Progress on the definition of asymptotically flat and de Sitter spacetimes

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Paris, April 8th, 2024

Outline

1. The 5 boundaries of Minkowski: "Penrose" versus "Puzzle piece" diagram
2. Unified BMS group acting simultaneously on the 5 boundaries. Boundary conditions consistent with the logarithmic corrections to the subleading soft graviton theorem.
3. Complete set of non-radiative charges : Geroch-Hansen multipoles + generalised BMS + non-stationary multipole moments
4. Properties of quadrupolar linear fields on dS_4: Memory effects, $\Lambda-$BMS transitions, and breaking of the conformal group which invalidates the dS_4/CFT_3 conjecture

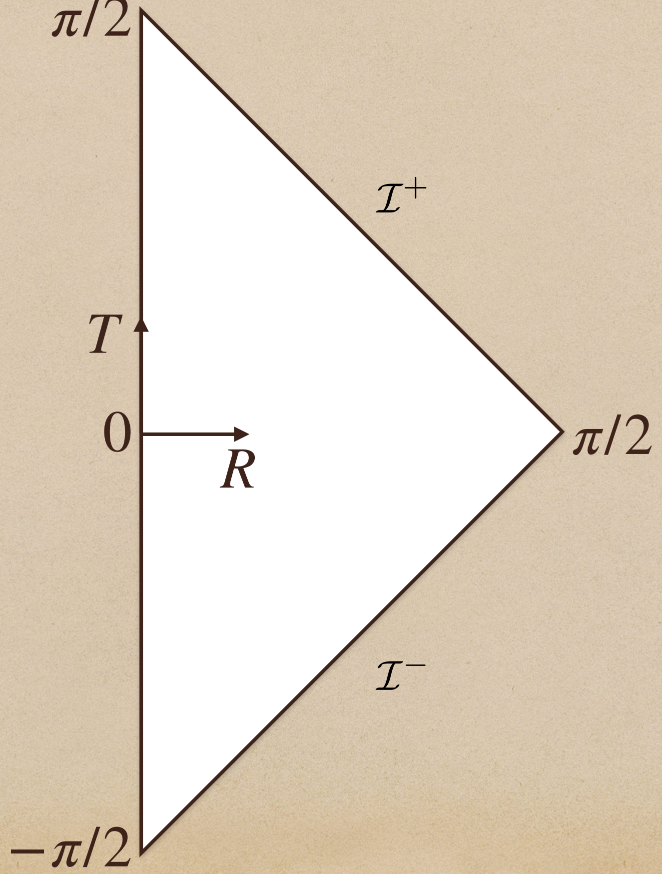
References

- "An asymptotic framework for gravitational scattering" with Samuel Gralla & Hongji Wei, 2303.17124
- "Metric reconstruction from celestial multipoles" with Roberto Oliveri & Ali Seraj, 2206.12597
- "Multipole expansion of gravitational waves: memory effects and Bondi aspects" with Luc Blanchet, Guillaume Faye, Ali Seraj & Roberto Oliveri, 2303.07732
- "Quadrupolar radiation in de Sitter: Displacement memory and Bondi metric" with Jahanur Hoque & Emine Kutluk, 2309.02081

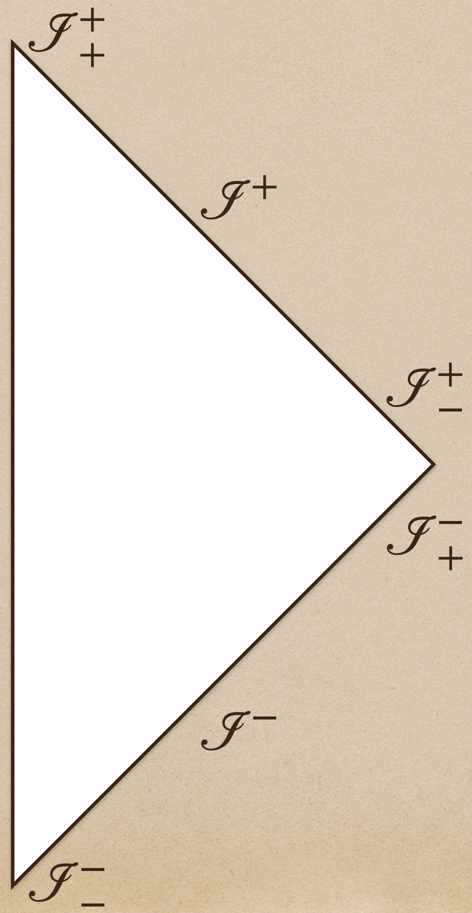
1. The 5 boundaries of Minkowski: “Penrose” versus “Puzzle piece” diagram

The standard Penrose(-Carter) diagram

$$u = t - r = \tan U \qquad U = T - R$$
$$v = t + r = \tan V \qquad V = T + R$$



Issues with the Penrose-Carter diagram



- ✓ Resolves $\mathscr{I}^+$ and $\mathscr{I}^-$
- ✗ No peeling?
- ✗ No resolution of timelike and spatial infinity
- ✗ No intuition on the detector frame

i^+ as a unit hyperboloid ($EAdS_3$)

Consider a particle emanating from $t = r = 0$ with direction of motion $x^A = (\theta, \phi)$ and constant velocity $v = r/t$.

We choose as coordinates the rapidity $\rho = \mathrm{arctanh}(r/t)$ and the proper time $\tau = \sqrt{t^2 - r^2}$.

Starting from Minkowski spacetime and applying the inverse transformation $t = \tau \cosh \rho$, $r = \tau \sinh \rho$, we obtain

$$ds^2 = -\, d\tau^2 + \tau^2 h_{ab} dx^a dx^b, \qquad h_{ab} dx^a dx^b \equiv d\rho^2 + \sinh^2 \rho \, \gamma_{AB} dx^A dx^B$$

where γ_{AB} is the unit metric over the sphere.

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where γ_{AB} is the unit metric over the sphere.

The coordinates (ρ, θ, ϕ) span a unit (one-sheet) hyperboloid also known as Euclidean AdS_3 spacetime whose points represent the velocities of outgoing massive particles.

A spacetime is asymptotically flat at i^+ if it exists coordinates such that $ds^2 = -d\tau^2 + \tau^2 h_{ab} dx^a dx^b + O(\tau^{-1}) d\tau^2 + O(\tau^1) dx^a dx^b$.

All finite-size bodies (particles, black holes, ...) become effectively point-like at large proper time. The subleading metric field has poles at the points where the finite-size bodies hit i^+. This can be called the skeletonization effect at i^+.

i^- as a unit hyperboloid ($EAdS_3$)

- i^- records the end state of massive bodies.
- We imagine massive bodies (particles, black holes, ...) entering the distant past and converging towards the origin of coordinates. We introduce the (negative) proper time $\tau = -\sqrt{t^2 - r^2}$ and the (positive) rapidity $\rho = \text{arctanh}(-t/r)$.
- Starting from Minkowski spacetime and applying $t = \tau \cosh \rho$, $r = -\tau \sinh \rho$, we obtain again

$$ds^2 = -d\tau^2 + \tau^2 h_{ab} dx^a dx^b, \qquad h_{ab} dx^a dx^b \equiv d\rho^2 + \sinh^2 \rho \, \gamma_{AB} dx^A dx^B$$

where γ_{AB} is the unit metric over the sphere. Therefore, i^- can also be modelled as a unit hyperboloid.

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- i^-can be obtained from i^+ by a combination of time-reversal (viewed actively) and a passive coordinate transformation that reverses back time towards the future.
- A spacetime is asymptotically flat at i^- if it exists coordinates such that

$$ds^2 = -d\tau^2 + \tau^2 h_{ab} dx^a dx^b + O(\tau^{-1}) d\tau^2 + O(\tau^1) dx^a dx^b \,.$$

i^0 as unit dS_3

It is causality disconnected from any physical process. But it allows to match physical quantities between the past and future.

Spatial infinity is the manifold of superluminal asymptotic luminosities. The relevant definitions are $\tau = \text{arctanh}(t/r)$ and $\rho = \sqrt{r^2 - t^2}$. Starting from Minkowski spacetime and applying inverse transformations are $t = \rho \sinh \tau, \; r = \rho \cosh \tau$, we obtain

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where now h^0_{ab} is the metric on the unit timelike hyperboloid otherwise known as Lorentzian dS_3.

