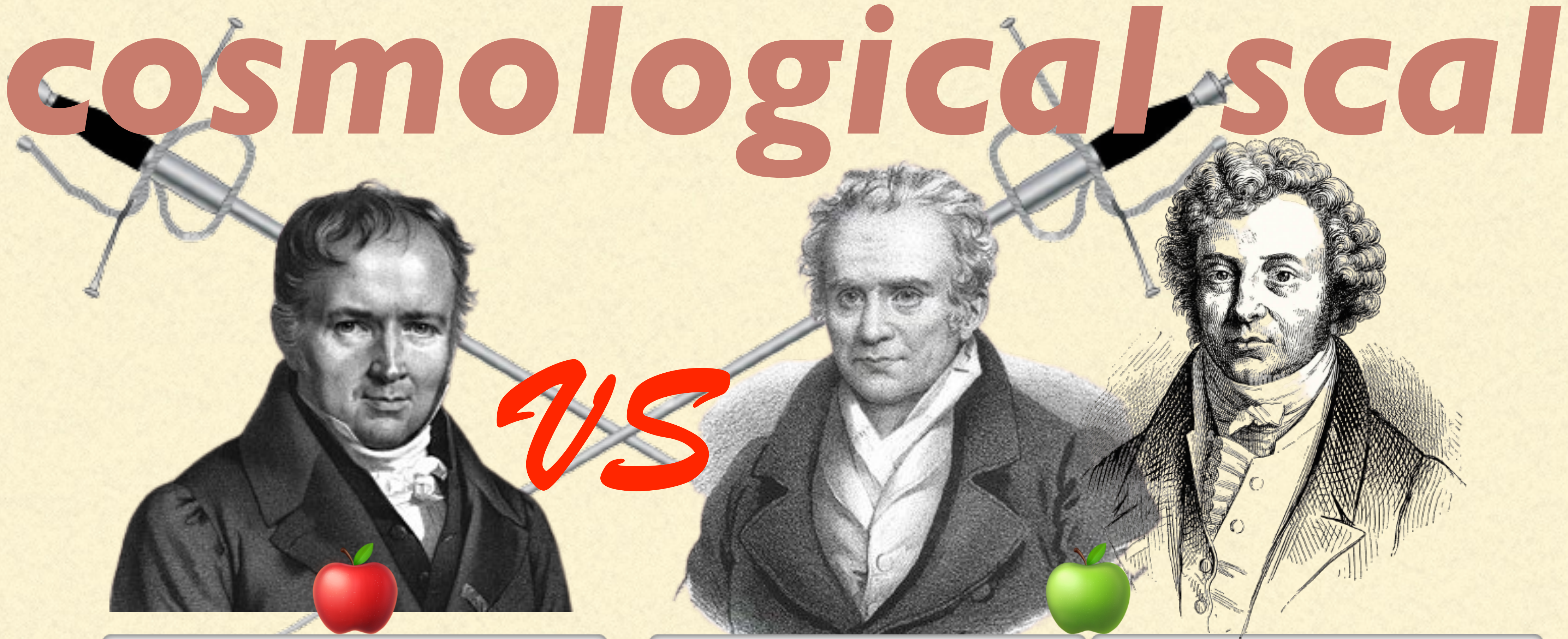
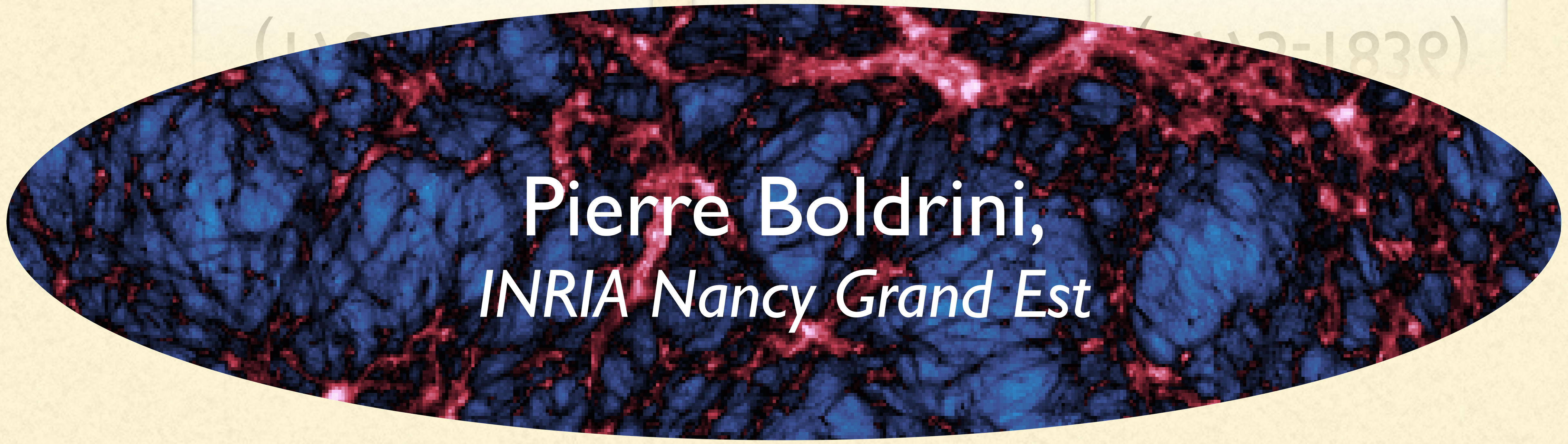
Monge-Ampère gravity at cosmological scales



Siméon Denis Poisson (1781-1840)	Gaspard Monge (1746-1818)	André-Marie Ampère (1775-1836)



Pierre Boldrini,
INRIA Nancy Grand Est

An interdisciplinary project
(*Mathematics, Computer science, Astrophysics*)

in collaboration with:

Yann Brenier — Roya Mohayaee — Bruno Lévy

ENS | PSL — IAP — Inria

based on:

Boldrini et al. 2022, in prep.

Atelier Univers, Théorie et Gravitation, Montpellier 2022

From Poisson to Monge-Ampère

$$\frac{d^2x(t)}{dt^2} = -\nabla\phi(t)$$

$$\Delta\phi = 4\pi G(\rho - \bar{\rho})$$

Poisson equation

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Poisson equation

$$\downarrow$$

$$det(\mathbb{I} + \frac{1}{4\pi G\bar{\rho}} D^2\phi) = \frac{\rho}{\bar{\rho}}$$

Monge-Ampère equation

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↓

$$det(\mathbb{I} + \frac{1}{4\pi G\bar{\rho}} D^2\phi) = \frac{\rho}{\bar{\rho}}$$ Monge-Ampère equation

$$D^2 \rightarrow \left(\frac{d^2}{dx_i dx_j}\right)_{i,j}$$

From Poisson to Monge-Ampère

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$$\left(\frac{d^2}{dx_i dx_j} \right)_{i,j}$$

In **one dimension**, Monge-Ampère is **equivalent** to Poisson

From Poisson to Monge-Ampère

A discret set of
N particles

$$i = 0,1,\ldots,N$$

Poisson

$$\Delta\phi = 4\pi G(\rho - \bar{\rho})$$

Monge-Ampère

$$det(\mathbb{I} + \frac{1}{4\pi G\bar{\rho}} D^2\phi) = \frac{\rho}{\bar{\rho}}$$

$$F_g = -m\nabla_x\phi(x)$$

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$$F_g = ?$$

From Poisson to Monge-Ampère

$$det(\mathbb{I} + \frac{1}{4\pi G\bar{\rho}}\ D^2\phi) = \frac{\rho}{\bar{\rho}} \qquad \& \qquad F_{\mathrm{g}} = -\, m\,\nabla_x \phi(x)$$

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With the following change of variable,

$$\psi(x) = \frac{|x|^2}{2} + \frac{1}{4\pi G\bar{\rho}}\phi(x)$$

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From Poisson to Monge-Ampère

Monge-Ampère equation

$$det(D^2\psi) = \frac{\rho}{\bar{\rho}} \quad \Longleftrightarrow$$

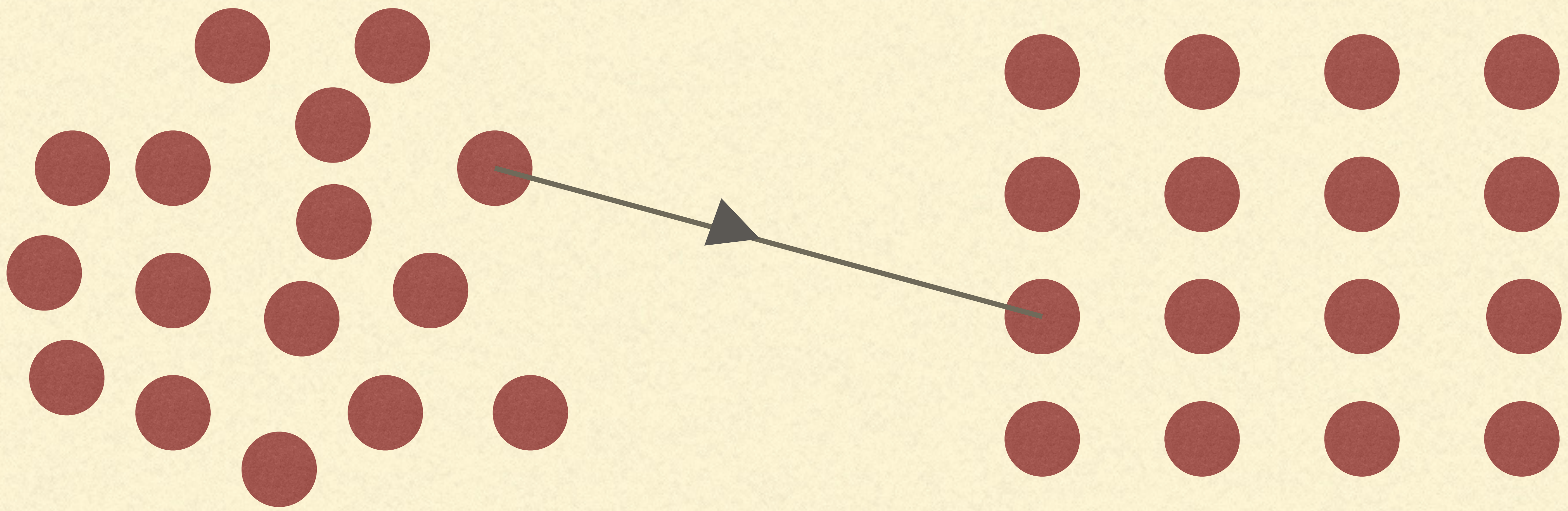
From Poisson to Monge-Ampère

Monge-Ampère equation

$$det(D^2\psi) = \frac{\rho}{\bar{\rho}}$$

$\Longleftrightarrow$

Monge problem
or
What is the most efficient way of transporting one distribution of mass into another?



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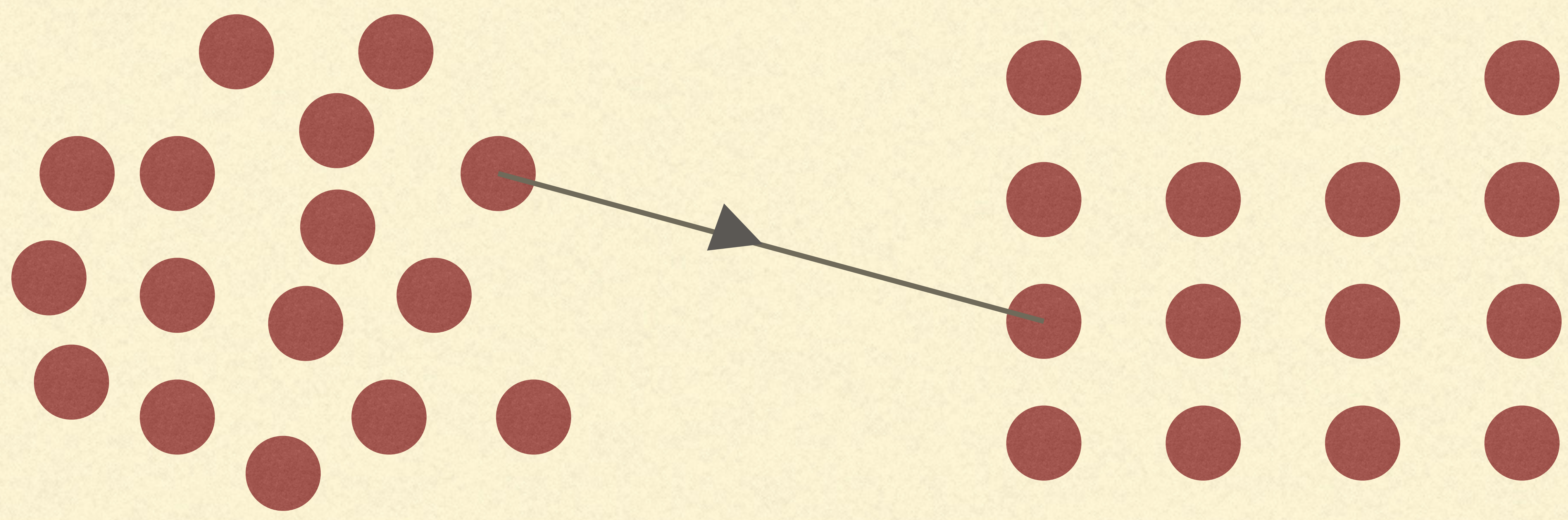

