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Lecture Notes:
The Time Flow of Cosmological Correlators

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Abstract: These lecture notes offer a comprehensive (and hopefully pedagogical) introduction to the cosmological flow, an approach to compute cosmological correlators. We begin with a broad overview of the physics underlying cosmological correlators and review the in-in operator formalism used for their computation. Then, we derive universal flow equations that govern the time evolution of correlators in phase space and show how these equations can be systematically solved. We dedicate the remainder of the lecture engaged in hands-on tutorial lessons on using CosmoFlow, illustrated through a series of detailed examples. Each application is accompanied by a ready-to-use Jupyter notebook. Additionally, many exercises are included to help in the assimilation of the material presented.

These notes are based on a series of lectures given in Spring 2025 at the *Institut Pascal* during the Cosmology Beyond the Analytic Lamppost three-week programme.

Contents

1 Introduction

Have you ever wondered what is the probability to find a galaxy at a certain distance from another one? If that sounds like an interesting question, then these lecture notes are for you! Actually, trying to answer it leads us to think about something even deeper: where does all the matter in the Universe come from? In fact, everything we see in the sky today (galaxies, stars, etc) started out as tiny fluctuations in the hot, dense plasma that filled the early Universe right after the Big Bang. Over time, gravity caused these small differences in density to grow and collapse, eventually forming the large-scale structures we observe today. Pretty amazing, right? But it gets even crazier. These initial fluctuations are actually quantum, and were generated out of the vacuum during a period of accelerated expansion called inflation. So, understanding these primordial fluctuations (and especially how they're correlated and how they evolve in time) is essential if we want to understand everything that came after.

Quite astonishingly, these cosmological correlators encode the physics of the early Universe. But as you might expect, the physics at play at that time is tremendously complicated, and there's in general little hope for being able to compute cosmological correlators exactly. However, as we'll learn during the course of the lectures, these correlators hide a beautiful structure: they are solutions to universal flow equations that govern their time evolution. Solving this flow is the core of the so-called *cosmological flow*, a numerical approach to compute cosmological correlators. The goal of these lectures is to introduce this approach and to get hands on the corresponding Python package: CosmoFlow. In these series of lectures, we will try to cover the following topics.

- **Lecture I: Computing Cosmological Correlators**, where we gently go through the basics of primordial fluctuations and work out in detail the intricacies of the canonical in-in formalism to compute cosmological correlators. We will apply this formalism to concrete (simple enough) examples.
- **Lecture II: Flow Equations**, in which we derive, from first principles using the in-in formalism, universal differential equations in time satisfied by tree-level cosmological correlators.
- **Lecture III: Implementing the Cosmological Flow** is rather short, and essentially discusses how these flow equations can be systematically solved in practice.
- **Lecture IV: CosmoFlow Tutorials** is going to introduce you to the Python package CosmoFlow with a series of concrete (hopefully interesting) examples with fill-in-the-blank Jupyter notebooks. It will be the opportunity for you to get your hands dirty!

Organisation

These notes are based on a series of four lectures, each lasting an hour and a half, specifically designed for graduate students. The prerequisites are quite minimal: a basic understanding of primordial cosmology and quantum field theory is sufficient. A significant part of these lectures will focus on using CosmoFlow, a Python package for cosmological correlators. Therefore, a working knowledge of Python is helpful. If you're not really into programming, I would encourage you to use chatbot assistants such as Le Chat, the French/European alternative to ChatGPT, or feel free to reach out to me directly. Do not let coding intimidate you or deter you from engaging with this material. Remember, the most important is to have fun!

Please note that these lecture notes have been written in a record time, so kindly forgive any potential typos and mistakes.

Useful resources

We warn the reader that there won't be any references throughout these lecture notes, and that these notes are *not* meant to be a review. As such, we might have overlooked important points and only present a biased exposition of the subject. Having said this, here are some useful references:

- **Inflation.** Refs. [1, 2] are the classic lecture notes on the physics of inflation written by two leading maestros in the field: Daniel and Leonardo. An excellent book (which by now has also become a classic) is [3], which reviews high-energy aspects of inflation.

- **Non-Gaussianities.** Theoretical and observational aspects of inflation, with a focus on non-Gaussianities, are very nicely summarised in the following white paper and review: [4, 5]. Do not hesitate to look at the references therein. Ref. [6] gives a clear and concise overview of non-Gaussianities where you can find all the necessary formula (although without much detail).

- **In-in formalism.** The classic historial papers on applying the in-in formalism to cosmology are [7, 8], written by the two titans (if not legends) of modern physics. The in-in formalism is nicely reviewed in [9–11].

- **Transport approach.** The possibility of using differential equations in time to follow cosmological correlators on super-horizon scales has been first recognised in [12–17]. This method has been extended to sub-horizon scales in [18] where its equivalence with the in-in formalism has been demonstrated. The modern transport approach has been implemented and publicly released as freely-available software packages in [19–23].

- **Cosmological flow.** We invite the readers to refer to [24–26] for further details on the cosmological flow. Most of the material presented in these lecture notes are taken from these papers.

Also see this goldmine of Cosmology Resource Materials for additional resources.

Acknowledgements

I would like to thank all the participants of the lecture courses given at the *Institut Pascal* in Saclay and the organisers of the three-week programme. In particular, let me thank Sébastien Renaux-Petel for giving me opportunity to give these lectures. This material is based upon work supported by the *Contrat Doctoral Spécifique Normalien* (CDSN), awarded by the French Ministry of Higher Education and Research, and by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 758792, Starting Grant project GEODESI).

2 Lecture I: Computing Cosmological Correlators

One of the most remarkable features of inflation (apart from solving the standard ΛCDM problems) is that it naturally provides a mechanism for generating the primordial density fluctuations that later seeded cosmic structure formation: they arose from quantum zero-point fluctuations of the inflaton field which drives the accelerated expansion. However, primordial fluctuations can only be described in a statistical sense, and a natural question to ask is: what is the probability to find a point with a certain primordial density at a given distance from another point (at equal time). By varying the distance, we construct the two-point correlation function, or equivalently the corresponding power spectrum in Fourier space. However, we can also ask the question for three points, four points, etc, building this way higher-order correlation functions. These so-called *cosmological correlators* are the main interest in these lectures.

In this first lecture, we will first quickly recall important properties of primordial fluctuations and discuss primordial non-Gaussianities (especially their shapes). Then, we will dive into the details of the canonical in-in formalism to compute cosmological correlators. We will make use of this formalism later when deriving the flow equations.

2.1 Initial cosmological fluctuations

Primordial fluctuations are in general very complicated to describe. In principle, we should describe each component of the primordial plasma and take into account all interactions between these fluctuations. However, from observations, these primordial fluctuations have three very important properties:

- **Adiabatic.** All fluctuating components (photons, baryons, dark matter, neutrinos, etc) oscillate in phase, i.e. they are synchronised, and are proportional to each other: $\delta_X(\boldsymbol{x}) \propto \delta_Y(\boldsymbol{x})$ where X and Y are two fluctuating components. We say primordial fluctuations are adiabatic. This means we can describe all primordial fluctuations with a single scalar degree of freedom that we choose to be the (gauge invariant) curvature perturbation $\zeta(t, \boldsymbol{x})$, which is defined as a local rescaling of the spatial metric

$$g_{ij} = a^2 e^{2\zeta(t,\boldsymbol{x})}\delta_{ij}\,, \tag{2.1}$$

  in the so-called comoving gauge where inflaton fluctuations are set to zero $\delta\phi = 0$. As such, the curvature perturbation can be interpreted as the longitudinal mode of the metric after having eaten the scalar inflaton fluctuation (we neglect tensor modes). The name curvature perturbation comes from the fact this field really describes the local curvature $R^{(3)} \propto \partial_i^2\zeta$, where $R^{(3)}$ is the three-dimensional Ricci scalar. A very important property of the curvature perturbation ζ is that it is conserved on superhorizon scales $\dot{\zeta}_{\boldsymbol{k}} = 0$ for $k \ll aH$

if the matter perturbations are adiabatic. We therefore describe primordial density perturbations in terms of the field $\zeta(t, \boldsymbol{x})$. In practice, it means that any new physics during inflation should be coupled and ultimately decay to this field to leave distinctive signatures in the sky.

- **Near scale-invariant.** The (dimensionless) power spectrum of primordial fluctuations $\mathcal{P}_\zeta(k) \equiv \frac{k^3}{2\pi^2} P_\zeta(k) = \frac{k^3}{2\pi^2} \langle \zeta_{\boldsymbol{k}} \zeta_{-\boldsymbol{k}} \rangle'$ has been well measured on CMB scales by the Planck satellite, and is very simple. It is described by only its amplitude and tilt

$$\mathcal{P}_\zeta(k) = A_s \left(\frac{k}{k_\star} \right)^{n_s - 1} , \tag{2.2}$$

with $A_s = 2.2 \times 10^{-9}$ and $n_s = 0.974 \pm 0.003$. These values are measured at the pivot scale $k_\star = 0.05\,\mathrm{Mpc}^{-1}$. The detection of a non-zero red tilt $n_s - 1 \neq 0$, i.e. statistically significant deviation from perfect scale invariance, is a major hint for slow-roll inflation. Perfect scale invariance translates to $\langle \zeta(\lambda \boldsymbol{x}) \zeta(\lambda \boldsymbol{y}) \rangle = \langle \zeta(\boldsymbol{x}) \zeta(\boldsymbol{y}) \rangle$ in real space. It is though a good approximation to model inflation as period of (quasi) de Sitter expansion, as deviation from perfect scale invariance is slow-roll suppressed.

- **Almost Gaussian.** Primordial fluctuations are mostly described by their two-point correlation function: they are almost Gaussian. This means that fluctuations during inflation are described by a weakly coupled (quantum field) theory dominated by a quadratic free action. Therefore, interactions are small. But not zero! Actually, because gravity is intrinsically non-linear, a small amount of non-Gaussianity—i.e. non-zero power in higher-order correlation functions—is guaranteed! We have just not reached the precision to detect them in the sky. These cosmological correlators encode the interacting (and therefore interesting) physics of the early Universe.

Connecting primordial fluctuations to cosmological observables. We learn about the Universe by detecting photons travelling through the cosmos. This includes temperature anisotropies of the Cosmic Microwave Background (CMB), or the location of galaxies in the Large Scale Structure of the Universe (LSS). In this sense, cosmology is an observational science, rather than experimental. Given that all these structures were generated during inflation and provided we understand well the subhorizon evolution of fluctuations (which we do!), one can trace these observables back in time to learn about the initial conditions. In fact, as long as fluctuations are small, they can be linearly evolved from horizon entry to e.g. the CMB. This is done by solving the coupled Einstein-Boltzmann system and is precisely what codes like CLASS and CAMB are doing. In the end, we can linearly relate primordial fluctuations to cosmological observables, e.g. temperature anisotropies $\mathcal{O} = \delta T / \bar{T}$ or the dark matter

density contrast $\mathcal{O} = \delta\rho/\bar{\rho}$ via

$$\mathcal{O}(z, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \, \Delta_{\mathcal{O}}(z, k) \, \zeta_k \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \,, \tag{2.3}$$

where the transfer function $\Delta_{\mathcal{O}}$ encodes all the late-time physics (sound waves in the primordial plasma, dark matter clustering, etc). The physics described by these transfer functions is very rich and is far beyond the scope of these lectures. We invite the reader to wade through standard cosmology textbooks to learn more about this fascinating topic. Remember only that this physics is well-understood, so that all what we observe can be traced back to primordial fluctuations described by (correlation functions of) ζ.

2.2 Primordial non-Gaussianities

Cosmological correlators provide an exciting probe of the physics governing the earliest moments of our Universe, and—for connected n-point correlators with $n \geq 3$—are conventionally called *non-Gaussianities*. Let us now review the basics of phenomenological aspects of cosmological correlators, setting up all the relevant definitions. We primarily focus on the standard shapes of non-Gaussianities and the physical processes they conceal.

The first non-trivial information about deviations from Gaussianity is encoded in the thee-point correlation function, which is also known as the *bispectrum* in Fourier space. The (equal-time) three-point function defines the bispectrum B_ζ as

$$\langle \zeta_{\boldsymbol{k}_1} \zeta_{\boldsymbol{k}_2} \zeta_{\boldsymbol{k}_3} \rangle = (2\pi)^3 \delta^{(3)}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3) B_\zeta(k_1, k_2, k_3) \,, \tag{2.4}$$

where the momentum conserving delta function is a consequence of global homogeneity of space: the wavevectors $\boldsymbol{k}_i$ $(i = 1, 2, 3)$ are enforced to close and form a triangle in Fourier space. Global isotropy implies that the only momentum dependency is through rotational invariant contractions involving δ_{ij} and ϵ_{ijk} (and various combinations of their tensor products), which in the case of the bispectrum can be fully expressed in terms of the three magnitudes $k_i = |\boldsymbol{k}_i|$. Note that for a n-point correlator, imposing homogeneity and isotropy implies that correlation functions—after stripping off the momentum conserving delta function—depend on $3n - 6$ variables (with only one variable for $n = 2$). In a nutshell, the bispectrum is a function that takes as input a triangle shape and outputs a number, i.e. the correponding amplitude.

Shape dependence. Scale dependence is encoded in the size of the triangle, for example in the total perimeter $k_t \equiv k_1 + k_2 + k_3$, or the size of the largest side in Fourier space. For a fixed scale, the triangle can also be deformed: for different *shapes* of triangles, the amplitude of the bispectrum may be very different. In order to

investigate these dependencies, it is customary to define the so-called dimensionless bispectrum shape function S as

$$S(k_1, k_2, k_3) \equiv \frac{(k_1 k_2 k_3)^2}{(2\pi)^4 \mathcal{P}_\zeta^2} B_\zeta(k_1, k_2, k_3)\,, \tag{2.5}$$

where $\mathcal{P}_\zeta$ is the dimensionless power spectrum evaluated at a fiducial momentum scale. Without loss of generality, we order the momenta such that $k_3 \leq k_2 \leq k_1$ and fix the largest Fourier mode k_1 by virtue of scale invariance. Therefore, the shape information only depends on the two dimensionless ratios k_2/k_1 and k_3/k_1 that satisfy $0 \leq k_3/k_1 \leq k_2/k_1 \leq 1$ and $k_1 \leq k_2 + k_3$. The second condition follows from the triangle inequality.

Scale dependence. Information about the scale dependence of the bispectrum is found by fixing the ratios k_2/k_1 and k_3/k_1 and varying the overall scale k_1 (or equivalently k_t). Although it may be relevant to explore it for arbitrary fixed momentum ratios, it is customary to parameterise the scale-dependence in equilateral triangle configurations via $f_{\rm NL}(k) \equiv S(k, k, k)$. In pure de Sitter, $f_{\rm NL}(k) = f_{\rm NL}$ is scale-independent and measures the amplitude of non-Gaussianity as the overall size of the bispectrum. During slow-roll inflation, the scale dependence (running) is proportional to logarithmic time derivatives of the background functions, and therefore remains negligible within any small range of k. It is therefore usual to discard the study of the bispectrum scale dependence at the largest cosmological scales and to focus on its shape dependence.

Standard bispectrum shapes. The bispectrum shape function encodes a lot of information about the physics of inflation. Let us now review the standard shapes, each one corresponding to a specific type of physical processes.

- **Local shape.** The local shape is the one that results from a local definition of non-Gaussianity (in real space), where the primordial curvature perturbation $\zeta(t, \boldsymbol{x})$ can be expressed in terms of a set of Gaussian fluctuations at the exact same location in spacetime. The resulting bispectrum shape, normalised to unity in the equilateral configuration is given by

$$S^{\rm loc}(k_1, k_2, k_3) = \frac{1}{3}\left(\frac{k_3^2}{k_1 k_2} + 2\,\text{perm.}\right)\,. \tag{2.6}$$

  The local shape $S^{\rm loc}$ is typically found in models of inflation with very light scalar fields interacting on super-horizon scales as gradients are suppressed and interactions must happen locally. Note that the local shape is divergent in the squeezed limit $k_3/k_1 \to 0$ as k_1/k_3.

- **Equilateral shape.** The equilateral shape differs sensibly from the local one: it is the largest at configurations of the triangle in Fourier space that are equilateral. The equilateral shape is typically encountered in models that feature derivative interactions, e.g. $\dot{\varphi}(\partial_i\varphi)^2$ or $\dot{\varphi}^3$, which are non-negligible only around horizon crossing of the three modes. Since the field φ self-interacts locally in the bulk of spacetime during inflation, the amplitudes of their wavevectors must be similar $k_1 \sim k_2 \sim k_3$. Specifically, the equilateral shape is given by

$$S^{\rm eq}(k_1,k_2,k_3) = \left(\frac{k_1}{k_2} + 5\,\text{perm.}\right) - \left(\frac{k_3^2}{k_1 k_2} + 2\,\text{perm.}\right) - 2\,. \tag{2.7}$$

- **Cosmological collider physics.** Although the local and equilateral templates are often sufficient to describe the bulk of the primordial bispectrum shapes, a plethora of different functional dependencies have been found and described, and in particular in the squeezed limit $k_3/k_1 \to 0$. Behaviours in the squeezed limit that are *intermediate* between the ones of the local shape $\propto (k_3/k_1)^{-1}$, and the equilateral shape $\propto k_3/k_1$, have been found. The scaling is $\propto (k_3/k_1)^{1/2-\nu}$ where $0 < \nu \equiv \sqrt{9/4 - m^2/H^2} < 3/2$ is a dimensionless measure of the mass of a light ($m < 3H/2$) scalar field coupled to the adiabatic curvature fluctuation. This result has been generalised to heavy ($m > 3H/2$) fields which feature oscillations in the squeezed limit $\propto (k_3/k_1)^{1/2\pm i\mu}$ with $\mu \equiv -i\nu > 0$. Probing early Universe physics through the bispectrum squeezed limit (or more generally soft limits of cosmological correlators) is now referred to as the "cosmological collider programme". We will solve an exercise related to cosmological collider signals at the end of this section, and dedicate an entire CosmoFlow tutorial session (with the corresponding Jupyter notebook) on this later (see Sec. 5.4.4).

The usual local and equilateral templates are shown in Fig. 1. Note that there are other shapes (e.g. flattened or orthogonal) that are also used as templates to look for non-Gaussian signals in e.g. the CMB data. The physics of primordial non-Gaussianities is extremely rich and still is an active field of research.

2.3 In-in formalism

We have a well-established method for computing cosmological correlators: the celebrated in-in operator formalism, which resembles the operator approach to standard flat-space QFT to compute the S-matrix. In a nutshell, this formalism is based on the Hamiltonian, and requires performing field contractions after expanding the Dyson series in perturbation theory.