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where now h^0_{ab} is the metric on the unit timelike hyperboloid otherwise known as Lorentzian dS_3.

A spacetime is asymptotically flat at i^0 if there exists coordinates such that

$$ds^2 = d\rho^2 + \rho^2(-d\tau^2 + \cosh^2 \tau \;\gamma_{AB} dx^A dx^B) + O(\rho^{-1}) d\rho^2 + O(\rho) dx^a dx^b.$$

It exists an analytic continuation between i^+ and i^0. One first complexify the metric. Then maximally extend it. Then apply a complex diffeomorphism and field redefinitions. For Minkowski spacetime it is given by

$$\rho\,|_{i^0} = i\tau\,|_{i^+}, \qquad \tau\,|_{i^0} = \rho\,|_{i^+} - \frac{i\pi}{2}, \qquad h^0 = -\,h^+ .$$

One finally restrict to real metrics and fields. The analytic continuation extends to asymptotically flat spacetimes.

$\mathscr{I}^+$ as a singular limit of a timelike boundary

- Starting from Minkowski spacetime and applying $t = u + r$, we obtain $ds^2 = -du^2 - 2dudr + r^2\gamma_{AB}dx^A dx^B$. The coordinates (u, θ, ϕ) span $\mathscr{I}^+$.

- At fixed large r the metric is timelike (u is the time). When $r \to \infty$, $\mathscr{I}^+$ has topology $\mathbb{R} \times S^2$ but has no metric.

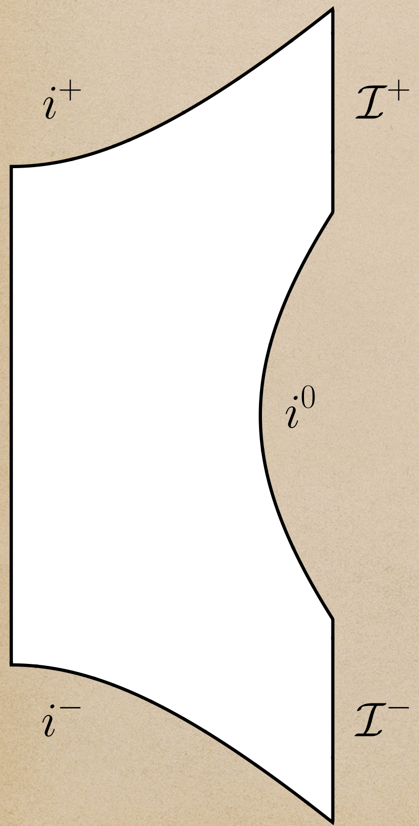
- We can introduce a non-invertible metric $\gamma_{ab}dx^a dx^b = 0\, du^2 + \gamma_{AB}dx^A dx^B$ of signature $(0, +, +)$ and the vector $n^a\partial_a = \partial_u$ such that $n^a\gamma_{ab} = 0$. The couple (γ_{ab}, n^a) forms a Carrollian structure.

- A spacetime is asymptotically flat at $\mathscr{I}^+$ if there exists coordinates such that

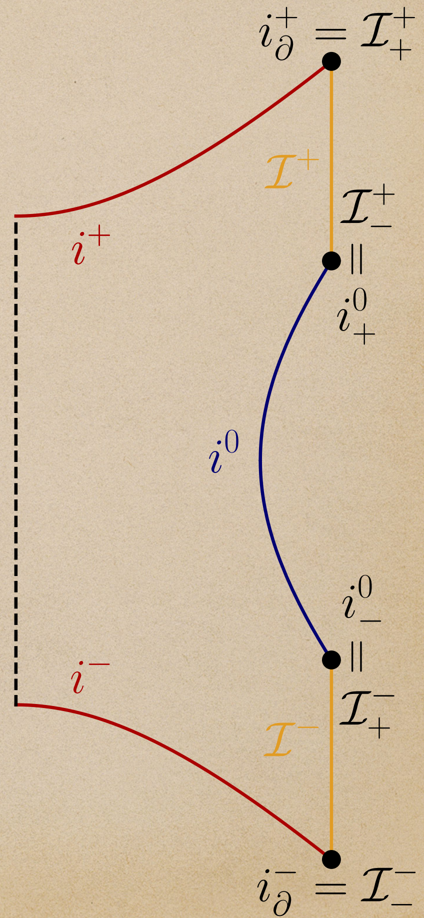
$$ds^2 = -du^2 - 2dudr + r^2\gamma_{AB}dx^A dx^B + (\frac{2m}{r} + O(r^{-2}))du^2 + (rC_{AB} + O(r^0))dx^A dx^B$$
$$+(\nabla^B C_{AB} + \frac{4}{3r}(N_A + u\partial_A m - \frac{3}{32}\partial_A(C_{BC}C^{BC}))dudx^A + O(r^{-2})dudr$$

- The field $C_{AB}(u, x^C)$ is the Bondi shear and its time derivative $N_{AB} \equiv \partial_u C_{AB}$ is the Bondi news.

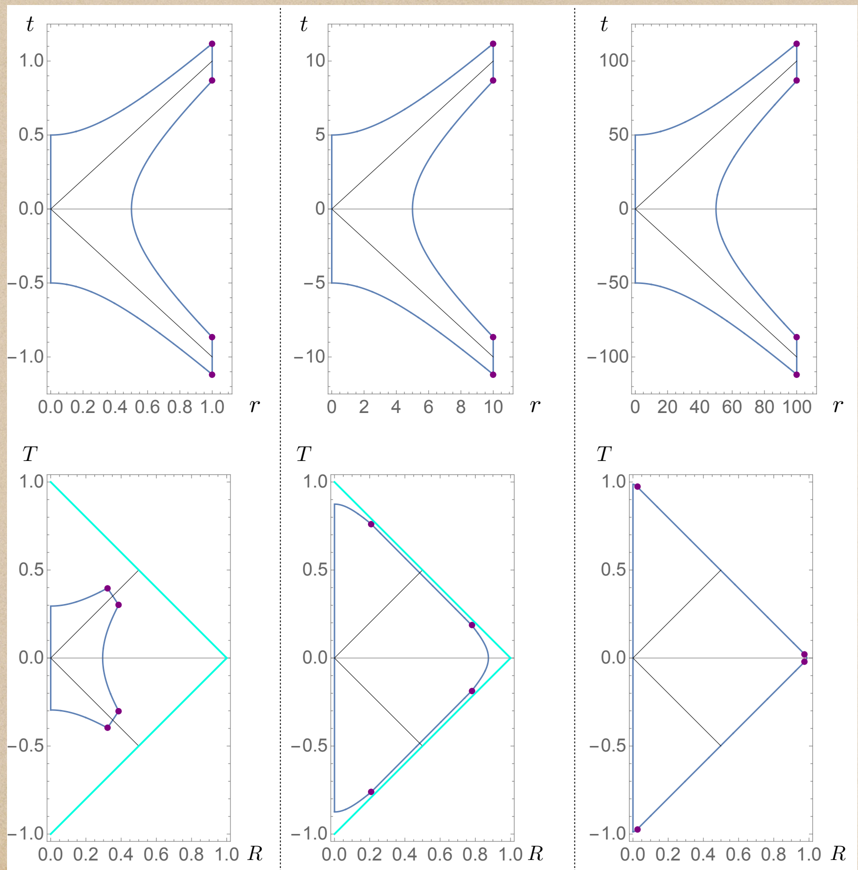
The “Puzzle piece” diagram for asymptotically flat spacetimes



- Resolves all 5 boundaries and the 4 corners.
- ∃ asymptotic solutions to Einstein equations
- Intuition at all boundaries
- Consistent with known infrared structure
- Single coordinate frame for all boundaries



Puzzle piece versus conformal diagram



(a) (b) (c)

$(a)\quad |t^2 - r^2| < 0.25, \qquad r < 1$

$(b)\quad |t^2 - r^2| < 2.5, \qquad r < 10$

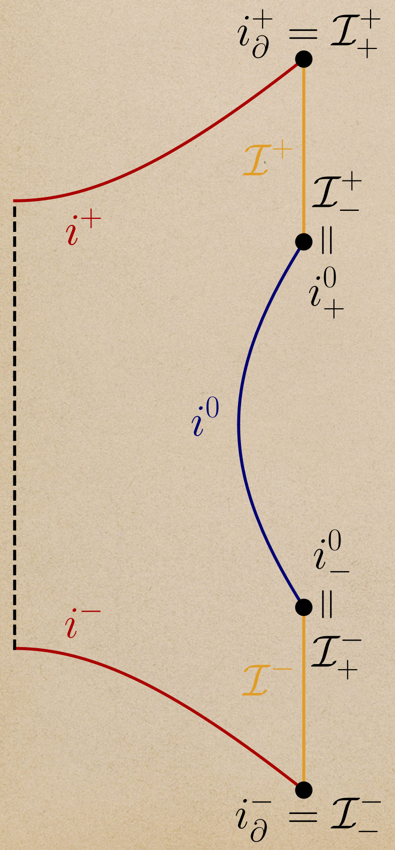
$(c)\quad |t^2 - r^2| < 25, \qquad r < 100$

2.