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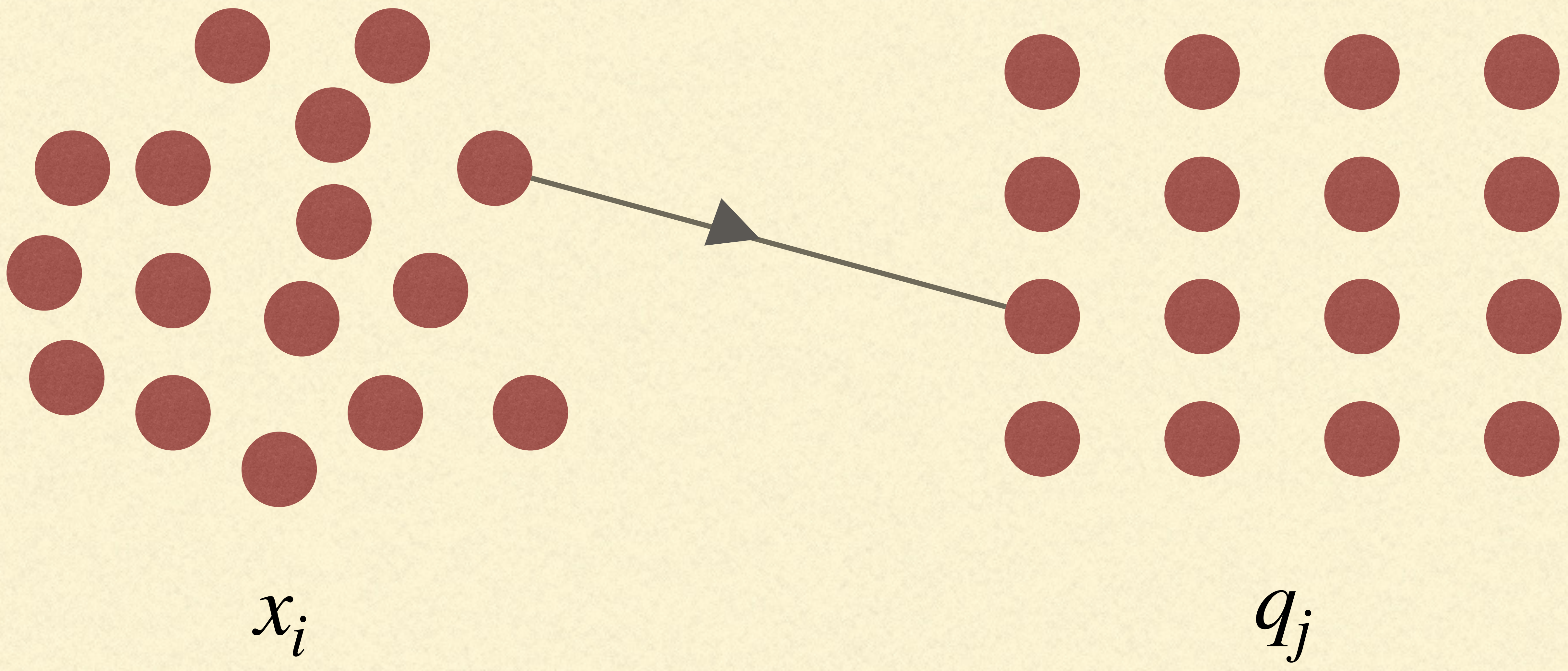

$$\inf \sum_i |x_i - q_j|^2$$

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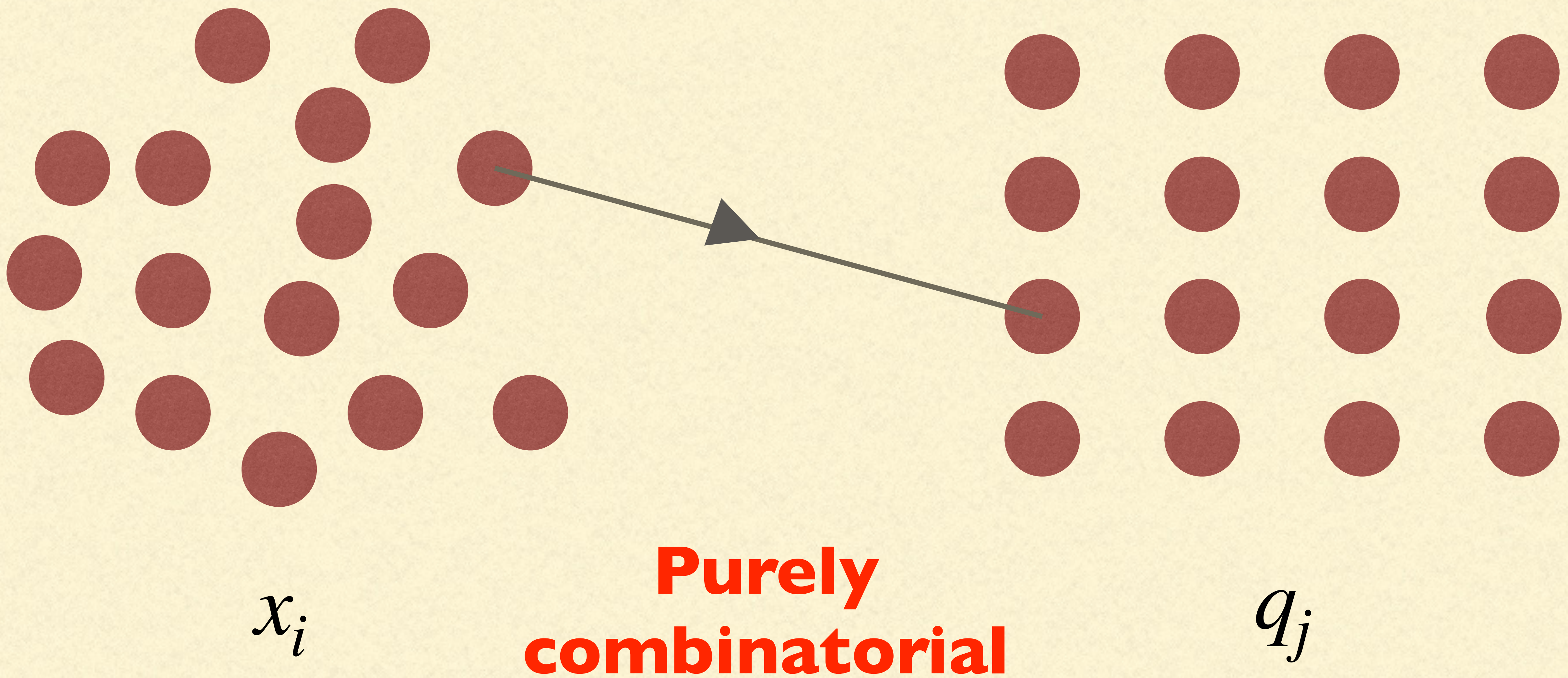

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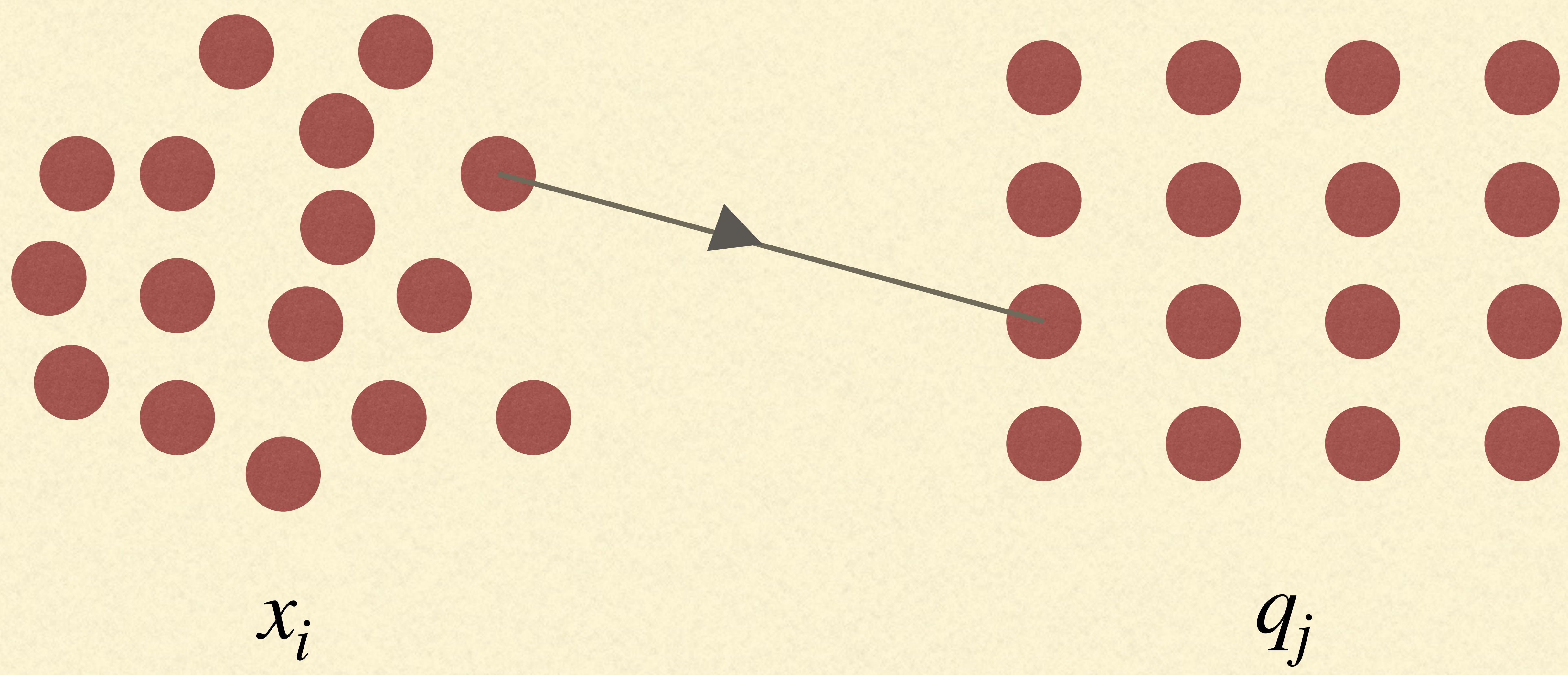

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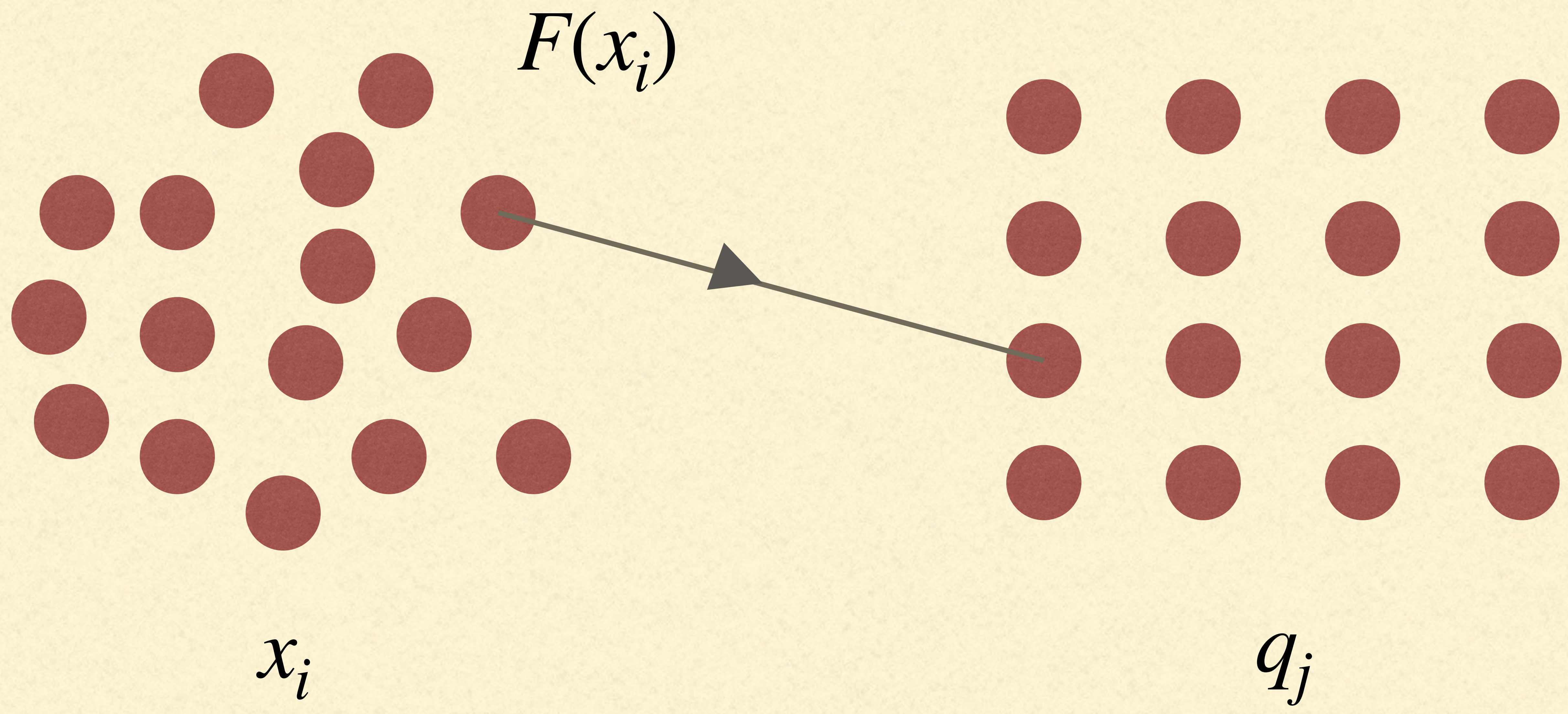

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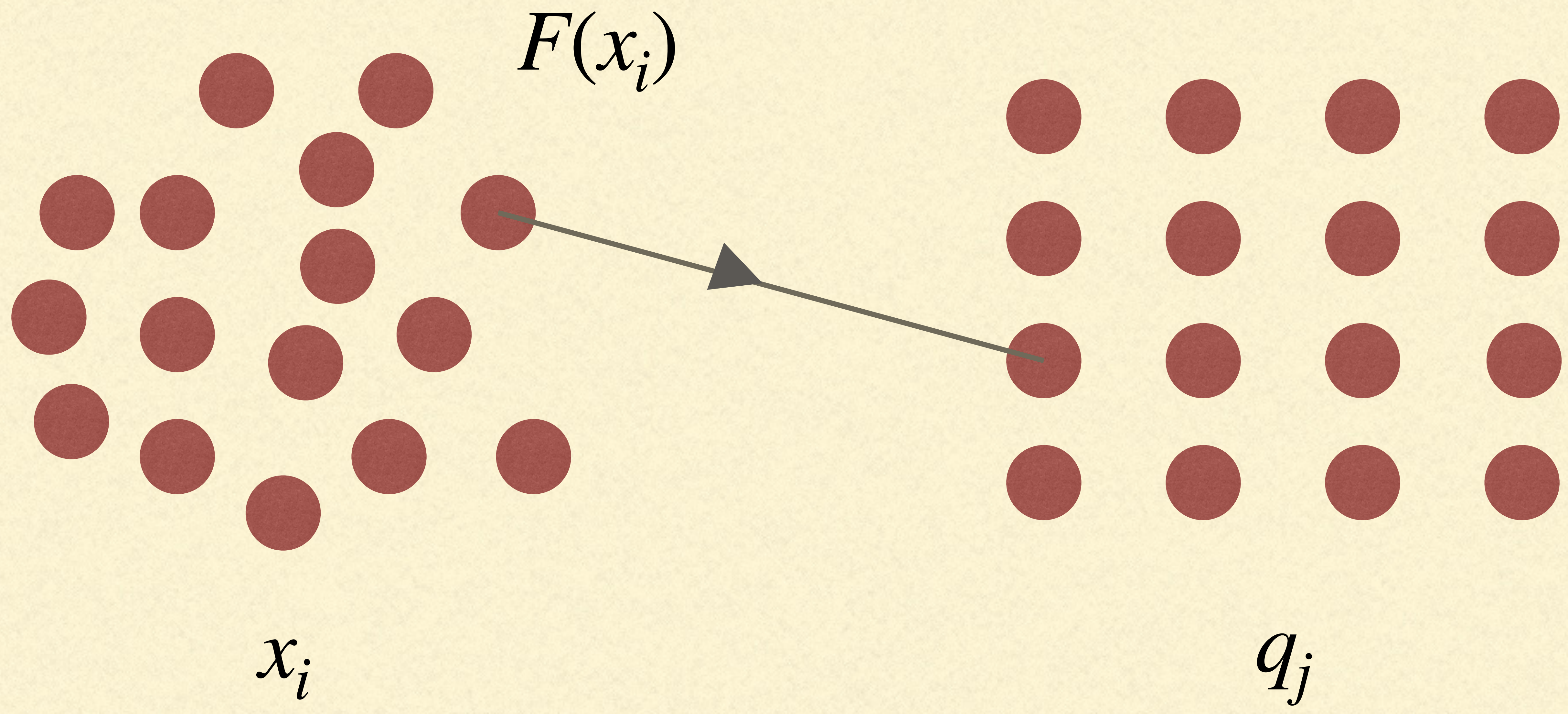
$$\bar{\rho} d^3 q = \rho(\mathbf{x}) d^3 x$$



