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Ex 2 • 2-sphere $\equiv \{ (x,y,z) / x^2+y^2+z^2=1 \}$ in 3D Euclidean space

1) • Cartesian coordinates $g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2 + dz^2$

• 2-sphere $\equiv \{ (r,\theta,\phi) / r=1 \}$
$\Longrightarrow$ 1 coordinate fixed
the other are free

(T) Metric is a (0,2) tensor

$$\cdot \ \underbrace{g'_{ij}}_{\text{Spherical coordinates}} = \underbrace{(J^{-1})_i{}^k}_{\text{Jacobian}} (J^{-1})_j{}^l \underbrace{g_{ij}}_{\text{Cartesian coordinates}}$$

$$\boxed{g'_{ij} dx'^i dx'^j = dr^2 + r^2 (d\theta^2 + \sin^2(\theta) d\phi^2)}$$

• 2-sphere $\Longrightarrow r$ is fixed $\leadsto$ "$dr=0$"

$$\boxed{g^{(2S)}_{ij} dx'^i dx'^j = r^2 (d\theta^2 + \sin^2(\theta) d\phi^2)}$$

2) $\boxed{\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\eta}\left(\partial_\mu g_{\nu\eta} + \partial_\nu g_{\mu\eta} - \partial_\eta g_{\mu\nu}\right)}$

- $\lambda = \theta$: $\Gamma^{\theta}_{\theta\theta}$, $\Gamma^{\theta}_{\theta\phi} = \Gamma^{\theta}_{\phi\theta}$, $\Gamma^{\theta}_{\phi\phi}$
- $\lambda = \phi$: $\Gamma^{\phi}_{\theta\theta}$, $\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta}$, $\Gamma^{\phi}_{\phi\phi}$

$$g^{ij} = \begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & \frac{1}{r^2 \sin^2(\theta)} \end{pmatrix}$$

$\times$ $\Gamma^{\theta}_{\theta\theta} = \frac{1}{2} g^{\theta\eta}\left(\partial_\theta g_{\theta\eta} + \partial_\theta g_{\theta\eta} - \partial_\eta g_{\theta\theta}\right)$

$= \frac{1}{2} g^{\theta\theta}\left(\partial_\theta g_{\theta\theta} + \partial_\theta \cancel{g_{\theta\theta}} - \partial_\theta \cancel{g_{\theta\theta}}\right)$

$= \frac{1}{2} g^{\theta\theta}\left(\underbrace{\partial_\theta(g_{\theta\theta})}_{=0}\right) = 0$ $\qquad g_{\theta\theta} = r^2 = 1$

$\times$ $\Gamma^{\phi}_{\phi\theta} = \frac{1}{2} g^{\phi\eta}\left(\partial_\phi g_{\theta\eta} + \partial_\theta g_{\phi\eta} - \partial_\eta g_{\phi\theta}\right)$

$= \frac{1}{2} g^{\phi\phi}\left(\partial_\phi(\cancel{g_{\theta\phi}}) + \partial_\theta(g_{\phi\phi}) - \cancel{\partial_\phi(g_{\phi\theta})}\right)$

$= \frac{1}{2} \times \frac{1}{r^2 \sin^2(\theta)} \times \partial_\theta\left(r^2 \sin^2(\theta)\right)$

$\boxed{\Gamma^{\phi}_{\phi\theta} = \frac{\cos(\theta)}{\sin(\theta)}} = \cot(\theta)$

$\boxed{\underline{\text{Summary}}: \quad \Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \cot(\theta) \; ; \; \Gamma^{\theta}_{\phi\phi} = -\sin(\theta)\cos(\theta)}$

3) - In any local chart : $\boxed{R_{ijkl} = g_{ik} g_{jl} - g_{jk} g_{il}}$

· Both are $(0,4)$ tensors $\Rightarrow$ Transform the same way $\rightarrow$ Check it in a given chart

× "Naive way": Compute <u>all</u> components of both

× $\underbrace{R_{ijkl}} \equiv g_{im} \underbrace{R^{m}{}_{jkl}}$ Simple expression in terms of $\Gamma^{\lambda}_{\mu\nu}$

