Quantumness of inflationary cosmological perturbations

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École Polytechnique

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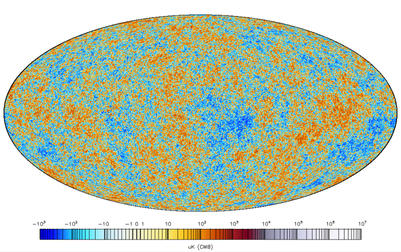
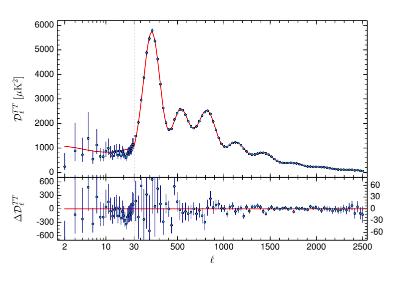
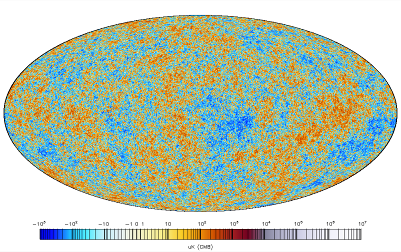
Introduction

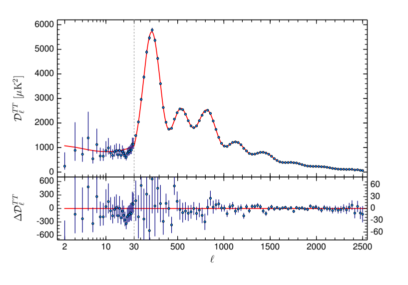
Introduction





Introduction





How quantum are the fluctuations ? Can it be seen in the CMB ?

Tools of quantum information

Information for classical systems

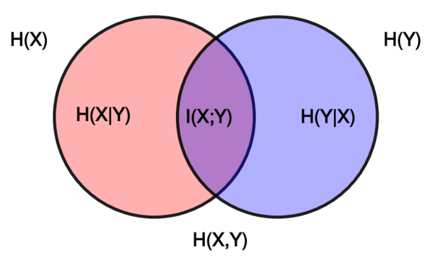
How much information does one lack about a variable X ?

$$\mathcal{H}(X) = -\sum_x p(X=x)\log_2\left[p(X=x)\right] \tag{1}$$

Maximal for uniform distribution, zero for an almost sure variable

Information for classical systems

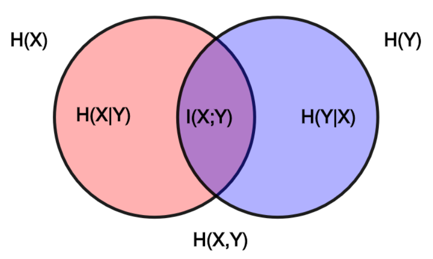
How much information does one lack about a variable X knowing Y already ?



$$\mathcal{H}(X|Y) = \sum_y p(Y=y)\mathcal{H}(X|Y=y)\,, \tag{2}$$

where $\mathcal{H}(X|Y=y) = -\sum_x p(X=x|Y=y)\log_2 p(X=x|Y=y)$.

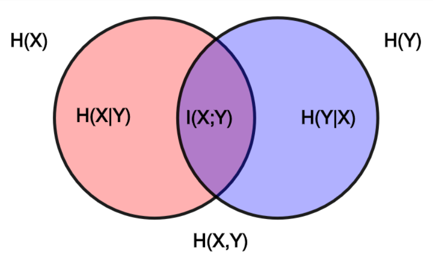
Mutual Information for classical systems



Mutual information :

$$\mathcal{I}(X, Y) = \mathcal{H}(X) + \mathcal{H}(Y) - \mathcal{H}(X, Y)\,. \tag{3}$$

Mutual Information for classical systems



Mutual information :

$$\mathcal{I}(X, Y) = \mathcal{H}(X) + \mathcal{H}(Y) - \mathcal{H}(X, Y)\,. \tag{3}$$

For classical system $\mathcal{H}(X|Y) = \mathcal{H}(X, Y) - \mathcal{H}(Y)$, then

$$\mathcal{I}(X, Y) = \mathcal{J}(X, Y) = \mathcal{H}(X) - \mathcal{H}(X|Y) \tag{4}$$

Mutual Information for quantum systems

Von-Neumann entropy of S

$$\mathcal{H}(\mathrm{S}) = -\mathrm{Tr}\left[\hat{\rho}_{\mathrm{S}} \log_2(\hat{\rho}_{\mathrm{S}})\right] . \tag{5}$$

If $\mathrm{S} = X \cup Y$, then one has two mutual information

$$\mathcal{I}(X, Y) = \mathcal{H}(X) + \mathcal{H}(Y) - \mathcal{H}(X, Y) , \tag{6}$$

$$\mathcal{J}\left(X, Y, \{\Pi_j^Y\}\right) = \mathcal{H}(X) - \mathcal{H}\left(X|Y, \{\Pi_j^Y\}\right) . \tag{7}$$

Quantum Discord

Quantum Discord is defined by

$$\boxed{\delta(X,Y) = \mathcal{I}(X,Y) - \max_{\{\Pi_j^Y\}} \mathcal{J}\left(X, Y, \{\Pi_j^Y\}\right)} \tag{8}$$

A Bell Inequality

A Bell Inequality

One considers two 2-valued spins in the entangled state (Bell state)

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)\,. \tag{9}$$

where $\hat{\boldsymbol{\sigma}}^{A/B}.\boldsymbol{u}_z\,|\pm\rangle = \pm 1/2$.