Before delving deeper, it is essential to pause and reflect on what exactly we aim to compute. In cosmology, we are interested in predicting—and observing—the (time-dependent) correlation functions of cosmological fluctuations. These differ

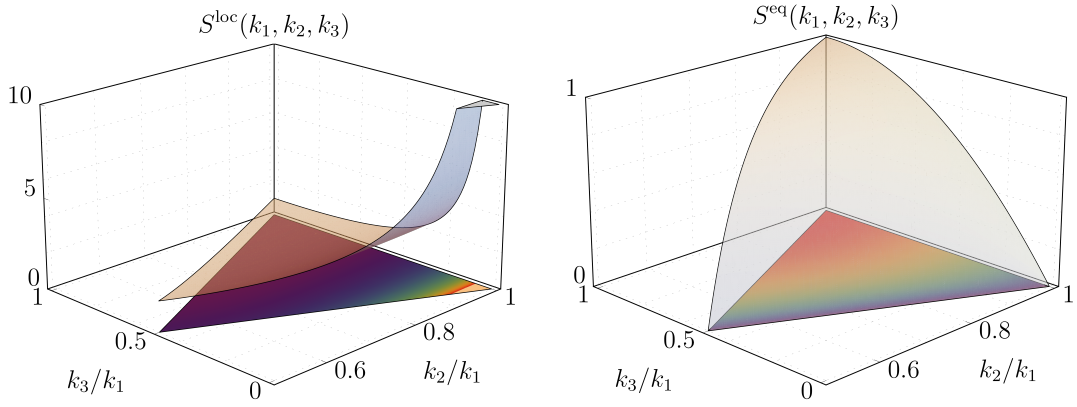


Figure 1: The dimensionless shape functions $S(k_1, k_2, k_3)$ for the standard local and equilateral templates. The local shape diverges as $\propto k_1/k_3$ in the squeezed limit $k_3/k_1 \to 0$ (and $k_2 \to k_1$). The equilateral shape is normalised to unity in the equilateral configuration ($k_1 = k_2 = k_3$) where it peaks, and vanishes in the squeezed limit.

fundamentally from cross sections in particle physics, which describe the probability of an event *localised* in spacetime. In contrast, our aim is to compute the *vacuum expectation value* of operators $\hat{\mathcal{O}}\left[\hat{\varphi}(t,\boldsymbol{x}),\hat{p}(t,\boldsymbol{y})\right]$, made up of products of various fields and their conjugate momenta at a given same time t,

$$\langle\Omega|\hat{\mathcal{O}}(t)|\Omega\rangle \equiv \langle\Omega|\hat{\mathcal{O}}\left[\hat{\varphi}(t,\boldsymbol{x}),\hat{p}(t,\boldsymbol{y})\right]|\Omega\rangle \ , \tag{2.8}$$

where the state $|\Omega\rangle$ is taken to be the vacuum of the fully interacting theory. The object defined in (2.8) is the answer to the question: what is the quantum-mechanical expectation value of (product of) fluctuations at some time t given that the Universe was prepared in some initial state in the infinite past that has been evolved to t? Note that the actual link between quantum expectation values of operators we compute and statistical averages of fluctuations we observe in the sky is far from obvious and is still not completely understood.

2.3.1 Interaction picture & Dyson's formula

In quantum field theory, states and operators are evolved in time using the evolution unitary operator $\hat{U}(t,t')$. This operator satisfies the following propoerties:

$$\hat{U}(t,t')\hat{U}(t',t'') = \hat{U}(t,t'') \, , \quad \hat{U}(t,t) = \mathbb{1} \, . \tag{2.9}$$

Note that the usual S-matrix operator in flat space is given by $\hat{S} = \hat{U}(+\infty,-\infty)$, and once sandwiched between two states gives the corresponding S-matrix elements $S_{\beta\alpha} \equiv \langle\beta|\hat{S}|\alpha\rangle$. These complex numbers, once squared, give the probability for a system initially prepared in the state $|\alpha\rangle$ at $t = -\infty$ to evolve in a state $|\beta\rangle$ at $t = +\infty$.

We now derive the explicit expression for the operator $\hat{U}$. We assume that the dynamics of operators $\hat{\mathcal{O}}(t, \boldsymbol{x})$ is determined by a Hamiltonian $\hat{H}$ through the Heisenberg equation of motion $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathcal{O}} = i[\hat{H}, \hat{\mathcal{O}}]$. This is the dynamical equation in the so-called *Heisenberg picture* where states (including the vacuum state of the full theory $|\Omega\rangle$) are time independent and operators are time dependent. The full Hamiltonian is in general too complicated to be solved exactly. Therefore, we need to resort to perturbation theory. To define a new picture in time-dependent perturbation theory, we write the Hamiltonian as

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} \,, \tag{2.10}$$

where the time evolution induced by $\hat{H}_0$ can be solved exactly. Importantly, this splitting is completely arbitrary as long as the interactions induced by $\hat{H}_{\text{int}}$ can be treated perturbatively. We now define the *interaction picture* where the leading time evolution generated by $\hat{H}_0$ is factored out. From the operators $\hat{\mathcal{O}}$, we define the interaction-picture operators $\hat{\mathcal{O}}_{\text{I}}$ as evolving according to the interaction-picture exactly solvable Hamiltonian $\hat{H}_0^{\text{I}}$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathcal{O}}_{\text{I}} = i[\hat{H}_0^{\text{I}}, \hat{\mathcal{O}}_{\text{I}}] \,. \tag{2.11}$$

Exercise 2.1 Derive Eq. (2.12).

Let us now find the time evolution operator $\hat{U}(t, t_0)$ such that is relates the full Heisenberg-picture operators to the interaction-picture ones at the *same time* t, $\hat{\mathcal{O}}(t) = \hat{U}^\dagger(t, t_0)\hat{\mathcal{O}}_{\text{I}}(t)\hat{U}(t, t_0)$. Note that we choose a fiducial time t_0 to be that when the interaction-picture operators $\hat{\mathcal{O}}_{\text{I}}$ are matched onto the Heisenberg ones: $\hat{\mathcal{O}}_{\text{I}}(t_0) = \hat{\mathcal{O}}(t_0)$. Differentiating this relation with respect to t and using the dynamical equations for $\hat{\mathcal{O}}_{\text{I}}$ and $\hat{\mathcal{O}}$, one finds that a necessary condition is that the operator $\hat{U}(t, t_0)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}(t, t_0) = -i\hat{H}_{\text{int}}^{\text{I}}(t)\hat{U}(t, t_0) \,, \tag{2.12}$$

and where $\hat{H}_{\text{int}}^{\text{I}}$ is the interaction Hamiltonian expressed in the interaction picture. If everything commutes, the solution to Eq. (2.12) would trivially be $\hat{U}(t, t_0) = \exp\left(-i\int_{t_0}^{t} \hat{H}_{\text{int}}^{\text{I}}(t')\mathrm{d}t'\right)$. However, $\hat{H}_{\text{int}}^{\text{I}}(t)$ does not necessarily commute with $\hat{H}_{\text{int}}^{\text{I}}(t')$ at different times $t \neq t'$. The correct solution is given, at least formally, by

$$\hat{U}(t, t_0) = \mathcal{T}\left\{\exp\left(-i\int_{t_0}^{t} \hat{H}_{\text{int}}^{\text{I}}(t')\mathrm{d}t'\right)\right\} \,, \tag{2.13}$$

where $\mathcal{T}$ (with $\mathcal{T}^\dagger = \bar{\mathcal{T}}$) is the time ordering operator. The expression (2.13) is known as the Dyson's formula, and works because time ordering effectively makes everything inside commute. Upon the correct boundary condition $\hat{U}(t_0, t_0) = \mathbb{1}$, this solution is unique.

2.3.2 Master formula

At some later time, the solution for the operator $\hat{\mathcal{O}}(t)$ in terms of the free fields and momenta in the interaction picture is then given by

$$\hat{\mathcal{O}}(t) = \bar{\mathcal{T}}\left\{\exp\left(i\int_{t_0}^{t}\hat{H}^{\rm I}_{\rm int}(t')\mathrm{d}t'\right)\right\}\hat{\mathcal{O}}_{\rm I}(t)\mathcal{T}\left\{\exp\left(-i\int_{t_0}^{t}\hat{H}^{\rm I}_{\rm int}(t')\mathrm{d}t'\right)\right\}. \tag{2.14}$$

We now want to express the expectation value of the operator $\hat{\mathcal{O}}(t)$ as evaluated in the vacuum of the full theory $|\Omega\rangle$. To do this, one needs to specify the time t_0 at which interactions are shut down and the theory is free, therefore trading the $|\Omega\rangle$ for the vacuum of the free theory $|0\rangle$. This allows concrete computations using Wick's theorem. In cosmological applications, the free theory can only be unambiguously defined in the asymptotic past $t_0 \to -\infty$. Reaching this limit requires introducing the $i\epsilon$ prescription. This amounts to tilt the time integral contour in the imaginary direction as it approaches the asymptotic past

$$t_0 = -\infty \to t_0 = -\infty(1 \pm i\epsilon)\,, \tag{2.15}$$

where $-i\epsilon$ (resp. $+i\epsilon$) is for the $\hat{U}$ (resp. $\hat{U}^\dagger$). After including the $i\epsilon$ prescription to project the full theory vacuum $|\Omega\rangle$ onto the vacuum of the free theory $|0\rangle$, we obtain the master in-in formula

$$\langle\Omega|\hat{\mathcal{O}}(t)|\Omega\rangle = \langle 0|\bar{\mathcal{T}}\left\{\exp\left(i\int_{-\infty^-}^{t}\hat{H}^{\rm I}_{\rm int}(t')\mathrm{d}t'\right)\right\}\hat{\mathcal{O}}_{\rm I}(t)\mathcal{T}\left\{\exp\left(-i\int_{-\infty^+}^{t}\hat{H}^{\rm I}_{\rm int}(t')\mathrm{d}t'\right)\right\}|0\rangle\,, \tag{2.16}$$

where $-\infty^\pm \equiv -\infty(1 \mp i\epsilon)$. This notation matches the visualisation in the complex plane, in the sense that e.g. $-\infty^+$ is tilted to the positive imaginary direction.

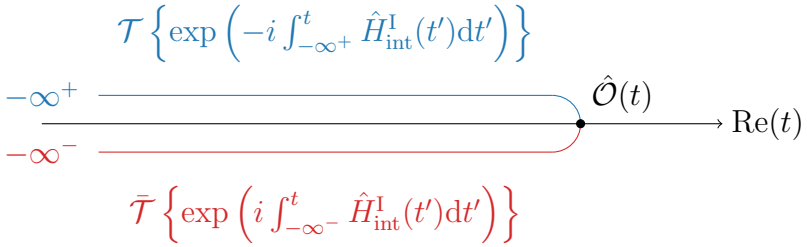


Figure 2: Illustration of the in-in contour in the complex time plane. As only the initial state is available, i.e. the "in" vacuum $|\Omega\rangle$, the dynamics is first evolved forward in time from t_0 to t using the $\hat{U}(t,t_0)$, and then backwards from t to t_0 with $\hat{U}^\dagger(t,t_0)$. In the limit $t_0 \to -\infty$, the correct $i\epsilon$ prescription on the two branches projects the interacting vacuum $|\Omega\rangle$ onto the free one $|0\rangle$.

Exercise 2.2 Consider the full Hamiltonian $\hat{H}$ to be time independent for simplicity.

- After inserting a complete set of states $\mathbb{1} = \sum_n |n\rangle \langle n|$ inside the state $e^{-i\hat{H}(t_0-t)}|0\rangle$ and isolate the ground state $|\Omega\rangle$, show that $|\Omega\rangle$ can be mapped to $|0\rangle$ via

$$|\Omega\rangle = \lim_{t\to-\infty(1-i\epsilon)} \left(e^{-i\omega_0(t_0-t)}\langle\Omega|0\rangle\right)^{-1} e^{-i\hat{H}(t_0-t)}|0\rangle \ . \tag{2.17}$$

- By splitting the full Hamiltonian $\hat{H}$ into $\hat{H}_0$ and $\hat{H}_{\rm int}$ and using $\hat{H}_0|0\rangle = |0\rangle$ (which defines the zero energy), show that

$$|\Omega\rangle = \lim_{\epsilon\to 0} \mathcal{N}_\epsilon \hat{U}^\dagger_\epsilon(t_0,-\infty)|0\rangle \ , \quad \text{with} \quad \mathcal{N}_\epsilon = \lim_{t\to-\infty(1-i\epsilon)} \left(e^{-i\omega_0(t_0-t)}\langle\Omega|0\rangle\right)^{-1} , \tag{2.18}$$

  and where we define the time evolution operator with a deformed contour as

$$\hat{U}^\dagger_\epsilon(t_0,-\infty) = \mathcal{T}\left\{\exp\left(-i\int_{-\infty(1-i\epsilon)}^{t_0} \hat{H}^{\rm I}_{\rm int}(t')\mathrm{d}t'\right)\right\} . \tag{2.19}$$

- In the limit $\epsilon \to 0$, compute $|\mathcal{N}_\epsilon|^2$.

- What crucial assumptions have we used here that strictly speaking do not apply to the cosmological scenario?

2.3.3 Perturbative expansion

In practice, the master in-in formula (2.16) cannot be solved as it is: the aim of splitting the full Hamiltonian into a free part and an interacting one was to treat interactions in a perturbative way. Assuming interactions are small, we can consistently expand the exponentials in powers of $\hat{H}^{\rm I}_{\rm int}$. This results in the perturbative expansion

$$\langle\Omega|\hat{\mathcal{O}}(t)|\Omega\rangle = \sum_{n=0}^{\infty} \langle\Omega|\hat{\mathcal{O}}(t)|\Omega\rangle^{(n)} \ , \tag{2.20}$$

where the superscript (n) refers to the number of $\hat{H}^{\rm I}_{\rm int}$ insertions.

- To zeroth order, the leading term in the series (2.20) easily reads

$$\langle\Omega|\hat{\mathcal{O}}(t)|\Omega\rangle^{(0)} = \langle 0|\hat{\mathcal{O}}_{\rm I}(t)|0\rangle \ . \tag{2.21}$$

- To first order, carefully accounting for the operator ordering, we obtain

$$\langle\Omega|\hat{\mathcal{O}}(t)|\Omega\rangle^{(1)} = i\int_{-\infty^-}^{t} \mathrm{d}t' \, \langle 0|\hat{H}^{\rm I}_{\rm int}(t')\hat{\mathcal{O}}_{\rm I}(t)|0\rangle - i\int_{-\infty^+}^{t} \mathrm{d}t' \, \langle 0|\hat{\mathcal{O}}_{\rm I}(t)\hat{H}^{\rm I}_{\rm int}(t')|0\rangle \ . \tag{2.22}$$

Exercise 2.3 The first-order term in the series expansion (2.20) can be written in different forms.

- Assuming that the operator $\hat{\mathcal{O}}$ is Hermitian, i.e. $\hat{\mathcal{O}}^\dagger = \hat{\mathcal{O}}$, show that the first-order expression can be recast in the following compact form

$$\langle\Omega|\hat{\mathcal{O}}(t)|\Omega\rangle^{(1)} = 2\,\mathrm{Im}\left[\int_{-\infty^-}^{t} \mathrm{d}t'\,\langle 0|\hat{\mathcal{O}}_{\mathrm{I}}(t)\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t')|0\rangle\right] . \tag{2.23}$$

- Show that this expression can also be written

$$\langle\Omega|\hat{\mathcal{O}}(t)|\Omega\rangle^{(1)} = i\int_{-\infty}^{t} \mathrm{d}t'\,\langle 0|[\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t'),\hat{\mathcal{O}}(t)]|0\rangle\ . \tag{2.24}$$

  This is known as the commutator form. What happens to the integration contour?

- In practice, tilting the integration contour $-\infty \to -\infty^{\pm}$ is equivalent to deforming the Hamiltonian instead of time $\hat{H} \to \hat{H}_\epsilon = \hat{H}e^{\epsilon t}$, hence preserving unitarity and integrating on the real time axis. To convince yourself with a simple example, show that

$$I_\pm(E) \equiv \int_{-\infty^\pm}^{0} \mathrm{d}t\, e^{\pm iEt} = \int_{-\infty}^{0} \mathrm{d}t\, e^{\pm Et}\, e^{\epsilon t}\,, \tag{2.25}$$

  where $E, \epsilon > 0$.

- The second-order contribution to the expansion (2.16) can be written in a compact form

$$\begin{aligned}\langle\Omega|\hat{\mathcal{O}}(t)|\Omega\rangle^{(2)} = &\int_{-\infty^-}^{t}\mathrm{d}t'\int_{-\infty^+}^{t}\mathrm{d}t''\,\langle 0|\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t')\hat{\mathcal{O}}_{\mathrm{I}}(t)\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t'')|0\rangle \\ &- 2\,\mathrm{Re}\left[\int_{-\infty^+}^{t}\mathrm{d}t'\int_{-\infty^+}^{t'}\mathrm{d}t''\,\langle 0|\hat{\mathcal{O}}_{\mathrm{I}}(t)\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t')\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t'')|0\rangle\right] .\end{aligned} \tag{2.26}$$

Exercise 2.4 Similarly to the first-order contribution, there are various ways to write the second-order term.

- Prove the so-called factorised form (2.26).

- Show that this expression can also be written with commutators

$$\langle\Omega|\hat{\mathcal{O}}(t)|\Omega\rangle^{(2)} = -\int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t'' \, \langle 0|[\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t''), [\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t'), \hat{\mathcal{O}}(t)]]|0\rangle \ . \quad (2.27)$$

Expanding (2.16) to higher-order is somehow cumbersome but can be done following the same procedure.