Enjoy many symmetries

In general, n dimensions · $R_{ijkl} \rightarrow n^4$ coeffs

· $\boxed{R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}}$

Count independent coeffs = Nbr of unordered couples $\{(i,j),(k,l)\}$ of unordered pairs

$$\mathcal{N} = \frac{\frac{n\times(n-1)}{2}\times\frac{n\times(n-1)}{2} - \frac{n\times(n-1)}{2}}{2} + \frac{n(n-1)}{2}$$

$$= \frac{n\times(n-1)}{4}\left(\frac{n\times(n-1)}{2} - 1\right) + \frac{n(n-1)}{2}$$

$$= \frac{n\times(n-1)}{2}\left[\frac{n\times(n-1)-2}{4} + 1\right]$$

$$\boxed{\mathcal{N} = \frac{n\times(n-1)\left[n\times(n-1)+2\right]}{8}}$$

· Here $n=2$ $\qquad \mathcal{N} = \frac{2\times(2-1)\times\left[2\times(2-1)+2\right]}{8} = \underline{1}$!

- $\boxed{R^\lambda{}_{\sigma\mu\nu} = \partial_\mu(\Gamma^\lambda_{\nu\sigma}) - \partial_\nu(\Gamma^\lambda_{\mu\sigma}) + \Gamma^\lambda_{\mu\eta}\Gamma^\eta_{\nu\sigma} - \Gamma^\lambda_{\nu\eta}\Gamma^\eta_{\mu\sigma}}$

Arbitrary choice

- $R^\phi{}_{\theta\phi\theta} = \partial_\phi(\underbrace{\Gamma^\phi_{\theta\theta}}_{=0}) - \partial_\theta(\underbrace{\Gamma^\phi_{\phi\theta}}_{=\cot(\theta)}) + \underbrace{\Gamma^\phi_{\phi\eta}\Gamma^\eta_{\theta\theta}}_{=\Gamma^\phi_{\phi\theta}\Gamma^\theta_{\theta\theta},\ \Gamma^\theta_{\theta\theta}=0} - \underbrace{\Gamma^\phi_{\theta\eta}\Gamma^\eta_{\phi\theta}}_{=\Gamma^\phi_{\theta\phi}\Gamma^\phi_{\phi\theta}}$

$$= -\partial_\theta(\cot(\theta)) - \cot^2(\theta)$$

$$= -\partial_\theta\left(\frac{\cos(\theta)}{\sin(\theta)}\right) - \cot^2(\theta) = -\left(\frac{\overbrace{-\cos^2(\theta)-\sin^2(\theta)}^{=-1}}{\sin^2(\theta)}\right) - \frac{\cos^2(\theta)}{\sin^2(\theta)}$$

$$= \frac{1-\cos^2(\theta)}{\sin^2(\theta)} = 1 \checkmark$$

$$\boxed{R^\phi{}_{\theta\phi\theta} = 1}$$

- $R_{\phi\theta\phi\theta} = g_{\phi\lambda}R^\lambda{}_{\theta\phi\theta} = g_{\phi\phi}R^\phi{}_{\theta\phi\theta} = r^2\sin^2(\theta)$

- $g_{ik}g_{jl} - g_{jk}g_{il} \longrightarrow g_{\phi\phi}g_{\theta\theta} - \overbrace{g_{\theta\phi}}^{=0}g_{\theta\phi} = r^2\sin^2(\theta)\times r^2$

$$= r^4\sin^2(\theta)$$

- $\boxed{R=1}$ $\Longrightarrow$ $\boxed{R_{\phi\theta\phi\theta} = \sin^2(\theta) = g_{\phi\phi}g_{\theta\theta} - g_{\theta\phi}g_{\theta\phi}}$

By sym $\longrightarrow$ True for any components!