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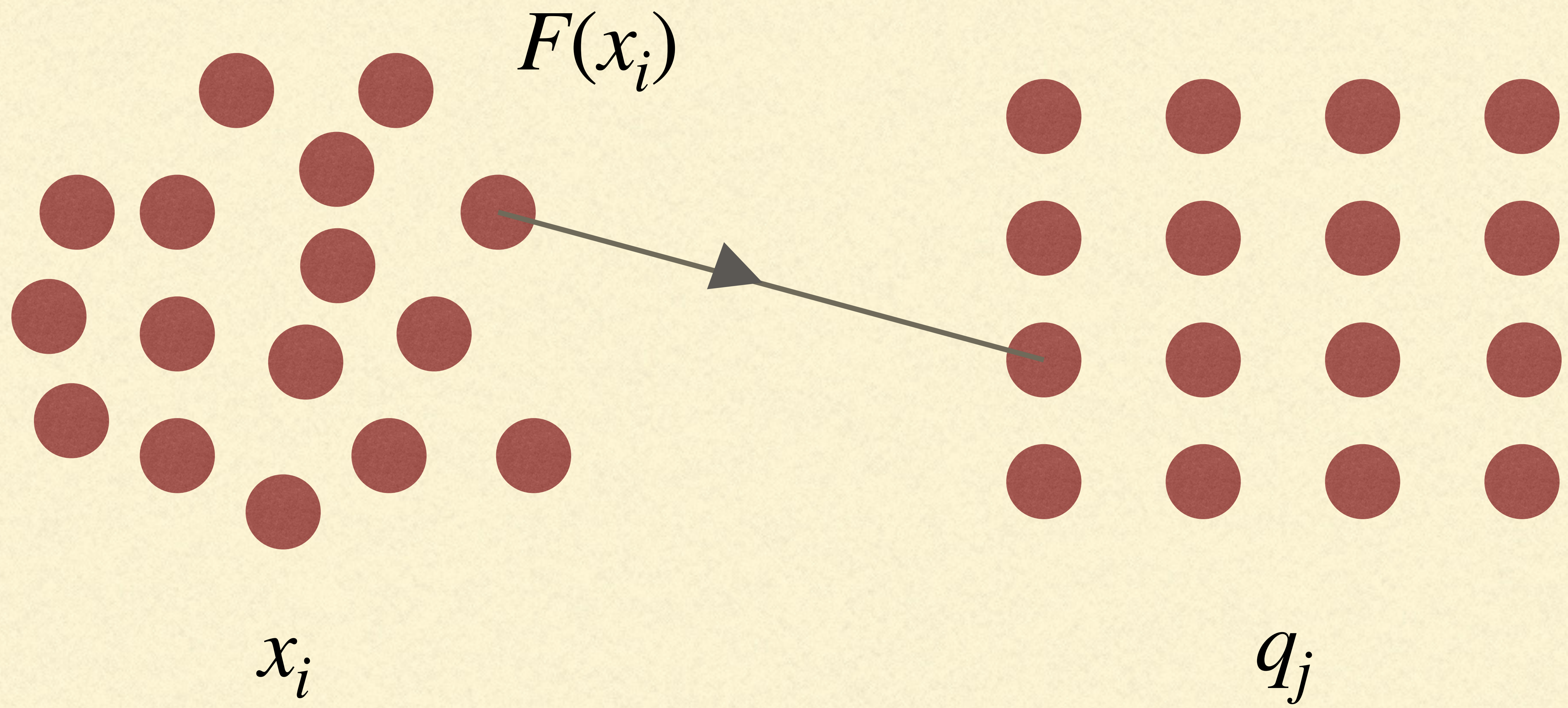

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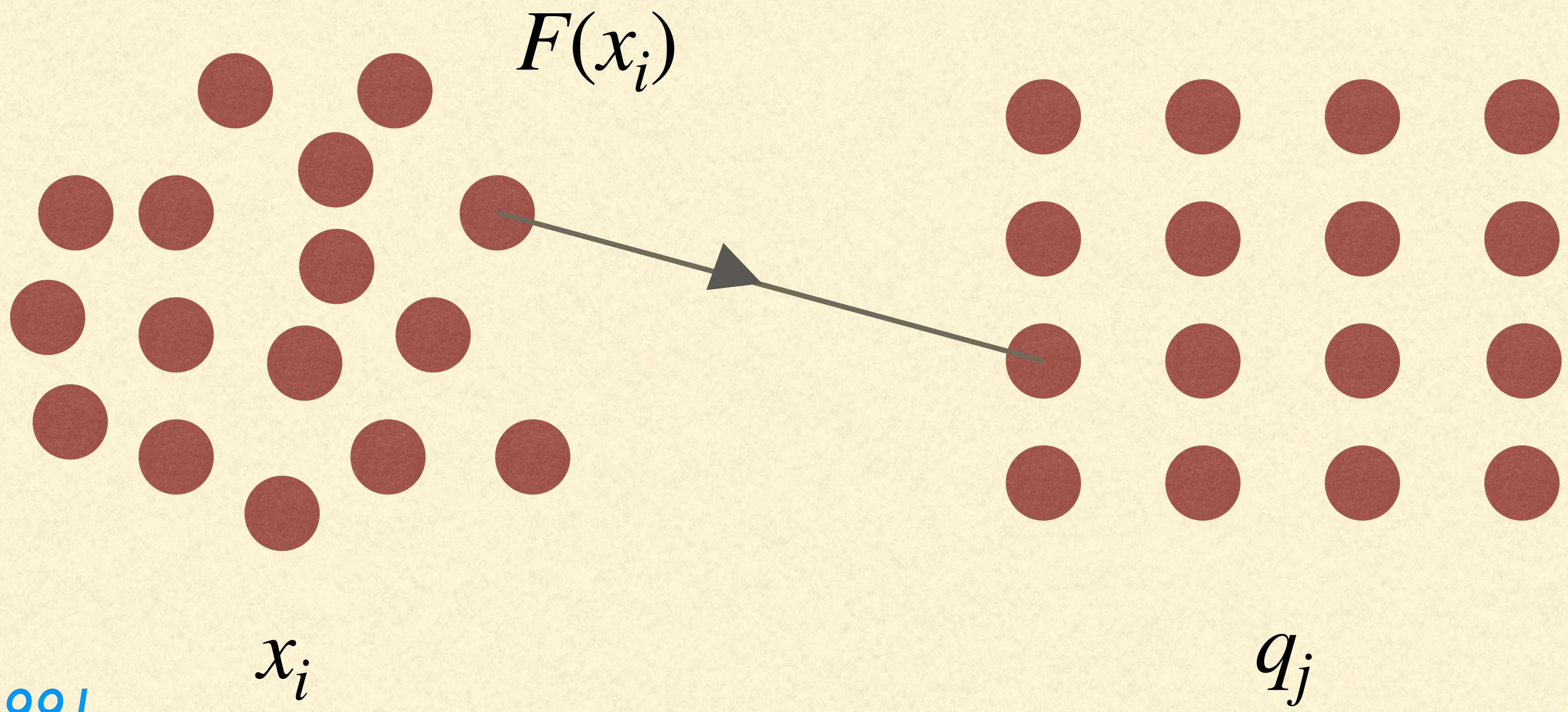

According to **Optimal Transport** theory, *Brenier 1991*

$$F = \nabla_x \psi(x)$$

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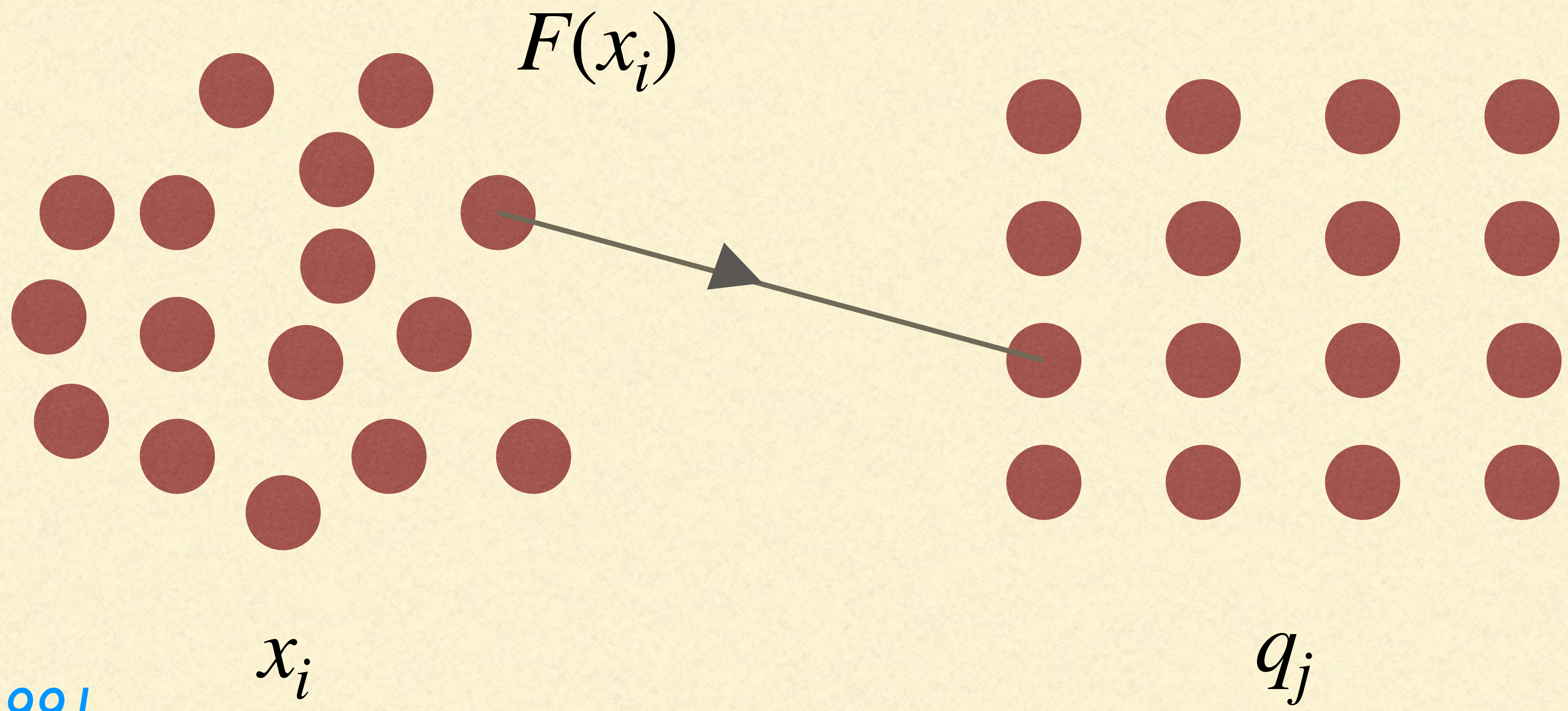

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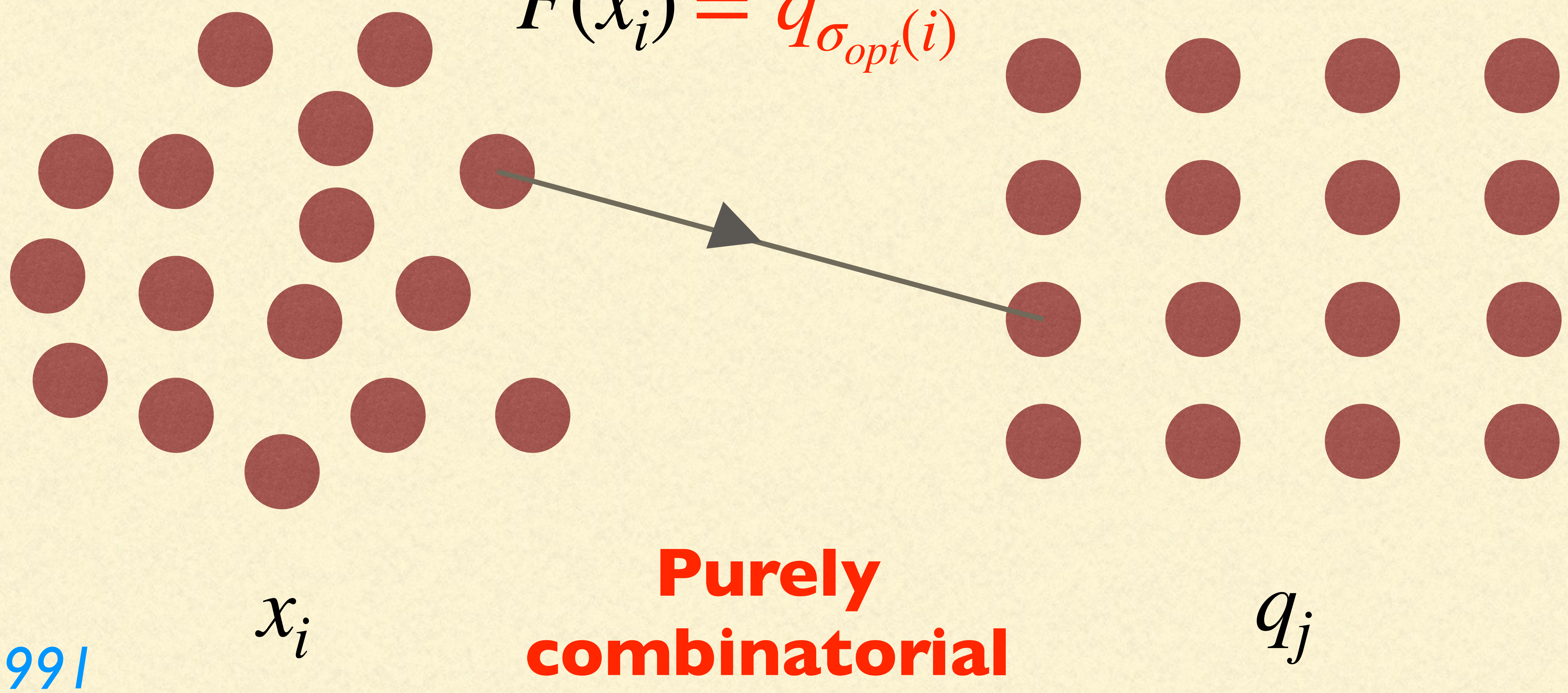

