Solution to binary black hole dynamics

Sashwat Tanay
LUTH, Observatoire de Paris

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In collaboration with L. C. Stein, G. Cho, J. T. Gálvez Ghersi, and R. Samanta

Plan of the talk

- Introduction and theory

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- 1.5PN: solution to the BBH system

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- 2PN: two new constants of motion
- Conclusions and future avenues

Introduction and theory

Gravitational waves (GWs) from binary black holes

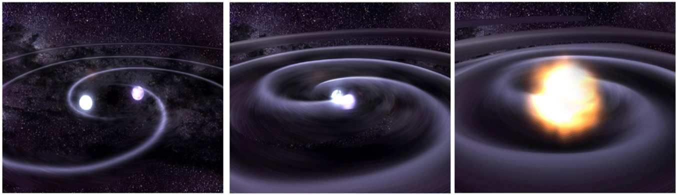

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Image credit: www.eoportal.org

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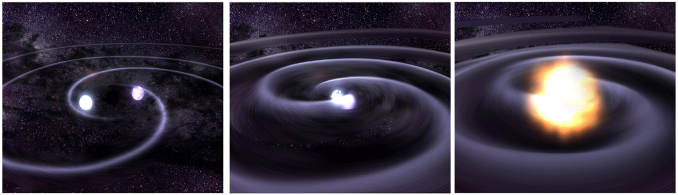

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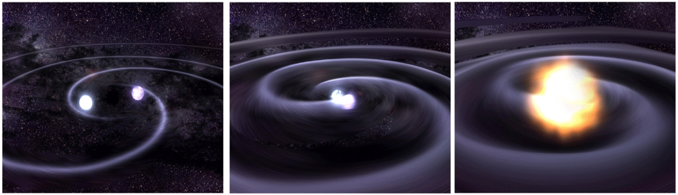

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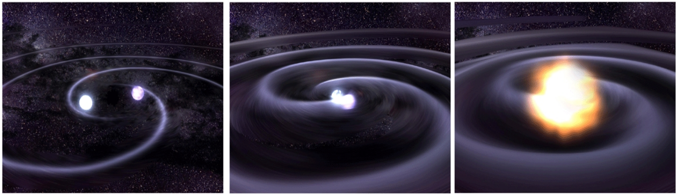

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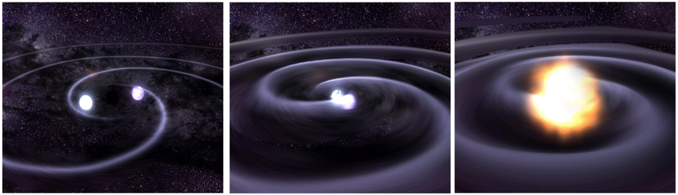

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- GWs are functions of black hole trajectories *(focus of the talk)*.

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- 0PN $\sim$ Newtonian order. The rest are relativistic corrections.

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- **Poisson bracket:** $\{f, g\} = \left(\frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} \right)$.

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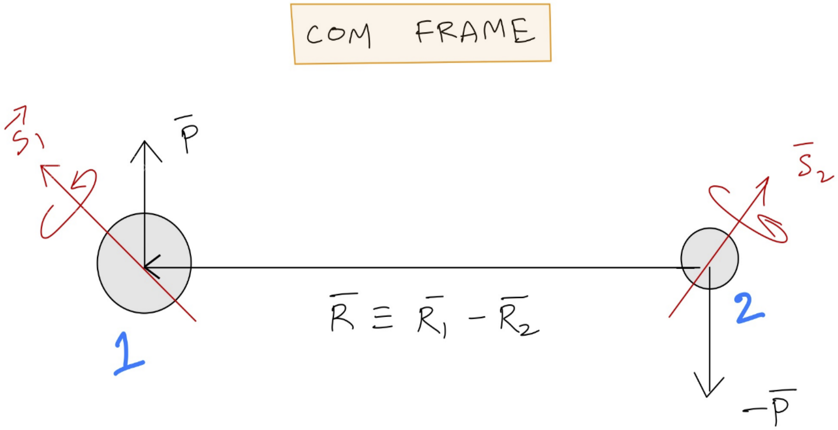

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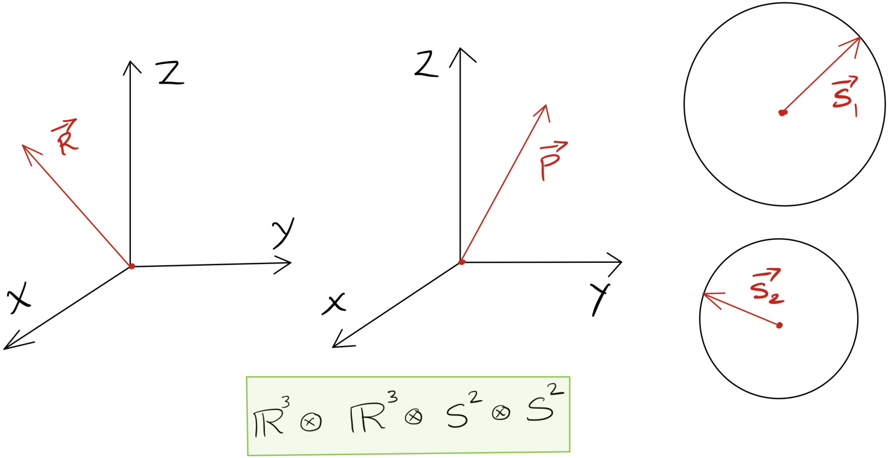