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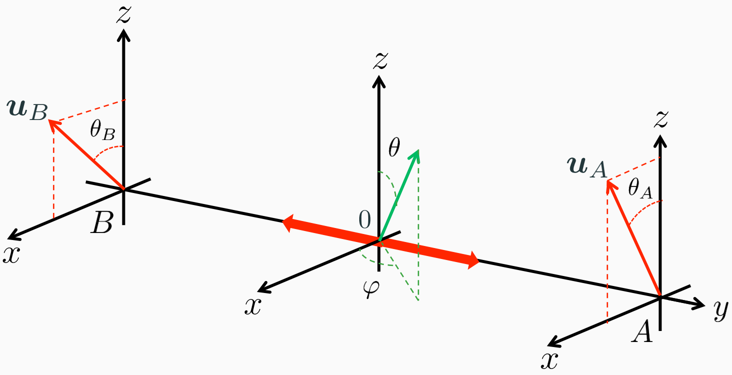
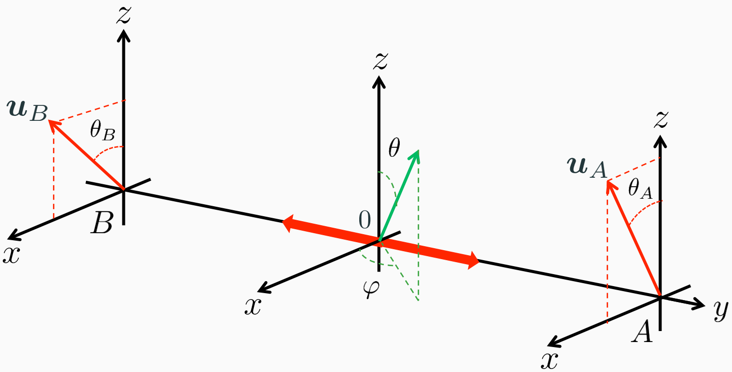
One measures each spin in a direction given by a vector $\boldsymbol{u}$



A Bell Inequality



One can then compute the mean value of the Bell operator $\hat{B}$ in the state $|\Psi\rangle$

$$\hat{B} = \boldsymbol{n}.\hat{\boldsymbol{\sigma}}_A \otimes \boldsymbol{m}.\hat{\boldsymbol{\sigma}}_B + \boldsymbol{n}.\hat{\boldsymbol{\sigma}}_A \otimes \boldsymbol{m}'.\hat{\boldsymbol{\sigma}}_B + \boldsymbol{n}'.\hat{\boldsymbol{\sigma}}_A \otimes \boldsymbol{m}.\hat{\boldsymbol{\sigma}}_B - \boldsymbol{n}'.\hat{\boldsymbol{\sigma}}_A \otimes \boldsymbol{m}'.\hat{\boldsymbol{\sigma}}_B\,. \quad (9)$$

A Bell Inequality

$$\left\langle \hat{B} \right\rangle = -\cos(\theta_n - \theta_m) - \cos(\theta_n - \theta_{m'}) - \cos(\theta_{n'} - \theta_m) + \cos(\theta_{n'} - \theta_{m'})\,. \tag{9}$$

If $\theta_n - \theta_m = \pi/4$, $\theta_{n'} - \theta_m = \theta_n - \theta_{m'} = -\pi/4$ and $\theta_{n'} - \theta_{m'} = 3\pi/4$:

$$\boxed{\left\langle \hat{B} \right\rangle = -2\sqrt{2}\,.} \tag{10}$$

One can show that in a local classical theory one would have

$$\boxed{|\left\langle \hat{B} \right\rangle_{\mathrm{class.}}| \leq 2} \tag{11}$$

A Bell Inequality - Conclusion

If one can find pseudo-spin operators for our system for which $\left\langle \hat{B} \right\rangle > 2$, one can show that the system cannot be described by a classical state !

A Bell Inequality - Conclusion

If one can find pseudo-spin operators for our system for which $\left\langle \hat{B} \right\rangle > 2$, one can show that the system cannot be described by a classical state !

Question :

Quantum Discord and Bell inequality both measure the quantumness of a state by measuring the quantumness of the correlations. Do they give a compatible account of the quantumness of a state ?

Can a system of CMB size still be a quantum one ?

Let us consider

$$|\Psi\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}} \tag{12}$$

Density matrix

$$\hat{\rho} = |\Psi\rangle\langle\Psi| = \frac{1}{2}(|+-\rangle\langle+-| - |+-\rangle\langle-+| - |-+\rangle\langle+-| + |-+\rangle\langle-+|)\,. \tag{13}$$

Not a statistical superposition of the states $|+-\rangle$ and $|-+\rangle$ but a *quantum* one.

Classical superposition :

$$\hat{\rho}_{\mathrm{class.}} = \frac{1}{2}|+-\rangle\langle+-| + \frac{1}{2}|-+\rangle\langle-+|\,. \tag{14}$$

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Not a statistical superposition of the states $|+-\rangle$ and $|-+\rangle$ but a *quantum* one.

Classical superposition :

$$\hat{\rho}_{\mathrm{class.}} = \frac{1}{2}|+-\rangle\langle+-| + \frac{1}{2}|-+\rangle\langle-+| . \tag{14}$$

Decoherence dynamically removes the non-diagonal terms.

Decoherence - Questions

Typically occurs when a system is coupled with an environment.

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Typically occurs when a system is coupled with an environment.

