General Relativity and Cosmology 2021 -2022
Questions and answers during tutorials

Amaury Micheli amaury.micheli@ijclab.in2p3.fr

http://www.iap.fr/useriap/micheli/

1er février 2022

Here some answers to the questions asked during the tutorials of General Relativity and Cosmology given for the M1 General Physics at Université Paris-Saclay in 2022.

1 Session 1

Questions :

— How to compute the Jacobian and then performed the transformation to write down the metric coordinates in spherical coordinates ?

— The Jacobian matrix is defined as the matrix of partial derivatives of the new coordinates x'_i with respect to the old ones x_i hece

$$J^i_j = dx'_i/dx_j \,. \tag{1}$$

see https://en.wikipedia.org/wiki/Jacobian_matrix_and_determinant. In this case the new coordinates are the spherical in which we want to write the metric and the old ones are the cartesian ones in which the metric is just the identity matrix.

— Knowing how a (0,2) tensor transforms we actually have to find the inverse of the jacobian of the above transform, or, and it's the same matrix, the jacobian of the inverse transform : from new to old coordinates :

$$J^{-1,i}_j = dx_i/dx'_j \,. \tag{2}$$

— To compute these we just need the expression x_i as a function of the x'_j e.g. x as a function of r, θ and ϕ, and then take the partial derivatives. These formula cand be derived from a simply drawing (important to know how to do that) and are available here https://en.wikipedia.org/wiki/Spherical_coordinate_system#Integration_and_differentiation_in_spherical_coordinates.

- What is the difference between the $(0,2)$ metric tensor $\mathbf{g}$ and the so-called line-element that give an expression for the distance in between two infinitesimally close points in a certain set of coordinates ?
  - In GR we use the metric as a pseudo-norm for vectors define in the tangent bundle of the manifold. In particular the length of curve between two points is given by the integral of the (opposite of the) pseudo-norm of its tangent vector along the curve.
  - For two infinitesimally close points on a curve parametrized by λ, the distance $\mathrm{d}\lambda^2$ will then be given by :

$$\mathrm{d}\lambda^2 = -g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \,. \tag{3}$$

  - Hence the coordinate of the metric in a given chart give access to the line-element in this chart.
- Did we use an intrinsic or extrinsic definition of the sphere in the exercise 2 ?
  - We used an extrinsic definition of the sphere since it is viewed as an hyper-surface embedded in the Euclidian 3D space. However all the quantities we computed are "intrinsic" to the 2-sphere in the sense that can be computed without any reference to the 3D space.
- When is the relation of question 3) for the Riemann tensor true ?
  - It is actually true for any 2D Rieamannian manifold ! Why ?
  - First, the proof that we have done is always valid in fact. Since the Rieamann and the tensor built out of the metric given in the text both enjoy the same symmetries, and because of them have only a single independent component : they have to be propotional

$$R_{\mu\nu\lambda\sigma} = \alpha \left(g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda} \right) \,. \tag{4}$$

    Now the coefficient α depends on the manifold but by contracting both tensors, we can express it as a function of the scalar curvature only.
  - For a deeper reason on why it has to be so you can look at the Ricci decomposition of the Riemann tensor https://en.wikipedia.org/wiki/Ricci_decomposition. In any dimension there is a piece of the Riemann tensor that can be written this way, as a function of the metric and the scalar curvature only. Another piece is written in term of the Ricci tensor and the metric. Finally a last piece is the Weyl tensor.
  - By a counting argument we can see that in 2D the Riemann has only a single independent component, as many as the scalar curvature built from it and contraction with the metric, hence it makes sense that the Riemann can be expressed in terms of scalar curvature and metric only. The other pieces in teh Ricci decomposition vanish.

- In 3D the Riemann has as many independent components as the Ricci hence we expect the Weyl tensor to vanish, and it does.

- Still about the same question 3, is there a physical meaning in GR of $R_{ij} = g_{ij}$?
  - To start with, in this exercice there is no gravity. We are only looking at geometry so we could only try to give a geometrical meaning using the meaning of the Ricci tensor see next question.
  - Still let's imagine that we find such result in a GR context where this is a solution of Einstein's equation. First, for a 2D space the general relation is rather $R_{ij} = \frac{R}{2} g_{ij}$ where for us $R = 2$ hence the simplification. Then this result exactly means that in 2D the Einstein tensor is identically 0 !
  - The consequence is that any metric is a solution of Einstein equation in 2D and the Einstein's equation in 2D are simply the fact that the stress-energy tensor has to be 0 everywhere, hence we have a vacuum solution see https://physics.stackexchange.com/questions/303999/derive-einsteins-field-equations-in-one-spatial-and-one-time-dimensions

- Can we give a geometrical meaning to the Ricci tensor itself ?
  - I did not dig too much but it seems the Ricci is related to the part of the Rieamann describing how the volume of a ball of test particles following a congruence of geodesics would be modified, in which direction it is distorted and how much. The scalar curvature, built from the Ricci, contains the information on the overall volume e.g. if a ball is distorted in an ellipsoid of the same volume then you won't see it in the scalar curvature. Finally the Weyl contains the information about other type of deformation, the shear. See https://www.physicsforums.com/threads/geometrical-meaning-of-weyl-tensor.708383/.