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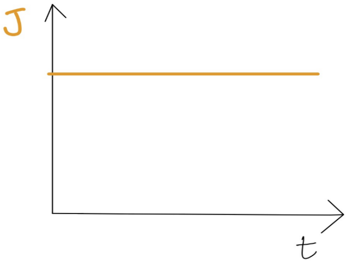

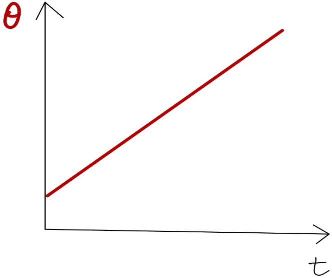

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It's nice to have integrable systems (they occur rarely), and extra nice to have action-angles.

1.5PN: solution to the BBH system

The calculations of this section were long and arduous, but as it turns out, they were merely child's play. At the time of writing, the gravitational waves for binary systems in circular motion have been calculated all the way out to 3.5PN order, and this is a much, much larger challenge. At 2PN order, for example, one finds not only the expected "standard" corrections of order β^4, but also tail contributions generated by the 0.5PN order terms. At 2.5PN order one finds tails generated by the 1PN terms, 1PN corrections to the 1.5PN tail terms, as well as standard 2.5PN terms. At 3PN order there are, in addition to the standard terms, tails generated by the normal 1.5PN terms, 1.5PN corrections to the 1.5PN tail terms, and completely new "tails of tails" terms: tails generated by the 1.5PN tails. These formidable calculations have been carried out by a number of groups around the world, at an enormous cost of labor and sweat (perhaps even blood). There was a strong motivation

Book: Gravity (Eric Poisson & Clifford Will), pg. 614

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- **1966:** 1.5PN Hamiltonian given in Barker et. al (1966).

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General relativistic celestial mechanics of binary systems I. The post-Newtonian motion

by

T. DAMOUR
Groupe d'astrophysique relativiste, E. R. nº 176 du CNRS,
Observatoire de Paris-Meudon, 92195 Meudon Principal Cedex (France)

and

N. DERUELLE
Laboratoire de gravitation et cosmologie relativistes, E. R. A. nº 533 du CNRS,
Institut Henri Poincaré, 11 rue Pierre et Marie Curie, 75005 Paris (France)

Higher-Order Relativistic Periastron Advances and Binary Pulsars.

T. Damour and G. Schäfer (*)
Groupe d'Astrophysique Relativiste, CNRS
DARC, Observatoire de Paris, Section de Meudon - 92195 *Meudon Principal Cedex,*

(ricevuto il 14 Marzo 1988)

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- $m \equiv m_1 + m_2, \qquad \mu \equiv m_1 m_2/m, \qquad \nu \equiv \mu/m, \qquad \vec{L} \equiv \vec{R} \times \vec{P},$
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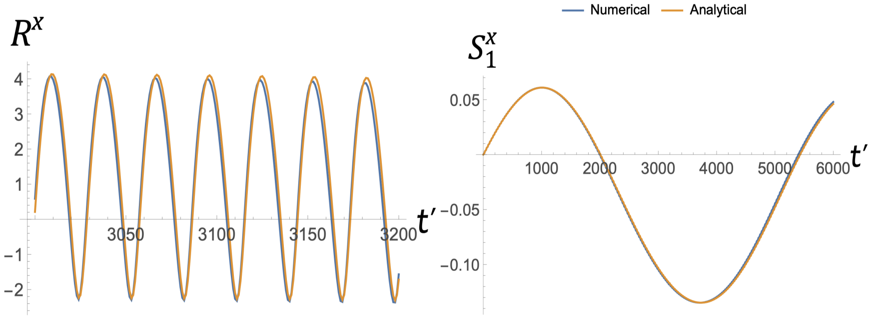
- $\mathcal{J}_5$ is very lengthy.

Moment of truth



(a) (b)

FIG. 2: Comparison of the analytical solutions with the numerical one. For a system with $(m_1, m_2) = (5/2, 1)$ and the initial values of the phase-space variables being $\vec{R} = (2,\, 2, 2)$, $\vec{P} = (1/2,\, -1/2, 1/3)$, $\vec{S_1} = \sqrt{\epsilon}\;\; (0, 1,\, 1)$, $\vec{S_2} = \sqrt{\epsilon}\; (1,\, -3/10,\, 0)$. Subfigures (a) and (b) show evolution of x-component of $\vec{R}$ and $\vec{S}_1$, respectively. We choose $\epsilon = 0.003$ for (a) and $\epsilon = 0.01$ for (b). All this results in a Newtonian-orbital time period of $T_N \sim 29$ for both (a) and (b), and the PN parameter ~ 0.0036 for (a) ~ 0.012 for (b) respectively. Throughout we keep $G = 1$.