Will decoherence destroy the quantum correlations ?

Compatible with Bell Inequality and Quantum Discord ?

Inflationary Cosmological Perturbations

Mukhanov-Sasaki variable

The perturbations can be described by a single variable, the Mukhanov-Sasaki variable

$$\hat{v}(\eta, x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{k}\hat{v}_k(\eta)e^{i\boldsymbol{k}.\boldsymbol{x}} \,. \tag{15}$$

Makes sense to consider $\hat{v}$ since for CMB temperature anisotropies, on large scales

$$\frac{\delta T}{T} \propto \hat{v} \,. \tag{16}$$

Hamiltonian : Sum of parametric oscillators

$$\hat{H} = \int_{\mathbb{R}^2 \times \mathbb{R}^+} \mathrm{d}^3\, \boldsymbol{k} \left[\hat{p}_{\boldsymbol{k}} \hat{p}_{\boldsymbol{k}}^{\dagger} + \omega^2(\eta, \boldsymbol{k}) \hat{v}_{\boldsymbol{k}} \hat{v}_{\boldsymbol{k}}^{\dagger} \right] , \tag{17}$$

Commutation relations :

$$[\hat{v}_{\boldsymbol{p}}, \hat{p}_{\boldsymbol{q}}] = i\delta(\boldsymbol{p} + \boldsymbol{q}) \tag{18}$$

One has independent systems indexed by $\mathbb{R}^2 \times \mathbb{R}^+$.

Hamiltonian : Sum of parametric oscillators

$$\hat{H} = \int_{\mathbb{R}^2\times\mathbb{R}^+} \mathrm{d}^3\,\boldsymbol{k} \left[\hat{p}_{\boldsymbol{k}}\hat{p}_{\boldsymbol{k}}^\dagger + \omega^2(\eta,\boldsymbol{k})\hat{v}_{\boldsymbol{k}}\hat{v}_{\boldsymbol{k}}^\dagger\right] , \tag{17}$$

Introducing real and imaginary parts

$$\hat{v}_{\boldsymbol{k}} = \frac{\hat{v}_{\boldsymbol{k}}^R + i\hat{v}_{\boldsymbol{k}}^I}{\sqrt{2}} , \quad \hat{p}_{\boldsymbol{k}} = \frac{\hat{p}_{\boldsymbol{k}}^R + i\hat{p}_{\boldsymbol{k}}^I}{\sqrt{2}} . \tag{18}$$

One has

$$\hat{H}_v = \int_{\mathbb{R}^2\times\mathbb{R}^+} \mathrm{d}^3\boldsymbol{k} \sum_{s=R,I}\left[\frac{(\hat{p}_{\boldsymbol{k}}^s)^2}{2} + \omega^2(\eta,\boldsymbol{k})\frac{(\hat{v}_{\boldsymbol{k}}^s)^2}{2}\right] , \tag{19}$$

$$= \int_{\mathbb{R}^{3+}} \mathrm{d}^3\boldsymbol{k} \sum_{s=R,I}\hat{\mathcal{H}}_{\boldsymbol{k}}^s \tag{20}$$

Evolution of the system

Liouville-von Neumann equation for each of the $\boldsymbol{k} \in \mathbb{R}^{3+}$ system density matrix

$$\frac{\mathrm{d}\hat{\rho}_{\boldsymbol{k}}^{\mathrm{S}}}{\mathrm{d}\eta} = -i\left[\hat{\mathcal{H}}_{\boldsymbol{k}}^{\mathrm{S}}, \hat{\rho}_{\boldsymbol{k}}^{\mathrm{S}}\right]. \tag{21}$$

Evolution of the system with decoherence

Lindblad equation for each of the $\boldsymbol{k} \in \mathbb{R}^{3+}$ system density matrix

$$\frac{\mathrm{d}\hat{\rho}^{S}_{\boldsymbol{k}}}{\mathrm{d}\eta} = -i\left[\hat{\mathcal{H}}^{S}_{\boldsymbol{k}}, \hat{\rho}^{S}_{\boldsymbol{k}}\right] - \frac{\gamma}{2}(2\pi)^{\frac{3}{2}}\tilde{C}_{R}(\boldsymbol{k})[\hat{v}^{S}_{\boldsymbol{k}}, [\hat{v}^{S}_{\boldsymbol{k}}, \hat{\rho}^{S}_{\boldsymbol{k}}]]\,. \tag{21}$$

Systems are still decoupled !

Previous calculations and Objectives

Previous calculations :

- Martin and Vennin have studied the quantumness of these systems without decoherence in Refs [2, 3].
- They found an explicit solution for Eq. 21 in [4].

Goals :

- Take into account decoherence to assess the quantumness of CMB with respect to Quantum Discord and Bell inequality.
- Calibrate the different quantum criteria : Quantum Discord, Bell inequalities and decoherence compatible ?

Preliminaries

Splitting

Two ways of representing the systems the R/I splitting and the $\pm\boldsymbol{k}$ splitting define two different sets of subsystems

$$\hat{H} = \int_{\mathbb{R}^{3+}} \mathrm{d}^3\,\boldsymbol{k}\left[\hat{p}_{\boldsymbol{k}}\hat{p}_{\boldsymbol{k}}^{\dagger} + \omega^2(\eta,\boldsymbol{k})\hat{v}_{\boldsymbol{k}}\hat{v}_{\boldsymbol{k}}^{\dagger}\right], \tag{22}$$

$$= \int_{\mathbb{R}^{3+}} \mathrm{d}^3\boldsymbol{k}\sum_{s=R,I}\left[\frac{(\hat{p}_{\boldsymbol{k}}^{s})^2}{2} + \omega^2(\eta,\boldsymbol{k})\frac{(\hat{v}_{\boldsymbol{k}}^{s})^2}{2}\right]. \tag{23}$$

We want to generalize to compute the quantum discord for an arbitrary splitting of the system in subsystems.