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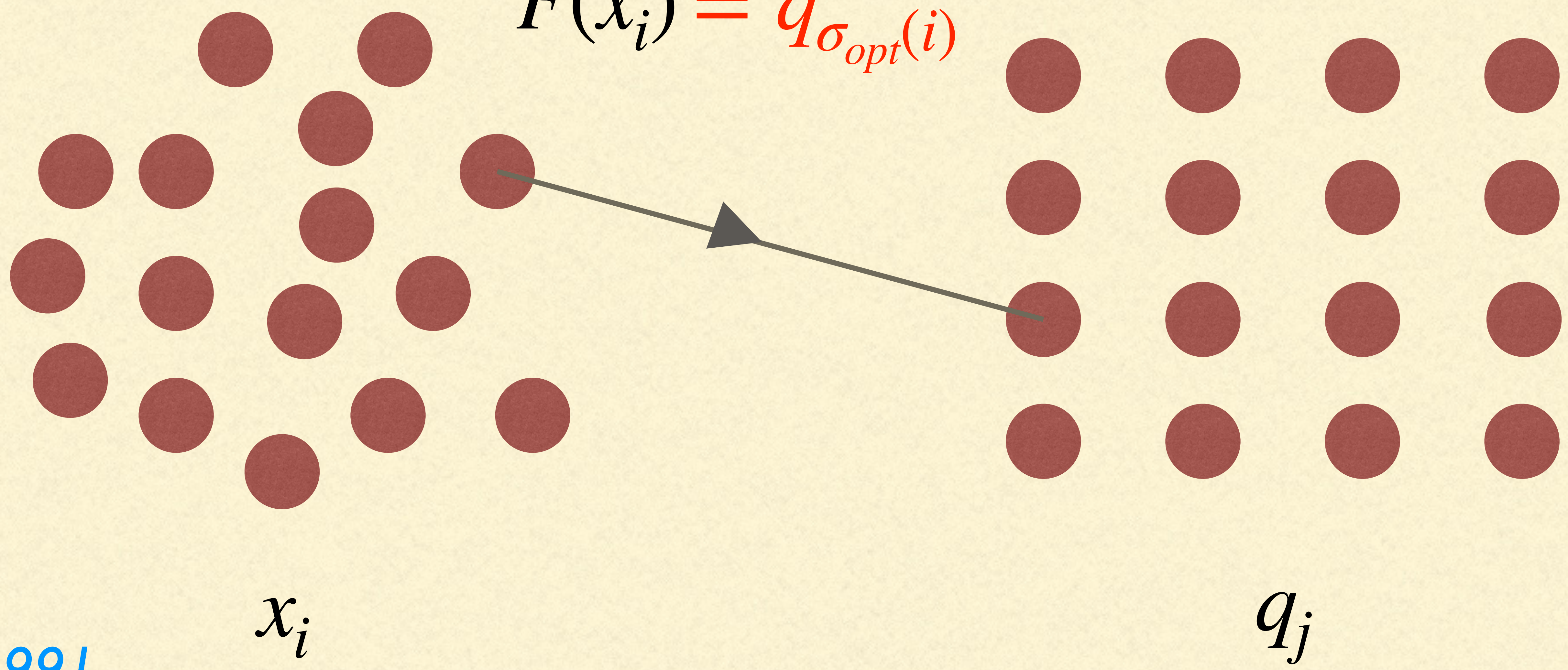

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Why Monge-Ampère gravity?

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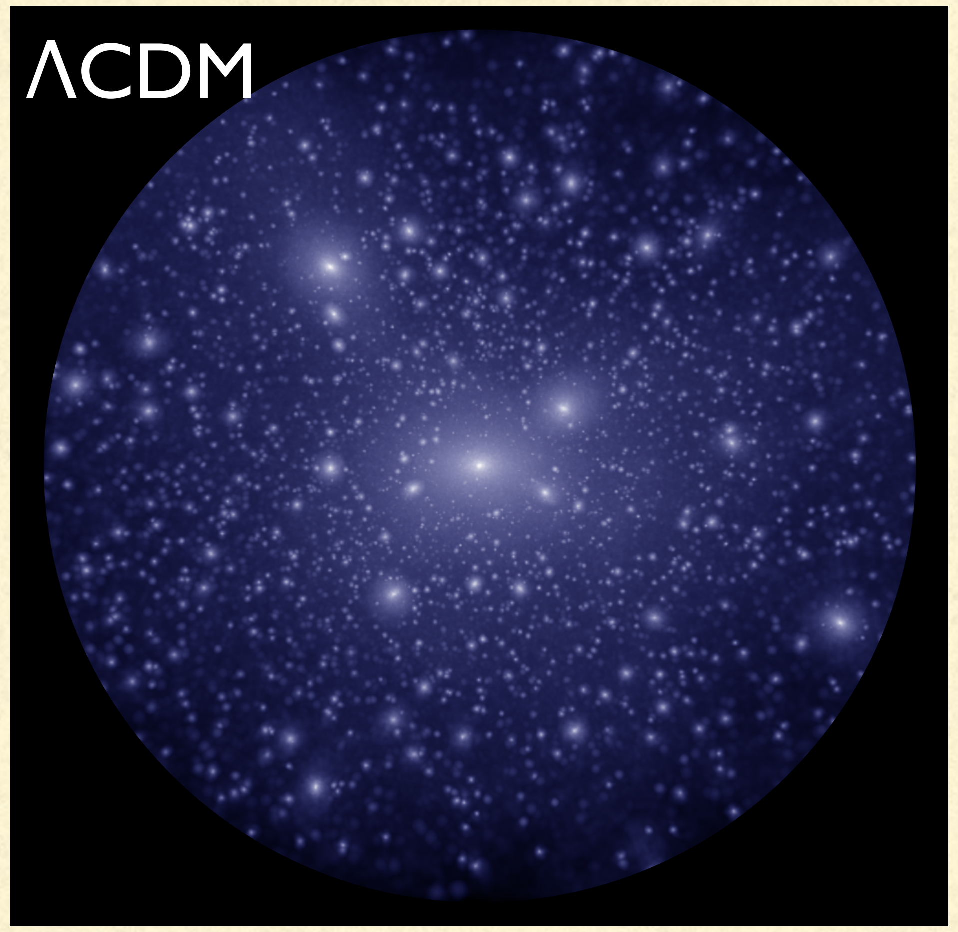
1. Challenges to the ΛCDM Paradigm

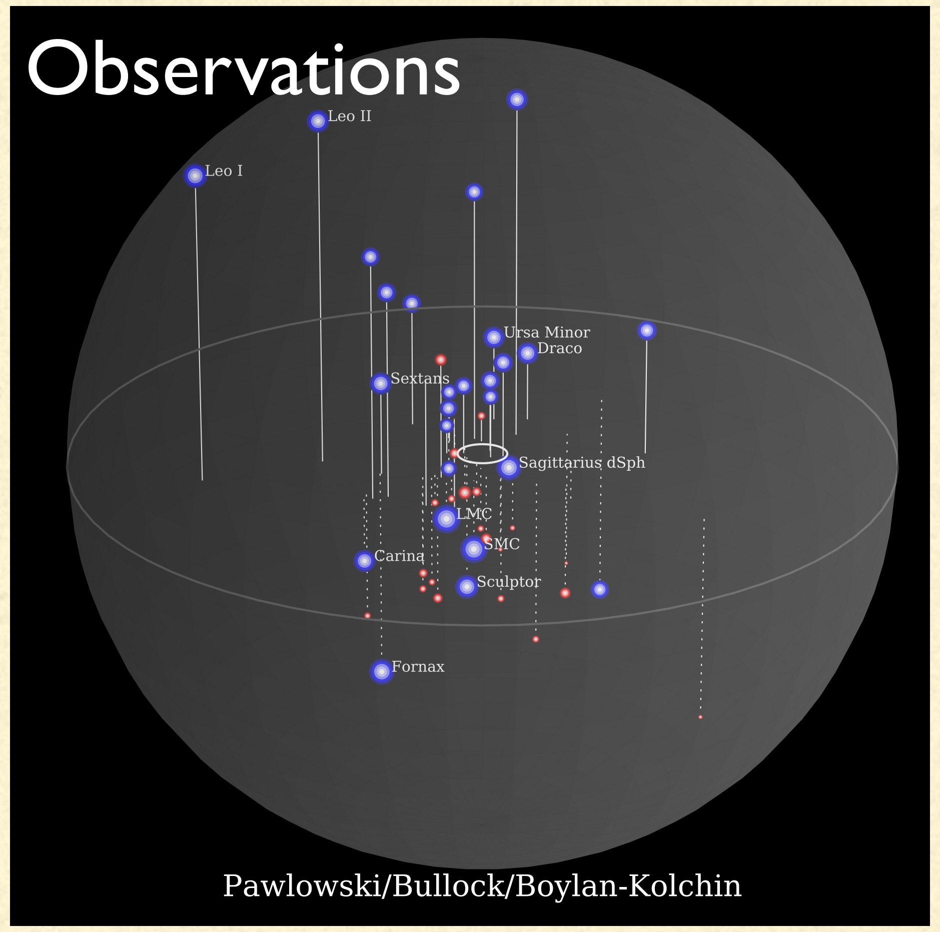
Why Monge-Ampère gravity?

1 Challenges to the ΛCDM Paradigm

At small scales,

Missing satellite problem





$r \leq 250$ kpc | ~1000-2000 satellites | ~ 50 satellites

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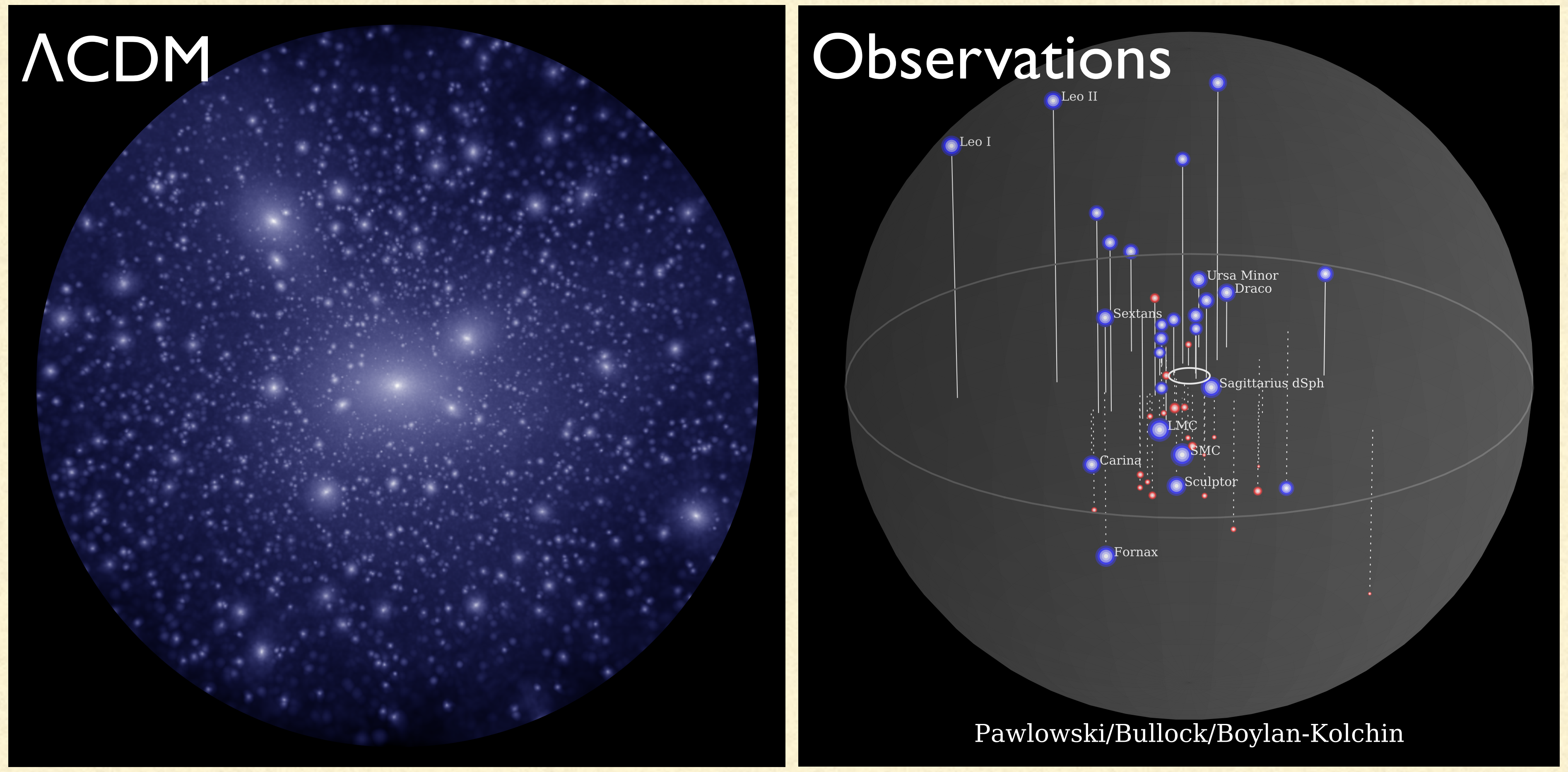