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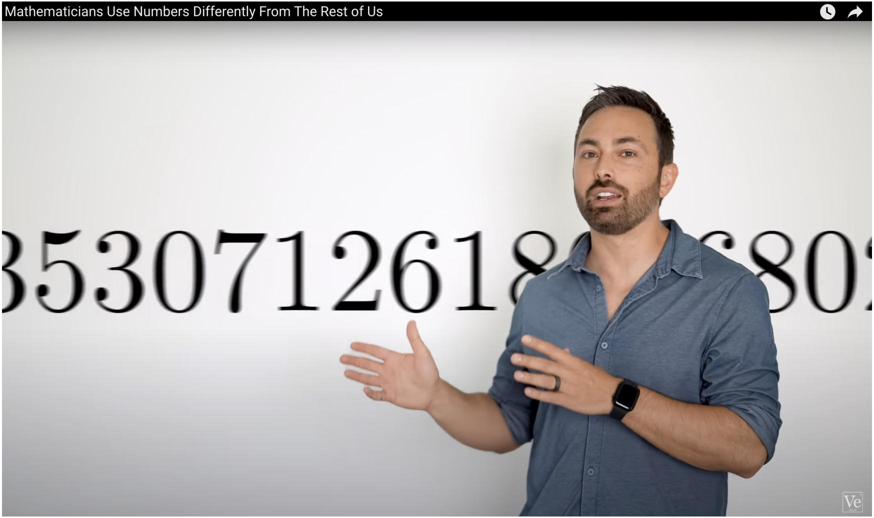
- **(In)famous example:** Fermat's last theorem [YouTube:Veritasium].

Veritasium on p-adic numbers



Link: youtu.be/tRaq4aYPzCc

2PN: two new constants of motion

History: are PN BBHs chaotic or integrable?

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- See the Introduction of [gr-qc:0511009] and [2012.06586] for details.

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The fourth commuting constant of motion

With the definitions:

$$\begin{aligned} \sigma_1 &:= (2 + 3m_2/m_1) \\ \sigma_2 &:= (2 + 3m_1/m_2) \\ \vec{S}_{\rm eff} &:= \sigma_1 \vec{S_1} + \sigma_2 \vec{S_2} \\ \vec{L} &:= \vec{R} x \vec{P} \\ \epsilon &:= 1/c^2 \end{aligned}$$

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\end{aligned}
$$

Conclusions and future avenues

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- **2PN:** Found 2 new (PN perturbative) constants of motion, thereby establishing the integrable nature of the BBH.

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Summary

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Afterthoughts: (1) Nature is complex; elaborate math unavoidable
(2) Using classical mechanics to do GW research.

Future avenues

- Find 2PN action-angles using canonical pert. theory $\to$ extend QKP elements (a, e_t, e_r, e_ϕ, n) to 2PN spinning systems.

Hint: $(H^{1.5\mathrm{PN}}, J^2, J_z, L^2, \vec{S}_{\mathrm{eff}} \cdot \vec{L}) \to (H^{2\mathrm{PN}}, J^2, J_z, \widetilde{L^2}, \widetilde{\vec{S}_{\mathrm{eff}} \cdot \vec{L}})$

$$
\begin{aligned}
a_r &= -\frac{1}{2h}\left(1 - \frac{1}{2}(\nu - 7)\frac{h}{c^2} - 2\frac{s_{\mathrm{eff}} \cdot l}{l^2}\frac{h}{c^2}\right), \\
e_r^2 &= 1 + 2hl^2 - 2(6 - \nu)\frac{h}{c^2} - 5(3 - \nu)\frac{h^2 l^2}{c^2} \\
&\quad + 8\left(1 + hl^2\right)\frac{s_{\mathrm{eff}} \cdot l}{l^2}\frac{h}{c^2}, \\
n &= (-2h)^{3/2}\left(1 + \frac{2h}{8c^2}(15 - \nu)\right),
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- Add radiation reaction via $\dot{\vec{C}} = \vec{f}(\vec{C})$. $\vec{C} \equiv (H^{1.5\mathrm{PN}}, J^2, J_z, L^2, \vec{S}_{\mathrm{eff}} \cdot \vec{L})$.
  Hint: Spins don't shrink. We may need only $\dot{L^2}, \dot{H}$. But how to integrate?

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- Prove integrability at 3PN.

Refs:

- Papers: 2012.06586, 2110.15351, 2210.01605.
- Lecture notes: 2206.05799
- Mathematica package: github.com/sashwattanay/BBH-PN-Toolkit
- YouTube video on the package
- Contact: sashwat.tanay@obspm.fr

Thank you!
Questions?