Unified BMS group acting simultaneously on the 5 boundaries.

Boundary conditions consistent with the logarithmic corrections to the subleading soft graviton theorem.

Boundary conditions. Matching of the four overlap regions



A spacetime is asymptotically flat in the global sense if it is asymptotically flat in the 5 asymptotic regions and if for each of the 4 overlap regions

$$i_\partial^+ = \mathscr{I}_+^+, \qquad \mathscr{I}_-^+ = i_+^0, \qquad i_-^0 = \mathscr{I}_+^-, \qquad i_\partial^- = \mathscr{I}_-^-$$

there exists a coordinate transformation that relate the two asymptotic expansions that are both valid around the overlap region.

Asymptotically flat spacetimes are therefore defined as a result of 4 asymptotically matched expansions.

We prove that this is possible given the 4 (in fact 3) sets of boundary conditions at $\mathscr{I}^+$ and $\mathscr{I}^-$ as $u \to \pm\infty$:

$$m(u, x^A) = m^{(0)}(x^A) + m^{(1)}(x^A)u^{-1} + o(u^{-1})$$

$$C_{AB}(u, x^A) = C_{AB}^{(0)}(x^A) + C_{AB}^{(1)}(x^A)u^{-1} + o(u^{-1}) \qquad \text{with } C_{AB}^{(0)} \text{ electric (parity even)}$$

$$N_A(u, x^A) = N_A^{\log}(x^A)\log u + N_A^{(0)}(x^A) + o(u^0)$$

Consistent with the logarithmic correction to the subleading classical soft graviton theorem [Laddha-Sen, 2018].

Remarks on logarithmic divergences

For the experts !

The Laddha-Sen behaviour $N_{AB} \sim u^{-2}$ is consistent with and slightly more restrictive than Christodoulou-Klainerman falloff $N_{AB} = O(1 + |u|)^{-3/2}$.

$N_{AB} \sim u^{-2}$ and Einstein's equations imply $N_A \sim \log u$.

The logarithmic divergence prevents to define the super-Lorentz charges at spatial infinity as $\int d^2x \;\; Y^A N_A$.

The conjectured map $\Upsilon^* N_A|_{\mathcal{I}^+_-} = N_A|_{\mathcal{I}^-_+}$ [Hawking, Perry, Strominger, 2016] cannot be demonstrated.

However, we proved that $N_A^{(\ell=1)} \sim u^0$ as $u \to \pm\infty$ and $\Upsilon^* N_A^{(\ell=1)}|_{\mathcal{I}^+_-} = - N_A^{(\ell=1)}|_{\mathcal{I}^-_+}$. This implies that the total Lorentz charges are finite and obey a conservation law.

Frames in Special and General Relativity

- In Special Relativity, a Poincaré frame needs to be specified to set up an experiment.
- In General Relativity and in the presence of radiation, a BMS frame needs to be specified which include in addition to the Poincaré frame a pure supertranslation frame. Pure supertranslations are 4-dimensional spacelike transformations.
- At $\mathscr{I}^+$, the shear has two degrees of freedom and can be decomposed into electric (parity-even) and magnetic (parity-odd) parts as $C_{AB} = (-2\nabla_A\nabla_B + \gamma_{AB}\nabla^2)C + \epsilon_{C(A}\nabla_{B)}\nabla^C\Psi$. Only the $\ell \geq 2$ spherical harmonics of $C(u,\theta,\phi)$ are defined. Pure supertranslations act as $C \mapsto C + T(\theta,\phi)$ where $T(\theta,\phi)$ have $\ell \geq 2$ harmonics. For $u \to \pm\infty$, $C = C^{(0)}(\theta,\phi) + O(u^{-1})$.
- In General Relativity, in an asymptotic region where the Bondi news $N_{AB} = \partial_u C_{AB}$ asymptotically vanishes, one could require as a boundary condition that the electric part of the shear asymptotically vanishes, which selects a particular Poincaré subgroup of BMS. However, after the passage of radiation, the final and initial frames will differ by a supertranslation, encoding the displacement memory.

The BMS$_4$ algebra

Take the sphere with metric γ_{AB}, measure ϵ_{AB} and covariant derivative ∇_A.

We consider the arbitrary function $T(\theta, \phi)$ and the 2 functions $Y^A(\theta, \phi)$ such that Y^A obeys the conformal Killing equation over the sphere: $\nabla_A Y_B + \nabla_B Y_A = \gamma_{AB} \nabla_C Y^C$. There are 6 solutions to that equation.

The BMS algebra can be presented using such covariant 2-dimensional generators:

$$[T_1, T_2] = 0$$

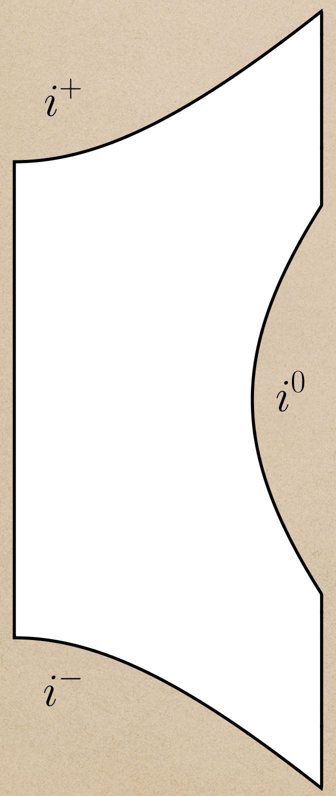
$$[Y, T] = Y(T) \qquad \text{where} \qquad Y(T) \equiv Y^A \partial_A T - \frac{1}{2} \nabla_A Y^A T$$

$$[Y_1, Y_2] = Y_1^A \partial_A Y_2 - Y_2^A \partial_A Y_1$$

The $\ell = 0{,}1$ harmonics of T are called translations, the $\ell > 1$ harmonics are called the pure supertranslations. The generators Y^A are called the Lorentz transformations.

For each of the 5 boundaries, there is a map $2d \mapsto 3d \mapsto 4d$ that allows to reconstruct the 4 dimensional asymptotic BMS symmetries. The matching at the 4 junctions is performed as a match of 2-dimensional quantities.

BMS covariant framework



At $\mathscr{I}^+$, we have $C^{(0)}_{AB} = (-2\nabla_A\nabla_B + \gamma_{AB}\nabla^2)C^{(0)}$ with

$$C^{(0)}|_{\mathscr{I}^+_+}(\theta,\phi), \qquad C^{(0)}|_{\mathscr{I}^+_-}(\theta,\phi) = -\Upsilon^* C^{(0)}|_{\mathscr{I}^-_+}(\theta,\phi), \qquad C^{(0)}|_{\mathscr{I}^-_-}(\theta,\phi)$$

(only $\ell \geq 2$ harmonics).

The antipodal map $\Upsilon : (\theta,\phi) \mapsto (\pi-\theta; \phi+\pi)$ at spatial infinity is a consequence of an evolution equation on dS_3 obtained from Einstein's equations. [Strominger, 2013]

We specify a unique BMS frame by specifying matching conditions at the four matching corners.

First subleading structure at i^+

For the experts !