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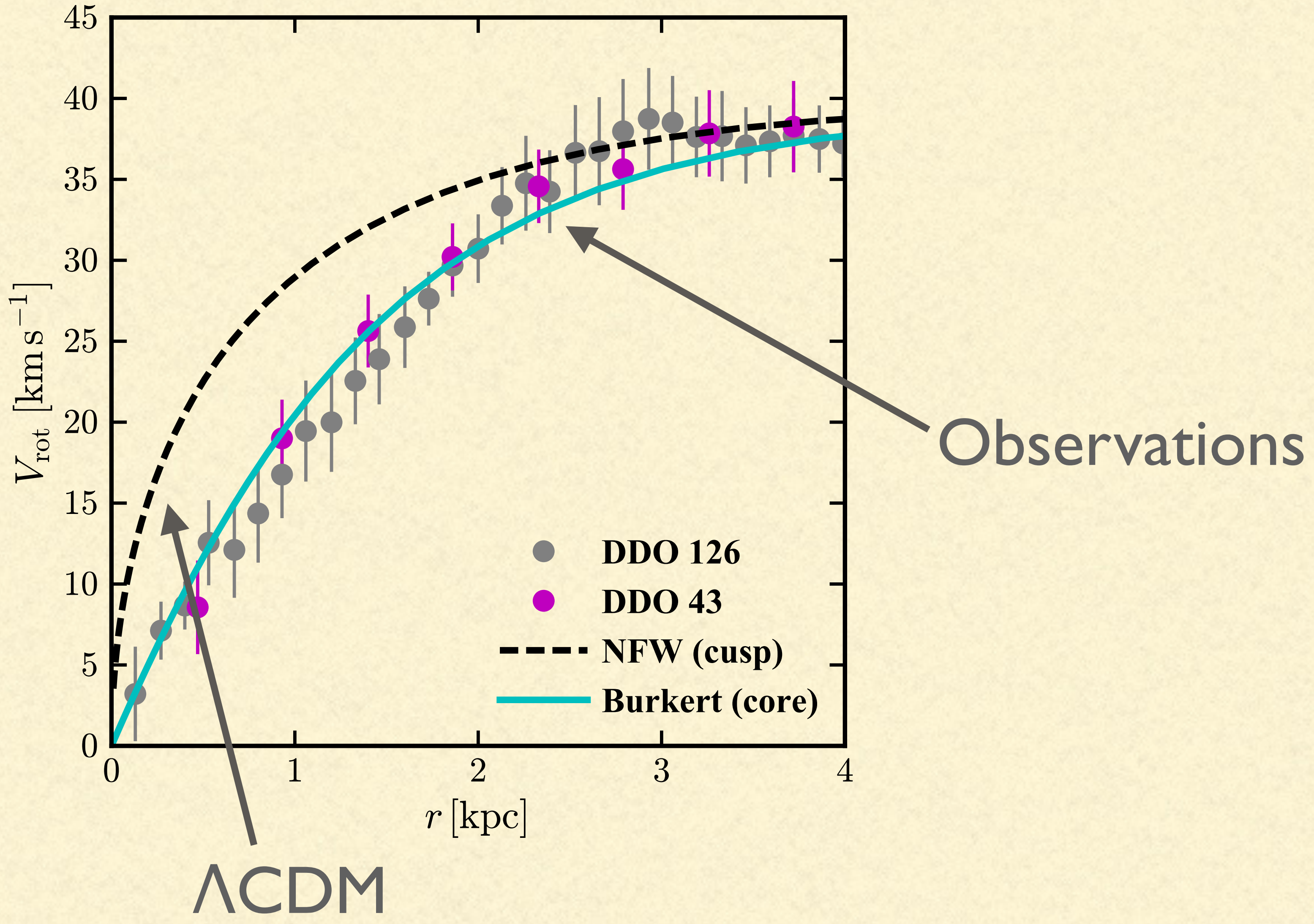
Cusp-core problem



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$$det(\mathbb{I} + \gamma D^2 \phi) = \frac{\rho}{\bar{\rho}}$$

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1. Challenges to the ΛCDM Paradigm

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$$det(\mathbb{I} + \gamma D^2\phi) = \frac{\rho}{\bar{\rho}} \longrightarrow 1 + \gamma\Delta\phi + \mathcal{O}(\gamma^2) = \frac{\rho}{\bar{\rho}}$$

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Large deviation principle + Brownian motion ⟶

$$\frac{d^2 x_i}{dt^2} = 4\pi G\bar{\rho}\left(x_i - g_i\right)$$

Brenier et al. 2012

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Monge-Ampère gravitational force

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VS

$$F_g = 4\pi G \bar{\rho} \left(x_i - g_i \right)$$

Cosmological simulation of Monge-Ampère gravity

Cosmological simulation of Monge-Ampère gravity

- Initial conditions
- Equations of motion in comoving coordinates
- How it works numerically?
- Comparing with Poisson *N*-body cosmological simulations
- Results

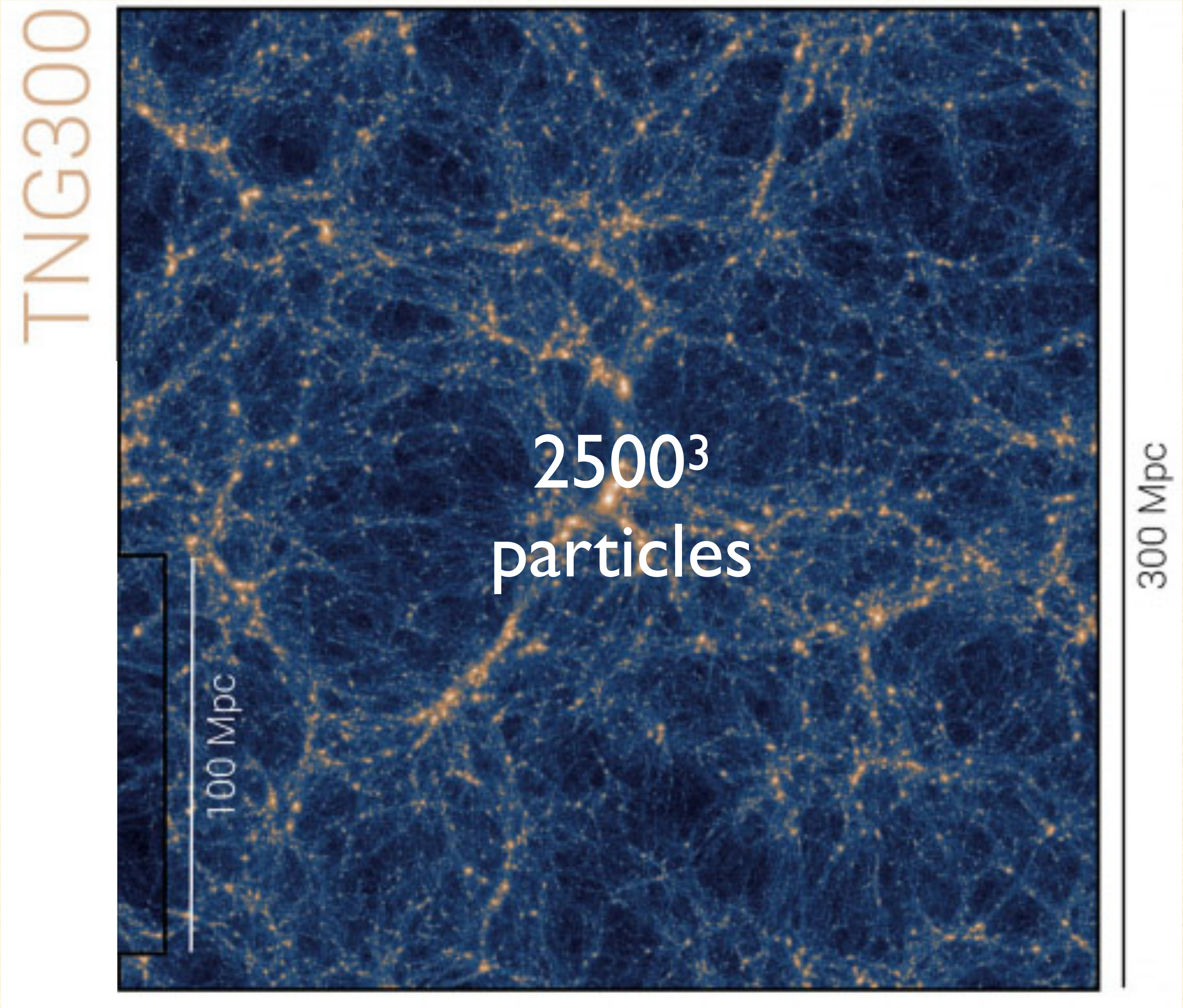
pyMAG 1.0

```
pip install pyMAG
```

Soon ….

Cosmological simulation of Monge-Ampère gravity

Initial conditions



[Springel et al. 2018]
(arXiv:1707.03397)

512^3
particles

205 Mpc/h ~ 300 Mpc

$z = 49 \longrightarrow z = 0$

$\Omega_{\rm m} = 0.3089$

$\Omega_\Lambda = 0.6911$

$H_0 = 67.74 \text{ km s}^{-1} \text{ Mpc}^{-1}$

$m_{DM} \sim 10^{10} M_\odot$

Cosmological simulation of Monge-Ampère gravity

- Comparing with standard N-body cosmological simulation

Poisson	Monge-Ampère

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GADGET - 2 A code for cosmological simulations of structure formation	pyMAG 1.0
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Cosmological simulation of Monge-Ampère gravity

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Poisson

GADGET - 2

A code for cosmological simulations of structure formation

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$$\mathbf{F}_{\mathrm{g}} = \sum_{j=0, i\neq j}^{N-1} \frac{G m_i m_j (\mathbf{x_j} - \mathbf{x_i})}{a^2(|\mathbf{x_j} - \mathbf{x_i}|^2 + \epsilon^2)^{3/2}}$$

Monge-Ampère

pyMAG 1.0

Boldrini et al. 2022, in prep

$$\mathbf{F}_{\mathrm{g}} = \frac{3 H_0^2 \Omega_{\mathrm{m}} m}{2a^2} \left(\mathbf{x_i} - \mathbf{g_i}\right)$$

Cosmological simulation of Monge-Ampère gravity

Comparing with standard N-body cosmological simulation

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Tree-Code

Barnes and Hut, 1986

$$\mathcal{O}(N \log N)$$

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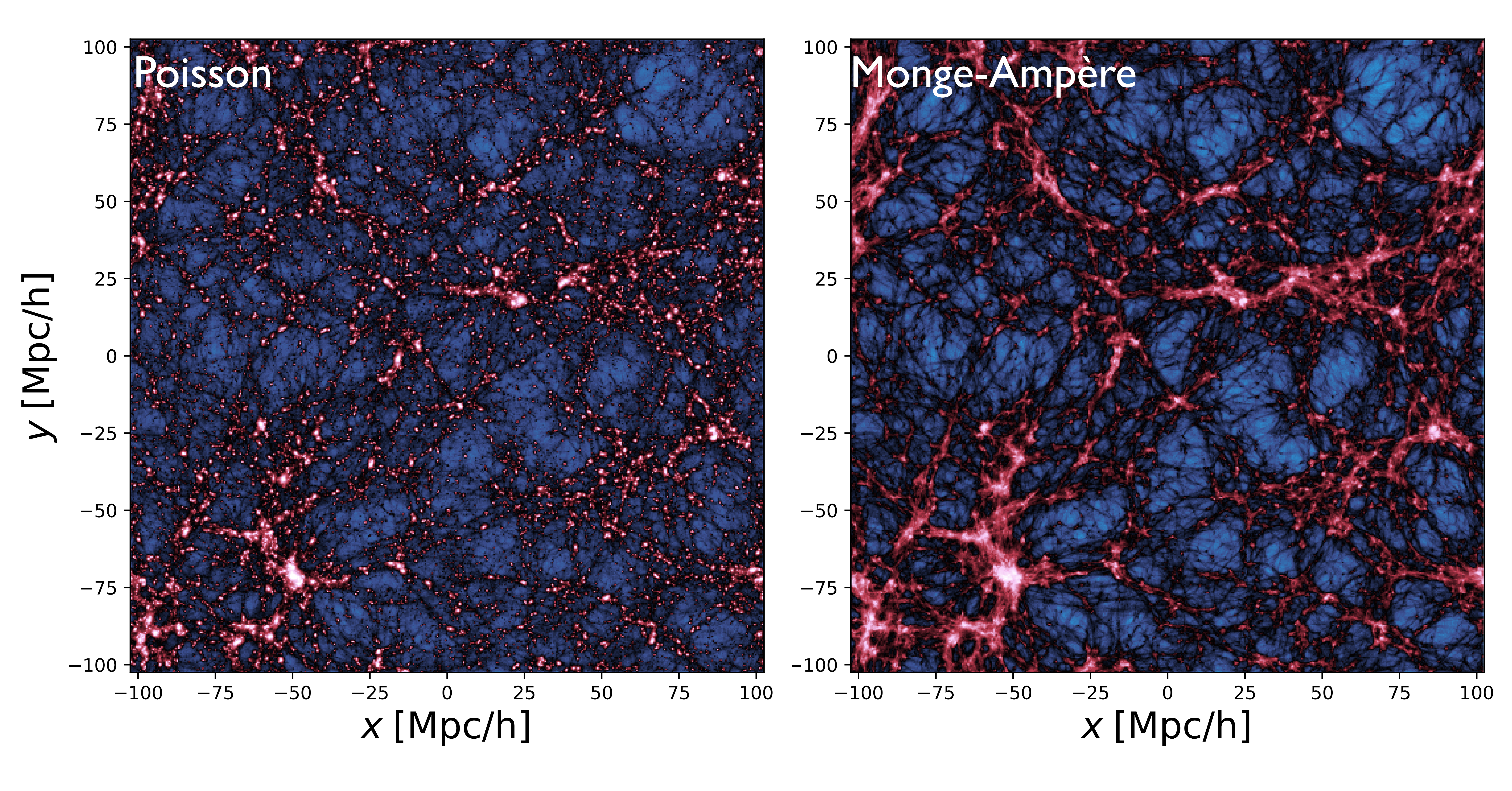
Tree-Code	Optimal transport algorithm
Barnes and Hut, 1986	*Lévy 2022*
$\mathcal{O}(N \log N)$	$\mathcal{O}(N^2 \log N)$

