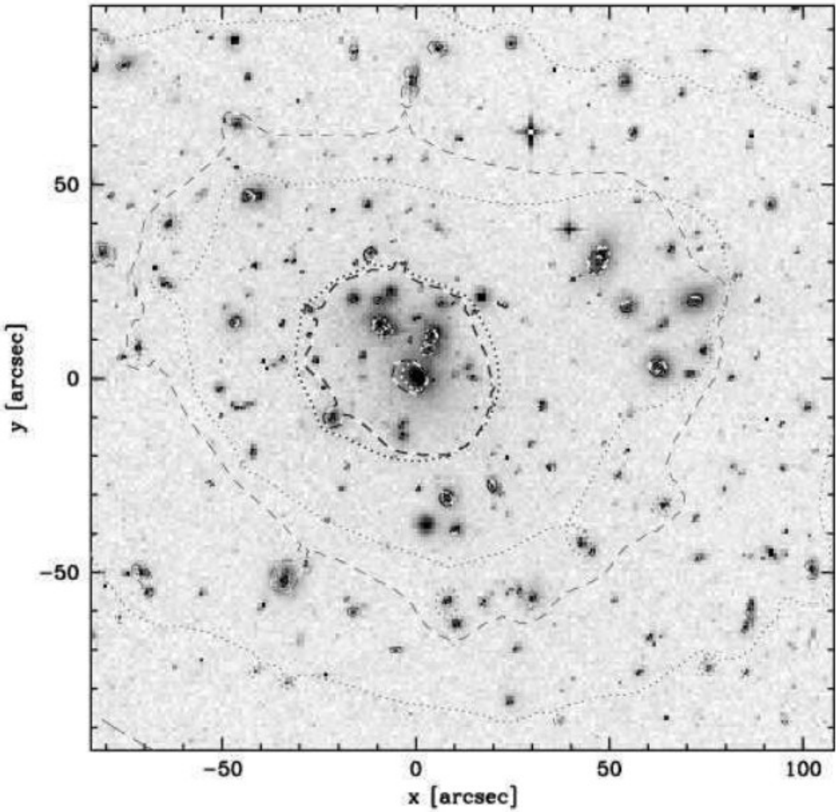
Strong lensing in clusters of galaxies

Reconstruction for parametric potential model
Or general description by the singular perturbative method

For Abell 1689

Halkola etal. 2006
identified 107 mutiples images
And 32 image systems

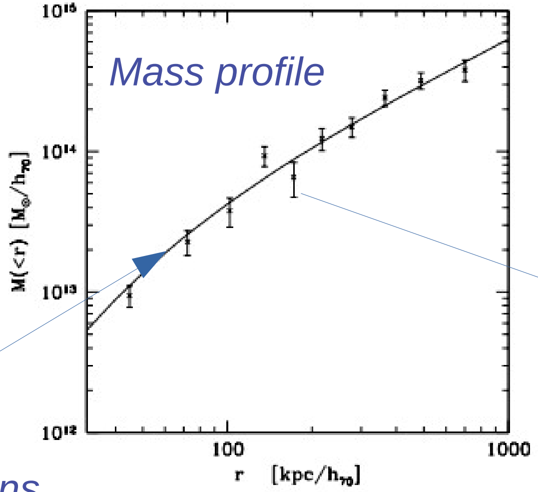
Strong lensing in clusters of galaxies



Surface density solution

For Abell 1689, Halkola etal. 2006

Could reproduce all the images systems by assigning NSIE or NFW dark matter halo's to individual galaxies



Lensing

X ray observations

Strong lensing in clusters of galaxies

Important asset of lensing in clusters:

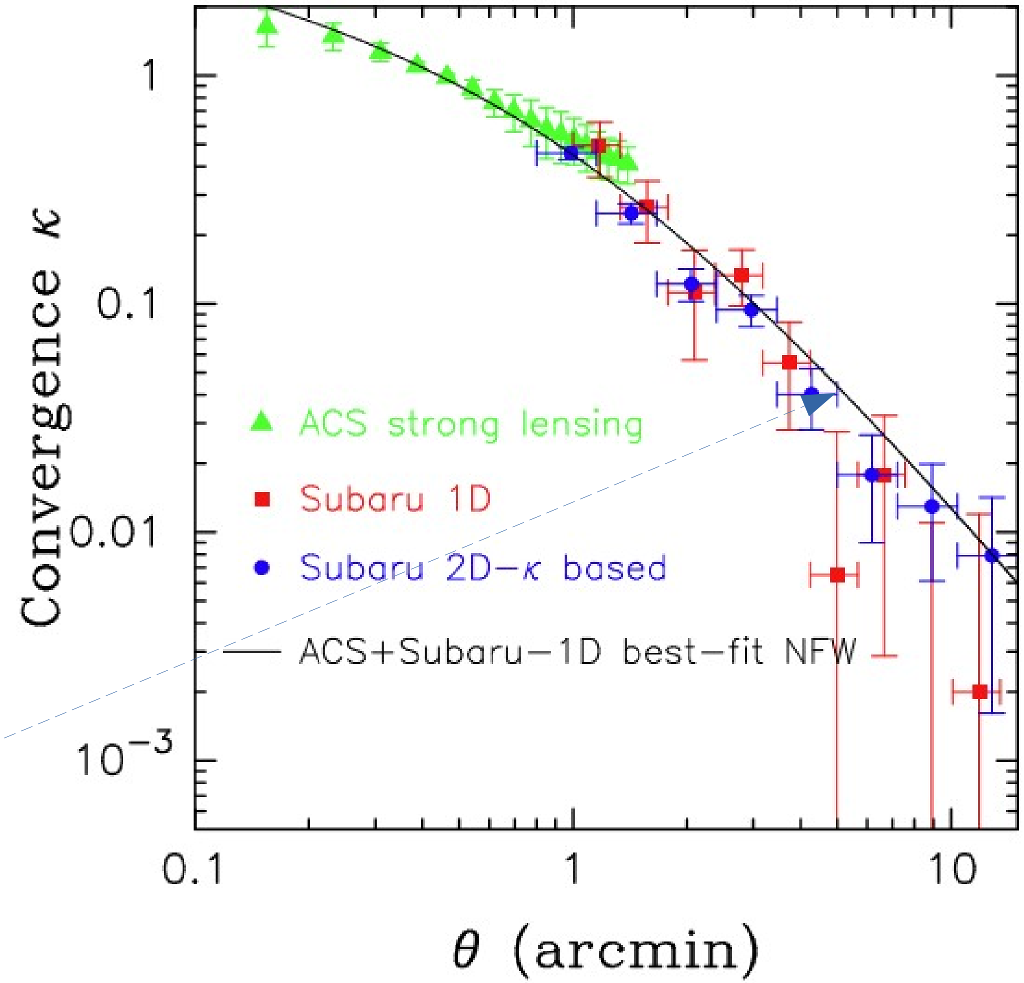
several sources with different
Redshifts (additional constraints)

When combined with weak lensing data
The mass-sheet degeneracy may be broken

See Bradac etal. (2004)

An example of combined strong+weak lensing data

Oguri etal. (2007)
Subaru data



NFW profile

Send me your questions or demand on specific part of the course.
Let me know if you will attend the course at IAP.

alard@iap.fr or christophe.alard@gmail.com

(send before March 1rst)

On March 21rst

I will make a short summary of the course
I will then pose some problems with simple solutions
We will also look at some simple numerical applications and propose some basic programming.

On March 28th

The last session of the course will be dedicated to a discussions on numerical methods and applications to real data. This will be also the time to answer The last questions.