Wick contractions. To evaluate the integrands, we use Wick's theorem. Here, we quickly review it. The integrands take the general form $\langle 0|\hat{A}_{\mathrm{I}}\hat{B}_{\mathrm{I}}\ldots|0\rangle$, which is hard to evaluate. On the other hand, the vacuum expectation value of a *normal-ordered* string of operators $\langle 0|\mathcal{N}\{\hat{A}_{\mathrm{I}}\hat{B}_{\mathrm{I}}\ldots\}|0\rangle$ identically vanishes, as the normal ordering operation $\mathcal{N}$ places all creation operators $\hat{a}^\dagger_{\boldsymbol{k}}$ on the left of all annihilation operators $\hat{a}_{\boldsymbol{k}}$. Let us now give the simplest non-trivial example of a string of two (same) field operators. In Fourier space, the mode expansion of an operator contains two parts, an annihilation part and a creation part, and so can be written

$$\hat{\varphi}_{\mathrm{I}}(\boldsymbol{k}, t) \equiv \hat{\varphi}^-_{\boldsymbol{k}}(t) + \hat{\varphi}^+_{\boldsymbol{k}}(t) = u_k(t)\hat{a}_{\boldsymbol{k}} + u^*_k(t)\hat{a}^\dagger_{-\boldsymbol{k}} \, . \quad (2.28)$$

Expressing both the simple product and the normal-ordered version of the string $\hat{\varphi}_{\mathrm{I}}(\boldsymbol{k}, t)\hat{\varphi}_{\mathrm{I}}(\boldsymbol{q}, t')$ in terms of $\hat{\varphi}^-$ and $\hat{\varphi}^+$, one can notice that these expressions only differ by a commutator

$$\hat{\varphi}_{\mathrm{I}}(\boldsymbol{k}, t)\hat{\varphi}_{\mathrm{I}}(\boldsymbol{q}, t') = \mathcal{N}\{\hat{\varphi}_{\mathrm{I}}(\boldsymbol{k}, t)\hat{\varphi}_{\mathrm{I}}(\boldsymbol{q}, t')\} + [\hat{\varphi}^-_{\boldsymbol{k}}(t), \hat{\varphi}^+_{\boldsymbol{q}}(t')] \, . \quad (2.29)$$

Subtracting out a normal-ordered string of operators from its simple product version defines a *contraction*

$$\begin{aligned} \overbrace{\hat{\varphi}_{\mathrm{I}}(\boldsymbol{k}, t)\hat{\varphi}_{\mathrm{I}}(\boldsymbol{q}, t')} &\equiv \hat{\varphi}_{\mathrm{I}}(\boldsymbol{k}, t)\hat{\varphi}_{\mathrm{I}}(\boldsymbol{q}, t') - \mathcal{N}\{\hat{\varphi}_{\mathrm{I}}(\boldsymbol{k}, t)\hat{\varphi}_{\mathrm{I}}(\boldsymbol{q}, t')\} \\ &= u_k(t)u^*_k(t') \, (2\pi)^3\delta^{(3)}(\boldsymbol{k} + \boldsymbol{q}) \, , \end{aligned} \quad (2.30)$$

where we have used $[\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{-\boldsymbol{q}}] = (2\pi)^3\delta^{(3)}(\boldsymbol{k} + \boldsymbol{q})$. Therefore, contractions turn field operators into simple complex numbers. This generalises to an arbitrary string of field operators

$$\langle 0|\hat{\varphi}_{\mathrm{I}}(\boldsymbol{k}_1, t_1)\ldots\hat{\varphi}_{\mathrm{I}}(\boldsymbol{k}_n, t_n)|0\rangle \ = \ \begin{pmatrix} \text{all possible combinations involving} \\ \text{contractions of every pair of operators} \end{pmatrix} \, , \quad (2.31)$$

with $n \geq 1$. Note that all terms must be contracted. If the string is composed of an odd number of operators, fully contracting the chain gives zero, as only one operator having zero vacuum expectation value is left. Now is the time to practice with concrete examples!

Exercise 2.5 Consider a scalar field φ in a general curved spacetime that self-interacts with $\mathcal{L}_{\rm int}/\sqrt{-g} = -\frac{g}{3!}\varphi^3$ where g is a coupling constant.

- Explicitly write the equal-time three-point correlator $\langle\varphi_{\boldsymbol{k}_1}(t)\varphi_{\boldsymbol{k}_2}(t)\varphi_{\boldsymbol{k}_3}(t)\rangle$ in terms of a single time integral and the field mode function $u_k(t)$.
- Show that this integral at $t = 0$, and in flat space for which $u_k(t) = \frac{e^{-ikt}}{\sqrt{2k}}$, evaluates to
$$\langle\varphi_{\boldsymbol{k}_1}\varphi_{\boldsymbol{k}_2}\varphi_{\boldsymbol{k}_3}\rangle' = \frac{g}{4k_1k_2k_3}\frac{1}{k_1+k_2+k_3}\,, \tag{2.32}$$
where the prime is defined to mean that we drop the momentum conserving delta function.
- What would be the result if the upper limit of the time integral were extended to $t \to +\infty$?

Exercise 2.6 Consider the equal-time four-point correlator of $\varphi_{\boldsymbol{k}}(t)$ coming from the exchange of a field σ. We place ourselves in FLRW and take the interacting Lagrangian to be $\mathcal{L}_{\rm int}/a^3(t) = -\frac{g}{2}\varphi^2\sigma$, which results in a tree-level exchange diagram.

- In the s-channel, for which the exchanged momentum is $\boldsymbol{s} = \boldsymbol{k}_1 + \boldsymbol{k}_2$, show that both contributions in (2.26) are given by
$$\begin{aligned}
&\langle\varphi_{\boldsymbol{k}_1}(t)\varphi_{\boldsymbol{k}_2}(t)\varphi_{\boldsymbol{k}_3}(t)\varphi_{\boldsymbol{k}_4}(t)\rangle' = g^2u^*_{k_1}(t)u^*_{k_2}(t)u_{k_3}(t)u_{k_4}(t)\\
&\times\int_{-\infty^-}^{t}\mathrm{d}t'a^3(t')u_{k_1}(t')u_{k_2}(t')v_s(t')\times\int_{-\infty^+}^{t}\mathrm{d}t''a^3(t'')u^*_{k_3}(t'')u^*_{k_4}(t'')v^*_s(t'')\\
&-2g^2\,\mathrm{Re}\left[u_{k_1}(t)u_{k_2}(t)u_{k_3}(t)u_{k_4}(t)\int_{-\infty^+}^{t}\mathrm{d}t'a^3(t')u^*_{k_1}(t')u^*_{k_2}(t')v_s(t')\right.\\
&\left.\times\int_{-\infty^+}^{t'}\mathrm{d}t''a^3(t'')u^*_{k_3}(t'')u^*_{k_4}(t'')v^*_s(t'')\right]\,.
\end{aligned} \tag{2.33}$$
- Argue that in the soft limit $s \to 0$, we can extend the upper bound of the second nested time integral to $t' \to t$, therefore factorising both time integrals.
- In de Sitter, consider that φ is a conformally coupled scalar field whose mode function is given by $u_k(\tau) = i\frac{H\tau}{\sqrt{2k}}e^{-ik\tau}$, and σ is a massive field whose

mode function is given by

$$v_k(\tau) = \frac{i\sqrt{\pi}}{2} e^{-\pi\mu/2} (-\tau)^{3/2} H^{(1)}_{i\mu}(-k\tau)$$
$$\to \sqrt{\frac{2\pi}{k^3}} \frac{H}{\sinh(\pi\mu)} \left[\frac{e^{+\pi\mu/2}}{\Gamma(1+i\mu)} \left(\frac{-k\tau}{2}\right)^{\frac{3}{2}+i\mu} - \frac{e^{-\pi\mu/2}}{\Gamma(1-i\mu)} \left(\frac{-k\tau}{2}\right)^{\frac{3}{2}-i\mu} \right], \tag{2.34}$$

at late times $\tau \to 0$, where τ is conformal time. By only considering non-local contributions (non-analytic in $s^2/(k_{12}k_{34})$), show that

$$\langle \varphi_{\boldsymbol{k}_1} \varphi_{\boldsymbol{k}_2} \varphi_{\boldsymbol{k}_3} \varphi_{\boldsymbol{k}_4} \rangle' = \frac{g^2 H^2 \tau_0^4}{32\pi k_1 k_2 k_3 k_4 (k_{12}k_{34})^{1/2}}$$
$$\times \left[(1 + i\sinh\pi\mu) \left(\frac{s^2}{4k_{12}k_{34}}\right)^{i\mu} \Gamma(-i\mu)^2 \Gamma\left(\tfrac{1}{2}+i\mu\right)^2 + \text{c.c.} \right], \tag{2.35}$$

where τ_0 is a small late-time regulator, and $k_{ij} \equiv k_i + k_j$. A remarkable property of this correlator is that it oscillates in the momentum ratio $s^2/(k_{12}k_{34})$ at a frequency set by μ. This is the cosmological collider signal.

Hint: Use the following integral formula

$$\int_{-\infty^\pm}^{0} \mathrm{d}\tau e^{\pm ik\tau} \tau^\Delta = -(\pm i)^{\Delta+1} k^{-\Delta-1} \Gamma(\Delta+1), \quad \text{for} \quad \mathrm{Re}(\Delta) > -1. \tag{2.36}$$

3 Lecture II: Flow Equations

As seen in the previous lecture, cosmological correlators can be computed from first principles to arbitrary orders of perturbation theory. However, in practice, computations hide a daunting complexity, and analytical results can only be found in particular cases and for simple-enough theories. The root of this difficulty resides in the challenge to track the detailed time evolution of the physics in the bulk of spacetime. Yet, a remarkable underlying structure is hidden within in-in perturbation theory, that is fundamentally shaped by time itself.

In this lecture, we recast the in-in time integrals as a closed system of differential equations in time satisfied by tree-level cosmological correlators. Systematically solving these universal flow equations is the essence of the cosmological flow. Notably, as we will see, this method: (i) does not require UV (early-time) regulators as correlators are evolved on the real-time axis, (ii) eliminates the need for mode functions as it directly focuses on observables, and (iii) completely bypasses the computation of intricate bulk time integrals.

Notations

We start by setting up the relevant notations and definitions. Compared to the previous lecture, from now on we remove hats on operators to avoid heavy notations. We consider a generic theory of N degrees of freedom $\boldsymbol{\varphi}^\alpha$ (with Greek letters running from 1 to N) and remain for now voluntarily agnostic on the precise form of the theory. The theory is assumed to be described by a Hamiltonian $H(\boldsymbol{\varphi}^\alpha, \boldsymbol{p}^\beta)$ which is a functional of the phase-space coordinates i.e. the field-space coordinates $\boldsymbol{\varphi}^\alpha$ and their corresponding conjugate momenta $\boldsymbol{p}^\beta$. Interaction-picture operators φ^α and p^β are defined in terms of the Heisenberg-picture operators $\boldsymbol{\varphi}^\alpha$ and $\boldsymbol{p}^\beta$ by

$$\varphi^\alpha \equiv U^\dagger \boldsymbol{\varphi}^\alpha U\,, \quad \text{and} \quad p^\alpha \equiv U^\dagger \boldsymbol{p}^\alpha U\,, \tag{3.1}$$

where U is the usual time evolution operator defined in Sec. 2 that is constructed out of the interacting Hamiltonian. For practical purposes, we gather all fields $\boldsymbol{\varphi}^\alpha$ and conjugate momenta $\boldsymbol{p}^\beta$ in a phase-space vectore $\boldsymbol{X}^a \equiv (\boldsymbol{\varphi}^\alpha, \boldsymbol{p}^\beta)$, and we do the same for the interaction-picture operators $X^a \equiv (\varphi^\alpha, p^\beta)$. The Latin index a runs over all phase-space coordinates from 1 to $2N$, and are organised so that a block of field labels is followed by a block of momentum labels, in the same order, i.e.

$$\boldsymbol{X}^a \equiv (\boldsymbol{\varphi}^1, \ldots, \boldsymbol{\varphi}^N, \boldsymbol{p}^1, \ldots, \boldsymbol{p}^N)\,. \tag{3.2}$$

Extended Fourier summation

We will entirely work in (spatial) Fourier space

$$\mathcal{O}_{\boldsymbol{k}} = \int \mathrm{d}^3x\, \mathcal{O}(\boldsymbol{x})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,, \quad \mathcal{O}(\boldsymbol{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, \mathcal{O}_{\boldsymbol{k}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,, \tag{3.3}$$

and adopt the extended Fourier summation convention that consists in using sans serif indices (underlined if hand-written) to denote a sum including integrals over Fourier modes. In what follows, an index contraction reads

$$A_{\mathsf{a}}B^{\mathsf{a}} = \sum_a \int \frac{\mathrm{d}^3k_a}{(2\pi)^3}\, A_a(\boldsymbol{k}_a)B^a(\boldsymbol{k}_a)\,. \tag{3.4}$$

Note that we position indices to respect the normal rules for covariant expressions. There is an extra complexity with $\boldsymbol{k}$-labels included because Fourier-space expressions sometimes produce the δ-function $\delta_{\mathsf{ab}} = (2\pi)^3\delta^{(3)}(\boldsymbol{k}_a + \boldsymbol{k}_b)\,\delta_{ab}$. Within a Fourier integral, this reverses the sign of a $\boldsymbol{k}$-label, which we indicate by decorating the label with a bar as in $B^{\bar{\mathsf{a}}}$. Hence,

$$A_{\mathsf{a}}B^{\bar{\mathsf{a}}} = \sum_a \int \frac{\mathrm{d}^3k_a}{(2\pi)^3} A_a(\boldsymbol{k}_a)B^a(-\boldsymbol{k}_a)\,. \tag{3.5}$$

When dealing with field Greek indices (α, β) running from 1 to N, we also indicate the existence of an extended Fourier summation by typesetting the labels in sans serif face $(\mathsf{\alpha}, \mathsf{\beta})$.

Exercise 3.7 Explicitly show that contracting A_{a} with $\delta^{\mathsf{a}}{}_{\mathsf{b}}$ bars an index, equivalently flips the sign of its corresponding momentum

$$A^{\bar{\mathsf{a}}} = \delta^{\mathsf{ab}}A_{\mathsf{b}}\,. \tag{3.6}$$

3.1 Equations of motion

In full generality, and using the previously introduced notations, a general Hamiltonian can be written

$$H = \frac{1}{2!}H_{\mathsf{ab}}\boldsymbol{X}^{\mathsf{a}}\boldsymbol{X}^{\mathsf{b}} + \frac{1}{3!}H_{\mathsf{abc}}\boldsymbol{X}^{\mathsf{a}}\boldsymbol{X}^{\mathsf{b}}\boldsymbol{X}^{\mathsf{c}} + \frac{1}{4!}H_{\mathsf{abcd}}\boldsymbol{X}^{\mathsf{a}}\boldsymbol{X}^{\mathsf{b}}\boldsymbol{X}^{\mathsf{c}}\boldsymbol{X}^{\mathsf{d}} + \dots\,. \tag{3.7}$$

Without loss of generality, we take $H_{\mathsf{ab}}, H_{\mathsf{abc}}, \dots$ to be symmetric under the exchange of any indices. The tensor components are in general functions of time and of the various momenta, and depend on the theory. We give an explicit example as an exercise below.

Exercise 3.8 Consider a free massive scalar field (in flat space for simplicity) whose action is given

$$S = -\frac{1}{2}\int \mathrm{d}t\mathrm{d}^3x\left[(\partial_\mu\varphi)^2 + m^2\varphi^2\right]\,. \tag{3.8}$$

Show that the corresponding Hamiltonian is given by $H = \frac{1}{2!}H_{\mathsf{ab}}\boldsymbol{X}^{\mathsf{a}}\boldsymbol{X}^{\mathsf{b}}$, where

the corresponding matrix H_{ab} is explicitly given by

$$H_{ab} = (2\pi)^3 \delta^{(3)}(\boldsymbol{p}+\boldsymbol{q}) \begin{bmatrix} \boldsymbol{p}\cdot\boldsymbol{q} + m^2 & 0 \\ 0 & 1 \end{bmatrix} . \tag{3.9}$$

Explain why we can write $H = \frac{1}{2!} H_{\mathsf{a}} \boldsymbol{X}^{\mathsf{a}} \boldsymbol{X}^{\bar{\mathsf{a}}}$, with an implicit sum over a, and explicitly write the corresponding Hamiltonian vector H_a.

The fully non-linear equations of motion for the variables $\boldsymbol{X}^{\mathsf{a}}$ reads

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{X}^{\mathsf{a}}}{\mathrm{d}t} &= i\,[H, \boldsymbol{X}^{\mathsf{a}}] \\ &= \epsilon^{\mathsf{ac}} H_{\mathsf{cb}} \boldsymbol{X}^{\mathsf{b}} + \frac{1}{2!} \epsilon^{\mathsf{ad}} H_{\mathsf{dbc}} \boldsymbol{X}^{\mathsf{b}} \boldsymbol{X}^{\mathsf{c}} + \frac{1}{3!} \epsilon^{\mathsf{ae}} H_{\mathsf{ebcd}} \boldsymbol{X}^{\mathsf{b}} \boldsymbol{X}^{\mathsf{c}} \boldsymbol{X}^{\mathsf{d}} + \dots \\ &= u^{\mathsf{a}}{}_{\mathsf{b}} \boldsymbol{X}^{\mathsf{b}} + \frac{1}{2!} u^{\mathsf{a}}{}_{\mathsf{bc}} \boldsymbol{X}^{\mathsf{b}} \boldsymbol{X}^{\mathsf{c}} + \frac{1}{3!} u^{\mathsf{a}}{}_{\mathsf{bcd}} \boldsymbol{X}^{\mathsf{b}} \boldsymbol{X}^{\mathsf{c}} \boldsymbol{X}^{\mathsf{d}} + \dots , \end{aligned} \tag{3.10}$$

where in the second line we have introduced the tensor ϵ^{ab} that is defined so that the commutation relation is written in the following compact form

$$[\boldsymbol{X}^{\mathsf{a}}, \boldsymbol{X}^{\mathsf{b}}] = i\epsilon^{\mathsf{ab}} , \tag{3.11}$$

where $\epsilon^{\mathsf{ab}} \equiv (2\pi)^3 \delta^{(3)}(\boldsymbol{k}_a + \boldsymbol{k}_b)\, \epsilon^{ab}$ and the $2N \times 2N$ matrix can be written in block form

$$\epsilon^{ab} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} . \tag{3.12}$$

This matrix defines the real symplectic group $\mathrm{Sp}(2N, \mathbb{R}) \equiv \{M \in \mathcal{M}_{2N\times 2N}(\mathbb{R}) \,|\, {}^{\mathrm{t}}M\epsilon M = \epsilon\}$ encompassing the symmetries of canonical variables that preserve the canonical commutation relation. As such, the time evolution of the operators $\boldsymbol{X}^{\mathsf{a}}$ is equivalent to an action of the real symplectic group on the phase space. Fundamentally, the cosmological flow can therefore be viewed as the Hamiltonian vector flow in phase space directly implemented at the level of correlators.

Exercise 3.9 Show that $\mathrm{Sp}(2N, \mathbb{R}) \equiv \{M \in \mathcal{M}_{2N\times 2N}(\mathbb{R}) \,|\, {}^{\mathrm{t}}M\epsilon M = \epsilon\}$ is a group. Determine: (i) ϵ^2, (ii) $\det(\epsilon)$, and (iii) ϵ^{-1}.

Consider $\boldsymbol{X} = (\boldsymbol{\varphi}, \boldsymbol{p}_\varphi)$ a phase-space vector. Suppose we define a new set of variables $\tilde{\boldsymbol{X}} = (\tilde{\boldsymbol{\varphi}}, \tilde{\boldsymbol{p}}_\varphi)$ that are linearly related to the original ones by

$$\tilde{\boldsymbol{X}} = A\boldsymbol{X} , \tag{3.13}$$

with A a constant 2×2 matrix. Show that the new variables satisfy the commutation relation

$$[\tilde{\boldsymbol{\varphi}}, \tilde{\boldsymbol{p}}_\varphi] = i , \tag{3.14}$$

if and only if the matrix A is symplectic.