Curvature $R^\lambda{}_{\sigma\lambda\nu} \equiv R_{\sigma\nu}$, R_{ijkl}

Ricci $R_{ij} = R^k{}_{ikj} = g^{k\eta}\underbrace{R_{\eta ikj}}$ $\longrightarrow$ Simple expression?

$$\delta_\mu{}^\nu = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= g^{k\eta}\left(g_{\eta k} g_{ij} - g_{ik} g_{\eta j}\right)$$

$$= \underbrace{(g^{k\eta} g_{\eta k})}_{=\delta^k{}_k} g_{ij} - g_{ik} \underbrace{g^{k\eta} g_{\eta j}}_{=\delta^k{}_j}$$

$$= 2\, g_{ij} - \underbrace{g_{ik}\,\delta^k{}_j}_{= g_{ij}}$$

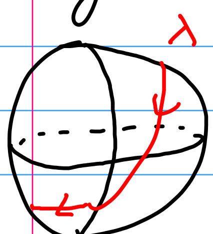
$$\boxed{R_{ij} = g_{ij}}$$

• $R = R^i{}_i = g^{ij} g_{ji} = \delta^i{}_i = 2 \implies \boxed{R = 2}$ $\Downarrow$

Scalar curvature

Constant curvature

4) • Geodesic parametrized affinely by λ



Tangent vector $U = U^\mu \partial_\mu$

$$\nabla_{\underline{U}} \underline{U} = \cancel{f(\underline{U})}\, \underline{U}$$

Affine geodesic $= 0$

$$\boxed{\nabla_U(\underline{U}) = 0} \implies U^\mu \nabla_\mu(\underline{U}) = 0$$

Correct

Wrong

$\mu = \theta$ $\quad \nabla_\theta(\underline{U}) = 0 \iff \boxed{\partial_\theta(U^\nu) + \Gamma^\nu_{\theta\lambda} U^\lambda = 0}$

$\mu = \phi$ $\quad \boxed{\partial_\phi(U^\nu) + \Gamma^\nu_{\phi\lambda} U^\lambda = 0}$

⚠ The following is wrong. It corresponds to looking for a cst vector $\nabla \underline{U} = 0$

* $\partial_\theta(v^\theta) + \underbrace{\Gamma^\theta_{\theta\lambda}}_{=0} v^\lambda = 0 \iff \boxed{\partial_\theta(v^\theta) = 0}$

* $\partial_\theta(v^\phi) + \underbrace{\Gamma^\phi_{\theta\lambda} v^\lambda}_{=\Gamma^\phi_{\theta\phi} v^\phi} = 0 \iff \boxed{\partial_\theta(v^\phi) + \cot(\theta)\, v^\lambda = 0}$

* $\partial_\phi(v^\theta) + \Gamma^\theta_{\phi\lambda} v^\lambda = 0 \iff \boxed{\partial_\phi(v^\theta) - \sin(\theta)\cos(\theta)\, v^\phi = 0}$

* $\partial_\phi(v^\phi) + \Gamma^\phi_{\phi\lambda} v^\lambda = 0 \iff \boxed{\partial_\phi(v^\phi) + \cot(\theta)\, v^\phi = 0}$

• Check that if $\theta = \frac{\pi}{2}$ then one can find a solut°:

$$\partial_\theta(v^\theta) = 0\ ;\ \partial_\theta(v^\phi) = 0\ ;\ \partial_\phi(v^\theta) = 0\ ;\ \partial_\phi(v^\phi) = 0$$

$v = $ cst vector is a solut° Wrong

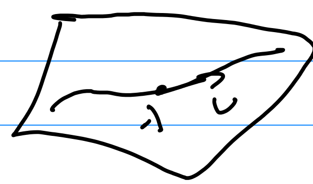
Correct

• $v^\mu \nabla_\mu(\underline{v}) = 0 \iff \boxed{v^\mu \nabla_\mu(v^\nu) = 0}$

$$\iff v^\mu\left(\partial_\mu(v^\nu) + \Gamma^\nu_{\mu\eta} v^\eta\right) = 0$$

$$\iff \boxed{v^\mu \partial_\mu(v^\nu) + \Gamma^\nu_{\mu\eta} v^\eta v^\mu = 0}$$

$$\boxed{v^\mu \partial_\mu(v^\nu) = \frac{d}{d\lambda}(v^\nu)}$$



$$\frac{d}{d\lambda}\left(f(\{x^\mu(\lambda)\})\right) = \underbrace{\frac{dx^\mu(\lambda)}{d\lambda}}_{= v^\mu} \partial_\mu f$$