General splitting

One can go from the R/I splitting to a general splitting by

$$\begin{pmatrix} \hat{q}_1 \\ \hat{\pi}_1 \\ \hat{q}_2 \\ \hat{\pi}_2 \end{pmatrix} = S(\alpha, \beta, \delta, \theta) \begin{pmatrix} \hat{v}_R \\ \hat{p}_R \\ \hat{v}_I \\ \hat{p}_I \end{pmatrix} \tag{24}$$

where

$$S(\alpha, \beta, \delta, \theta) = \tag{25}$$

$$\begin{pmatrix} \cos\alpha\cos\theta & -\sin\alpha\cos\theta & -\cos\delta\sin\theta & \sin\theta\sin\delta \\ \sin\alpha\cos\theta & \cos\alpha\cos\theta & -\sin\theta\sin\delta & \sin\theta\cos\delta \\ \cos\beta\sin\theta & -\sin\theta\sin\beta & \cos(\alpha-\beta-\delta)\cos\theta & -\cos\theta\sin(\alpha-\beta-\delta) \\ \sin\theta\sin\beta & \cos\beta\sin\theta & -\cos\theta\sin(\alpha-\beta-\delta) & \cos(\alpha-\beta-\delta)\cos\theta \end{pmatrix} \tag{26}$$

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Hypothesis : The change of splitting is symplectic, i.e. preserves the commutation relations, and preserves the vacuum.

$$S(\alpha, \beta, \delta, \theta)\Omega S(\alpha, \beta, \delta, \theta)^T = \Omega \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{25}$$

$(\hat{q}, \hat{\pi})$ are hermitian operators.

State of the perturbations

Defined by its Wigner function

$$W(R) = \frac{1}{(2\pi)^2} \int \int \mathrm{d}x \mathrm{d}y \, e^{-i\pi_1 x - i\pi_2 y} \left\langle q_1 + \frac{x}{2}, q_2 + \frac{y}{2} \middle| \hat{\rho} \middle| q_1 - \frac{x}{2}, q_2 - \frac{y}{2} \right\rangle , \tag{26}$$

with $\hat{R} = (\hat{q}_1, \hat{\pi}_1, \hat{q}_2, \hat{\pi}_2)^T$.

<u>Gaussian state :</u>

$$W(R) = \frac{1}{(2\pi)^2 \sqrt{\det \Gamma}} e^{-2R^T \Gamma^{-1} R} , \tag{27}$$

where the covariance matrix is $\Gamma_{ij} = \mathrm{Re}\left[\left\langle \hat{R}_i \hat{R}_j \right\rangle\right] - \left\langle \hat{R}_i \right\rangle \left\langle \hat{R}_j \right\rangle$.

All the information about the state is contained in Γ.

State of the perturbations

In the R/I splitting one has

$$\Gamma_v = \begin{pmatrix} P_1 & P_2 & 0 & 0 \\ P_2 & P_3 & 0 & 0 \\ 0 & 0 & P_1 & P_2 \\ 0 & 0 & P_2 & P_3 \end{pmatrix} . \tag{28}$$

$$P_1 = |v_{\boldsymbol{k}}^R|^2 + \underbrace{\mathcal{J}_{\boldsymbol{k}}}_{= 0 \quad \text{when} \quad \gamma = 0} , \quad P_2 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta} P_1 , \quad P_3 = \left(\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + \omega(\eta, \boldsymbol{k})^2\right) P_1 . \tag{29}$$

Quantum Discord

Computing the quantum discord

Transformation of the covariance matrix under $\hat{R} = (\hat{v}_R, \hat{p}_R, \hat{v}_I, \hat{p}_I)^T \rightarrow \hat{R}(\alpha, \beta, \delta, \theta) = S(\alpha, \beta, \delta, \theta)\hat{R}$:

$$\Gamma(\alpha, \beta, \delta, \theta) = S(\alpha, \beta, \delta, \theta)\Gamma S(\alpha, \beta, \delta, \theta)^T = \begin{pmatrix} \Gamma_A & \Gamma_C \\ \Gamma_C & \Gamma_B \end{pmatrix} . \qquad (30)$$

Sub-matrices very complex $\rightarrow$ intermediate quantities to perform the calculations

Computing the quantum discord

Sub-matrices very complex $\rightarrow$ intermediate quantities to perform the calculations

$$\Gamma_A = \begin{pmatrix} L - M_{\mathrm{A}} & N_{\mathrm{A}} \\ N_{\mathrm{A}} & M_{\mathrm{A}} \end{pmatrix} . \tag{30}$$

where

$$L = P_1 + P_3 \,, \tag{31}$$

$$M_{\mathrm{A}} = \cos^2\theta\, O(\alpha) + \sin^2\theta\, O(\delta) \,, \tag{32}$$

$$N_{\mathrm{A}} = \cos^2\theta\, P(\alpha,\alpha) + \sin^2\theta\, P(\delta,\delta) \,, \tag{33}$$

$$O(x) \equiv P_1 \sin^2 x + P_2 \sin(2x) + P_3 \cos^2 x \,, \tag{34}$$

$$P(x,y) \equiv 2P_2 \cos(x+y)(P_1 - P_3)\sin(x+y) \,. \tag{35}$$

Similar definition for Γ_B with different angles.