The third line of (3.10) should be seen as a definition for the tensors $u^{\mathsf{a}}{}_{\mathsf{b}}, u^{\mathsf{a}}{}_{\mathsf{bc}}, u^{\mathsf{a}}{}_{\mathsf{bcd}}, \ldots$. Written in this form, it is clear that the equations of motion (3.10) encode both the full evolution of $\boldsymbol{X}^{\mathsf{a}}$ and the commutation relations. In practice, they are hard—if not impossible—to solve. One must therefore choose a simpler "free" Hamiltonian H_0 to evolve the interaction-picture operators, thus resorting to a perturbative description of the interactions.

Exercise 3.10 Go through all the steps to derive Eq. (3.10).

3.2 Tree-level correlator time integrals

We choose the free Hamiltonian to be the *full quadratic Hamiltonian*, $H_0 = \frac{1}{2!} H_{\mathsf{ab}} \boldsymbol{X}^{\mathsf{a}} \boldsymbol{X}^{\mathsf{b}}$, with the interacting part of the Hamiltonian being given by the cubic and higher orders, i.e. $H_{\text{int}} = \frac{1}{3!} H_{\mathsf{abc}} \boldsymbol{X}^{\mathsf{a}} \boldsymbol{X}^{\mathsf{b}} \boldsymbol{X}^{\mathsf{c}} + \frac{1}{4!} H_{\mathsf{abcd}} \boldsymbol{X}^{\mathsf{a}} \boldsymbol{X}^{\mathsf{b}} \boldsymbol{X}^{\mathsf{c}} \boldsymbol{X}^{\mathsf{d}} + \ldots$. Doing so, the equations of motion verified by the interaction-picture fields and momenta become *linear*. Still, all quadratic mixings between the various fields and momenta are taken into account in a non-perturbative manner. With this splitting choice, the equations of motion for the interaction-picture operators read

$$\frac{\mathrm{d}X^{\mathsf{a}}}{\mathrm{d}t} = u^{\mathsf{a}}{}_{\mathsf{b}} X^{\mathsf{b}} \,, \tag{3.15}$$

which makes it explicit that X^{a} evolves with the full quadratic Hamiltonian. By now, we have derived all the necessary fundamental elements to write all possible n-point correlation functions. Up to four-point correlators and working at tree-level yields

$$\begin{aligned}
\langle \boldsymbol{X}^{\mathsf{a}} \boldsymbol{X}^{\mathsf{b}} \rangle &= \langle X^{\mathsf{a}} X^{\mathsf{b}} \rangle \,, \\
\langle \boldsymbol{X}^{\mathsf{a}} \boldsymbol{X}^{\mathsf{b}} \boldsymbol{X}^{\mathsf{c}} \rangle &= \langle \frac{i}{3!} \int_{-\infty}^{t} \mathrm{d}t' H_{\mathsf{def}} \left[X^{\mathsf{d}} X^{\mathsf{e}} X^{\mathsf{f}}, X^{\mathsf{a}} X^{\mathsf{b}} X^{\mathsf{c}} \right] \rangle \,, \\
\langle \boldsymbol{X}^{\mathsf{a}} \boldsymbol{X}^{\mathsf{b}} \boldsymbol{X}^{\mathsf{c}} \boldsymbol{X}^{\mathsf{d}} \rangle &= \frac{i}{4!} \langle \int_{-\infty}^{t} \mathrm{d}t' H_{\mathsf{efgh}}(t') \left[X^{\mathsf{e}} X^{\mathsf{f}} X^{\mathsf{g}} X^{\mathsf{h}}(t'), X^{\mathsf{a}} X^{\mathsf{b}} X^{\mathsf{c}} X^{\mathsf{d}} \right] \rangle \\
&- \frac{1}{(3!)^2} \langle \int_{-\infty}^{t} \int_{-\infty}^{t'} \mathrm{d}t' \mathrm{d}t'' H_{\mathsf{efg}}(t'') H_{\mathsf{hij}}(t') \left[X^{\mathsf{e}} X^{\mathsf{f}} X^{\mathsf{g}}(t''), \left[X^{\mathsf{h}} X^{\mathsf{i}} X^{\mathsf{j}}(t'), X^{\mathsf{a}} X^{\mathsf{b}} X^{\mathsf{c}} X^{\mathsf{d}} \right] \right] \rangle \,.
\end{aligned} \tag{3.16}$$

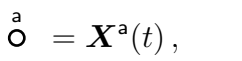
It should be understood that the left hand-side operators are evaluated in the vacuum of the fully interacting theory $|\Omega\rangle$ and the right hand-side operators in the vacuum of the free theory $|0\rangle$. Indeed, it is clear from the context whether the operators are in the Heisenberg ($\boldsymbol{X}^{\mathsf{a}}$) or interaction picture (X^{a}). It is also assumed that the time at which operators are evaluated is clear enough. We retain the commutator form of the correlators because it will be more convenient in the following development.

3.3 Deriving flow equations

We now have all the necessary ingredients to derive the flow equations. The idea is simple: differentiate these correlators with respect to time, use the linear equation of motion, and massage the obtained expressions to find a closed systems of differential equations in time.

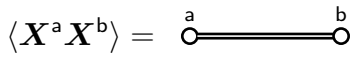
3.3.1 Two-point flow equations

We start by deriving the flow equations for the two-point functions $\langle \boldsymbol{X}^{\mathsf{a}} \boldsymbol{X}^{\mathsf{b}} \rangle$. This object is the equal-time tree-level two-point correlator, taking into account the full quadratic Hamiltonian (in particular possible non-trivial mixings). In what follows, it will be more convenient to adopt a diagrammatic representation of such objects. We use a white dot to denote an external field insertion at the time t



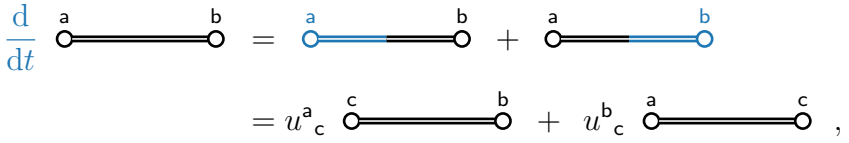
(3.17)

and a black dot to denote a single vertex insertion in the bulk at a time $t' < t$ which needs to be integrated over. The two-point correlator can be represented as



(3.18)

Since the states are time independent, one can formally differentiate (3.18) with respect to time by acting on the interaction-picture operators X^{a} inside the correlator in (3.16). We then use the equation of motion (3.15) to find the following closed system



(3.19)

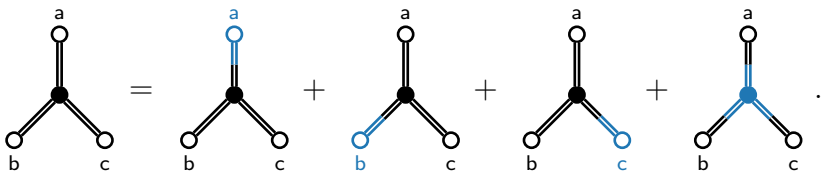
where we have colored in blue the parts of the correlator that have been differentiated with respect to time. Note that the introduced diagrammatic representation follows the usual Leibniz product rule of differentiation. The sum over repeated indices makes it explicit that—as usual in quantum physics—we sum over all possible diagrams. The flow equations for the two-point correlators are then

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{X}^{\mathsf{a}} \boldsymbol{X}^{\mathsf{b}} \rangle = u^{\mathsf{a}}{}_{\mathsf{c}} \langle \boldsymbol{X}^{\mathsf{c}} \boldsymbol{X}^{\mathsf{b}} \rangle + u^{\mathsf{b}}{}_{\mathsf{c}} \langle \boldsymbol{X}^{\mathsf{a}} \boldsymbol{X}^{\mathsf{c}} \rangle \,. \tag{3.20}$$

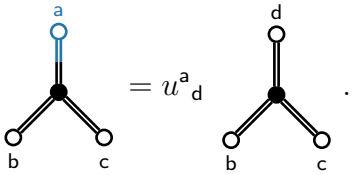
Of course, these equations are completely equivalent—in the sense that they encode the same physics—to the linear equations of motion supplemented by the quantisation condition. Consequently, they correctly capture all physical effects arising from quadratic operators in the theory. Note also that (3.20) couples all two-point correlators through the tensor $u^{\mathsf{a}}{}_{\mathsf{b}}$, including mixed propagators and correlators which contain conjugate momenta.

3.3.2 Three-point flow equations

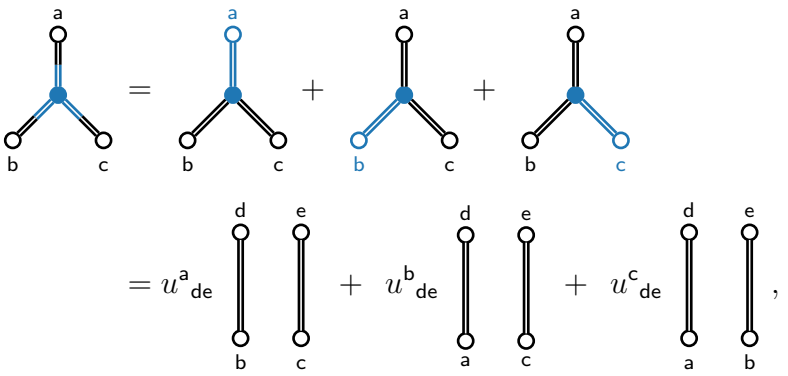
Let us now derive the flow equations for the three-point correlators. From (3.16), we see that such correlators carry two time dependencies: (i) the external operators, and (ii) the upper limit of the integral over a bulk vertex, in the following denoted by a black dot. Diagrammatically, differentiating the three-point correlators with respect to time is then represented by

$\frac{\mathrm{d}}{\mathrm{d}t}$  . (3.21)

Similar to the case of two-point correlators in the previous section, using the equation of motion (3.15), differentiating an external operator leads to

 (3.22)

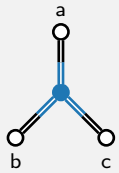
The other term—coming from deriving the cubic bulk vertex—requires a few manipulations. We give a formal derivation of this contribution in the insert below. In terms of diagrams, one needs to differentiate all combinations of a bulk vertex together with an external operator insertion, resulting in cutting the diagram in two-point correlators. The resulting two-point correlators should then be contracted with a $u^{\mathsf{a}}{}_{\mathsf{bc}}$ tensor. This leads to

 (3.23)

which should be regarded as the diagrammatic rule corresponding to differentiating a cubic bulk vertex. Collecting the various terms, the flow equations for the three-point correlators are

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{X}^{\mathsf{a}}\boldsymbol{X}^{\mathsf{b}}\boldsymbol{X}^{\mathsf{c}}\rangle = u^{\mathsf{a}}{}_{\mathsf{d}}\langle \boldsymbol{X}^{\mathsf{d}}\boldsymbol{X}^{\mathsf{b}}\boldsymbol{X}^{\mathsf{c}}\rangle + u^{\mathsf{a}}{}_{\mathsf{de}}\langle \boldsymbol{X}^{\mathsf{b}}\boldsymbol{X}^{\mathsf{d}}\rangle\langle \boldsymbol{X}^{\mathsf{c}}\boldsymbol{X}^{\mathsf{e}}\rangle + (2 \text{ perms})\,. \tag{3.24}$$

Exercise 3.11 Show that



$$= u^{\mathsf{a}}{}_{\mathsf{de}}\langle \boldsymbol{X}^{\mathsf{b}}\boldsymbol{X}^{\mathsf{d}}\rangle\langle \boldsymbol{X}^{\mathsf{c}}\boldsymbol{X}^{\mathsf{e}}\rangle + (2 \text{ perms})\,. \tag{3.25}$$

Hint: Rearrange the commutator and use Wick's theorem.

Similar to Eq. (3.20), these equations couple all correlators, including mixed correlators and those involving conjugate momenta. Note that

- These equations are quantum in nature, as they evolve correlators in time including the effects of quantum interactions.
- These equations are linear, which allows the flow of each kinematic configuration to be tracked independently.
- Solving for the three-point correlators requires first solving for the two-point ones. Intuitively, solving the two-point correlators is equivalent to solving the mode functions that are needed to compute higher-point correlators.
- The non-linearity of the source is related to the fact that fields are quantum operators, as we have used the commutation relation.

Exercise 3.12 In general, and especially in cosmology, a theory is described by an action supplemented by a boundary term. Show that a boundary term of the form

$$\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{3!}\,\Omega_{\mathsf{abc}}\boldsymbol{X}^{\mathsf{a}}\boldsymbol{X}^{\mathsf{b}}\boldsymbol{X}^{\mathsf{c}}\right)\,, \tag{3.26}$$

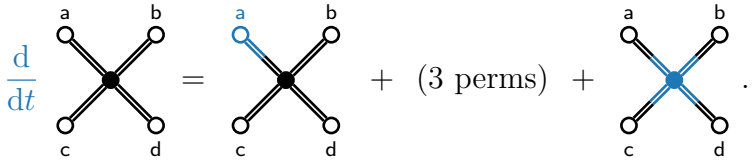
where the symmetric tensor $\Omega_{\mathsf{abc}} = \Omega_{(\mathsf{abc})}$ has time and momentum dependent elements, contributes to three-point correlators as

$$\begin{aligned}\langle \boldsymbol{X}^{\mathsf{a}}\boldsymbol{X}^{\mathsf{b}}\boldsymbol{X}^{\mathsf{c}}\rangle \supset -\Omega_{\mathsf{def}}\,\big(&\epsilon^{\mathsf{ad}}\langle \boldsymbol{X}^{\mathsf{e}}\boldsymbol{X}^{\mathsf{b}}\rangle\langle \boldsymbol{X}^{\mathsf{f}}\boldsymbol{X}^{\mathsf{c}}\rangle + \epsilon^{\mathsf{be}}\langle \boldsymbol{X}^{\mathsf{a}}\boldsymbol{X}^{\mathsf{d}}\rangle\langle \boldsymbol{X}^{\mathsf{c}}\boldsymbol{X}^{\mathsf{f}}\rangle \\ &+\epsilon^{\mathsf{cf}}\langle \boldsymbol{X}^{\mathsf{a}}\boldsymbol{X}^{\mathsf{d}}\rangle\langle \boldsymbol{X}^{\mathsf{e}}\boldsymbol{X}^{\mathsf{b}}\rangle\big)\,.\end{aligned} \tag{3.27}$$

3.3.3 Four-point flow equations

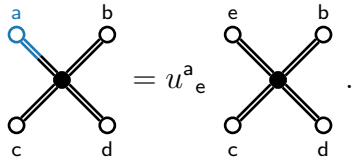
In full generality, a four-point correlator can be contact-like—with quartic interactions encoded in H_{abcd}—or exchange-like—with two cubic interactions from H_{abc}. Both contributions are given by the two contributions in Eq. (3.16). Here, we only treat the case of contact-like four-point correlators. The exchange-like correlator is left as an exercise. As for the three-point correlators, such correlators carry a time dependence

in external operator insertions, and in the upper limit of the integral over the bulk vertex. Following the Leibniz product rule of differentiation, the diagrammatic representation of differentiating a contact-like four-point correlator is



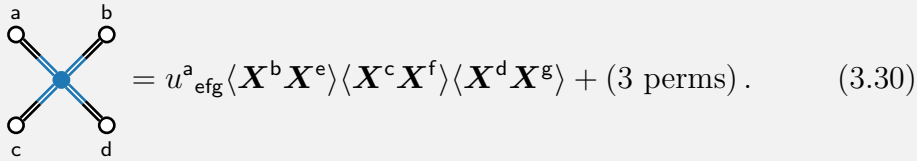
(3.28)

As usual, differentiating an external operator makes use of the equations of motions (3.15)



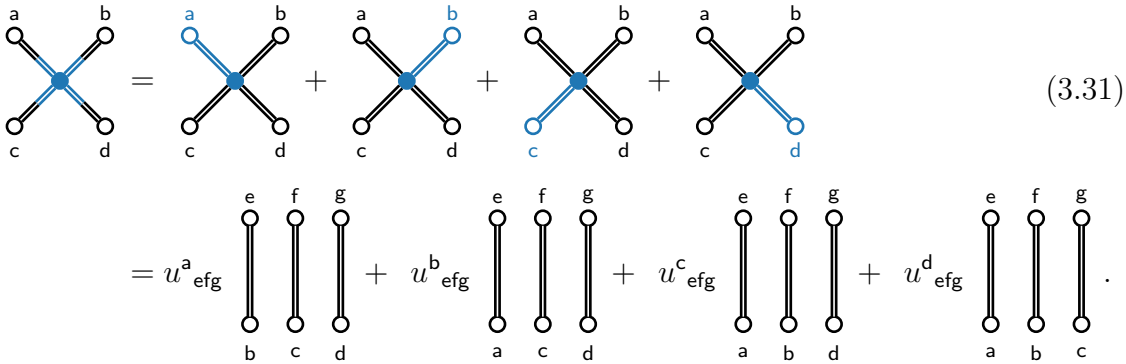
(3.29)

Exercise 3.13 Show that



(3.30)

Differentiating the bulk vertex, as previously done for the three-point correlators, results in cutting the diagram in two-point correlators contracted with the tensor $u^{\mathsf{a}}{}_{\mathsf{bcd}}$



(3.31)

Exercise 3.14 Show that the flow equations for the exchange-like four-point

correlators are

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{X}^{\mathsf{a}}\boldsymbol{X}^{\mathsf{b}}\boldsymbol{X}^{\mathsf{c}}\boldsymbol{X}^{\mathsf{d}}\rangle = u^{\mathsf{a}}{}_{\mathsf{e}}\langle \boldsymbol{X}^{\mathsf{e}}\boldsymbol{X}^{\mathsf{b}}\boldsymbol{X}^{\mathsf{c}}\boldsymbol{X}^{\mathsf{d}}\rangle + u^{\mathsf{a}}{}_{\mathsf{ef}}\langle \boldsymbol{X}^{\mathsf{b}}\boldsymbol{X}^{\mathsf{e}}\rangle\langle \boldsymbol{X}^{\mathsf{c}}\boldsymbol{X}^{\mathsf{d}}\boldsymbol{X}^{\mathsf{f}}\rangle + (11 \text{ perms})\,. \tag{3.32}$$

Comment on the complexity of solving these equations compared to solving the flow equations for contact-like four-point correlators.

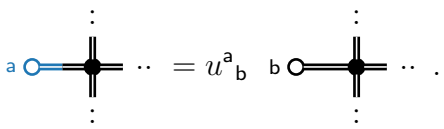
3.4 Diagrammatic rules for n-point correlators

Having treated the simplest exchange-like diagram appearing in four-point correlators, we now state—without formal proof—general diagrammatic rules to derive the flow equations for any tree-level n-point correlators. These rules are the following:

- Draw a tree-level graph assigning an operator $\boldsymbol{X}^{\mathsf{a}}(t)$ to each of the n external field, denoted with a white dot. Bulk vertices, denoted with black dots, should be at least of cubic order because quadratic interactions are fully resummed. Hence, the external operators are connected to bulk vertices with doubled lines. Time dependence appears in external operators and in the upper limit of the integral over bulk vertices, considered as a whole.