Cosmological simulation of Monge-Ampère gravity

- Large scale-structures $z = 0$



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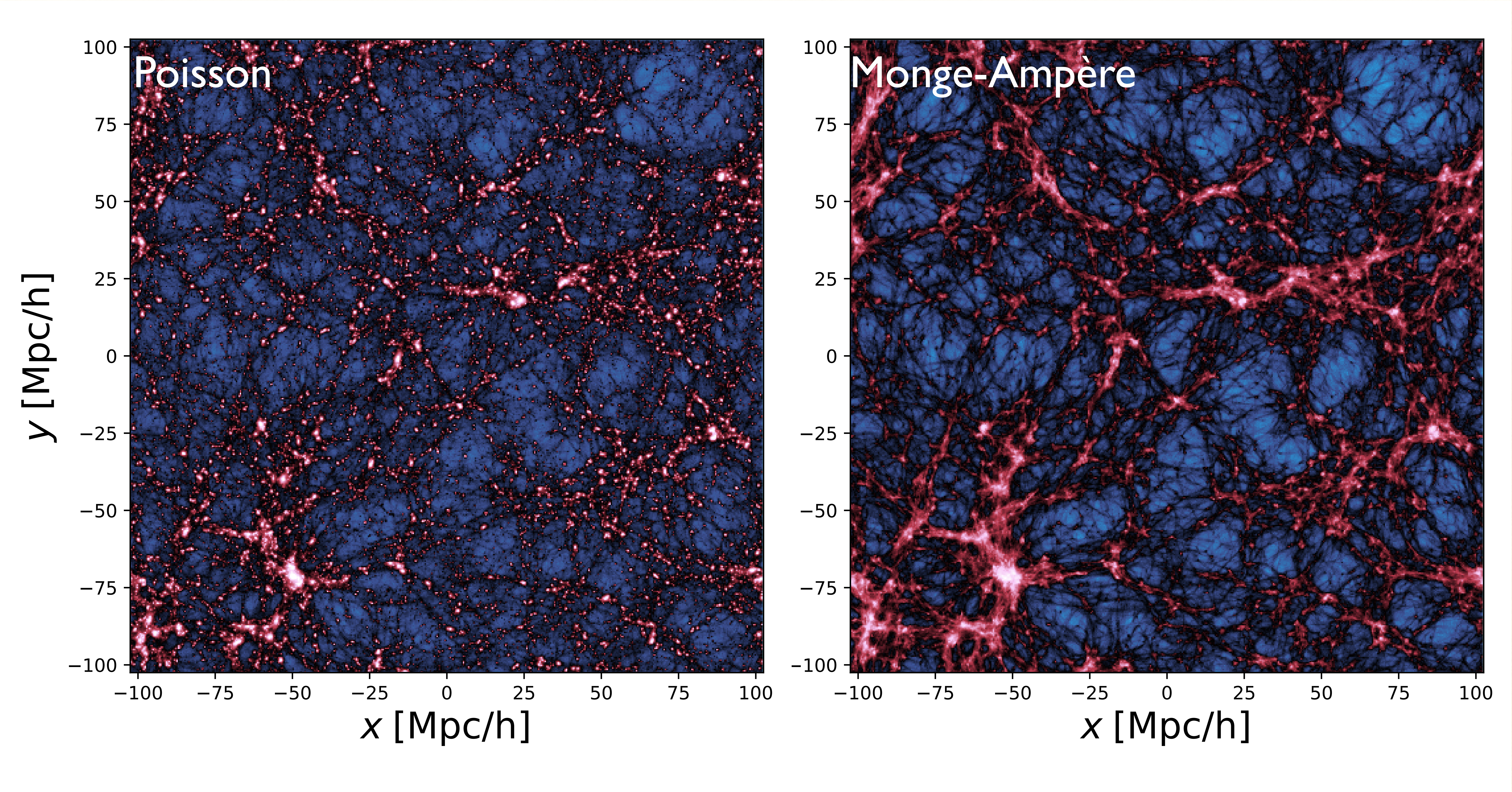
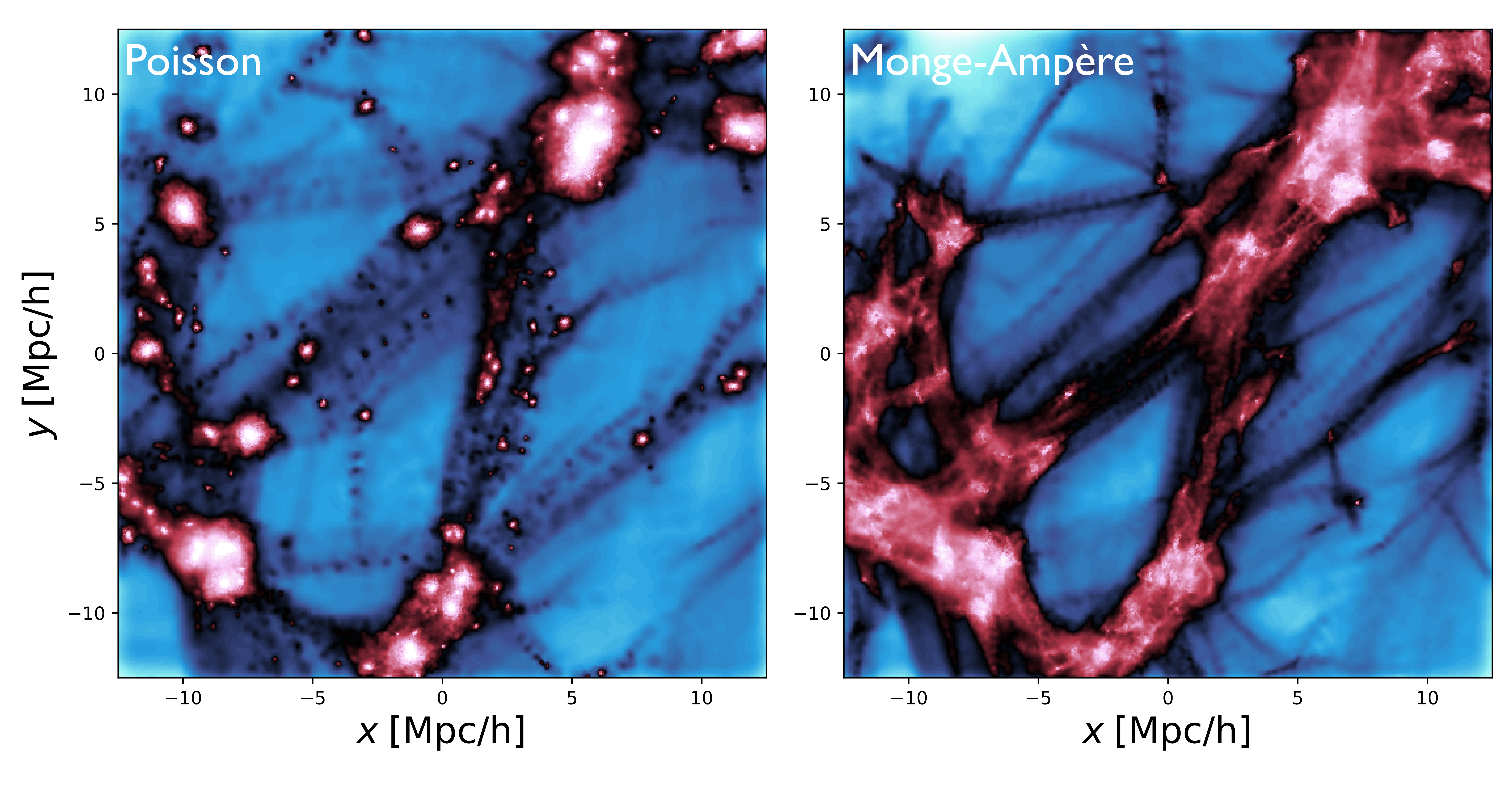


A **weaker gravitational** clustering

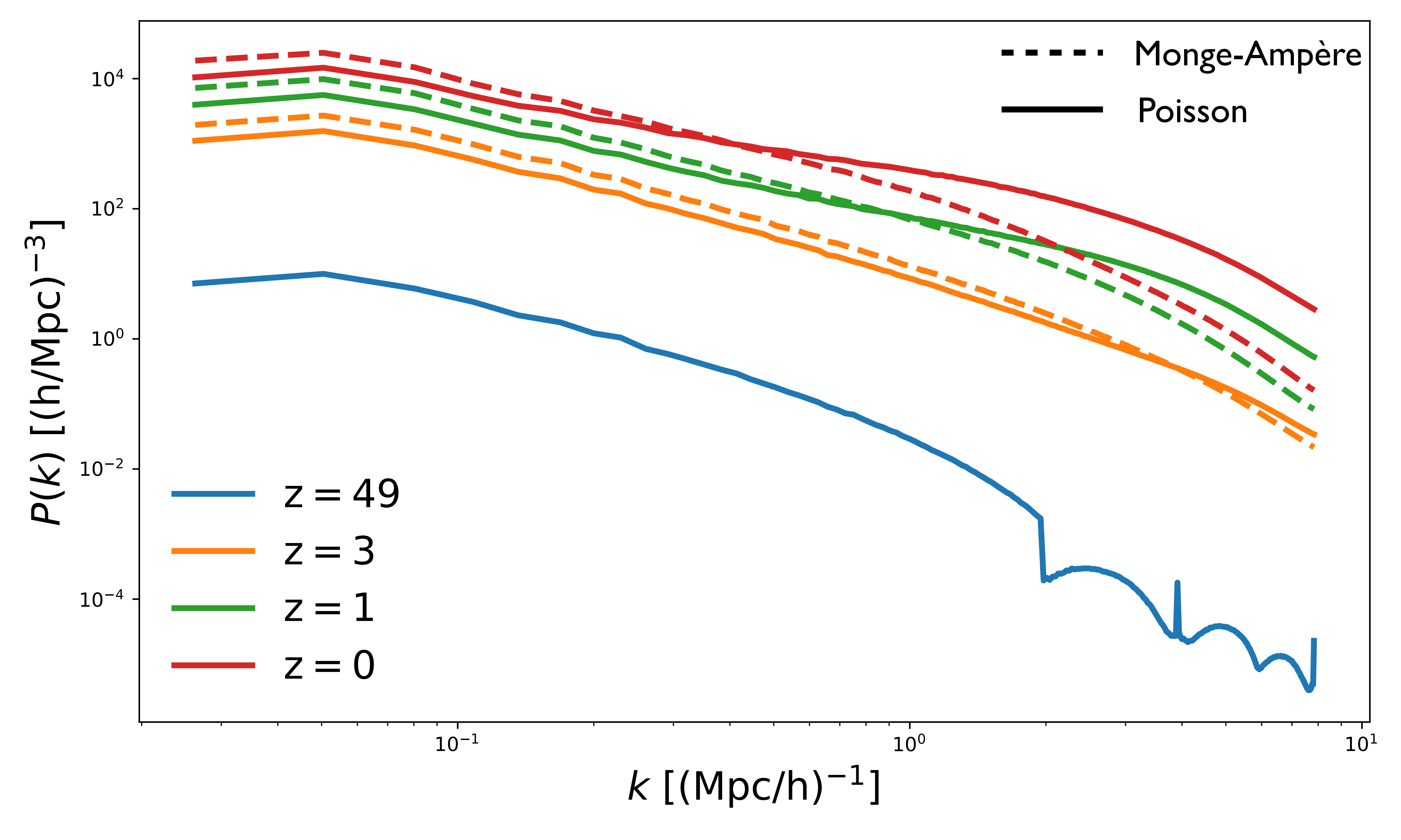
Cosmological simulation of Monge-Ampère gravity

- Zoom $z = 0$



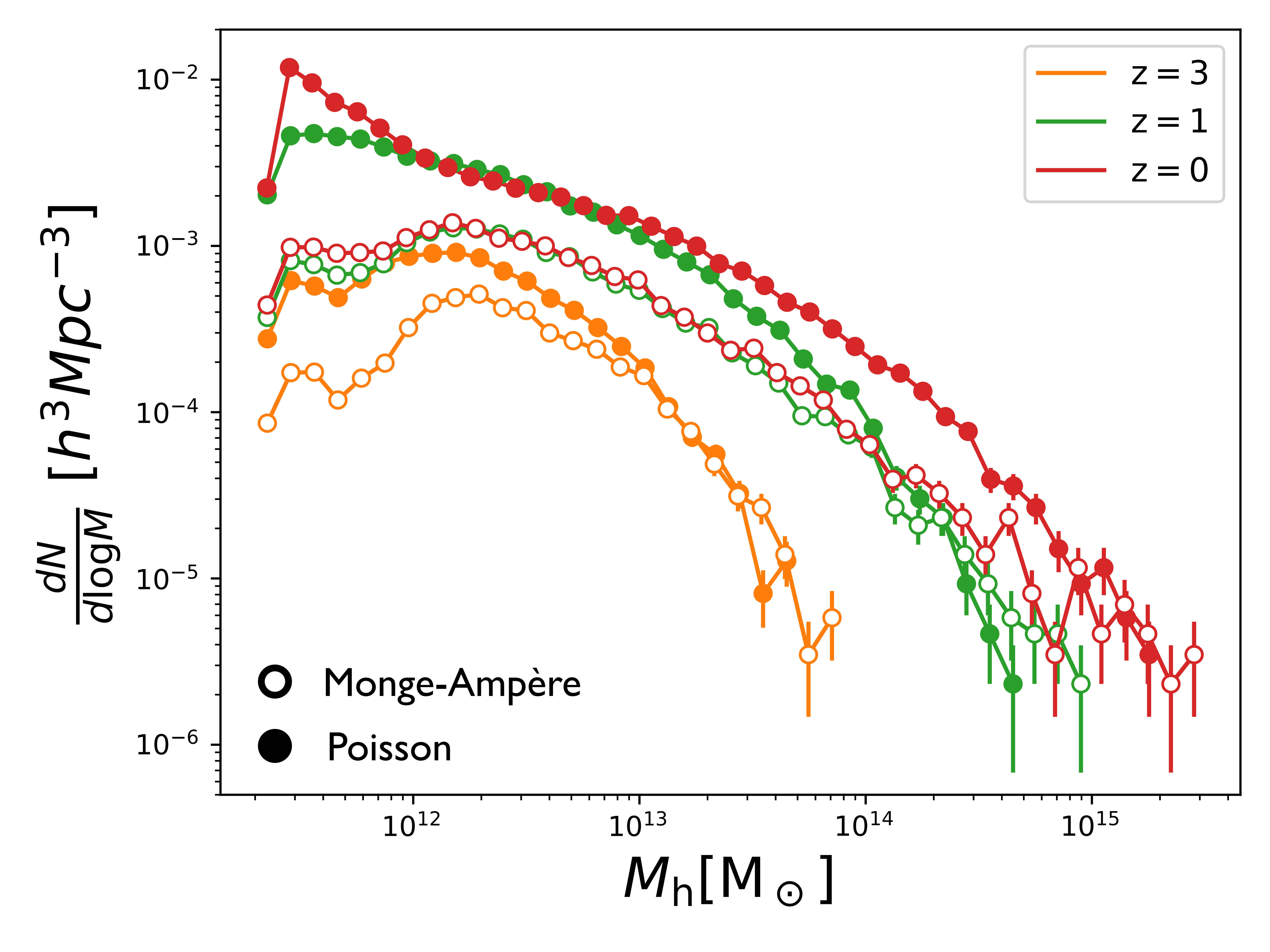
Cosmological simulation of Monge-Ampère gravity