Computing $\mathcal{I}$

$$\mathcal{I}(A,B) = \mathcal{H}(\hat{\rho}_A) + \mathcal{H}(\hat{\rho}_B) - \mathcal{H}(\hat{\rho})\,. \tag{36}$$

For a Gaussian state

$$\mathcal{H}(\hat{\rho}) = \sum_{i=1}^{n_{\mathrm{sub}}} f(2\sigma_i)\,, \tag{37}$$

where

$$f(x) = \left(\frac{x+1}{2}\right)\log_2\left(\frac{x+1}{2}\right) - \left(\frac{x-1}{2}\right)\log_2\left(\frac{x-1}{2}\right)\,, \tag{38}$$

$$\mathrm{Sp}(\Omega\Gamma) = \{\pm i\sigma_i\}\,, \tag{39}$$

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,. \tag{40}$$

One finds

$$\begin{aligned}\mathrm{Sp}(\omega\Gamma_A) =& \mathrm{Sp}(\omega\Gamma_B) = \{\pm i\sigma(\alpha,\beta,\delta,\theta)\}\\ =& \{\pm i\sqrt{P_1P_3 - P_2^2 + \left[\frac{\sin(2\theta)\sin(\alpha-\delta)}{2}\right]^2\left[(P_1-P_3)^2+4P_2^2\right]}\}\,,\end{aligned} \tag{41}$$

$$\mathrm{Sp}(\Omega\Gamma) = \{\pm i\sigma\} = \{\pm i\sqrt{P_1P_3 - P_2^2}\}\,. \tag{42}$$

So that

$$\boxed{\mathcal{I} = 2\left[f(2\sigma) - f(2\sigma(\alpha,\beta,\delta,\theta))\right]}\,. \tag{43}$$

In Ref. [1] it is shown that for gaussian state $\mathcal{J}$ is given by

$$\begin{aligned}\max_{\{\Pi_j^{\mathcal{Y}}\}} \mathcal{J}\left(X, Y, \{\Pi_j^{Y}\}\right) = f\left(\sqrt{\det(\Gamma_A)}\right) \\ - f\Bigg(\frac{2 \times \left[4\det(\Gamma_C)\right]^2 + (4\det(\Gamma_B) - 1)(16\det(\Gamma) - 4\det(\Gamma_A))}{(4\det(\Gamma_B) - 1)^2} \\ - \frac{2 \times 4\left[\det(\Gamma_C)\right]\sqrt{\left[4\det(\Gamma_C)\right]^2 + (4\det(\Gamma_B) - 1)(16\det(\Gamma) - 4\det(\Gamma_A))}}{(4\det(\Gamma_B) - 1)^2}\Bigg) .\end{aligned} \tag{44}$$

One has

$$\det(\Gamma_A) = \det(\Gamma_B) = \sigma^2 + c^2 , \tag{45}$$

$$\det(\Gamma) = \sigma^4 , \tag{46}$$

$$\det(\Gamma_C) = -c^2 , \tag{47}$$

$$c = -\sqrt{\left[\frac{\sin(2\theta)\sin(\alpha - \delta)}{2}\right]^2 \left[(P_1 - P_3)^2 + 4P_2^2\right]} . \tag{48}$$

One can simplify the result to

$$\boxed{\max_{\{\Pi_j^{\mathcal{Y}}\}} \mathcal{J}\left(X, Y, \{\Pi_j^Y\}\right) = f(2\sigma(\alpha, \beta, \delta, \theta))) - f\left(2\sigma(\alpha, \beta, \delta, \theta) - \frac{2c^2}{\sigma(\alpha, \beta, \delta, \theta) + 1/2}\right)}. \tag{49}$$

So that

$$\boxed{\delta = f(2\sigma(\alpha, \beta, \delta, \theta)) - f(2\sigma) + f\left(2\sigma(\alpha, \beta, \delta, \theta) - \frac{2c^2}{\sigma(\alpha, \beta, \delta, \theta) + 1/2}\right) - f(2\sigma)}. \tag{50}$$

Quantum Discord - A particular case

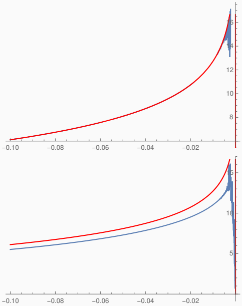
In the R/I splitting i.e $(\alpha, \beta, \delta, \theta) = (0,0,0,0)$ one has

$$\mathcal{I} = 2\left[f(2\sigma) - f(2\sigma(0,0,0,0))\right] = 0\,, \tag{51}$$

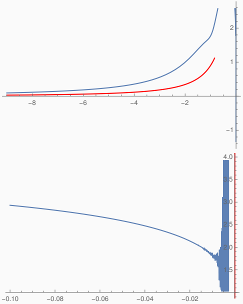
$$\max_{\{\Pi_j^{\mathcal{Y}}\}} \mathcal{J}\left(X, Y, \{\Pi_j^Y\}\right) = f(2\sigma(0,0,0,0))) - f\left(2\sigma(0,0,0,0) - 0\right) = 0 \tag{52}$$

Two independent susbystems $\rightarrow$ No mutual information $\rightarrow \delta = 0$