The asymptotic metric reads as

$$ds^2 = (-1 - \frac{2\sigma}{\tau} + O(\tau^{-2}))d\tau^2 + O(\tau^{-1})d\tau dx^a + \tau^2(h_{ab} + \frac{k_{ab} - 2\sigma h_{ab}}{\tau} + O(\tau^{-2}))dx^a dx^b$$

σ

Einstein's equations imply

$$(D^2 - 3)\sigma = \sum_{n=1}^{N} 4\pi M_n \frac{\delta^{(3)}(\phi - \phi_n)}{\sqrt{h}}.$$

$$\phi_n^a = (\rho_n, \theta_n, \phi_n)$$

→ Finite bodies are reduced to $\delta-$function sources

We impose the boundary condition: $\lim_{\rho\to\infty} \sigma = 0\,.$ (this fixes logarithmic translations)

Matching with $\mathscr{I}^+$ gives $\sigma = -2m^{(0)}e^{-3\rho} + o(e^{-4\rho})$. This matches the mass aspect between i^+ and $\mathscr{I}^+$.

k_{ab}

We assume that k_{ab} is traceless and determined from a scalar Φ as

$$k_{ab} = -2(D_a D_b - h_{ab})\Phi$$

$$\Phi(\rho, \theta, \phi) = \sum_{\ell\geq 2, m} C^{(0)}_{\ell m}\psi^{\mathcal{O}}_{\ell}(\rho)Y_{\ell m}(\theta, \phi)$$

Matching with $\mathscr{I}^+$ gives $k_{AB} = \frac{1}{2}e^{\rho}C^{(0)}_{AB} + o(e^{0\rho})$.

This matches the supertranslation frame between i^+ and $\mathscr{I}^+$.

Second subleading structure at i^+

For the experts !

The asymptotic metric reads as

$$ds^2 = \left(-1 - \frac{2\sigma}{\tau} - \frac{\sigma^2}{\tau^2} + o(\tau^{-2})\right) d\tau^2 + o(\tau^{-2})\tau d\tau d\phi^a$$
$$+ \tau^2\left(h_{ab} + \tau^{-1}(k_{ab} - 2\sigma h_{ab}) + \frac{\log\tau}{\tau^2} i_{ab} + \tau^{-2} j_{ab} + o(\tau^{-2})\right) d\phi^a d\phi^b.$$

i_{ab}

Matching with $\mathcal{I}^+$ gives $i_{\rho A} = -4e^{-2\rho}N_A^{(\log)} + o(e^{-3\rho})$.
This matches the logarithmic corrections between i^+ and $\mathcal{I}^+$.

j_{ab}

Matching with $\mathcal{I}^+$ gives $j_{\rho A} = -\nabla^B C^{(1)}_{AB} + (-4N^{(0)}_A + \ldots)e^{-2\rho} + o(e^{-3\rho})$.
This matches the Lorentz charge aspect between i^+ and $\mathcal{I}^+$.

Lorentz charges are defined locally around each body as $Q^i_Y = -\frac{1}{8\pi}\oint_{C^i} \sqrt{q}d^2x\,(j_{ab} + \ldots)r^a\xi^b_Y$

(the charge is finite only after a renormalization procedure)

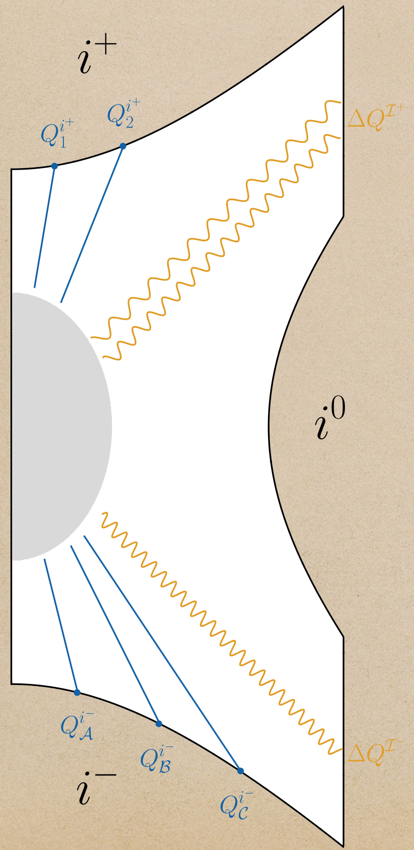
Why is a BMS covariant framework useful?

A. Formulate conservation laws

B. Clarify the choice of BMS frame

C. Define intrinsic spin of each massive body

A. Formulate conservation laws



The BMS group is the asymptotic symmetry group of all 5 infinities simultaneously.

There are therefore globally conserved BMS charges.

Let be N^- incoming bodies and N^+ outgoing bodies. The conservation laws are

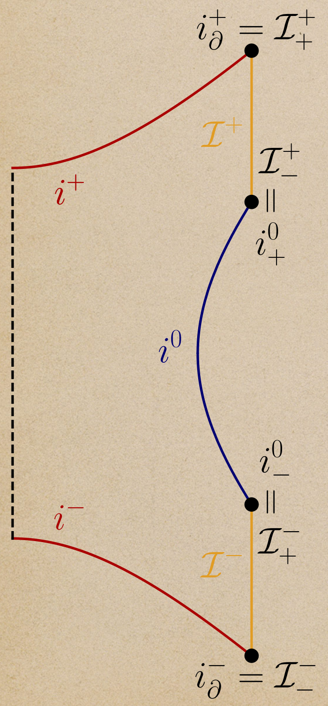
$$\sum_{n=1}^{N^+} Q_n^{i^+} + \Delta Q^{\mathcal{I}^+} = Q^{i^0} = \sum_{n=1}^{N^-} Q_n^{i^-} + \Delta Q^{\mathcal{I}^-}$$

for any charge associated with T and Y^A. This is a theorem under our assumptions.

Charge	name	generator
E	Energy	$T = 1$
P^i	Momentum	$T = n^i(\theta, \phi)$
$P_{\ell m}$	Supermomentum	$T = Y_{\ell m}(\theta, \phi)$
L^i	Angular Momentum	$Y^A = -\epsilon^{AB}\partial_B n^i(\theta, \phi)$
N^i	Mass Moment	$Y^A = \partial^A n^i(\theta, \phi)$

B. Clarify the choice of BMS frame

Even in a covariant theory, results are sometimes best derived in specific frames.



In special relativity:

$P^i|_{i_\partial^-} = 0$ "initial center of momentum frame" (fixes boosts)

$N^i|_{i_\partial^-} = P^i|_{i_\partial^-} = 0$ "initial center of energy frame" (fixes boosts and spatial translations)

One could further fix time translations and rotations.

In general relativity:

$C|_{\mathscr{I}^-_-} = 0$ "good cut" (fixes pure supertranslations)

In the absence of incoming radiation, it is equivalent to $C|_{\mathscr{I}^-_+} = 0$ and $C|_{\mathscr{I}^+_-} = 0$

[Alternative (used in PM/NR): $\left(\frac{1}{4}\nabla^2(\nabla^2+2)C + m_{\ell\geq 2}\right)|_{\mathscr{I}^+_-} = 0$ "nice cut".

This is equivalent to cancelling the Moreschi supermomenta at $\mathscr{I}^+_-$.

However, $C|_{\mathscr{I}^+_-}(\theta,\phi) = -\Upsilon^* C|_{\mathscr{I}^-_+}(\theta,\phi)$ while $m|_{\mathscr{I}^+_-}(\theta,\phi) = +\Upsilon^* m|_{\mathscr{I}^-_+}(\theta,\phi)$.

Therefore the relationship $\left(\frac{1}{4}\nabla^2(\nabla^2+2)C + m_{\ell\geq 2}\right)|_{\mathscr{I}^-_+} = 0$ would not be simultaneously true.]

Physical effects coded in shifts of BMS frames

Let us impose the BMS frame fixing conditions at i^- :

$N^i|_{i^-_\partial} = P^i|_{i^-_\partial} = 0$ "initial center of energy frame" (fixes boosts and spatial translations)

$C|_{\mathcal{I}^-_-} = 0$ "good cut" (fixes supertranslations)

In the absence of incoming radiation, it is equivalent to $C|_{\mathcal{I}^-_+} = 0$ and $C|_{\mathcal{I}^+_-} = 0$

The final BMS frame at i^+ will differ :

- When there is change in momentum, we say there was "recoil" equal to $P_i^{i^+_\partial}$
- When there is mass moment, we say there was "scoot" equal to $N_i^{i^+_\partial}$
- When there is change in shear, we say there was "displacement memory" equal to $C|_{i^+_\partial} = \Delta C^{\text{total}}(\theta, \phi)$

[There is no name for the other 2 effects of change of BMS frame (time translations and rotations).]

C. Define intrinsic spin of each massive body at i^+

In scattering problems, one needs to define the mass and (intrinsic) spin of individual bodies.
The mass of a body is defined as $M = \sqrt{E^2 - P^i P_i}$ which is fully BMS-invariant.