Power spectra



Cosmological simulation of Monge-Ampère gravity

- Halo mass function



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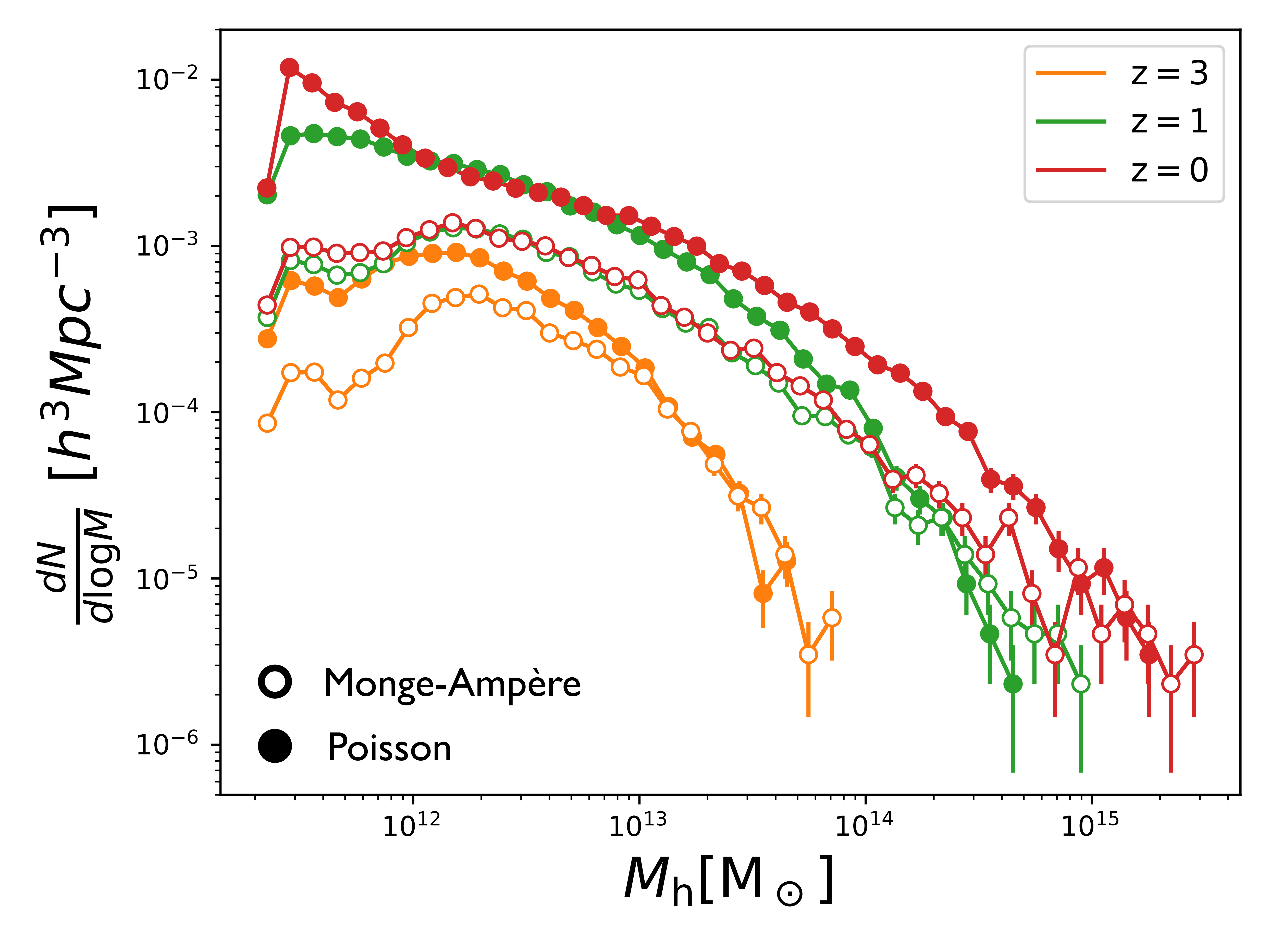


Poisson | Monge-Ampère

$N_h = 66091 \,|\, 16057$

4 times less halos at $z = 0$

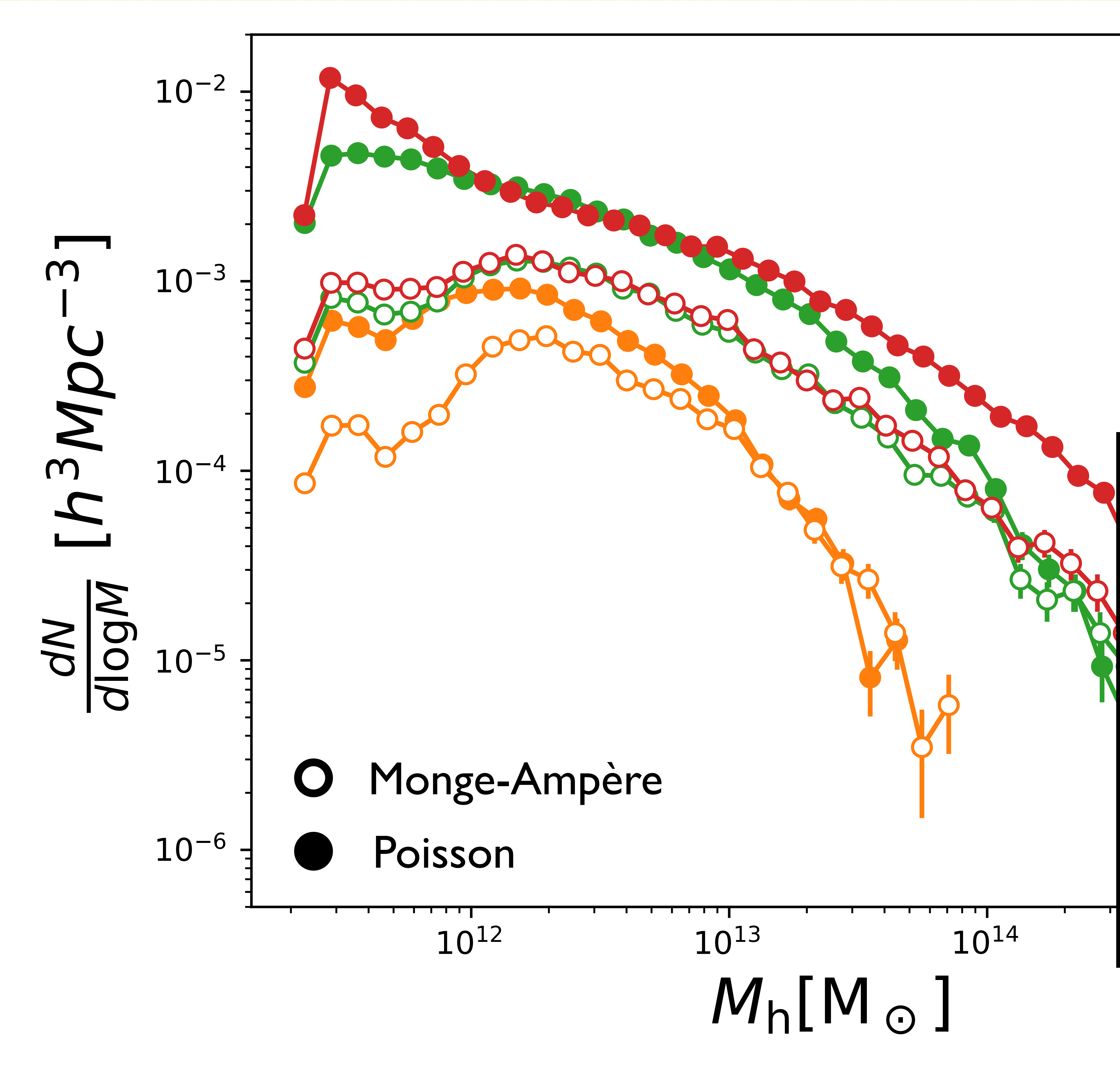
Cosmological simulation of Monge-Ampère gravity