• $\nu = \theta$

$$\frac{d}{d\lambda}(v^\theta) + \underbrace{\Gamma^\theta_{\mu\lambda} v^\lambda v^\mu}_{=\Gamma^\theta_{\phi\phi} v^\phi v^\phi} = 0$$

$$\iff \boxed{\frac{d}{d\lambda}(v^\theta) - \sin(\theta)\cos(\theta)(v^\phi)^2 = 0} \quad (1)$$

$\nu = \phi$ $\quad \frac{d}{d\lambda}(v^\phi) + \underline{\Gamma^\phi_{\mu\eta} v^\eta v^\lambda} = 0$

$= \Gamma^\phi_{\phi\theta} v^\theta v^\phi + \Gamma^\phi_{\theta\phi} v^\theta v^\phi = 2\Gamma^\phi_{\theta\phi} v^\theta v^\phi$

$= 2\cot(\theta) v^\theta v^\phi$

$$\boxed{\frac{d}{d\lambda}(v^\phi) + 2\cot(\theta)\, v^\theta v^\phi = 0} \quad (2)$$

× $\theta = \frac{\pi}{2}$ $\quad$ (1) : $\boxed{\frac{d}{d\lambda}(v^\theta) = 0}$ / (2) : $\boxed{\frac{d}{d\lambda}(v^\phi) = 0}$

Constraint

$\theta(\lambda) = \frac{\pi}{2} \Rightarrow \frac{d}{d\lambda}(\theta(\lambda)) = 0 \Rightarrow \boxed{v^\theta = 0} \longrightarrow$ ok (1) is satisfied

$$\boxed{v^\theta \equiv \frac{dx^\theta}{d\lambda} = \frac{d}{d\lambda}(\theta(\lambda))}$$

(2) : $\underset{\frac{d\phi}{d\lambda}}{\underset{\shortparallel}{v^\phi}} = \text{cst} = A \Longrightarrow \boxed{\begin{array}{l}\phi(\lambda) = A\lambda + B \\ \theta(\lambda) = \frac{\pi}{2}\end{array}}$

× $\phi = \text{cst}$ $\quad v^\phi = \frac{d}{d\lambda}(\phi(\lambda)) \Rightarrow v^\phi = 0$ $\quad$ Constraint

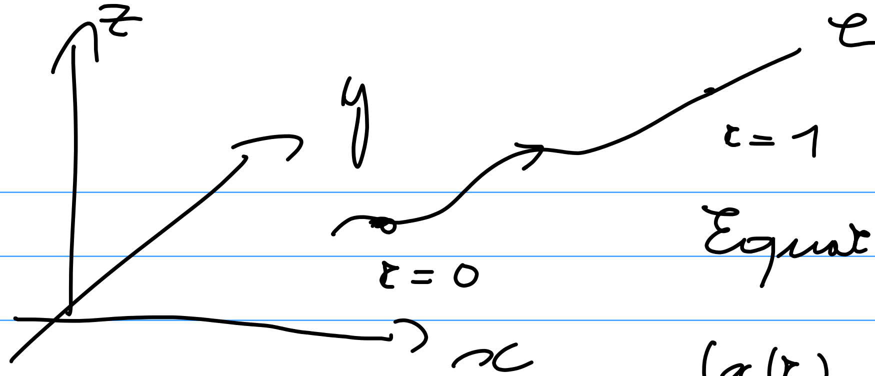
* (2) becomes $\quad \frac{d}{d\lambda}(\underbrace{v^\phi}_{=0}) + 2\cot(\theta)\, v^\theta \underbrace{v^\phi}_{=0} = 0$

* (1) becomes $\quad \frac{d}{d\lambda}(v^\theta) - \sin(\theta)\cos(\theta)\underbrace{(v^\phi)^2}_{=0} = 0$ ok

$\Longleftrightarrow \frac{d}{d\lambda}(v^\theta) = 0 \Longleftrightarrow \frac{d^2}{d\lambda^2}(\theta(\lambda)) = 0 \Longleftrightarrow \boxed{\theta(\lambda) = A \times \lambda + B}$

$$\boxed{\begin{cases}\phi = C \\ \theta(\lambda) = A\times\lambda + B\end{cases}}$$

Example



Equat° describing $\mathcal{C}$

$(x(t), y(t), z(t))$