Plotting the Quantum Discord



Numerical errors



Error that might come from the formula used for P_i

Effective-spin operators

The GKM operators

$$\hat{S}_z^i = -\int_{-\infty}^{\infty} |q_i\rangle \langle -q_i| \, \mathrm{d}q_i \,, \tag{53}$$

$$\hat{S}_x^i = \int_{-\infty}^{\infty} \mathrm{sign}(q_i) \, |q_i\rangle \langle q_i| \, \mathrm{d}q_i \,, \tag{54}$$

$$\hat{S}_y^i = i \int_{-\infty}^{\infty} \mathrm{sign}(q_i) \, |q_i\rangle \langle -q_i| \, \mathrm{d}q_i \,, \tag{55}$$

satisfy the commutation relation for spin operators

$$\left[\hat{S}_\mu^i, \hat{S}_\nu^i\right] = i\epsilon_{\mu\nu\eta}\hat{S}_\eta^i \,, \tag{56}$$

Computation of $\left\langle \hat{B} \right\rangle$

One need to compute the mean-values $\left\langle \hat{S}_z^1 \hat{S}_z^2 \right\rangle$, $\left\langle \hat{S}_x^1 \hat{S}_x^2 \right\rangle$, $\left\langle \hat{S}_x^1 \hat{S}_z^2 \right\rangle$ and $\left\langle \hat{S}_z^1 \hat{S}_x^2 \right\rangle$.

For instance

$$\left\langle \hat{S}_z^1 \hat{S}_z^2 \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle -q_1, -q_2 | \, \rho \, | q_1, q_2 \rangle \, \mathrm{d}q_1 \mathrm{d}q_2 \, . \tag{57}$$

By similar proceedings one also computes the other mean-values

$$\left\langle \hat{S}_z^1 \hat{S}_z^2 \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle -q_1, -q_2 | \, \rho \, | q_1, q_2 \rangle \, \mathrm{d}q_1 \mathrm{d}q_2 \,, \tag{57}$$

$$\left\langle \hat{S}_x^1 \hat{S}_x^2 \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{sign}(q_1)\mathrm{sign}(q_2) \, \langle q_1, q_2 | \, \rho \, | q_1, q_2 \rangle \, \mathrm{d}q_1 \mathrm{d}q_2 \tag{58}$$

$$\left\langle \hat{S}_x^1 \hat{S}_z^2 \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{sign}(q_1) \, \langle q_1, -q_2 | \, \rho \, | q_1, q_2 \rangle \, \mathrm{d}q_1 \mathrm{d}q_2 \tag{59}$$

$$\left\langle \hat{S}_z^1 \hat{S}_x^2 \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{sign}(q_2) \, \langle -q_1, q_2 | \, \rho \, | q_1, q_2 \rangle \, \mathrm{d}q_1 \mathrm{d}q_2 \,. \tag{60}$$

Computation of $\left\langle \hat{B} \right\rangle$

By similar proceedings one also computes the other mean-values

$$\left\langle \hat{S}_z^1 \hat{S}_z^2 \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle -q_1, -q_2 | \, \rho \, | q_1, q_2 \rangle \, \mathrm{d}q_1 \mathrm{d}q_2 \,, \tag{57}$$

$$\left\langle \hat{S}_x^1 \hat{S}_x^2 \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{sign}(q_1) \mathrm{sign}(q_2) \, \langle q_1, q_2 | \, \rho \, | q_1, q_2 \rangle \, \mathrm{d}q_1 \mathrm{d}q_2 \tag{58}$$

$$\left\langle \hat{S}_x^1 \hat{S}_z^2 \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{sign}(q_1) \, \langle q_1, -q_2 | \, \rho \, | q_1, q_2 \rangle \, \mathrm{d}q_1 \mathrm{d}q_2 \tag{59}$$

$$\left\langle \hat{S}_z^1 \hat{S}_x^2 \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{sign}(q_2) \, \langle -q_1, q_2 | \, \rho \, | q_1, q_2 \rangle \, \mathrm{d}q_1 \mathrm{d}q_2 \,. \tag{60}$$

Then one uses the Wigner function

$$\left\langle q_1 + \frac{x}{2}, q_2 + \frac{y}{2} \right| \hat{\rho} \left| q_1 - \frac{x}{2}, q_2 - \frac{y}{2} \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(R) e^{i\pi_1 x + i\pi_2 y} \mathrm{d}\pi_1 \mathrm{d}\pi_2 \tag{61}$$

One has

$$\Gamma = \begin{pmatrix} \frac{P_1+P_3}{2} & 0 & \frac{P_1-P_3}{2} & P_2 \\ 0 & \frac{P_1+P_3}{2} & P_2 & -\frac{P_1-P_3}{2} \\ \frac{P_1-P_3}{2} & P_2 & \frac{P_1+P_3}{2} & 0 \\ P_2 & -\frac{P_1-P_3}{2} & 0 & \frac{P_1+P_3}{2} \end{pmatrix}, \tag{62}$$

One can compute for instance

$$\langle q_1, -q_2 | \, \rho \, | q_1, q_2 \rangle = \frac{1}{2\pi\sqrt{P_1 P_3}} \exp\left(-\frac{(P_1 + P_3)(q_1^2 + 4iP_2 q_1 q_2 - 4(P_2^2 - P_1 P_3) q_2^2)}{4P_1 P_3} \right). \tag{63}$$

One finally obtains

$$
\begin{aligned}
\left\langle \hat{S}_x^1 \hat{S}_z^2 \right\rangle =& 0 \\
\left\langle \hat{S}_z^1 \hat{S}_x^2 \right\rangle =& 0 \\
\left\langle \hat{S}_z^1 \hat{S}_z^2 \right\rangle =& \frac{1}{4(P_1 P_3 - P_2^2)} \\
\left\langle \hat{S}_x^1 \hat{S}_x^2 \right\rangle =& \frac{2}{\pi} \arctan\left(\frac{P_1 - P_3}{2\sqrt{P_1 P_3}} \right) .
\end{aligned}
\tag{64}
$$

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\end{aligned}
\tag{64}
$$

Finally optimizing in a simple situation

$$
\boxed{\left\langle \hat{B} \right\rangle^{\max} = 2\sqrt{\left(\frac{1}{4(P_1 P_3 - P_2^2)} \right)^2 + \frac{4}{\pi^2} \arctan^2 \left(\frac{P_1 - P_3}{2\sqrt{P_1 P_3}} \right)}} . \tag{65}
$$

Is the bell inequality violated ?