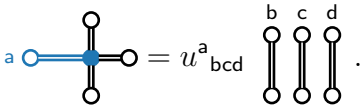
- Following the Leibniz product rule of differentiation, taking the time derivative of the drawn diagram equals the sum of initial-like diagrams where time dependence has been isolated, in our notations coloured in blue.

- Differentiating an external operator $\boldsymbol{X}^{\mathsf{a}}$ introduces the tensor $u^{\mathsf{a}}{}_{\mathsf{b}}$ which contracts—b being a dummy index here—with the external operator



(3.33)

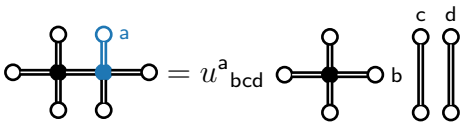
- Differentiating a contact-like bulk vertex of order n—hence exclusively connected to n external operators—leads to the product of $n-1$ two-point correlators contracted with the tensor $u^{\mathsf{a}}{}_{\mathsf{b}_1\mathsf{b}_2\ldots\mathsf{b}_{n-1}}$, where $\mathsf{b}_1, \mathsf{b}_2, \ldots, \mathsf{b}_{n-1}$ are dummy indices



(3.34)

- Differentiating an exchange-like bulk vertex of order m, attached to an external field, results in cutting the diagram in $m-1$ subdiagrams contracted with the

tensor $u^{\mathsf{a}}{}_{\mathsf{b}_1\mathsf{b}_2\ldots\mathsf{b}_{m-1}}$. One should consider all possible contractions. An example is



(3.35)

In general, the flow equations for an exchange-like diagram, together with those for the two-point correlators, do not form a closed system. One then needs to also find the flow equations for lower-order diagrams in order to close the system.

4 Lecture III: Implementing the Cosmological Flow

We have seen that tree-level cosmological correlators satisfy universal differential equations in time, that can be derived without specific knowledge of the bulk theory. This suggests that we can design an automated tool to solve these flow equations. This lecture is dedicated to implementing a systematic solver for these equations. For practical purposes, from now on, we will only focus on two- and three-point correlators.

4.1 Landscape of theories

Working at the fluctuations level, we assume that fluctuations are the fundamental degrees of freedom and the specific background mechanism determines the time-dependent parameters entering the Lagrangian. Up to cubic order, and specifying to FLRW spacetimes, the most general Hamiltonian in (spatial) Fourier space is

$$\begin{aligned} H = \frac{1}{2}\int \mathrm{d}t\, a^3(t)\, \big(&\Delta_{\alpha\beta}\boldsymbol{p}^\alpha\boldsymbol{p}^\beta - M_{\alpha\beta}\boldsymbol{\varphi}^\alpha\boldsymbol{\varphi}^\beta - 2I_{\alpha\beta}\boldsymbol{\varphi}^\alpha\boldsymbol{p}^\beta \\ &- A_{\alpha\beta\gamma}\boldsymbol{\varphi}^\alpha\boldsymbol{\varphi}^\beta\boldsymbol{\varphi}^\gamma - B_{\alpha\beta\gamma}\boldsymbol{\varphi}^\alpha\boldsymbol{\varphi}^\beta\boldsymbol{p}^\gamma - C_{\alpha\beta\gamma}\boldsymbol{p}^\alpha\boldsymbol{p}^\beta\boldsymbol{\varphi}^\gamma - D_{\alpha\beta\gamma}\boldsymbol{p}^\alpha\boldsymbol{p}^\beta\boldsymbol{p}^\gamma\big)\,, \end{aligned} \tag{4.1}$$

where $a(t)$ is the scale factor whose time evolution can be arbitrary. Recall that we use the extended Fourier summation notation, meaning that repeated sans serif indices $\mathsf{\alpha}, \mathsf{\beta}$ (running from 1 to the number of fields N) denote a sum that includes an integral over Fourier modes. All the introduced tensors carry hidden arbitrary momentum and time dependencies. The tensor $\Delta_{\alpha\beta}$ is real and symmetric and therefore can be considered diagonal without loss of generality (we also consider that $\bar{\Delta}_{\alpha\beta}$ is positive definite to avoid ghosts that would render the theory unstable). As such, this tensor only accounts for non-canonically normalised fields. The tensor $M_{\alpha\beta}$ is also taken to be symmetric $M_{\alpha\beta} = M_{(\alpha\beta)}$. It contains the kinetic gradient terms and the mass matrix. The non-symmetric $I_{\alpha\beta}$ tensor captures the remaining possible linear mixings among the fields. For the cubic part of the action, the tensors $A_{\alpha\beta\gamma} = A_{(\alpha\beta\gamma)}$ and $D_{\alpha\beta\gamma} = D_{(\alpha\beta\gamma)}$ should be symmetrised over all indices, and $B_{\alpha\beta\gamma} = B_{(\alpha\beta)\gamma}$ and $C_{\alpha\beta\gamma} = C_{(\alpha\beta)\gamma}$ over the first two indices, with corresponding exchange of momenta. No further assumptions than the ones stated above are made.

Of course, in most cases, we rather use the action to define the theory. In this case, one needs to first derive the conjugate momentum defined by the following functional derivative

$$\boldsymbol{p}_\alpha(t) = \frac{\delta S}{\delta\dot{\boldsymbol{\varphi}}^\alpha(t)}\,, \quad \text{with} \quad \frac{\delta\boldsymbol{\varphi}^\alpha(t)}{\delta\boldsymbol{\varphi}^\beta(t')} = \delta^\alpha_\beta\,\delta(t-t')\,, \tag{4.2}$$

and then perform the usual Legendre transform

$$H = \int \mathrm{d}t\, \left(\boldsymbol{p}_\alpha\dot{\boldsymbol{\varphi}}^{\bar{\alpha}} - \mathcal{L}\right)\,. \tag{4.3}$$

In practice, we always rescale the momenta $\boldsymbol{p}_\alpha \to a^3\boldsymbol{p}_\alpha$ to avoid an extra overall factor of a^3 coming from $\mathcal{L}$ once the field time derivatives have been replaced by their conjugate momenta. Importantly, note that at cubic order, it is enough to restrict to linear order to derive the conjugate momenta and use $\mathcal{H}^{(3)} = -\mathcal{L}^{(3)}$, where $\dot{\boldsymbol{\varphi}}^\alpha$ should be expressed in terms of the linear momenta. To convince yourself, solve the following small exercise.

Exercise 4.15 Consider a self-interacting massless scalar field φ in a FLRW background whose action is given by

$$S = \int \mathrm{d}t\mathrm{d}^3x a^3(t) \left[-\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{g}{3!}\dot{\varphi}^3 \right] . \tag{4.4}$$

Derive the corresponding Hamiltonian by considering only the linear momentum, and then with the fully non-linear momentum (inverse the expression perturbatively). Compare both expressions. Recast the Hamiltonian in the form (4.1) and identify the corresponding tensor components.

4.2 Explicit u-tensors

Given the explicit form of the Hamiltonian (4.1), it is an easy task to derive the $u^{\mathsf{a}}{}_{\mathsf{b}} = \epsilon^{\mathsf{ac}}H_{\mathsf{cb}}$ and $u^{\mathsf{a}}{}_{\mathsf{bc}} = \epsilon^{\mathsf{ad}}H_{\mathsf{dbc}}$ tensors entering in the flow equations (3.20) and (3.24). For the quadratic Hamiltonian, after carefully symmetrising the $H_{\mathsf{ab}} = H_{(\mathsf{ab})}$ tensor, we obtain

$$H_{\mathsf{ab}} = \begin{pmatrix} -M_{\alpha\beta} & -I_{\alpha\beta} \\ -I_{\beta\alpha} & \Delta_{\alpha\beta} \end{pmatrix} , \tag{4.5}$$

where the a, b indices are arranged in such a way that the field indices are followed by momentum indices in the same order. The desired $u^{\mathsf{a}}{}_{\mathsf{b}}$ tensor is then

$$u^{\mathsf{a}}{}_{\mathsf{b}} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} -M^{\bar{\alpha}}{}_{\beta} & -I^{\bar{\alpha}}{}_{\beta} \\ -I_{\beta}{}^{\bar{\alpha}} & \Delta^{\bar{\alpha}}{}_{\beta} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -3H\delta^{\bar{\alpha}}_{\beta} \end{pmatrix} = \begin{pmatrix} I_{\beta}{}^{\bar{\alpha}} & \Delta^{\bar{\alpha}}{}_{\beta} \\ M^{\bar{\alpha}}{}_{\beta} & I^{\bar{\alpha}}{}_{\beta} - 3H\delta^{\bar{\alpha}}_{\beta} \end{pmatrix} . \tag{4.6}$$

The extra factor $-3H\delta^{\bar{\alpha}}_{\beta}$ comes from the momentum rescaling. Recall that we have rescaled the momenta $\boldsymbol{p}_\alpha \to a^3\boldsymbol{p}_\alpha$. This results in a modified Heisenberg equation

$$\frac{\mathrm{d}\boldsymbol{p}_\alpha}{\mathrm{d}t} = i[H, \boldsymbol{p}_\alpha] - 3H\boldsymbol{p}_\alpha \, . \tag{4.7}$$

We have absorbed the additional non-canonical factor in the $u^{\mathsf{a}}{}_{\mathsf{b}}$ tensor. Similarly, the cubic Hamiltonian reads

$$H_{\mathsf{abc}} = \left\{ \begin{pmatrix} -3A_{\alpha\beta\gamma} & -B_{\alpha\gamma\beta} \\ -B_{\gamma\beta\alpha} & -C_{\alpha\beta\gamma} \end{pmatrix} \begin{pmatrix} -B_{\alpha\beta\gamma} & -C_{\gamma\beta\alpha} \\ -C_{\alpha\gamma\beta} & -3D_{\alpha\beta\gamma} \end{pmatrix} \right\} , \tag{4.8}$$

where the index c labels the first (resp. second) matrix within the braces $\{\ldots\}$ if it is a field (resp. a momentum) index, and similarly a (resp. b) labels the row (resp. column) in each 2×2 block matrix. We recall that H_{abc} is constructed to be symmetric under the exchange of any pair of indices hence the multiple presence of some tensors. The $u^{\mathsf{a}}{}_{\mathsf{bc}}$ tensor is obtained by explicitly performing the matrix multiplication

$$u^{\mathsf{a}}{}_{\mathsf{bc}} = \left\{ \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} -3A^{\bar{\alpha}}{}_{\beta\gamma} & -B^{\bar{\alpha}}{}_{\gamma\beta} \\ -B_{\gamma\beta}{}^{\bar{\alpha}} & -C^{\bar{\alpha}}{}_{\beta\gamma} \end{pmatrix} \right\} = \left\{ \begin{pmatrix} -B_{\gamma\beta}{}^{\bar{\alpha}} & -C^{\bar{\alpha}}{}_{\gamma\beta} \\ 3A^{\bar{\alpha}}{}_{\beta\gamma} & B^{\bar{\alpha}}{}_{\gamma\beta} \end{pmatrix} \right\} . \tag{4.9}$$

We further notice that the particular arrangement of barred indices—namely the upper $\bar{\alpha}$ index for each tensors—enables us to extract a delta function, hence defining $\boldsymbol{k}$-dependent tensor coefficients $u^a{}_b$ and $u^a{}_{bc}$ that we label with normal indices $a, b, \ldots$ in the following way

$$\begin{aligned} u^{\mathsf{a}}{}_{\mathsf{b}} &= (2\pi)^3 \delta^{(3)}(\boldsymbol{k}_a - \boldsymbol{k}_b)\, u^a{}_b(\boldsymbol{k}_a, \boldsymbol{k}_b)\,, \\ u^{\mathsf{a}}{}_{\mathsf{bc}} &= (2\pi)^3 \delta^{(3)}(\boldsymbol{k}_a - \boldsymbol{k}_b - \boldsymbol{k}_c)\, u^a{}_{bc}(\boldsymbol{k}_a, \boldsymbol{k}_b, \boldsymbol{k}_c)\,. \end{aligned} \tag{4.10}$$

It is important to stress that in writing expressions with explicit momentum dependence, the index labelling the momentum is associated with the corresponding tensor index. Explicitly, $\boldsymbol{k}_a$ is associated with a and so forth. For example if one would want to exchange b and c in $u^a{}_{bc}$, the corresponding momenta should be exchanged too. Furthermore, due to the presence of the delta function and by isotropy, the $u^a{}_b$ tensor coefficients only depend on the magnitude of the momentum, $u^a{}_b(k)$ where $k = |\boldsymbol{k}_a| = |\boldsymbol{k}_b|$.

4.3 Differential equations in time

Turning to the two- and three-point correlation functions, statistical isotropy and homogeneity imply that one can write

$$\begin{aligned} \langle \boldsymbol{X}^{\mathsf{a}} \boldsymbol{X}^{\mathsf{b}} \rangle &= (2\pi)^3 \delta^{(3)}(\boldsymbol{k}_a + \boldsymbol{k}_b)\, \Sigma^{ab}(k)\,, \\ \langle \boldsymbol{X}^{\mathsf{a}} \boldsymbol{X}^{\mathsf{b}} \boldsymbol{X}^{\mathsf{c}} \rangle &= (2\pi)^3 \delta^{(3)}(\boldsymbol{k}_a + \boldsymbol{k}_b + \boldsymbol{k}_c)\, B^{abc}(k_a, k_b, k_c)\,, \end{aligned} \tag{4.11}$$

where for the two-point correlator we have set $k = |\boldsymbol{k}_a| = |\boldsymbol{k}_b|$. We do not explicitly write the time dependency of the above correlators but it should be understood that they are evaluated at the same time t. The three-point correlators $B^{abc}(k_a, k_b, k_c)$ are real (see exercise below), while Σ^{ab} is complex and can be split into real and imaginary parts $\Sigma^{ab} = \Sigma^{ab}_{\mathrm{Re}} + i\Sigma^{ab}_{\mathrm{Im}}$. Because the flow equation (3.20) is linear and the $u^a{}_c$ have real elements, one can solve independently for $\Sigma^{ab}_{\mathrm{Re}}$ and $\Sigma^{ab}_{\mathrm{Im}}$, obeying

$$\frac{\mathrm{d}\Sigma^{ab}_{\mathrm{Re,Im}}(k)}{\mathrm{d}t} = u^a{}_c(k)\, \Sigma^{cb}_{\mathrm{Re,Im}}(k) + u^b{}_c(k)\, \Sigma^{ac}_{\mathrm{Re,Im}}(k)\,. \tag{4.12}$$

Exercise 4.16 The commutation relation, which is valid at all times, actually fully fixes the imaginary part of the two-point correlators. To see this:

- Explain why $\boldsymbol{X}^{\mathsf{a}\dagger} = \boldsymbol{X}^{\bar{\mathsf{a}}}$.
- Show that $\Sigma^{ab}_{\rm Re} = \Sigma^{ba}_{\rm Re}$ and $\Sigma^{ab}_{\rm Im} = -\Sigma^{ba}_{\rm Im}$. In other words, the real part is symmetric and the imaginary part is anti-symmetric.
- Show that $\Sigma^{ab}_{\rm Im} = \frac{\epsilon^{ab}}{2a^3}$.

For three-point (connected) correlators, show that $\langle \boldsymbol{X}^{\mathsf{a}} \boldsymbol{X}^{\mathsf{b}} \boldsymbol{X}^{\mathsf{c}} \rangle^* = \langle \boldsymbol{X}^{\bar{\mathsf{a}}} \boldsymbol{X}^{\bar{\mathsf{b}}} \boldsymbol{X}^{\bar{\mathsf{c}}} \rangle$. Conclude that all three-point correlators must be real. Why this is not true for higher-point correlators? What would a non-zero detection of an imaginary trispectrum imply?

For three-point correlators, it is important to keep track of both $\Sigma^{ab}_{\rm Re}$ and $\Sigma^{ab}_{\rm Im}$ because the flow equations—being sourced by a non-linear term—requires the knowledge of the *complex* two-point correlators. Taking into account that the tensor coefficients $u^a{}_{bc}$ are real, we arrive at

$$\begin{aligned}
\frac{\mathrm{d}B^{abc}(k_a,k_b,k_c)}{\mathrm{d}t} &= u^a{}_d(k_a)B^{dbc}(k_a,k_b,k_c) + u^b{}_d(k_b)B^{adc}(k_a,k_b,k_c) + u^c{}_d(k_c)B^{abd}(k_a,k_b,k_c) \\
&+ u^a{}_{de}(\boldsymbol{k}_a,\boldsymbol{k}_b,\boldsymbol{k}_c)\Sigma^{db}_{\rm Re}(k_b)\Sigma^{ec}_{\rm Re}(k_c) - u^a{}_{de}(\boldsymbol{k}_a,\boldsymbol{k}_b,\boldsymbol{k}_c)\Sigma^{db}_{\rm Im}(k_b)\Sigma^{ec}_{\rm Im}(k_c) \\
&+ u^b{}_{de}(\boldsymbol{k}_b,\boldsymbol{k}_a,\boldsymbol{k}_c)\Sigma^{ad}_{\rm Re}(k_a)\Sigma^{ec}_{\rm Re}(k_c) - u^b{}_{de}(\boldsymbol{k}_b,\boldsymbol{k}_a,\boldsymbol{k}_c)\Sigma^{ad}_{\rm Im}(k_a)\Sigma^{ec}_{\rm Im}(k_c) \\
&+ u^c{}_{de}(\boldsymbol{k}_c,\boldsymbol{k}_a,\boldsymbol{k}_b)\Sigma^{ad}_{\rm Re}(k_a)\Sigma^{be}_{\rm Re}(k_b) - u^c{}_{de}(\boldsymbol{k}_c,\boldsymbol{k}_a,\boldsymbol{k}_b)\Sigma^{ad}_{\rm Im}(k_a)\Sigma^{be}_{\rm Im}(k_b)\,,
\end{aligned} \tag{4.13}$$

where we have deliberately written all the permutations to avoid ambiguity in the index ordering. It is clear that B^{abc} is fully symmetric.

Exercise 4.17 Consider the following cubic interactions

$$\mathcal{L}/a^3 = -\frac{g_1}{2}\varphi\frac{(\partial_i\varphi)^2}{a^2} - \frac{g_2}{2}\varphi^2\dot{\varphi}\,. \tag{4.14}$$

Derive the corresponding tensors entering the Hamiltonian (4.1).

Exercise 4.18 How many coupled differential equations one needs to solve for the two- and three-point correlators?