In special relativity, a body located at position x^i has (total) angular momentum $L^i = S^i + \epsilon^i{}_{jk} x^j P^k$ and mass moment $N^i = E x^i - P^i t$ where S^i is the spin. The formula for the spin in terms of the charges is thus

$$S^i = L^i - \frac{1}{E} \epsilon^i{}_{jk} N^j P^k.$$

In general relativity, we also define the spin as $S^i = L^i - \frac{1}{E} \epsilon^i{}_{jk} N^j P^k$. It is invariant under translations and it transforms in the expected way under rotations and boosts.

Using the representation of the BMS algebra on the conserved charges, we can prove that under a supertranslation T, the spin is invariant: $\delta_T S^i = 0$. This defines the supertranslation invariant spin in general relativity. The spin magnitude $S = \sqrt{S^i S_i}$ is BMS invariant.

3. Complete set of non-radiative charges : Geroch-Hansen multipoles + generalised BMS + non-stationary multipole moments

Multipolar post-Minkowskian formalism

Introduce a background Minkowski metric and define the non-linear field

$$h^{\alpha\beta} \equiv \sqrt{-g}\, g^{\alpha\beta} - \eta^{\alpha\beta}$$

Simple algebra leads to

$$\partial_\mu h^{\alpha\mu} = \sqrt{-g}\, \Box_g x^\alpha \,, \qquad \Box_g \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$$

Fix de Donder gauge, also known as harmonic gauge,

$$\partial_\mu h^{\alpha\mu} = 0 \,. \quad \longleftrightarrow \quad \Box_g x^\alpha = 0$$

The Einstein field equations then take the form

$$\Box_\eta h^{\mu\nu} = \frac{16\pi G}{c^4}|g|T^{\mu\nu} + \Lambda^{\mu\nu}(h, \partial h, \partial^2 h)$$

$$\begin{aligned}\Lambda^{\alpha\beta} &= -h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}\partial_\lambda h^{\mu\tau}\partial_\tau h^{\nu\lambda} \\ &- g^{\alpha\mu}g_{\nu\tau}\partial_\lambda h^{\beta\tau}\partial_\mu h^{\nu\lambda} - g^{\beta\mu}g_{\nu\tau}\partial_\lambda h^{\alpha\tau}\partial_\mu h^{\nu\lambda} + g_{\mu\nu}g^{\lambda\tau}\partial_\lambda h^{\alpha\mu}\partial_\tau h^{\beta\nu} \\ &+ \frac{1}{8}\left(2g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu}\right)\left(2g_{\lambda\tau}g_{\epsilon\pi} - g_{\tau\epsilon}g_{\lambda\pi}\right)\partial_\mu h^{\lambda\pi}\partial_\nu h^{\tau\epsilon}\,.\end{aligned}$$

These equations are amenable to the post-Minkowskian (PM) expansion

$$h^{\mu\nu} = \sum_{n=1}^{+\infty} G^n h_n^{\mu\nu}\,.$$

In addition, a spherical harmonic decomposition or, equivalently, a decomposition in multipole moments is performed. This is motivated by the fact that GW emitted from compact sources mainly depend upon the lowest multipoles.

At linear order, $\Box_\eta h_1^{\mu\nu} = 0$

We impose the boundary condition no incoming radiation from $\mathfrak{I}^-$

The most general solution (up to residual gauge transformations) is

$$h_1^{00} = -4 \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \tilde{\partial}_L \left(\frac{M_L(\tilde{u})}{\tilde{r}} \right),$$

$$h_1^{0j} = 4 \sum_{\ell=1}^{+\infty} \frac{(-)^\ell}{\ell!} \left[\tilde{\partial}_{L-1} \left(\frac{M^{(1)}_{jL-1}(\tilde{u})}{\tilde{r}} \right) + \frac{\ell}{\ell+1} \tilde{\partial}_{pL-1} \left(\frac{\varepsilon_{jpq} S_{qL-1}(\tilde{u})}{\tilde{r}} \right) \right],$$

$$h_1^{jk} = -4 \sum_{\ell=2}^{+\infty} \frac{(-)^\ell}{\ell!} \left[\tilde{\partial}_{L-2} \left(\frac{M^{(2)}_{jkL-2}(\tilde{u})}{\tilde{r}} \right) + \frac{2\ell}{\ell+1} \tilde{\partial}_{pL-2} \left(\frac{\varepsilon_{pq(j} S^{(1)}_{k)qL-2}(\tilde{u})}{\tilde{r}} \right) \right],$$

We defined: de Donder coordinates: $\tilde{x}^\mu = (\tilde{t}, \tilde{\mathbf{x}})$ or $(\tilde{t}, \tilde{r}, \tilde{\theta}^a)$. $L = i_1 i_2 \dots i_\ell$ a multi-index made of ℓ spatial indices.

the multi-derivative operator $\partial_L = \partial_{i_1} \dots \partial_{i_\ell}$, the product of vectors $n_L = n_{i_1} \dots n_{i_\ell}$ $x_L = x_{i_1} \dots x_{i_\ell} = r^\ell n_L$.

$M_L(u),\ S_L(u)$ are the mass and current canonical multipole moments, respectively

They are STF (symmetric trace-free) tensors.

[Thorne, 1980] [Blanchet, Liv.Rev.Rel.]

We can perform a coordinate transformation to Bondi gauge and read off the Bondi data in terms of canonical multipole moments

We read :

$$G^{-1}m = \sum_{\ell=0}^{+\infty} \frac{(\ell+1)(\ell+2)}{2\ell!}\, n_L M_L^{(\ell)} + \mathcal{O}(G)\,,$$

$$G^{-1}N_a = e_a^i \sum_{\ell=1}^{+\infty} \frac{(\ell+1)(\ell+2)}{2(\ell-1)!}\, n_{L-1}\left[M_{iL-1}^{(\ell-1)} + \frac{2\ell}{\ell+1}\varepsilon_{ipq} n_p S_{qL-1}^{(\ell-1)}\right] + \mathcal{O}(G)\,.$$

$$G^{-1}C_{ab} = e^i_{\langle a} e^j_{b\rangle} H^{ij}_{\mathrm{TT}} - 2D_{\langle a}D_{b\rangle} f\,.$$

$$H^{ij}_{\mathrm{TT}} = 4\perp^{ijkl}_{\mathrm{TT}} \sum_{\ell=2}^{+\infty} \frac{n_{L-2}}{\ell!}\left[M_{klL-2}^{(\ell)} - \frac{2\ell}{\ell+1}\varepsilon_{kpq} n_p S_{lqL-2}^{(\ell)}\right] + \mathcal{O}(G)\,,$$

ℓ derivatives of the canonical multipoles

[Blanchet, G.C., Faye, Oliveri, Seraj, 2013]

Additional data: the "$n \geq 2$ Bondi aspects"

$$g_{ab} = r^2\gamma_{ab} + rC_{ab} + r^{-1}\sum_{n=2}^{\infty} r^{2-n} \underset{(n)}{E}_{ab} + O(G^2)$$

[BMS, 1962] [Tambourino, Winicour, 1966] [...] [Barnich-Troessaert, 2011]

We read :

$$\underset{(n)}{E}_{ab} = G e^i_{\langle a} e^j_{b\rangle} 4 \frac{n-1}{n+1} \sum_{\ell=n}^{\infty} \frac{1}{\ell!} a_{n\ell} n_{L-2} \left[M^{(\ell-n)}_{ijL-2} + \frac{2\ell}{\ell+1} \epsilon_{ipq} n_p S^{(\ell-n)}_{jqL-2} \right] + O(G^2),$$

$$a_{n\ell} = \frac{(\ell+n)!}{2^n n!(\ell-n)!}.$$

Less than ℓ derivatives of the canonical multipoles

[Blanchet, G.C., Faye, Oliveri, Seraj, 2013]

Non-radiative spacetimes (linear level)

$$N_{ab} = \partial_u C_{ab} = 0$$

$$C_{ab} = e^i_{\langle a} e^j_{b\rangle} H^{ij}_{\rm TT} - 2 D_{\langle a} D_{b\rangle} f \,.$$

$$H^{ij}_{\rm TT} = 4 \perp^{ijkl}_{\rm TT} \sum_{\ell=2}^{+\infty} \frac{n_{L-2}}{\ell!} \left[M^{(\ell)}_{klL-2} - \frac{2\ell}{\ell+1} \varepsilon_{kpq} n_p S^{(\ell)}_{lqL-2} \right] + \mathcal{O}(G)\,,$$

Equivalently, in terms of canonical multipole moments : $M_L^{(\ell+1)} = S_L^{(\ell+1)} = 0$

General solution in terms of conserved charges :

$$M_L(u) = \sum_{k=0}^{\ell} M_{L,k} u^k, \qquad S_L(u) = \sum_{k=0}^{\ell} S_{L,k} u^k$$

What are those conserved charges?