🍋 Halo mass function

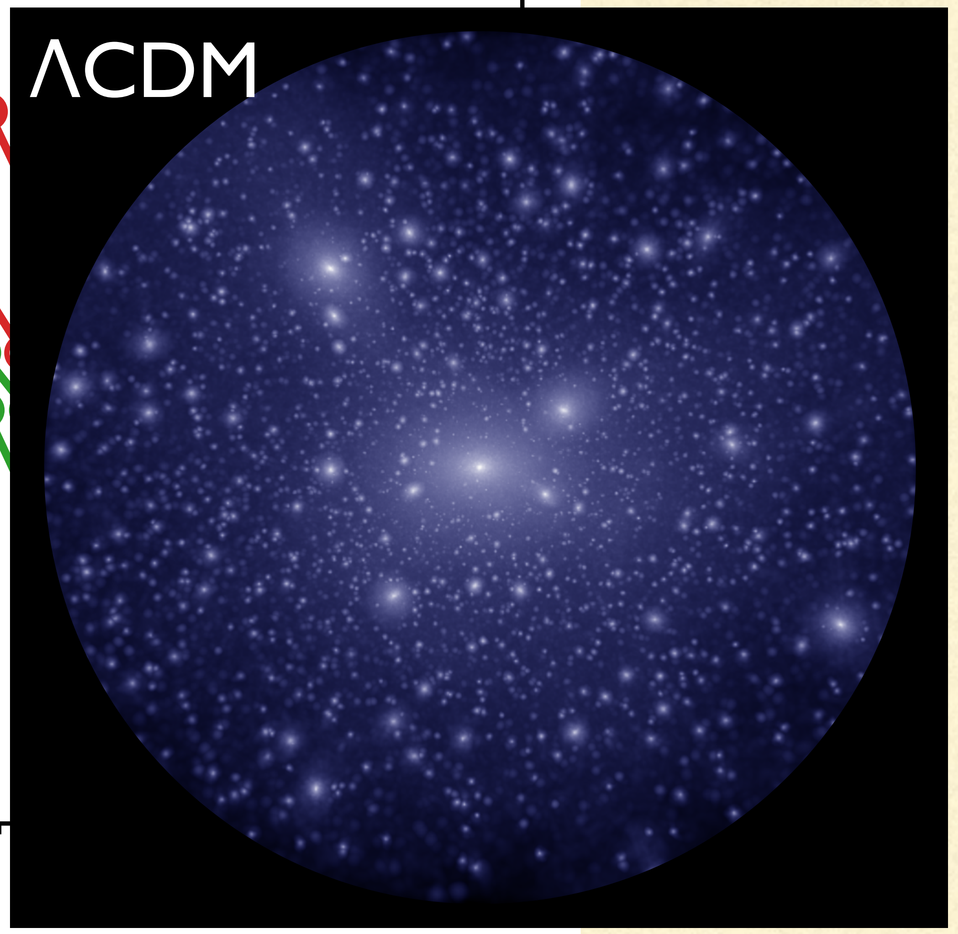
Poisson | Monge-Ampère

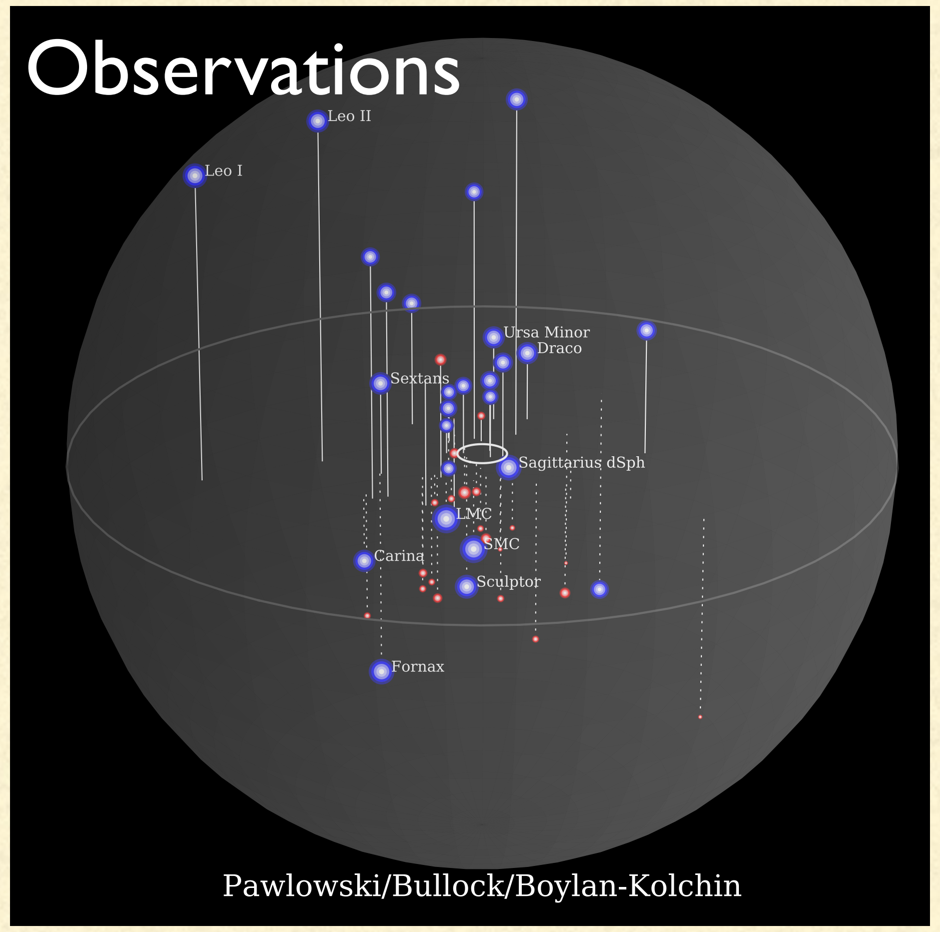
$N_{\rm h} = 66091 \,|\, 16057$

4 times less halos at $z = 0$



Missing satellite problem





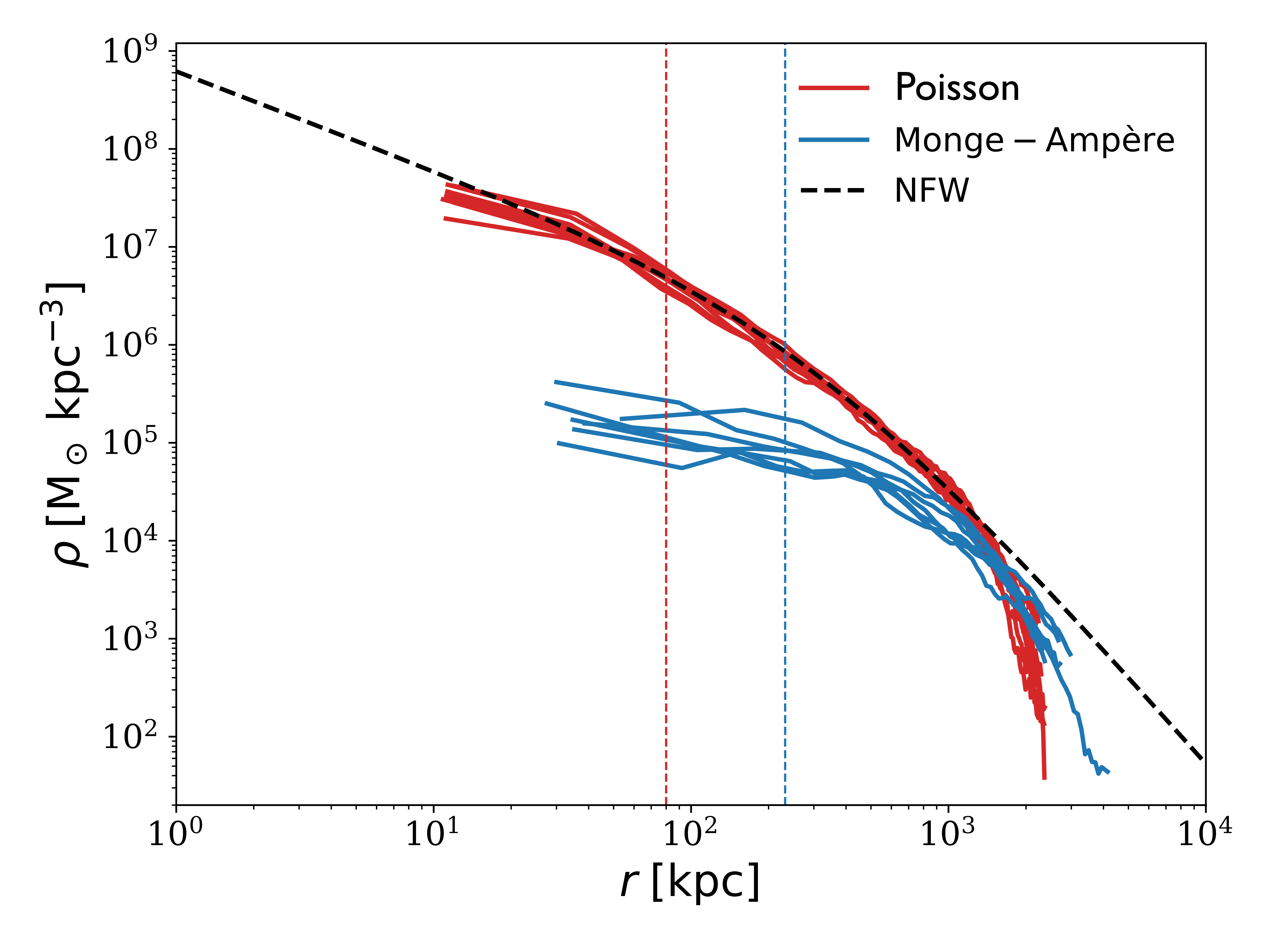
~1000-2000 satellites

~ 50 satellites

A potential solution to the missing satellite problem

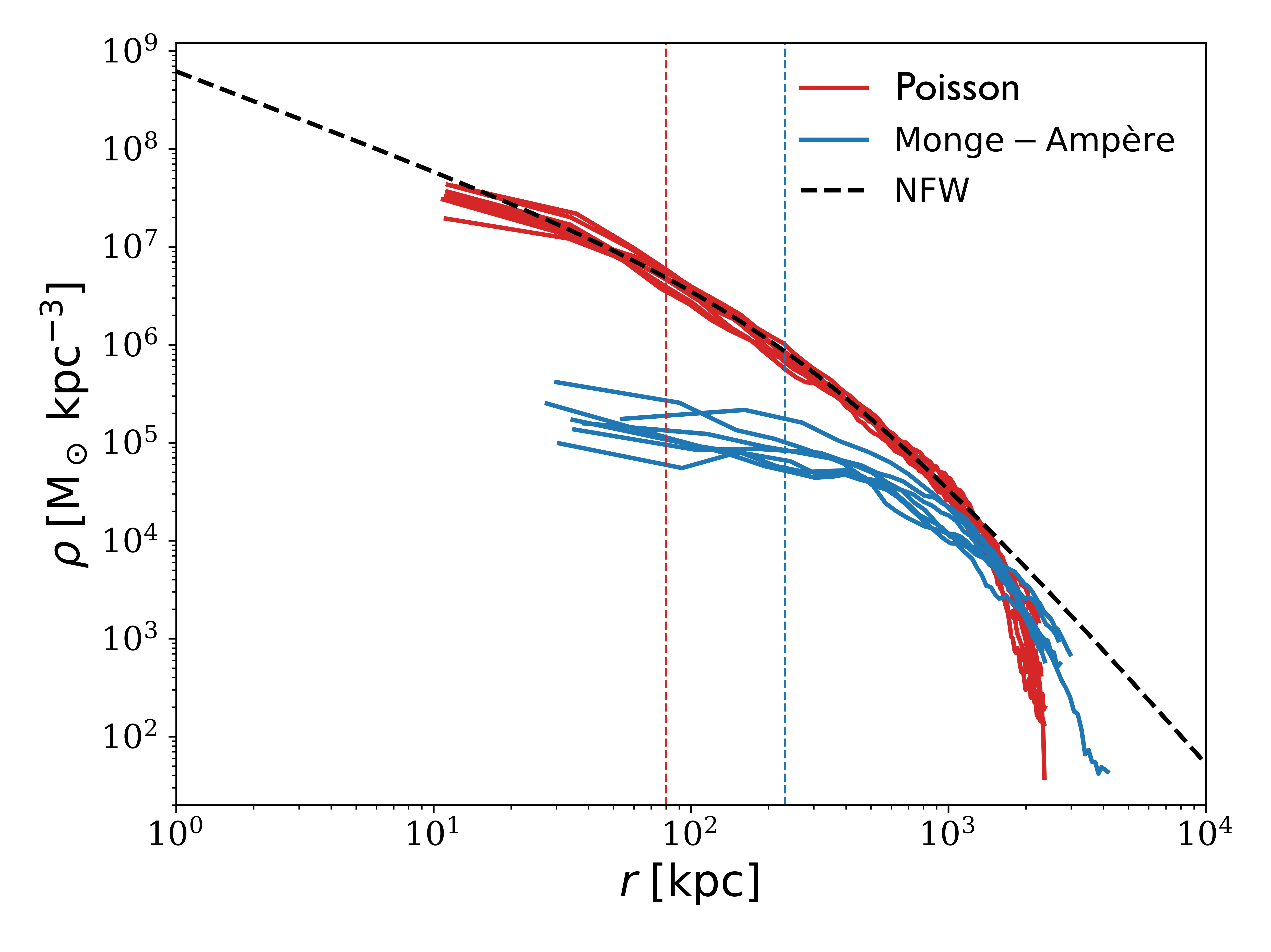
Cosmological simulation of Monge-Ampère gravity

- Dark matter density profiles



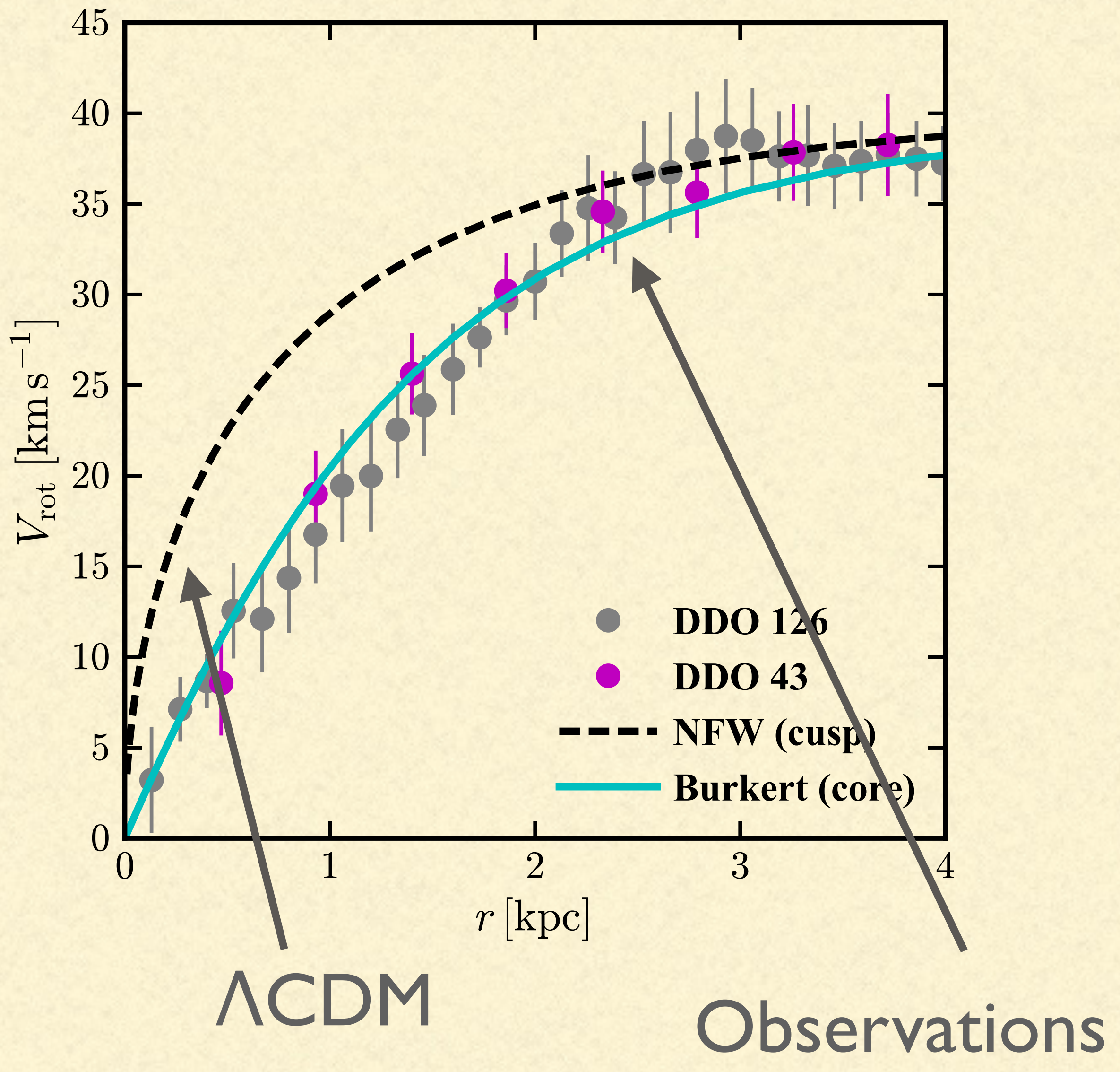
Cosmological simulation of Monge-Ampère gravity

- Dark matter density profiles



A potential solution to the cusp-core problem

Cusp-core problem



Conclusion

A **motivated** gravity theory

First N-body cosmological simulation
of **Monge-Ampère gravity**

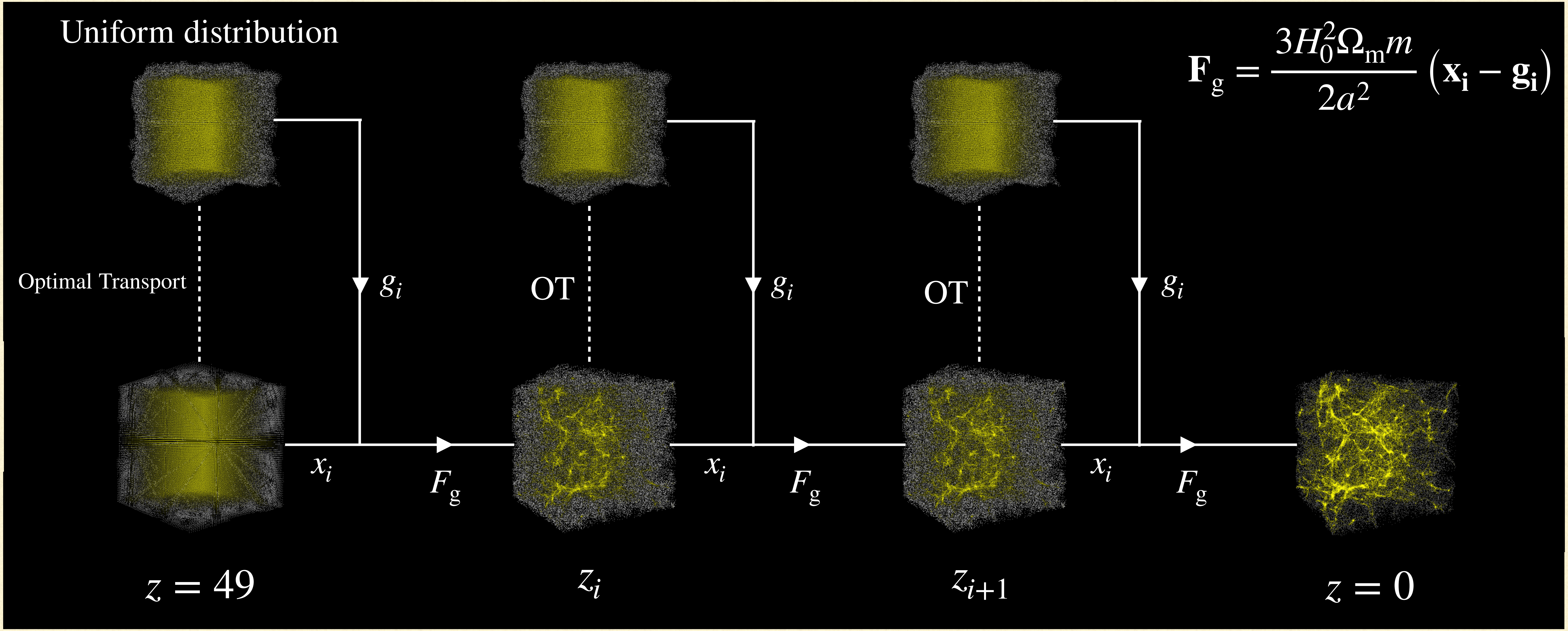
State-of-the-art algorithms
from computer science for **Optimal Transport** problem

A **weaker gravitational** clustering
for dark matter particles

Promising alternative theory of gravity

Cosmological simulation of Monge-Ampère gravity

- Solving Monge-Ampère equation with Optimal Transport



Cosmological simulation of Monge-Ampère gravity

- Equations of motion in comoving coordinates

$$\frac{d\mathbf{x}}{da} = \frac{\mathbf{p}}{ma^2S(a)} \quad \& \quad \frac{d\mathbf{p}}{da} = \frac{1}{aS(a)}\left[a^2\mathbf{F}_\mathrm{g} + \frac{mH_0^2\Omega_\mathrm{m}}{2}\mathbf{x}\right]$$

$$S(a) = aH_0\sqrt{\Omega_\mathrm{m}a^{-3} + (1 - \Omega_\mathrm{m} - \Omega_\Lambda)a^{-2} + \Omega_\Lambda}$$

Poisson

$$\mathbf{F}_\mathrm{g} = \sum_{j=0,i\neq j}^{N-1} \frac{Gm_im_j(\mathbf{x_j} - \mathbf{x_i})}{a^2(|\mathbf{x_j} - \mathbf{x_i}|^2)^{3/2}}$$

Monge-Ampère

$$\mathbf{F}_\mathrm{g} = \frac{3H_0^2\Omega_\mathrm{m}m}{2a^2}\left(\mathbf{x_i} - \mathbf{g_i}\right)$$