4.4 Initial conditions

To render the system complete, one must provide initial conditions for $\Sigma^{ab}_{\rm Re}, \Sigma^{ab}_{\rm Im}$ and B^{abc}. Within the cosmological flow formalism, these initial conditions can be

derived analytically provided one initialises the correlators sufficiently in the deep past, namely when all modes of interest are well inside the horizon. Essentially, the quadratic theory is dominated by spatial gradients, mass terms can be neglected, and mixing interactions become irrelevant so that the system decouples and approaches that of a set uncoupled massless degrees of freedom, i.e. it asymptotically reaches the vacuum state.

Exercise 4.19 Consider a massless free scalar field in de Sitter whose action is given by

$$S = -\frac{1}{2}\int \mathrm{d}t\mathrm{d}^3x a^3 \left(\partial_\mu \varphi\right)^2 , \tag{4.15}$$

with $a(t) = e^{Ht} = -1/(H\tau)$ where τ is conformal time, defined by $\mathrm{d}\tau = \mathrm{d}t/a$. Imposing the Bunch-Davies vacuum, show that the two-point correlators are given by

$$\Sigma^{ab}_{\mathrm{Re}}(k) = \frac{H^2}{2k^3}\begin{pmatrix} 1+\left(\frac{k}{aH}\right)^2 & -\left(\frac{k}{aH}\right)^2 \\ -\left(\frac{k}{aH}\right)^2 & \left(\frac{k}{aH}\right)^4 \end{pmatrix} , \quad \Sigma^{ab}_{\mathrm{Im}}(k) = \frac{1}{2a^3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{4.16}$$

What happens on super-horizon scales $k \ll aH$?

$i\epsilon$ prescription. The standard approach of computing cosmological correlators requires deforming the time integration contour to the imaginary direction at early times, with fields appearing in the integral defined by analytical continuation $t \to t(1 \pm i\epsilon)$, to achieve UV convergence. This procedure maps the vacuum of the free theory to the full theory vacuum in the infinite past and is at the core of the in-in formula. The cosmological flow approach completely bypasses this complex deformation of the integrand as the correlators are evolved on the *real time axis*. The $i\epsilon$ prescription is transferred to the derivation of analytical initial conditions. However, an alternative elementary approach is to realise that the $i\epsilon$ defines the asymptotic vacuum state in which all interactions are turned off. Therefore, for the numerical implementation of the cosmological flow, we also propose a *numerical* $i\epsilon$ prescription that consists in initialising the three-point correlators to zero and adiabatically switching on interactions, be they cubic interactions or quadratic mixings, allowing various correlators to build at early times before reaching the attractor solution given by the flow equations. We will see this in action when using CosmoFlow.

5 Lecture IV: CosmoFlow Tutorials

Built upon the cosmological flow approach, CosmoFlow automatically computes cosmological correlators. Given a certain theory, i.e. knowledge of the tensors $u^{\mathsf{a}}{}_{\mathsf{b}}$ and $u^{\mathsf{a}}{}_{\mathsf{bc}}$, it is straightforward to implement a routine that numerically solves the flow equations. In this lecture, we present a detailed overview of how CosmoFlow is implemented and guide you through hands-on tutorials. You'll play with the code yourself and see it in action. But no stress, everyone learns at their own pace. Most importantly, enjoy the process and have fun!

5.1 Prerequisites & Installation

Aiming for simplicity above all, CosmoFlow has minimal prerequisites.

Python. The code CosmoFlow needs a working Python installation.[1] During the course of the development, we used Python 3 (which we recommend) although it should run without problem on Python 2.7 (or below). For convenience, we recommend to use a complete Python distribution such as Anaconda, which comes with all the packages used by the code. It also contains a nice interactive development environment and includes Jupyter that enables to run all the CosmoFlow notebooks.

Packages. We have precisely designed CosmoFlow so that it only uses elementary and commonly-used Python packages, as well as employs no sophisticated Python implementation nor hard coding. The Python packages used in CosmoFlow and/or in the numerous notebooks accompanying the code are Numpy, Matplotlib, and Scipy. The modules time and tqdm are also used in tutorial notebooks only, and are for timing the computation and displaying a progress bar, respectively. These modules are not essential and their use can be avoided by commenting out the appropriate code lines. All these packages can be installed using pip. If you use Conda or similar Python environments, they usually come with their own package manager to install packages. Have in mind that the web is a goldmine of useful resources to help installing and using these packages.

Installation. CosmoFlow is distributed via its GitHub public repository. The easy way to import the code on your machine is to click the green button "code" and then "download ZIP". The zipped folder contains the entire repository. The code can also be downloaded as a complete git repository with

```
$ git clone https://github.com/deniswerth/CosmoFlow.git CosmoFlow
```

The material for these lectures (lecture notes and the corresponding Jupyter notebooks) are stored in the "Lectures on the Cosmological Flow" folder.

[1] To see if you have Python on your laptop (which by default should be the case), you can just open the terminal and write python.

5.2 Code architecture

The CosmoFlow implementation has a clear structure, separated in modules that fully exploit the object-oriented programming paradigm of Python (coded as Python classes) with well defined tasks:

- **Parameters.py** contains the parameters class that defines all the necessary variables and arrays, including the time grid array in e-folds N_load on which the time-dependent parameters are defined, and parameters entering the theory, e.g. Hubble parameter and coupling constants. This class defines interpolated continuous functions out of the given inputs using the function interp1d from Scipy. It enables to treat both analytical or numerical (given by an array evaluated on a discrete set of points) time-dependent parameters. It is important to define all parameters in this class, even the constant ones, as some of them (typically coupling constants) will be switched on adiabatically at early times to implement the numerical $i\epsilon$ prescription. For every argument of this class, say variable, one should: (i) define it as an argument of the class variable_load, (ii) state it as a variable within the class using self.variable_load, (iii) create a continuous function self.variable_f, and (iv) add variable_f to the interpolated list that contains all interpolated continuous functions, in the same order as given by the arguments of the class.

- **Theory.py** contains the theory class that defines the u-tensors given the tensors appearing explicitly in the Hamiltonian (4.1). All tensor elements of this class are continuous functions that can be evaluated at a given time. This class will be called at each integration step of the flow equations. Theory takes as inputs N (time in e-folds at which all the parameters are defined), Nfield (number of fields), and interpolated (list of all the interpolated functions given by parameters).

  For each continuous function, say variable_f, one needs to define a variable within the class self.variable and a function variable_f of N using the list interpolated. It is important to keep the same continuous parameter indexing as defined in Parameters. The quadratic-theory tensors $\Delta_{\alpha\beta}, M_{\alpha\beta}$ and $I_{\alpha\beta}$ and the cubic-theory tensors $A_{\alpha\beta\gamma}, B_{\alpha\beta\gamma}, C_{\alpha\beta\gamma}$ and $D_{\alpha\beta\gamma}$ must be fed according to the considered theory. Recall that Python indexing starts with 0 and that these tensors must satisfy specific symmetry properties that should be well implemented. Defining the tensors $u^{\mathsf{a}}{}_{\mathsf{b}}$ and $u^{\mathsf{a}}{}_{\mathsf{bc}}$ is done automatically.

- **Solver.py** contains the solver class that defines the initial conditions for the two- and three-point correlators, and the flow equations. Being universal regardless the theory, these functions are hard-coded. This class takes as inputs Nspan (time grid array in e-folds on which the flow equations are solved), Nfield (number

of fields), interpolated (list of all the interpolated functions given by parameters), Rtol (relative tolerance for the numerical integrator), and Atol (absolute tolerance for the numerical integrator). The function f_solution contains the solve_ivp solver to integrate the flow equations. Various integrators can be used as a method option, see the solve_ivp documentation for more details.

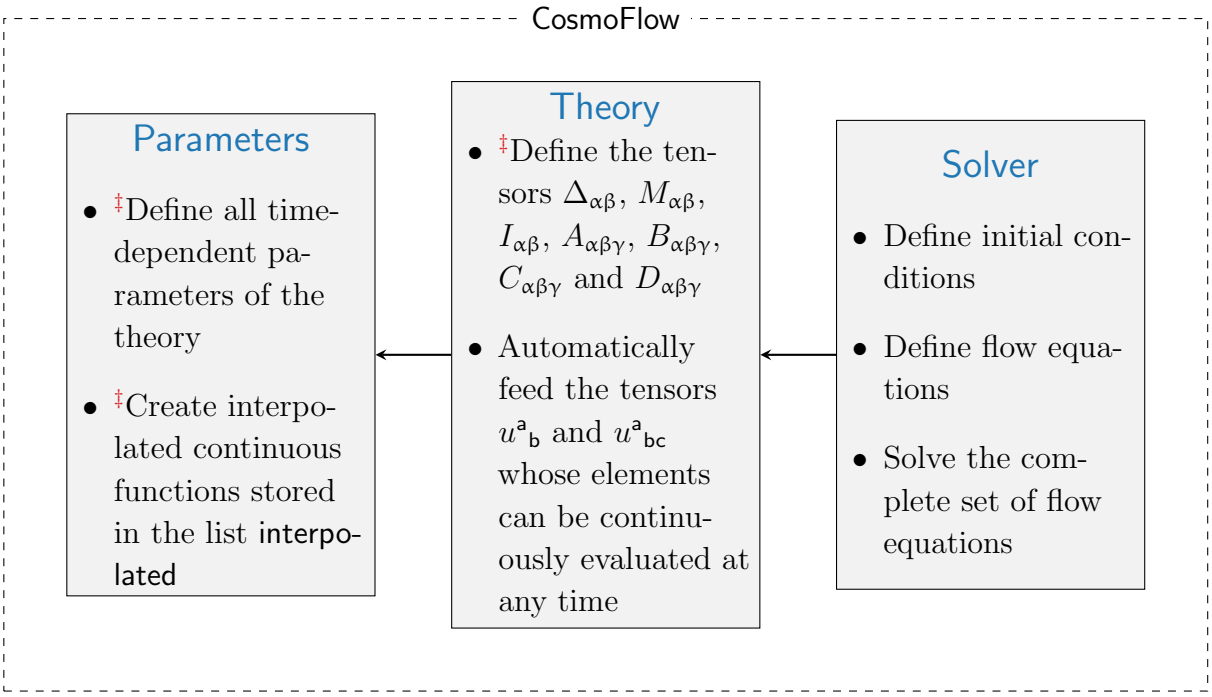


Figure 3: General overview of the CosmoFlow architecture with the main features. The class Solver calls Theory at each integration step, and Theory uses continuous parameters defined in Parameters. Tasks decorated with ‡ are theory dependent and must be implemented manually.

Aiming for flexibility and user-friendliness, physical quantities (number of fields, coupling constants, etc) or numerical parameters (integrator, relative tolerance, etc) that are likely to be modified in different theories or usages are localised in a small number of well-identified places in the code that the user can access easily. A general overview of the CosmoFlow architecture is given in Fig. 3. Once these files have been implemented, they do not need to be modified further.[2] Executing CosmoFlow for various applications is performed in a separate file (Python script or notebook).

5.3 My first run (Jupyter Notebook) + (Jupyter Notebook)

Let us now introduce the basic functionalities of CosmoFlow and lead the user through its first step-by-step implementation of a concrete theory. In this section, we first

[2]Although it may seem evident, do not forget to save these files after implementation.

introduce a toy theory that serves as a simple example of how to use CosmoFlow. We then explicitly show how to implement the modules related to this theory and run the code.

We consider a simple massless $\dot{\varphi}^3$ theory in de Sitter. The action written in cosmic time is

$$S = \int \mathrm{d}t\mathrm{d}^3x\, a^3 \left[-\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{g}{3!}\,\dot{\varphi}^3 \right] , \tag{5.1}$$

where g is a coupling constant and an overdot indicates a derivative with respect to cosmic (physical) time. We have derived the corresponding tensors in exercise 4.15, and they are given by

$$\Delta_{\varphi\varphi} = 1\,, \quad M_{\varphi\varphi} = -\frac{k^2}{a^2}\,, \quad D_{\varphi\varphi\varphi} = -\frac{g}{3}\,. \tag{5.2}$$

This theory is implemented in the "MyFirstRun" folder. All the parameters of the theory should be specified and defined in the parameters class in the Parameters.py file. In our case, we have the time grid array on which the parameters are defined N_load, the Hubble parameter H_load, and the cubic coupling constant g_load. We use the suffix _load to indicate that the parameters are imported and not yet interpolated (as opposed to _f which means that the corresponding variables are continuous functions). The Parameters.py file is the following[3]

```
from scipy.interpolate import interp1d # import interp1d from Scipy
    package

class Parameters():
        """
        This class takes as inputs the parameters of the theory and
            creates interpolated continuous functions
        """

        def __init__(self, N_load, H_load, g_load):

                #Importing pre-computed parameters of the theory
                self.N_load = N_load
                self.H_load = H_load
                self.g_load = g_load

                #Creating continuous functions out of the imported
                    parameters
                self.H_f = interp1d(self.N_load, self.H_load, bounds_error
                    = False, kind = 'cubic', fill_value = "extrapolate")
```

[3]The character # is used for comments (in green) so everything following it in a given line should be ignored.

```
            self.g_f = interp1d(self.N_load, self.g_load, bounds_error
                = False, kind = 'cubic', fill_value = "extrapolate")

            #Creating interpolating list containing continuous functions
            self.interpolated = [self.H_f, self.g_f]

    def output(self):
            """
            Returns the interpolated continous functions that can be
                evaluated at any N (time in e-folds)
            """
            return self.interpolated
```

Once all parameters have been defined, one needs to define the theory class in the Theory.py file. We start by defining the class and all variables with the following lines

```
import numpy as np # import Numpy package for vectorisation
from scipy.misc import derivative # import derivative from Scipy package

class theory():
    """
    This class defines the theory: (i) time-dependent functions, (ii)
        tensors Delta, M, I, A, B, C and D, and (iii) tensors u^A_B and
        u^A_BC
    All functions have been optimised to be computed once at each
        integration step
    """

    def __init__(self, N, Nfield, interpolated):
            self.N = N
            self.Nfield = Nfield
            self.interpolated = interpolated # (H_f, g_f)

            #Evaluate the parameters at N
            N = self.N
            self.H = self.H_f(N)
            self.g = self.g_f(N)
```

Additional functions that have explicit dependence on N (e.g. the scale factor) or that depend on the previously defined parameters (e.g. derivative of the Hubble rate with respect to cosmic time) can also be defined. Because correlators containing conjugate momenta to the fields decay with different rates on super-horizon scales

depending on the number of momenta, we rescale the corresponding correlators using the function scale. This procedure is not necessary but improves numerical stability. In order for the parameters to be evaluated continuously at each integration step, all defined quantities are defined as functions in the following way

```
        #Define continuous functions
        def H_f(self, N):
                return self.interpolated[0](N)

        def g_f(self, N):
                return self.interpolated[1](N)
```

It is important to keep the same order for continuous variables encoded in the list interpolated. These functions allow us to feed the quadratic and cubic tensors appearing in the Hamiltonian (4.1), e.g.

```
        def M_ab(self, k):
                Nfield = self.Nfield
                Mab = np.eye(Nfield)
                Mab[0, 0] = -k**2/(self.a**2)
                return Mab
```

Note that if a tensor index exceeds Nfield, an error message would pop up while executing the code. Given these tensors, the u-tensors are defined automatically. We invite new users to go through these files in the repository while reading these lines at the same time. The Solver.py file containing the numerical integrator should not be modified. The philosophy of CosmoFlow is that once these files have been implemented for a specific theory, the user would never have to modify them again. At this point, we are now ready to execute the code for the first time.

First run. We now show how to run the code and pass different parameters to CosmoFlow to obtain the first computed correlator. This is done in a separate Python script or Jupyter notebook. The following Python script (MyFirstRun.py in the associated folder) demonstrates the basic execution of CosmoFlow.

```
import numpy as np # import Numpy package for vectorisation
import matplotlib.pyplot as plt # import matplotlib for visualisation

# Import CosmoFlow modules
from Parameters import parameters
from Theory import theory
from Solver import solver
```

```
# Define the numerical i\epsilon prescription
def adiabatic(N_load, DeltaN):
    return (np.tanh((N_load + DeltaN - 1)/0.1) + 1)/2

n = 10000 # Number of points for the parameter evaluation
N_load = np.linspace(-10, 20, n) # Time grid array in e-folds for the
    parameters
DeltaN = 4 # Number of e-folds before horizon crossing

# Theory
g_load = 1 * np.ones(n) * adiabatic(N_load, DeltaN) # Cubic coupling
    constant
H_load = np.ones(n) # Hubble scale

# Load the parameters and define continuous functions
param = parameters(N_load, H_load, g_load) # Load the class parameters
interpolated = param.output() # Define list with continuous parameters

# Numerical parameters
Nspan = np.linspace(-10, 20, 500) # Time span in e-folds for the
    numerical integration
Nfield = 1 # Number of fields
Rtol, Atol = 1e-4, 1e-180 # Relative and absolute tolerance of the
    integrator
N_exit = 0 # Horizon exit for a mode
Ni, Nf = N_exit - DeltaN, 10 # Sets initial and final time for integration
N = np.linspace(Ni, Nf, 1000) # Define the time array for output
    correlators

# Initialise the integrator
theo = theory(N = Nspan, Nfield = Nfield, interpolated = interpolated)

# Kinematic configuration
k = theo.k_mode(N_exit) # Mode corresponding to N = 0 horizon exit
k1, k2, k3 = k, k, k # Kinematic configuration for 3-pt function (here
    equilateral)

# Solve flow equations
s = solver(Nspan = N, Nfield = Nfield, interpolated = interpolated, Rtol
    = Rtol, Atol = Atol)
f = s.f_solution(k1 = k1, k2 = k2, k3 = k3)
```

```
# Plot correlators
plt.semilogy(N, np.absolute(f[0][0, 0])) # field-field correlator
plt.semilogy(N, np.absolute(f[6][0, 0, 0])) # field-field-field correlator
plt.show()
```

If everything went smoothly, by executing this script using the following lines

```
$ cd ... # Go MyFirstRun directory
$ python MyFirstRun.py # Execute the script
```

you should see appearing a window with the correlators $\Sigma^{\varphi\varphi}_{\rm Re}(k)$ and $B^{\varphi\varphi\varphi}(k,k,k)$ (equilateral kinematic configuration) as function of time, expressed in number of e-folds with respect to horizon crossing of the mode k. Let us have a look at this script and give some explanations. The first thing to do is to import the three CosmoFlow modules. They should be stored in the same folder as the executing script. Otherwise their location on your machine should be specified. The function adiabatic encodes the numerical $i\epsilon$ prescription in a simple manner: it is a window function starting at zero and ending at unity centered around one e-fold after initialising the correlators with width $\sigma_N = 0.1$ e-folds. The parameter DeltaN is the number of e-folds before horizon crossing, and specifies the initial time for the correlators. Turning on interactions can be considered an adiabatic process when the function adiabatic varies on a timescale much longer than the relevant timescale of the dynamics inside the horizon. Considering a unit sound speed without loss of generality, hence with modes behaving as $e^{ik\tau}$, one should require $1/k \ll (\Delta\tau)^{\rm adiabatic} \simeq 3\sigma_N/(aH)$, at the central time of the adiabatic function. One thus obtain the criterion $\sigma_N \gg e^{-\Delta N}$. Sharper transitions would excite negative frequency modes and would not correspond to the Bunch-Davies vacuum. The function adiabatic should multiply all cubic coupling constants. This is the very reason why all parameters are defined as functions on a time grid, even though certain coupling constants are actually taken to be constant, with e.g. np.ones(n). After defining the parameters of the theory and numerical parameters, these are loaded using parameters and the list interpolated is created. The flow equation integrator is initialised by calling theory which takes interpolated as an argument. Before solving the flow equations, a specific kinematic configuration needs to be chosen. Here, we take an equilateral one $k_1 = k_2 = k_3 = k$ for simplicity. We use the convention that the scale factor as a function of the number of e-folds is given by $a(N) = e^N$, that arbitrarily fixes $a = 1$ at $N = 0$. This way, the default k is chosen such that the corresponding mode crosses its horizon at N_exit$= 0$, i.e. $k = aH$. Therefore, in scale-invariant theories, there is a one-to-one correspondence between a mode k and the time $t_\star$ at which it crosses the horizon. The corresponding scale is determined in this way by the use of the function k_mode in the Theory class. It is important to keep this in mind when varying the kinematic configuration, i.e. when

all modes are not the same. Finally, solving the flow equations is done by calling solver and executing the function f_solution.