Without decoherence: Always !

$$\boxed{\left\langle \hat{B} \right\rangle^{\max} = 2\sqrt{1 + \frac{4}{\pi^2}\arctan^2\left(\frac{P_1 - P_3}{2\sqrt{P_1 P_3}}\right)} > 2}. \tag{66}$$

Is the bell inequality violated ?

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With decoherence : Still need to plot the curves to assess the evolution of the two terms...

Conclusion

We derived two formulas for continuous system in a Gaussian state : Quantum Discord and Bell inequality for the most general splitting.

Conclusion and prospects

We derived two formulas for continuous system in a Gaussian state : Quantum Discord and Bell inequality for the most general splitting.

Soon by plotting the curves we will be able

- To assess the quantumness of the CMB in a realistic context
- To calibrate the different criterion in this situation

Thank you for your attention !

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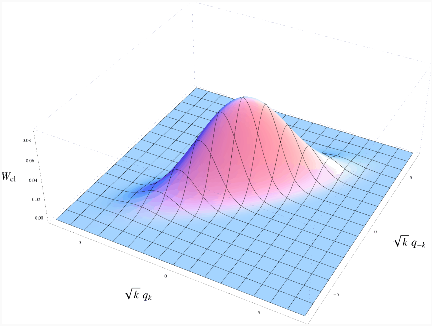
State of the perturbations

Without decoherence : squeezed state

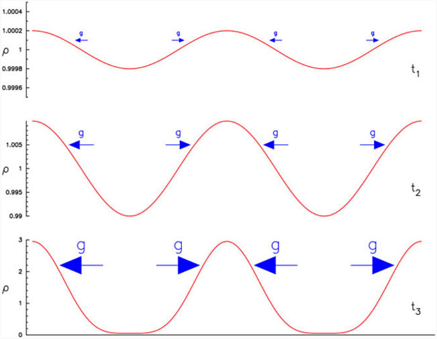


Highly quantum !

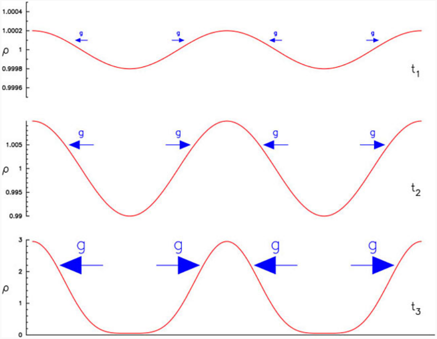
With decoherence : still Gaussian

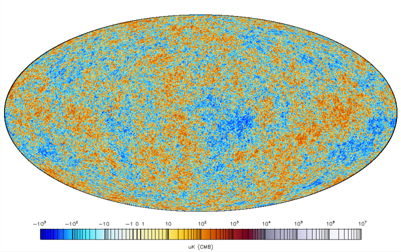
Destroys the squeezing ?

Introduction

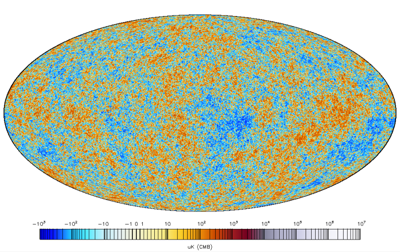


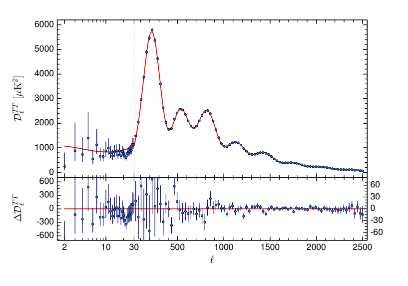
Introduction





Introduction





Quantum origin of primordial inhomogeneities *vs.* Large-scale structures

- ”How much” quantum are the fluctuations ? Are they still today ? What tools to estimate the quantumness ?

Quantum origin of primordial inhomogeneities *vs.* Large-scale structures

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- How ”wrong” are the classical predictions ? Can you measure the discrepancies ?

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How ”wrong” are the classical predictions ?

See “Quantum Discord of Cosmic Inflation: Can we Show that CMB Anisotropies are of Quantum-Mechanical Origin?”

Classical framework can not fully mimic quantum correlations.

The involved correlations are not (yet) measurable.

Quantum origin of primordial inhomogeneities *vs.* Large-scale structures

- ”How much” quantum are the fluctuations ? Are they still today ? What tools to estimate the quantumness ?
- How wrong are the classical predictions ? Can you measure the discrepancies ?
- Can a system of CMB size still be a quantum one ?