Conserved charges are built from "dressed Bondi data"

$n = 0$

$$m_{ab} \equiv m\gamma_{ab} + \frac{1}{2} D_{[a} D^c C_{b]c} = m\gamma_{ab} + m^- \epsilon_{ab}, \qquad m^- \equiv \frac{1}{4} D_c D_d \widetilde{C}^{cd},$$

[Godazgar, Godazgar, Pope, 2018]

$n = 1$

$$\mathcal{N}_a \equiv N_a - \frac{1}{4} C_{ab} D_c C^{bc} - \frac{1}{16} \partial_a (C_{bc} C^{bc}) - u D^b m_{ab};$$

[Hawking, Perry, Strominger, 2016]
[Compère, Oliveri, Seraj, 2019]

$n = 2$

$$\underset{(2)}{\mathcal{E}_{ab}} = \underset{(2)}{E_{ab}} - \frac{u}{2} C_{(a}{}^c m_{b)c} - \frac{u}{3} D_{\langle a} \mathcal{N}_{b\rangle} - \frac{u^2}{6} D_{\langle a} D^c m_{b\rangle c},$$

[Freidel, Pranzetti, 2021]

$n = 3$

$$\underset{(3)}{\mathcal{E}_{ab}} = \underset{(3)}{E_{ab}} - u \left\{ \mathcal{D}_0 \underset{(2)}{E_{ab}} + D^c \left[\left(\frac{1}{4} D_e C^{de} C_{d\langle a} - \frac{3}{32} D_{\langle a} C^2 + \frac{5}{32} C^2 D_{\langle a} - \frac{1}{3} N_{\langle a} \right) C_{b\rangle c} \right] \right\}$$
$$+ \frac{u^2}{2} \left[-\frac{1}{3} D^c \left(D^d m_{d\langle a} C_{b\rangle c} \right) + \frac{1}{2} \mathcal{D}_0 \left(m_{ac} C^c_{\ b} \right) + \frac{1}{3} \mathcal{D}_0 D_{\langle a} N_{b\rangle} \right] - \frac{u^3}{18} \mathcal{D}_0 D_{\langle a} D^c m_{b\rangle c} .$$

[Grant, Nichols, 2022]
[Blanchet, Compère, Faye, Oliveri, Seraj, 2022]

$n \geq 4$ $\quad \ldots$

The dressed quantities are conserved in the absence of news.

Uniqueness?

All local flux-balance laws at $\mathscr{I}^+$

$$
\begin{aligned}
n=0 \quad &: \quad \frac{1}{4} D_b D_c N^{bc} = -\mathcal{F}(u) + \partial_u m, \\
n=1 \quad &: \quad -\frac{u}{2} D_c D_{\langle a} D_{b\rangle} N^{bc} = -\mathcal{F}_a(u) + \partial_u \mathcal{N}_a, \\
n=2 \quad &: \quad \frac{u^2}{12} \mathrm{STF}_{ab}[D_a D_c D_{\langle b} D_{d\rangle} N^{cd}] = -\underset{(2)}{\mathcal{F}_{ab}}(u) + \partial_u \underset{(2)}{\mathcal{E}_{ab}}, \\
n \geq 3 \quad &: \quad \frac{(-u)^n}{6\, n!} \mathcal{D}_{n-3} \cdots \mathcal{D}_0\, \mathrm{STF}_{ab}[D_a D_c D_{\langle b} D_{d\rangle} N^{cd}] = -\underset{(n)}{\mathcal{F}_{ab}}(u) + \partial_u \underset{(n)}{\mathcal{E}_{ab}}\,.
\end{aligned}
$$

$$\mathcal{F} \equiv -\frac{1}{8} N_{ab} N^{ab}$$

$$\mathcal{D}_n \equiv -\frac{n+2}{2(n+1)(n+4)} (D^a D_a + n^2 + 5n + 2).$$

[Grant, Nichols, 2021] [Freidel, Pranzetti, Raclariu, 2021]

Charges of non-radiative spacetimes

BMS Supermomenta and BMS dual supermomenta

$$\mathcal{P}_L = \oint_S m\,\hat{n}_L$$

$$\mathcal{P}_L^- = \oint_S m^-\,\hat{n}_L = \frac{1}{2}\oint_S m_{ab}\epsilon^{ab}\,\hat{n}_L$$

Generalized BMS Super-Lorentz charges

$$-\mathcal{J}_L = \frac{1}{2}\oint_S \epsilon^{ab}\partial_b\hat{n}_L\,\mathcal{N}_a$$

$$\mathcal{K}_L = \frac{1}{2}\oint_S \partial^a\hat{n}_L\,\mathcal{N}_a$$

$n \geq 2$

Bondi charges

$$\mathcal{Q}^+_{n,L}(u) \equiv \oint_S \mathop{\mathcal{E}^{ab}}_{(n)}\,D_aD_b\hat{n}_L,$$

Non-stationary moments

Stationary Geroch-Hansen moments

$$M_L,\,S_L \qquad M_L = m_{L,\ell}u^\ell + m_{L,\ell-1}u^{\ell-1} + \cdots + m_{L,1}u^1 + m_{L,0}u^0$$

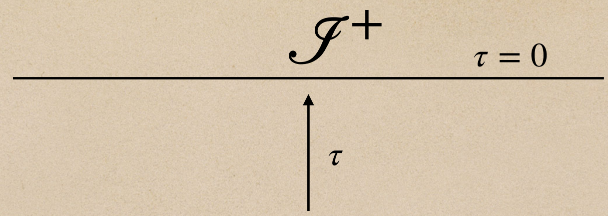
They occur in the metric components

$$g_{uu},\ rg_{ua},\ r^2g_{ab} \qquad \frac{1}{r} \qquad \frac{1}{r^2} \qquad \cdots \qquad \frac{1}{r^\ell} \qquad \frac{1}{r^{\ell+1}}$$

Can be converted to $Lw_{1+\infty}$ basis [Freidel, Raclariu, Pranzetti, 2021]

4. Properties of quadrupolar linear fields on dS_4: Memory effects, $\Lambda-$BMS transitions, and breaking of the conformal group which invalidates the dS_4/CFT_3 conjecture

The asymptotic structure of $\mathscr{I}^+$ for $\Lambda > 0$ in a nutshell



$$H = \sqrt{\frac{\Lambda}{3}}$$

Starobinsky / Fefferman-Graham gauge :

$$ds^2 = -d\tau^2 + \tau^2 (g^{(0)}_{ab}(x^c) + \ldots + \tau^{-3} T_{ab}(x^c) + \ldots) dx^a dx^b \qquad T^a_a = 0, \qquad D^a_{(0)} T_{ab} = 0$$

The residual gauge transformations consist in 4 functions of x^a (“integration constants” after gauge fixing)

We can further gauge fix the boundary metric :

$$g^{(0)}_{ab} dx^a dx^b = H^2 du + q_{AB}(u, x^C) dx^A dx^B \qquad \det(q_{AB}) = \det(\mathring{q}_{AB})$$

The residual gauge transformations are spanned by 3 functions of $x^A = (\theta, \phi)$.
They form the Λ-BMS algebroid whose structure constants depend upon the phase space field q_{AB} .