All correlators are stored in the object f in the following way:

- Setting the first index to $0, 1$ or 2 selects the modes k_1, k_2 or k_3, respectively, corresponding to the *real part of the two-point correlators*. The next layer of indices selects the correlator, e.g. f[1][0, 1] would give $\Sigma^{\varphi p_\varphi}_{\rm Re}(k_2)$.

- Setting the first index to $3, 4$ or 5 selects the modes k_1, k_2 or k_3, respectively, corresponding to the *imaginary part of the two-point correlators*. The next layer of indices selects the correlator, e.g. f[3][0, 0] would give $\Sigma^{\varphi\varphi}_{\rm Im}(k_1)$.

- Setting the first index to 6 corresponds to selecting the three-point correlators, e.g. f[6][0, 0, 0] encodes $B^{\varphi\varphi\varphi}(k_1, k_2, k_3)$. One can explicitly check that this object is fully symmetric.

These objects are single-dimensional arrays of the same size as N. The last element of these arrays, which in most applications is the desired value for the correlator at the end of inflation, are given by e.g. f[0][0, 0, 0][-1] for $B^{\varphi\varphi\varphi}(k_1, k_2, k_3)$. The rest of the script file is rather straightforward as each line is commented appropriately.

Exercise 5.20 Go through the Jupyter notebook BasicUsage.ipynb and modify it to find the scaling laws of the correlators $\Sigma^{\varphi\varphi}_{\rm Re}(k), \Sigma^{\varphi p_\varphi}_{\rm Re}(k), \Sigma^{p_\varphi p_\varphi}_{\rm Re}(k)$ and $B^{\varphi\varphi\varphi}(k)$ as function of the scale factor $\propto a^p$, both on sub-horizon and super-horizon scales.

Playing with parameters. At the core of the cosmological flow implementation, some things can go wrong if not done correctly and several tricks are regularly used. It is crucial for the new user to know them. I invite you to go through and fill in the blanks of the PlayingWithParameters.ipynb notebook to get a sense of what all parameters are controlling. The performed tests and generated figures of this notebook are meant to get an intuition on the following features.

- **Absolute and relative error.** Numerically solving the flow equations for a sufficiently complicated theory can be challenging and requires good accuracy. The absolute tolerance Δ_a should be set as low as possible as correlators involving massive fields or conjugate momenta decay outside the horizon until reaching extremely low values. Changing absolute tolerance does not affect the computational speed for the flow equation resolution. On the other hand, there is no general rule to choose the relative tolerance Δ_r, and the most important criterion to chose it is convergence of the final result, i.e. by decreasing Δ_r (hence increasing numerical precision) you should see the correlators (typically the one including only massless fields as they usually freeze on super-horizon scales) converge to some value. The integrator has converged if the solution does not vary when Δ_r is slightly modified.

- **Time discretisation and numerical $i\epsilon$ prescription.** It is important to have enough points in the discretisation of time (here in e-folds) to properly integrate the flow equations. Convince yourself by playing with the notebook. When lowering $n_{\rm disc}$, the numerical solution slightly gets off the attractor solution at a certain time. The Bunch-Davies state is no longer picked and excited states are visible in the form of negative frequency modes. Cancelling these contaminating oscillations is exactly the purpose of the numerical $i\epsilon$ prescription. Without it, the initial flow is deviated from the correct vacuum state and the solution picks up oscillations. On the contrary, correctly implementing the $i\epsilon$ prescription selects the correct vacuum state and the correlators appear smooth.

- **Massless e-folds.** In order to select the correct initial vacuum state for the correlators, they must be initialised sufficiently deep inside the horizon. Play with the corresponding notebook to find the optimal number of massless e-folds. Note that too many sub-horizon e-folds makes the code slow as integrating the flow equations under the horizon is computationally expensive (one must resolve the tiny oscillations resulting from a small error in the cancellation of positive and negative frequency modes). Finding an accurate ΔN for specific runs is done empirically, and is usually a trad-off between precision and speed, as usual when doing numerics.

5.4 Exploring the physics of cosmological correlators

Let us now do actual interesting science with CosmoFlow! The following worked examples serve both to illustrate the numerous possibilities that this code provides, and to show the user how to implement different calculations.

Theory. For this entire section, we consider the theory of a massless scalar field φ interacting with a massive scalar field Ψ in de Sitter. These fields can interact quadratically, allowing both particles to mix at the linear level (because Lorentz invariance is broken), and at the cubic level so that the theory can generate cubic correlations among the fields. The theory we consider is the following:

$$S = \int \mathrm{d}t\mathrm{d}^3x\, a^3 \left[\frac{1}{2}\dot{\varphi}^2 - \frac{c_s^2}{2}\frac{(\partial_i\varphi)^2}{a^2} - \frac{1}{2}\left[(\partial_\mu\Psi)^2 + m^2\Psi^2\right] + \rho\,\dot{\varphi}\Psi - \frac{g}{2}\frac{(\partial_i\varphi)^2}{a^2}\,\Psi \right] , \tag{5.3}$$

where overdots denote derivatives with respect to cosmic (physical) time, c_s is the speed of sound of φ—allowing a breaking of de Sitter boosts—, m is the mass of Ψ, ρ is the linear mixing strength, and g is a coupling constant. The quadratic theory leads to linear equations of motions whose analytical solutions are unknown. However, at weak mixing (typically when $\rho \ll H$ where H is the Hubble scale), the linear mixing can be treated perturbatively and (some) analytical computations are tractable. In the limit $\rho/H \to 0$, both sectors decouple at the linear level and the flow equations become block diagonal.

Exercise 5.21 Derive the form of the tensors $\Delta_{\alpha\beta}, M_{\alpha\beta}, I_{\alpha\beta}, A_{\alpha\beta\gamma}, B_{\alpha\beta\gamma}, C_{\alpha\beta\gamma}$ and $D_{\alpha\beta\gamma}$ appearing in the Hamiltonian (4.1), and properly implement this theory as a new CosmoFlow module.

5.4.1 Characteristic time scales (Jupyter Notebook)

The quadratic theory cannot be solved analytically in the entire parameter space, i.e. for all ranges for the speed of sound c_s, the mass m of the additional field Ψ, and the strength of the linear mixing between both fields ρ, and analytical solutions for three-point correlators are out of reach. Therefore, this example showcases how one can use CosmoFlow to gain intuition when studying complex theories. For a unity speed of sound $c_s = 1$ and a weak mixing $\rho/H \ll 1$ (in Hubble units), correlators of the massless field φ freeze at the usual *Hubble horizon* given by $|k\tau| \sim 1$, here at $N = 0$. At weak mixing, when the speed of sound is reduced, the massless mode freezes at the *sound horizon* $|k\tau| \sim c_s^{-1}$, before the usual Hubble horizon, and with a globally greater amplitude. At strong mixing, the coupled dynamics is drastically modified and massless modes freeze at the *ρ-horizon* $|k\tau| \sim \sqrt{\rho/H}$, here for $c_s = 1$. Note that for $c_s = 1$, the weak and strong mixing cases lead to the same two-point correlators deep inside the horizon. This is expected as the system asymptotically reaches the Bunch-Davies state in the infinite past, and the two fields are effectively decoupled. However, the mixing between both fields builds up progressively around ρ and Hubble crossing, leading to an amplification of the correlators in the strong mixing case. These might just sound like stated facts. In order to convince yourself, I invite you to go through the Jupyter notebook CharacteristicTimeScales.ipynb.

5.4.2 Phase diagrams (Jupyter Notebook)

CosmoFlow offers the possibility to easily navigate through a vast landscape of theories, e.g. varying coupling constant strengths. Indeed, contrary to C++, Python does not require compilation before execution. Moreover, the module Theory.py allows to pre-implement a large class of theories at the same time. This way, the structure of CosmoFlow makes it possible to scan different theories without the need to ever modify the source code.

Fixing $c_s = 1$, the quadratic theory only depends on two parameters, the mass m of the field Ψ and the linear mixing strength ρ. Therefore, these two parameters define a two-dimensional phase diagram for the correlators. Go through the Jupyter notebook PhaseDiagrams.ipynb to obtain the phase diagrams. Caution, in the ready-to-use notebook, we use parallelisation with the Python package joblib, that can be installed with pip. My advice is to follow the correct parallelisation implementation using the top command on the terminal, e.g. checking that several cores are used for the computation. In general, parallelisation allows to gain roughly a factor 10 on the runtime, and is regularly used.

5.4.3 Shape of non-Gaussianities (Jupyter Notebook)

For the bispectrum shape, it is convenient to plot the rescaled three-point function $(k_1k_2k_3)^2 \langle\varphi_{\boldsymbol{k}_1}\varphi_{\boldsymbol{k}_2}\varphi_{\boldsymbol{k}_3}\rangle'$ as it is scale-invariant and is related to the usual bispectrum shape function up to a normalisation. In practice, as can be noticed in the notebook, we arbitrarily fix the scale $k_3 = 1$ by virtue of scale-invariance, and choose the scale factor $a = e^N$ so that this mode exits its horizon at $N = 0$. This way, the number of massless e-folds ΔN can be unambiguously defined with respect to $N = 0$ (we set $H = 1$). To avoid changing ΔN as we scan over soft limits, these are reached by increasing $k_1 \sim k_2 \gg k_3$ (corresponding to the short modes) which therefore exit their horizon later. Of course, by scale invariance, this is completely equivalent to fixing k_1 and k_2 and decreasing k_3. However, the latter method is numerically less practical. Now play with the corresponding notebook to rediscover the cosmological low-speed collider resonance.

5.4.4 Cosmological collider physics (Jupyter Notebook)

Scanning soft limits of cosmological correlators is computationally very expensive, as a large hierarchy of modes requires to resolve more massless e-folds in the sub-horizon regime. Therefore, these calculations are long and need parallelisation or additional computational capabilities, such as the use of HPC clusters. Nevertheless, for mildly soft configurations, we show here that soft limits can be computed in a reasonable amount of time ($\sim$ 20min) on a laptop. In the last notebook CosmologicalCollider-Physics.ipynb, you will have fun with massive fields in de Sitter, and compute the famous cosmological collider signal.

Solutions to (some) exercises

Solution 2.2 The state $|\Omega\rangle$ asymptotically reaches the free theory vacuum $|0\rangle$ in the far past. Let us evolve $|0\rangle$, i.e. the ground state of $\hat{H}_0$, using the full Hamiltonian $\hat{H}$

$$e^{-i\hat{H}(t_0-t)}\,|0\rangle = e^{-i\hat{H}(t_0-t)}\sum_n |n\rangle\,\langle n|0\rangle = \sum_n e^{-i\omega_n(t_0-t)}\,|n\rangle\,\langle n|0\rangle\ , \tag{5.4}$$

where ω_n are eigenvalues of $\hat{H}$ and with $|n\rangle$ ($n \geq 0$) being a complete set of states of $\hat{H}$. We have introduced an arbitrary constant shift in time t_0 whose purpose will be clearer later on. In order for the free dynamics dictated by $\hat{H}_0$ to be the dominant one (otherwise signaling a breakdown of perturbation theory), $|\Omega\rangle$ must have some overlap with $|0\rangle$, i.e. $\langle\Omega|0\rangle \neq 0$. Therefore, we can isolate the state $|\Omega\rangle$

$$e^{-i\hat{H}(t_0-t)}\,|0\rangle = e^{-i\omega_0(t_0-t)}\,|\Omega\rangle\,\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-i\omega_n(t_0-t)}\,|n\rangle\,\langle n|0\rangle\ , \tag{5.5}$$

where $\hat{H}\,|\Omega\rangle = \omega_0\,|\Omega\rangle$. Since $\omega_n > \omega_0$ for $n \neq 0$, assuming that the ground state is not degenerate and that energy levels do not cross (which holds as long as interactions are switched off adiabatically), we can slightly tilt time in the imaginary direction

$$t \to t(1-i\epsilon)\,, \tag{5.6}$$

with $\epsilon > 0$ an infinitesimal constant, so that the limit $t \to -\infty$ effectively makes all terms with $n \neq 0$ in the series vanish. We obtain that $|\Omega\rangle$ can be mapped to $|0\rangle$ via

$$|\Omega\rangle = \lim_{t\to-\infty(1-i\epsilon)} \left(e^{-i\omega_0(t_0-t)}\,\langle\Omega|0\rangle\right)^{-1} e^{-i\hat{H}(t_0-t)}\,|0\rangle\ . \tag{5.7}$$

Splitting the full Hamiltonian $\hat{H}$ into $\hat{H}_0$ and $\hat{H}_{\rm int}$ and using $\hat{H}_0\,|0\rangle = |0\rangle$ (which defines the zero energy), a few elementary manipulations lead to

$$|\Omega\rangle = \lim_{\epsilon\to 0} \mathcal{N}_\epsilon\,\hat{U}_\epsilon(t_0,-\infty)\,|0\rangle\ , \quad \text{with} \quad \mathcal{N}_\epsilon = \lim_{t\to-\infty(1-i\epsilon)} \left(e^{-i\omega_0(t_0-t)}\,\langle\Omega|0\rangle\right)^{-1}\ , \tag{5.8}$$

and where we define the time evolution operator with a deformed contour as

$$\hat{U}_\epsilon(t_0,-\infty) = \mathcal{T}\left\{\exp\left(-i\int_{-\infty(1-i\epsilon)}^{t_0} \hat{H}^{\rm I}_{\rm int}(t')\mathrm{d}t'\right)\right\}\ . \tag{5.9}$$

Eventually, one obtains

$$\langle\Omega|\hat{\mathcal{O}}(t)|\Omega\rangle = \langle\Omega|\hat{U}^\dagger(t,t_0)\hat{\mathcal{O}}_{\rm I}(t)\hat{U}(t,t_0)|\Omega\rangle = \lim_{\epsilon\to 0} |\mathcal{N}_\epsilon|^2\,\langle 0|\hat{U}^\dagger_\epsilon(t,-\infty)\hat{\mathcal{O}}_{\rm I}(t)\hat{U}_\epsilon(t,-\infty)|0\rangle\ , \tag{5.10}$$

where in the limit $\epsilon \to 0$, we have $|\mathcal{N}_\epsilon|^2 = |\,\langle\Omega|0\rangle\,|^{-2}$. This normalisation can be evaluated by choosing $\hat{\mathcal{O}} = \mathbb{1}$, i.e. $1 = \langle\Omega|\mathbb{1}|\Omega\rangle = |\,\langle\Omega|0\rangle\,|^{-2}\,\langle 0|\hat{U}^\dagger_\epsilon(t,-\infty)\hat{U}_\epsilon(t,-\infty)|0\rangle = |\,\langle\Omega|0\rangle\,|^{-2}\,\langle 0|0\rangle = |\,\langle\Omega|0\rangle\,|^{-2}$.

- We have crucially assumed that the (interacting) Hamiltonian is time independent in the manipulations (5.8) (and that $\hat{H}_0$ commutes with $\hat{H}_{\rm int}$) by writing
$$\mathcal{T}\left\{\exp\left(-i\int_t^{t_0}\hat{H}^{\rm I}_{\rm int}(t')\mathrm{d}t'\right)\right\}\approx e^{-i\hat{H}^{\rm I}_{\rm int}(t_0-t)}\,, \tag{5.11}$$
in passing completely neglecting the time ordering. This hand-wavy manipulation is often justified with some adiabaticity condition when all modes are well inside the horizon, therefore reaching the flat-space limit where the standard derivation is applicable. This infinite past limit has also allowed us to replace $\hat{H}_{\rm int}\to\hat{H}^{\rm I}_{\rm int}$ to recover the evolution operator.

- After projection, the produced state is not correctly normalised since the time evolution operator is non-unitary, i.e. we do not in general have $\hat{U}^\dagger_\epsilon(t,-\infty)\hat{U}_\epsilon(t,-\infty)=\mathbb{1}$. This is because the contours of integration are different in $\hat{U}_\epsilon(t,-\infty)$ and $\hat{U}^\dagger_\epsilon(t,-\infty)$, producing a commutator and not the identity. Moreover, cancelling the overall phase in the normalisation implicitly assumes that both the limits $\epsilon\to 0$ and $t\to-\infty$ commute, which is not the case in general.