Mutual Information for quantum systems

Von-Neumann entropy of S

$$\mathcal{H}(\mathrm{S}) = -\mathrm{Tr}\left[\hat{\rho}_{\mathrm{S}} \log_2(\hat{\rho}_{\mathrm{S}})\right] . \tag{67}$$

If $\mathrm{S} = X \cup Y$, then one has two mutual information

$$\mathcal{I}(X, Y) = \mathcal{H}(X) + \mathcal{H}(Y) - \mathcal{H}(X, Y) , \tag{68}$$

$$\begin{aligned} \mathcal{J}\left(X, Y, \{\Pi_j^Y\}\right) &= \mathcal{H}(X) - \mathcal{H}\left(X|Y, \{\Pi_j^Y\}\right) \\ &= -\mathrm{Tr}\left[\hat{\rho}_X \log_2(\hat{\rho}_X)\right] + \sum_j p_j \,\mathrm{Tr}\left[\hat{\rho}_{Y|\Pi_j^Y} \log_2\left(\hat{\rho}_{Y|\Pi_j^Y}\right)\right] , \end{aligned} \tag{69}$$

where $\hat{\rho}_{Y|\Pi_j^Y} = \frac{\Pi_j^Y \hat{\rho} \Pi_j^Y}{p_j}$, $p_j = \mathrm{Tr}\left(\Pi_j^Y \hat{\rho}\right)$ and $\{\Pi_j^Y\}$ is a POVM.

$\mathcal{J}\left(X, Y, \{\Pi_j^Y\}\right)$ is non-symmetric under $X \leftrightarrow Y$

In Fourier space its dynamics is controlled by

$$\hat{H} = \int_{\mathbb{R}^2\times\mathbb{R}^+} \mathrm{d}^3\,\boldsymbol{k}\left(\hat{p}_{\boldsymbol{k}}\hat{p}_{\boldsymbol{k}}^\dagger + \omega^2(\eta,\boldsymbol{k})\hat{v}_{\boldsymbol{k}}\hat{v}_{\boldsymbol{k}}^\dagger\right)\,, \tag{70}$$

where $\omega^2(\eta,\boldsymbol{k}) = k^2 - \frac{(a\sqrt{\epsilon_1})''}{a\sqrt{\epsilon_1}}$ and $\hat{p}_{\boldsymbol{k}} = \frac{\delta\hat{\mathcal{L}}}{\delta\hat{v}_{\boldsymbol{k}}^{*\prime}} = \hat{v}_{\boldsymbol{k}}'$.

Commutation relation :

$$[\hat{v}_{\boldsymbol{k}},\hat{p}_{\boldsymbol{q}}] = i\delta(\boldsymbol{k}+\boldsymbol{q})\ \pm\boldsymbol{k}\,\text{not independent.} \tag{71}$$

One has independent systems indexed by $\mathbb{R}^2\times\mathbb{R}^+$.

Evolution of the system

Liouville-von Neumann equation for each of the $\boldsymbol{k} \in \mathbb{R}^{3+}$ system density matrix

$$\frac{\mathrm{d}\hat{\rho}_{\boldsymbol{k}}^{\mathsf{S}}}{\mathrm{d}\eta} = -i\left[\hat{\mathcal{H}}_{\boldsymbol{k}}^{\mathsf{S}}, \hat{\rho}_{\boldsymbol{k}}^{\mathsf{S}}\right]. \tag{72}$$

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One adds a decoherence term

$$\hat{H}_{\mathrm{int}}(\eta) = \int \mathrm{d}^3\boldsymbol{x}\, \hat{A}(\eta, \boldsymbol{x}) \otimes \hat{R}(\eta, \boldsymbol{x}). \tag{73}$$

$$\text{System} \quad \text{Environment} \tag{74}$$

Evolution of the system

Liouville-von Neumann equation for each of the $\boldsymbol{k} \in \mathbb{R}^{3+}$ system density matrix

$$\frac{\mathrm{d}\hat{\rho}^{S}_{\boldsymbol{k}}}{\mathrm{d}\eta} = -i\left[\hat{\mathcal{H}}^{S}_{\boldsymbol{k}}, \hat{\rho}^{S}_{\boldsymbol{k}}\right]. \tag{72}$$

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Total Hamiltonian

$$\hat{H}_{\mathrm{tot}} = \hat{H}_v \otimes \hat{\mathbb{I}}_{\mathrm{env}} + \hat{\mathbb{I}}_v \otimes \hat{H}_{\mathrm{env}} + \gamma \hat{H}_{\mathrm{int}}. \tag{74}$$

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Lindblad equation show that systems are still decoupled ! For each $\boldsymbol{k} \in \mathbb{R}^{3+}$

$$\frac{\mathrm{d}\hat{\rho}^s_{\boldsymbol{k}}}{\mathrm{d}\eta} = -i\left[\hat{\mathcal{H}}^s_{\boldsymbol{k}}, \hat{\rho}^s_{\boldsymbol{k}}\right] - \frac{\gamma}{2}(2\pi)^{\frac{3}{2}}\tilde{C}_{\mathrm{R}}(\boldsymbol{k})[\hat{v}^s_{\boldsymbol{k}}, [\hat{v}^s_{\boldsymbol{k}}, \hat{\rho}^s_{\boldsymbol{k}}]] \,, \tag{73}$$

where $\tilde{C}_{\mathrm{R}}(\boldsymbol{p}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{k} \left\langle \hat{R}(\eta, \boldsymbol{x})\hat{R}(\eta, \boldsymbol{y}) \right\rangle e^{i\boldsymbol{k}.(\boldsymbol{x}-\boldsymbol{y})} \,.$