In the presence of radiation, an observer located close to $\mathscr{I}^+$ cannot gauge fix the diffeomorphism group any further. The Λ-BMS symmetries reflect the freedom at setting up a detector at $\mathscr{I}^+$ in asymptotically de Sitter. (Same results in Bondi gauge)

[GC, Fiorucci, Ruzziconi, 2019]

The asymptotic structure of $\mathscr{I}^+$ for $\Lambda > 0$ in a nutshell

"2d" presentation of the Λ-BMS generators :

$$\xi^u = U(u, x^A) \qquad \partial_u U = -\frac{1}{2} D_A Y^A$$
$$\xi^A = Y^A(u, x^A) + O(r^{-1}) \qquad \partial_u Y^A = -H^2 q^{AB} \partial_B U$$

Algebroid :

$$[(U, Y^A), (U', Y'^A)] = (U'', Y''^A)$$

$$U'' = Y^A \partial_A U' + \frac{1}{2} U D_A Y'^A - (() \leftrightarrow ()')$$
$$Y''^A = Y^B \partial_B Y'^A - H^2 U q^{AB} \partial_B U' - (() \leftrightarrow ()')$$

In the flat limit, the algebra reduces to the generalized BMS algebra diff(S^2) + vect(S^2)

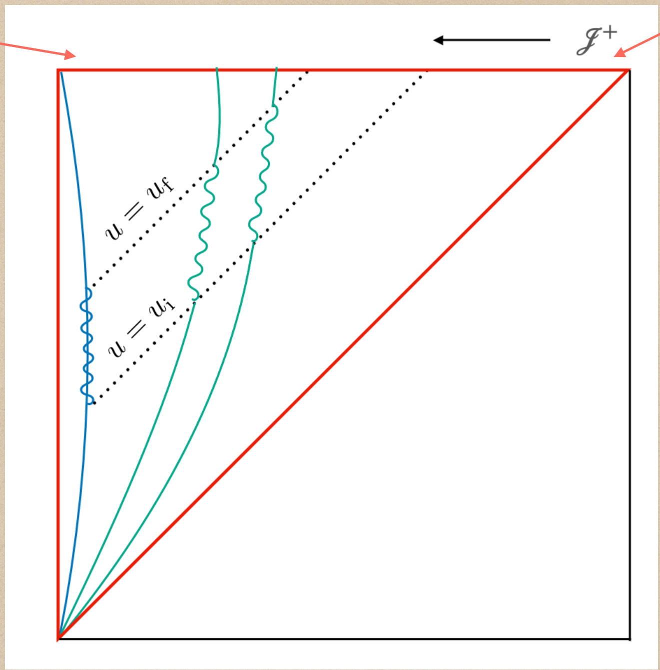
[GC, Fiorucci, Ruzziconi, 2019]
[Barnich, Troessaert, 2010]
[Campiglia, Laddha, 2015]

When $q_{AB}(u, x^A) = \mathring{q}_{AB}(x^A)$, the Λ-BMS algebroid becomes the Λ-BMS algebra that contains the $SO(4,1)$ algebra of exact symmetries of de Sitter.

What is the structure at $\mathscr{I}^+$ generated by a localized event?

No radiation at $\mathscr{I}^+_+$

No radiation at $\mathscr{I}^+_-$



Note: The topology at $\mathscr{I}^+$ is S^3 minus 2 points : $\mathbb{R} \times S^2$.

What is the metric $q_{AB}(u, x^A)$ resulting of a localized event below the Hubble scale?

Is there a Λ-BMS group transition after the passage of the gravitational wave strain?

Linear spin 2 field on de Sitter

Starting point: de Sitter in the Poincaré patch

$$\bar{g}_{\alpha\beta}dx^\alpha dx^\beta = a^2(-d\eta^2 + d\vec{x}^2), \qquad a(\eta) = -\frac{1}{H\eta} \qquad\qquad \eta = -\frac{1}{H}e^{-Ht}$$

Perturbations $h_{\alpha\beta}$ are described using good variables: $\chi_{\mu\nu} = a^{-2}(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\alpha_\alpha), \qquad \hat{\chi} = \chi_{00} + \chi_{ii} \;, \quad \chi_{0i}, \;\; \chi_{ij}$

and a good gauge “Generalized harmonic gauge” : $\partial^\alpha \chi_{\alpha\mu} + \frac{1}{\eta}(2\chi_{0\mu} + \delta^0_\mu \chi^\alpha_\alpha) = 0.$

[de Vega, Ramirez, Sanchez, 98]

Linear spin 2 field on de Sitter

Linear equations of motion:

$$\Box\left(\frac{\hat{\chi}}{\eta}\right) = -\frac{16\pi G\hat{T}}{\eta},$$
$$\Box\left(\frac{\chi_{0i}}{\eta}\right) = -\frac{16\pi G T_{0i}}{\eta},$$
$$\left(\Box+\frac{2}{\eta^2}\right)\left(\frac{\chi_{ij}}{\eta}\right) = -\frac{16\pi G}{\eta}T_{ij},$$

$$\hat{T} := T_{00} + T^i{}_i.$$

$$\Box = -\partial_\eta^2 + \partial_i^2$$

Scalar and vectors modes are similar to flat space.

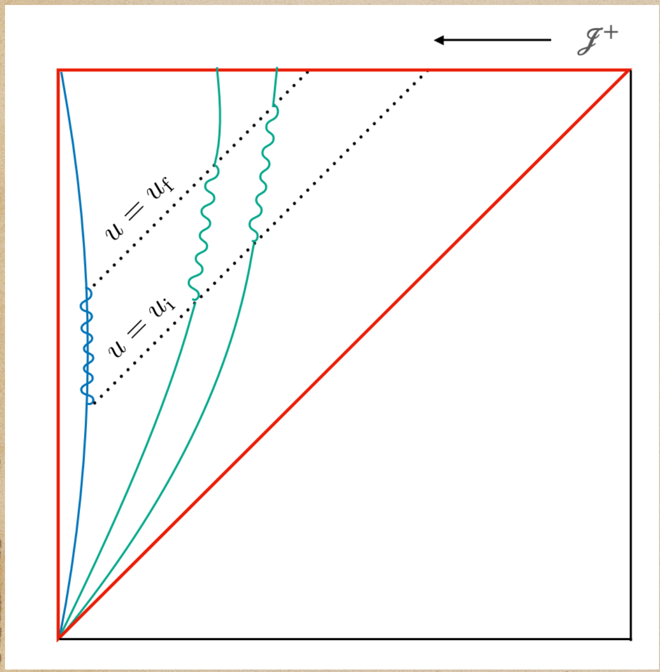
Tensor mode depends upon the de Sitter potential. There is propagation inside the lightcone.

We solve the equations in the quadrupolar truncation:

$$\int d^3x a^{\ell+1} T_{\mu\nu} x^L = 0, \qquad \forall \ell > 2.$$

(Distinct from [Ashtekar, Bonga, Kesavan, 2015] [Bunster, Perez, Bonga, 2023])

Result of the linear analysis



We define the even parity and odd parity quadrupolar moments of the stress-energy tensor as

$$Q_{ij}^{(\rho+p)}(\eta) \equiv \int d^3x a^3(\eta)(T_{00} + T_{kk})x_i x_j \qquad K_{ij}(\eta) \equiv \frac{4}{3}\int d^3x a^3(\eta)\epsilon_{kl(i}T_{j)k}x_l$$

We assume for simplicity staticity in addition to non-radiative boundary conditions at early and late times.

The boundary metric at $\mathscr{I}^+$ of the linear perturbation is given by

$$g_{ab}^{(0)}dx^a dx^b = H^2 du^2 + q_{AB}dx^A dx^B$$

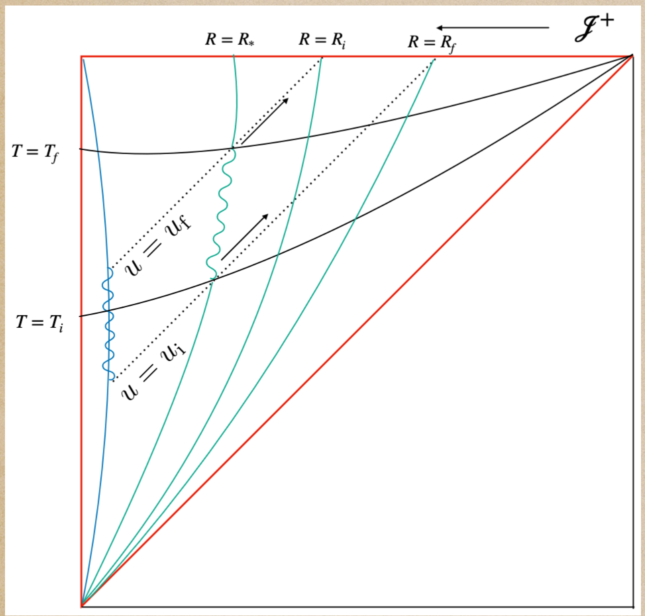
$$q_{AB} = \mathring{q}_{AB} + 2\mathring{q}_{C\langle A}\mathring{D}_{B\rangle}\xi^C + e^i_{\langle A}e^j_{B\rangle}\left(\partial_u\zeta_{ij} + 2H^2\partial_u Q_{ij}^{(\rho+p)} + 2H^2\epsilon_{ikl}n_k(K_{jl} + H\int^u du' K_{jl}(u'))\right),$$

where $\partial_u^2\zeta_{ij} - 3H^2\zeta_{ij} = -2H^4 Q_{ij}^{(\rho+p)}$

(Here, the multipoles are evaluated at $\mathscr{I}^+ : \eta = -H^{-1}e^{-Hu}$.)

[G.C., Hoque, Kutluk, 23]

Cosmological displacement memory effect



In the even sector, a fixed quadrupole can be re-absorbed at either $u = u_{\mathsf{f}}$ or $u = u_{\mathsf{i}}$ as a $\Lambda - BMS$ transformation.

In the even sector, the finite difference of the quadrupole between $u = u_{\mathsf{f}}$ or $u = u_{\mathsf{i}}$ leads to a finite displacement memory, which is gauge invariant.

In the odd sector, a fixed quadrupole cannot be absorbed into a residual gauge transformation. There also again a finite displacement memory.

Contrary to the flat case, the displacement memory is at leading order. In a sense it also arises from a flux-balance law, $\partial_u q_{AB} = HC_{AB}$, which becomes trivial in the flat limit. There are also subleading effects (which match the flat case).

Similarly to the flat case, there is a distinction between even and odd sectors with respect to memory.

[Chu, '16][G.C., Hoque, Kutluk, 23]

Consequences for holography

Localized sources in dS_4 lead to a non-trivial boundary metric, after using the retarded propagator.

Dirichlet boundary conditions are therefore generically violated. Even in the absence of radiation, in the presence of a static odd quadrupole.

(Sending advanced signals from the past cosmological horizon would induce non-linear interferences)

Dynamical gravity in dS_4 cannot be modelled by a CFT_3.

See also [Ashtekar, Bonga, Kesavan, 2015] [Bunster, Perez, Bonga, 2023]

Summary

1. The 5 boundaries of Minkowski: “Penrose” versus “Puzzle piece” diagram
2. Unified BMS group acting simultaneously on the 5 boundaries. Boundary conditions consistent with the logarithmic corrections to the subleading soft graviton theorem.
3. Complete set of non-radiative charges : Geroch-Hansen multipoles + generalised BMS + non-stationary multipole moments
4. Properties of quadrupolar linear fields on dS_4: Memory effects, $\Lambda-$BMS transitions, and breaking of the conformal group which invalidates the dS_4/CFT_3 conjecture

Non-decoupling feature of the BMS algebra

For the experts !

Spherical harmonic decomposition: $T = \sum_{\ell=0}^{\infty} T_L n_L = T_0 + T_i n_i + T_{ij} n_i n_j + \cdots$

$n_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ $\qquad Y^A = b^i \partial^A n_i - \kappa^i \epsilon^{AB} \partial_B n_i$

The BMS algebra can be written as

$$[\xi_{T_1}, \xi_{T_2}]_\star = 0\,,$$
$$[\xi_T, \xi_{\kappa_i}]_\star = \xi_{T'_i}\,, \qquad T'_i = \epsilon_{kij} n_k T_j + \sum_{\ell\geq 2} \ell \epsilon_{kij} n_k n_{L-1} T_{jL-1}\,,$$
$$[\xi_T, \xi_{b_i}]_\star = \xi_{T''_i}\,, \qquad T''_i = T_i + n_i T + \sum_{\ell\geq 2} (\ell T_{iL-1} n_{L-1} - (\ell-1) T_L n_i n_L)\,,$$

The pure supertranslations do not form an ideal because ordinary translations appear in the right-hand side of the commutator of a boost and a pure supertranslation with $\ell = 2$.

Decoupling BMS at i^+, i^0, i^- :

Supertranslation invariant Lorentz charges

For the experts !

BMS algebra:

$$\{Q_{T_1}, Q_{T_2}\} = 0$$

$$\{Q_Y, Q_T\} = Q_{Y(T)} \qquad \text{where} \qquad Y(T) \equiv Y^A \partial_A T - \frac{1}{2} \nabla_A Y^A T$$

$$\{Q_{Y_1}, Q_{Y_2}\} = Q_{[Y_1, Y_2]}, \qquad [Y_1, Y_2]^A \equiv Y_1^B \partial_B Y_2^A - Y_2^B \partial_B Y_1^A$$

Supertranslation frame at either i^+, i^0, i^- : $C^{(0)}|_{\mathscr{I}^+_+}(\theta,\phi)$, $C^{(0)}|_{\mathscr{I}^+_-}(\theta,\phi) = -\Upsilon^* C^{(0)}|_{\mathscr{I}^-_+}(\theta,\phi)$, $C^{(0)}|_{\mathscr{I}^-_-}(\theta,\phi)$

I will denote it in all three cases as $C(\theta,\phi)$. Under a supertranslation $\delta_{Y,T} C = T + Y^A \partial_A C - \frac{1}{2} D_A Y^A C = T + Y(C)$.

The supertranslation invariant Lorentz charge is defined as

$$\boxed{Q_Y^{\text{inv}} \equiv Q_Y - Q_{T=Y(C)}.}$$

Explicitly, at $\mathscr{I}^\pm$ as $u \to \pm\infty$, $Q_Y^{\text{inv}}|_{\mathscr{I}^\pm_\pm} = \frac{1}{8\pi} \int_{S^2} d\Omega Y^A \left(N_A - 3m\partial_A C - C\partial_A m\right)$.

By construction, under a pure supertranslation: $\{Q_T, Q_Y^{\text{inv}}\} = Q_{Y(T)} - Q_{Y(T)} = 0$

Under a translation, there is no change in the bracket: $\{Q_{T_\mu}, Q_Y^{\text{inv}}\} = Q_{Y(T_\mu)}$. One can check that $\{Q_{Y_1}^{\text{inv}}, Q_{Y_2}^{\text{inv}}\} = Q_{[Y_1,Y_2]}^{\text{inv}}$.

The Poincaré algebra is therefore decoupled from the pure supertranslation abelian algebra at i^+, i^0, i^-.

[Javadinezhad, Kol, Porrati, 2018&2022 ; Compère, Oliveri, Seraj, 2019 ;
Chen, Wang, Wang,, Yau, 2021 ; Compère,Nichols 2021 ; Fuentealba, Henneaux, Troessaert 2023]

Zero modes of the soft term

= Poincaré + Newman-Penrose charges

Poincaré	$4 : \ell = 0, 1$
	$2 \times 3 : \ell = 1$
	None
NP-type charges	$2(n^2 - 4) : 2 \leq \ell \leq n-1$

$$
\begin{aligned}
n = 0 \quad &: \quad \frac{1}{4} D_b D_c N^{bc} = -\mathcal{F}(u) + \partial_u m, \\
n = 1 \quad &: \quad -\frac{u}{2} D_c D_{\langle a} D_{b\rangle} N^{bc} = -\mathcal{F}_a(u) + \partial_u \mathcal{N}_a, \\
n = 2 \quad &: \quad \frac{u^2}{12} \mathrm{STF}_{ab}[D_a D_c D_{\langle b} D_{d\rangle} N^{cd}] = -\underset{(2)}{\mathcal{F}_{ab}}(u) + \partial_u \underset{(2)}{\mathcal{E}_{ab}}, \\
n \geq 3 \quad &: \quad \frac{(-u)^n}{6\, n!} \mathcal{D}_{n-3} \cdots \mathcal{D}_0 \, \mathrm{STF}_{ab}[D_a D_c D_{\langle b} D_{d\rangle} N^{cd}] = -\underset{(n)}{\mathcal{F}_{ab}}(u) + \partial_u \underset{(n)}{\mathcal{E}_{ab}} \,.
\end{aligned}
$$

$$
\mathcal{Q}^+_{n,L}(u) \equiv \oint_S \underset{(n-2)}{\mathcal{E}^{ab}} \, D_a D_b n_L, \qquad \mathcal{Q}^-_{n,L}(u) \equiv \oint_S \underset{(n-2)}{\mathcal{E}^{ab}} \, \epsilon_{ca} D_b D^c n_L.
$$