Solution 2.5 In Fourier space, the interacting Hamiltonian reads
$$\begin{aligned}H_{\rm int}(t)&=\frac{g}{3!}\int\mathrm{d}^3\boldsymbol{x}\sqrt{-g}\varphi^3(t,\boldsymbol{x})\\&=\frac{g}{3!}\sqrt{-g}\int\left(\prod_{i=1}^3\frac{\mathrm{d}^3\boldsymbol{q}_i}{(2\pi)^3}\right)\varphi_{\boldsymbol{q}_1}(t)\varphi_{\boldsymbol{q}_2}(t)\varphi_{\boldsymbol{q}_3}(t)\,(2\pi)^3\delta^{(3)}(\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{q}_3)\,,\end{aligned} \tag{5.12}$$
where the second line is obtained after performing the integral over space. The first contribution in Eq. (2.22) then reads
$$\begin{aligned}\langle\varphi_{\boldsymbol{k}_1}(t)\varphi_{\boldsymbol{k}_2}(t)\varphi_{\boldsymbol{k}_3}(t)\rangle\supset i\,\frac{g}{3!}\int\left(\prod_{i=1}^3\frac{\mathrm{d}^3\boldsymbol{q}_i}{(2\pi)^3}\right)(2\pi)^3\delta^{(3)}(\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{q}_3)\\\times\int_{-\infty^-}^t\mathrm{d}t'\sqrt{-g}\,\langle0|\varphi_{\boldsymbol{q}_1}(t')\varphi_{\boldsymbol{q}_2}(t')\varphi_{\boldsymbol{q}_3}(t')\varphi_{\boldsymbol{k}_1}(t)\varphi_{\boldsymbol{k}_2}(t)\varphi_{\boldsymbol{k}_3}(t)|0\rangle\,,\end{aligned} \tag{5.13}$$
where we have highlighted in blue the contributions coming from the single insertion of $H_{\rm int}$. Using Wick's theorem, we now perform all possible contractions of every pair of operators. However, being interested in the *connected* part of the correlator, we only contract internal legs ($\varphi_{\boldsymbol{q}_i}$) with external legs ($\varphi_{\boldsymbol{k}_i}$). An example of such contractions using (2.30) is
$$\langle0|\overbrace{\varphi_{\boldsymbol{q}_1}(t')\varphi_{\boldsymbol{q}_2}(t')\varphi_{\boldsymbol{q}_3}(t')\varphi_{\boldsymbol{k}_1}(t)\varphi_{\boldsymbol{k}_2}(t)\varphi_{\boldsymbol{k}_3}(t)}|0\rangle=\prod_{i=1}^3u_{k_i}(t')u^*_{k_i}(t)\,(2\pi)^3\delta^{(3)}(\boldsymbol{q}_i+\boldsymbol{k}_i)\,. \tag{5.14}$$

Integrating over the internal momenta $\boldsymbol{q}_i$ turns $\delta^{(3)}(\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{q}_3)$ into $\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)$ in (5.13). Noticing that all possible contractions lead to the same result (this simply corresponds to a relabelling of the $\boldsymbol{q}_i$), there are 3! ways of pairing contracted fields. Thus, we obtain

$$\langle\varphi_{\boldsymbol{k}_1}(t)\varphi_{\boldsymbol{k}_2}(t)\varphi_{\boldsymbol{k}_3}(t)\rangle' \supset ig\,u^*_{k_1}(t)u^*_{k_2}(t)u^*_{k_3}(t)\int_{-\infty^-}^{t}\mathrm{d}t'\sqrt{-g}u_{k_1}(t')u_{k_2}(t')u_{k_3}(t')\,, \tag{5.15}$$

where the prime denotes that we drop the overall momentum conserving delta function. Following the same manipulations for the second contribution in (2.22), it is possible to recast the final expression for the correlator as

$$\langle\varphi_{\boldsymbol{k}_1}(t)\varphi_{\boldsymbol{k}_2}(t)\varphi_{\boldsymbol{k}_3}(t)\rangle' = 2g\,\mathrm{Im}\left[u^*_{k_1}(t)u^*_{k_2}(t)u^*_{k_3}(t)\int_{-\infty^-}^{t}\mathrm{d}t'\sqrt{-g}u_{k_1}(t')u_{k_2}(t')u_{k_3}(t')\right]. \tag{5.16}$$

In order to do so, we have implicitly flipped the sign of momenta. For a given function $a(t)$ and a mode function $u_k(t)$ found by solving the linear equation of motion, the time integral in Eq. (5.16) can be explicitly evaluated. In flat space, we obtain

$$\langle\varphi_{\boldsymbol{k}_1}\varphi_{\boldsymbol{k}_2}\varphi_{\boldsymbol{k}_3}\rangle' = \frac{2g}{8k_1k_2k_3}\mathrm{Im}\int_{-\infty^-}^{0}\mathrm{d}t'\,e^{-i(k_1+k_2+k_3)t'} = \frac{g}{4k_1k_2k_3}\frac{1}{k_1+k_2+k_3}\,. \tag{5.17}$$

If the upper limit is taken to infinity, we would have a delta function $(2\pi)^3\delta(k_1+k_2+k_3)$ instead of a pole $1/(k_1+k_2+k_3)$. In this limit, we recover the conservation of energy.

Solution 2.6 Following the same steps as previously, the interacting Hamiltonian in Fourier space reads

$$H_{\rm int}(t) = \frac{g}{2}a^3(t)\int\left(\prod_{i=1}^{3}\frac{\mathrm{d}^3\boldsymbol{q}_i}{(2\pi)^3}\right)\varphi_{\boldsymbol{q}_1}(t)\varphi_{\boldsymbol{q}_2}(t)\sigma_{\boldsymbol{q}_3}(t)\,(2\pi)^3\delta^{(3)}(\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{q}_3)\,. \tag{5.18}$$

We directly use Eq. (2.26). The full correlator is composed of two contributions, a factorised part (first line of (2.26)) and a nested part (second line of (2.26)), which we evaluate in turn. The factorised contribution explicitly reads

$$\begin{aligned}\langle\varphi_{\boldsymbol{k}_1}(t)\varphi_{\boldsymbol{k}_2}(t)\varphi_{\boldsymbol{k}_3}(t)\varphi_{\boldsymbol{k}_4}(t)\rangle \supset{}& \frac{g}{2}\int\left(\prod_{i=1}^{3}\frac{\mathrm{d}^3\boldsymbol{q}_i}{(2\pi)^3}\right)\times\frac{g}{2}\int\left(\prod_{i=1}^{3}\frac{\mathrm{d}^3\boldsymbol{p}_i}{(2\pi)^3}\right)\\ &\times(2\pi)^3\delta^{(3)}(\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{q}_3)\,(2\pi)^3\delta^{(3)}(\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3)\int_{-\infty^-}^{t}\mathrm{d}t'a^3(t')\int_{-\infty^-}^{t}\mathrm{d}t''a^3(t'')\\ &\times\langle0|\varphi_{\boldsymbol{q}_1}(t')\varphi_{\boldsymbol{q}_2}(t')\sigma_{\boldsymbol{q}_3}(t')\varphi_{\boldsymbol{k}_1}(t)\varphi_{\boldsymbol{k}_2}(t)\varphi_{\boldsymbol{k}_3}(t)\varphi_{\boldsymbol{k}_4}(t)\varphi_{\boldsymbol{p}_1}(t'')\varphi_{\boldsymbol{p}_2}(t'')\sigma_{\boldsymbol{p}_3}(t'')|0\rangle\,,\end{aligned} \tag{5.19}$$

where we have highlighted in blue and red the contributions coming from the two insertions of $H_{\rm int}$. For simplicity, let us focus on the s-channel only in which the

exchanged momentum is $\boldsymbol{s} \equiv \boldsymbol{k}_1 + \boldsymbol{k}_2$. Using (2.30), an example of contractions contributing to this channel is

$$\begin{aligned}
&\langle 0|\varphi_{\boldsymbol{q}_1}(t')\varphi_{\boldsymbol{q}_2}(t')\sigma_{\boldsymbol{q}_3}(t')\varphi_{\boldsymbol{k}_1}(t)\varphi_{\boldsymbol{k}_2}(t)\varphi_{\boldsymbol{k}_3}(t)\varphi_{\boldsymbol{k}_4}(t)\varphi_{\boldsymbol{p}_1}(t'')\varphi_{\boldsymbol{p}_2}(t'')\sigma_{\boldsymbol{p}_3}(t'')|0\rangle \\
&= v_{q_3}(t')v^*_{q_3}(t'')(2\pi)^3\delta^{(3)}(\boldsymbol{q}_3+\boldsymbol{p}_3) \\
&\times u_{k_1}(t')u^*_{k_1}(t)(2\pi)^3\delta^{(3)}(\boldsymbol{q}_1+\boldsymbol{k}_1)\, u_{k_2}(t')u^*_{k_2}(t)(2\pi)^3\delta^{(3)}(\boldsymbol{q}_2+\boldsymbol{k}_2) \\
&\times u_{k_3}(t)u^*_{k_3}(t'')(2\pi)^3\delta^{(3)}(\boldsymbol{k}_3+\boldsymbol{p}_1)\, u_{k_4}(t)u^*_{k_4}(t'')(2\pi)^3\delta^{(3)}(\boldsymbol{k}_4+\boldsymbol{p}_2)\,.
\end{aligned} \tag{5.20}$$

There are 4 ways of doing such contractions keeping σ as the exchanged field. Performing the integrals over the $\boldsymbol{q}_i$ turns $(2\pi)^3\delta^{(3)}(\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{q}_3)$ into $(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{p}_3)$, and the integrals over $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ turn $(2\pi)^3\delta^{(3)}(\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3)$ into $(2\pi)^3\delta^{(3)}(\boldsymbol{k}_3+\boldsymbol{k}_4-\boldsymbol{p}_3)$. Eventually performing the integral over $\boldsymbol{p}_3$ gives the overall momentum conserving condition $(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4)$ with the exchanged mode carrying the momentum $s = |\boldsymbol{s}| = |\boldsymbol{k}_1+\boldsymbol{k}_2|$. For the factorised part, we obtain

$$\begin{aligned}
&\langle\varphi_{\boldsymbol{k}_1}(t)\varphi_{\boldsymbol{k}_2}(t)\varphi_{\boldsymbol{k}_3}(t)\varphi_{\boldsymbol{k}_4}(t)\rangle \supset g^2 u^*_{k_1}(t)u^*_{k_2}(t)u_{k_3}(t)u_{k_4}(t) \\
&\times \int_{-\infty^-}^{t} \mathrm{d}t' a^3(t')u_{k_1}(t')u_{k_2}(t')v_s(t') \times \int_{-\infty^+}^{t} \mathrm{d}t'' a^3(t'')u^*_{k_3}(t'')u^*_{k_4}(t'')v^*_s(t'')\,.
\end{aligned} \tag{5.21}$$

Notice that the two time integrals of this contribution factorise, hence the name. Following the same steps, the nested contribution is given by

$$\begin{aligned}
&\langle\varphi_{\boldsymbol{k}_1}(t)\varphi_{\boldsymbol{k}_2}(t)\varphi_{\boldsymbol{k}_3}(t)\varphi_{\boldsymbol{k}_4}(t)\rangle \supset -2g^2\,\mathrm{Re}\left[u_{k_1}(t)u_{k_2}(t)u_{k_3}(t)u_{k_4}(t)\right. \\
&\left.\times \int_{-\infty^+}^{t} \mathrm{d}t' a^3(t')u^*_{k_1}(t')u^*_{k_2}(t')v_s(t') \times \int_{-\infty^+}^{t'} \mathrm{d}t'' a^3(t'')u^*_{k_3}(t'')u^*_{k_4}(t'')v^*_s(t'')\right]\,.
\end{aligned} \tag{5.22}$$

In the soft limit $s \to 0$, each insertion of the interaction Hamiltonian is outside the light cone of the other so that time ordering does not matter anymore. We can safely factorise the nested time integrals in the second contribution. The rest is just computations. The first factorised piece gives

$$\langle\varphi_{\boldsymbol{k}_1}\varphi_{\boldsymbol{k}_2}\varphi_{\boldsymbol{k}_3}\varphi_{\boldsymbol{k}_4}\rangle' \supset \frac{-H^2}{2\pi(k_{12}k_{34})^{1/2}}\left[\left(\frac{s^2}{4k_{12}k_{34}}\right)^{i\mu}\Gamma(-i\mu)^2\Gamma\left(\tfrac{1}{2}+i\mu\right)^2 + \text{c.c.}\right]\,. \tag{5.23}$$

The second contribution is found to be

$$\begin{aligned}
\langle\varphi_{\boldsymbol{k}_1}\varphi_{\boldsymbol{k}_2}\varphi_{\boldsymbol{k}_3}\varphi_{\boldsymbol{k}_4}\rangle' \supset \frac{iH^2}{4\pi(k_{12}k_{34})^{1/2}}&\left[e^{-\pi\mu}\left(\frac{s^2}{4k_{12}k_{34}}\right)^{i\mu}\Gamma(-i\mu)^2\Gamma\left(\tfrac{1}{2}+i\mu\right)^2\right. \\
&\left.+e^{\pi\mu}\left(\frac{s^2}{4k_{12}k_{34}}\right)^{-i\mu}\Gamma(i\mu)^2\Gamma\left(\tfrac{1}{2}-i\mu\right)^2\right] + \text{c.c.}\,.
\end{aligned} \tag{5.24}$$

The final result is found after summing both terms.

Solution 3.2 Using the extended Fourier summation notations, we obtain

$$\delta^{\mathsf{ab}} A_{\mathsf{b}} = \sum_b \int \frac{\mathrm{d}^3 k_b}{(2\pi)^3} (2\pi)^3 \delta^{(3)}(\boldsymbol{k}_a + \boldsymbol{k}_b) \delta^{ab}\, A_b(\boldsymbol{k}_b) = A^a(-\boldsymbol{k}_a) = A^{\bar{\mathsf{a}}}\,. \tag{5.25}$$

Solution 3.3 The Lagrangian of a free massive scalar field in flat space is given by

$$\mathcal{L} = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\partial_i \varphi)^2 - \frac{1}{2} m^2 \varphi^2\,. \tag{5.26}$$

The conjugate momentum to the field φ is found to be $p \equiv \partial\mathcal{L}/\partial\dot{\varphi} = \dot{\varphi}$. Performing the Legendre transform, we obtain

$$\begin{aligned} H &= \frac{1}{2}\int \mathrm{d}^3x\, [p(\boldsymbol{x})\dot{\varphi}(\boldsymbol{x}) - \mathcal{L}] \\ &= \frac{1}{2}\int \mathrm{d}^3x\, \left[p(\boldsymbol{x})^2 + (\partial_i\varphi(\boldsymbol{x}))^2 + m^2\varphi(\boldsymbol{x})^2\right] \\ &= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\left\{p(\boldsymbol{p})p(\boldsymbol{q}) + [\boldsymbol{p}\cdot\boldsymbol{q} + m^2]\varphi(\boldsymbol{p})\varphi(\boldsymbol{q})\right\}(2\pi)^3\delta^{(3)}(\boldsymbol{p}+\boldsymbol{q}) \\ &= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}(\varphi(\boldsymbol{p}), p(\boldsymbol{p}))\underbrace{(2\pi)^3\delta^{(3)}(\boldsymbol{p}+\boldsymbol{q})\begin{bmatrix}(\boldsymbol{p}\cdot\boldsymbol{q}+m^2) & 0\\ 0 & 1\end{bmatrix}}_{H_{ab}}\begin{pmatrix}\varphi(\boldsymbol{q})\\ p(\boldsymbol{q})\end{pmatrix} \end{aligned} \tag{5.27}$$

Integrating over $\boldsymbol{q}$ using the momentum conserving delta function, we obtain

$$H = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left\{p(\boldsymbol{p})p(-\boldsymbol{p}) + [-p^2 + m^2]\varphi(\boldsymbol{p})\varphi(-\boldsymbol{p})\right\}\,, \tag{5.28}$$

which can be written in the compact form $H = \frac{1}{2}H_{\mathsf{a}}X^{\mathsf{a}}X^{\bar{\mathsf{a}}}$ (with an implicit sum over a) as the Hamiltonian is diagonal, with $H_a = (-p^2 + m^2, 1)$.

Solution 3.4 We will focus on the quadratic part of the Hamiltonian for simplicity $H = \frac{1}{2}H_{\mathsf{ab}}\boldsymbol{X}^{\mathsf{a}}\boldsymbol{X}^{\mathsf{b}}$. We have

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{X}^{\mathsf{a}}}{\mathrm{d}t} &= i\,[H, \boldsymbol{X}^{\mathsf{a}}] \\ &= \frac{i}{2}H_{\mathsf{bc}}[\boldsymbol{X}^{\mathsf{b}}\boldsymbol{X}^{\mathsf{c}}, \boldsymbol{X}^{\mathsf{a}}] \\ &= \frac{i}{2}H_{\mathsf{bc}}\left(\boldsymbol{X}^{\mathsf{b}}[\boldsymbol{X}^{\mathsf{c}}, \boldsymbol{X}^{\mathsf{a}}] + [\boldsymbol{X}^{\mathsf{b}}, \boldsymbol{X}^{\mathsf{a}}]\boldsymbol{X}^{\mathsf{c}}\right) \\ &= \frac{-1}{2}H_{\mathsf{bc}}\left(\epsilon^{\mathsf{ca}}\boldsymbol{X}^{\mathsf{b}} + \epsilon^{\mathsf{ba}}\boldsymbol{X}^{\mathsf{c}}\right) \\ &= \epsilon^{\mathsf{ac}}H_{\mathsf{cb}}\boldsymbol{X}^{\mathsf{b}}\,, \end{aligned} \tag{5.29}$$

where we have used the fact that H_{ab} is symmetric, ϵ^{ab} antisymmetric, and have relabelled dummy indices. Higher-order terms in the Hamiltonian can be treated in the same way.

Solution 4.13 The phase space is of dimension $2N$. Recall that a general rank-r tensor has $(2N)^r$ elements. However now we need to consider the symmetry properties. We first count the number of equations for the two-point correlators. The real part $\Sigma^{ab}_{\rm Re}$ is symmetric and of rank 2, so has

$$\binom{2N+1}{2} = \frac{2N(2N+1)}{2}\,, \tag{5.30}$$

components. (Counting the number of independent components of a symmetric rank-r tensor in d dimension is equivalent to distribute r socks into d bins, so we can write a sequence of r socks with $d-1$ bars separating them. In the end, we need to choose r slots for the socks, giving r out of $d+r-1$.) The imaginary part $\Sigma^{ab}_{\rm Im}$ is fully antisymmetric, so has

$$\binom{2N}{2} = \frac{2N(2N-1)}{2}\,, \tag{5.31}$$

components. (It can be found by subtracting symmetric indices from a general rank-3 tensor, or by realising that it actually boils down to choose 3 out of $2N$.) The three-point correlator B^{abc} is totally symmetric, so has

$$\binom{2N+2}{3} = \frac{2N(2N+1)(2N+2)}{3!}\,, \tag{5.32}$$

components. The number of equations to solve is found by summing all terms, and grows as $\mathcal{O}((2N)^3)$.

Solution 4.15 The rescaled conjugate momentum associated with the field φ is

$$p_\varphi \equiv \frac{1}{a^3}\frac{\delta S}{\delta\dot\varphi} = \dot\varphi - \frac{g}{2}\dot\varphi^2\,. \tag{5.33}$$

Inverting this relation perturbatively, and performing the Legendre transform gives[4]

$$\begin{aligned} H &= \int \mathrm{d}t\mathrm{d}^3x\, a^3\left[p_\varphi\dot\varphi - \mathcal{L}\right] \\ &= \int \mathrm{d}t\mathrm{d}^3x\, a^3\left[p_\varphi^2 + \frac{g}{2}p_\varphi^3 - \frac{1}{2}p_\varphi^2 - \frac{g}{2}p_\varphi^3 + \frac{1}{2}\frac{(\partial_i\varphi)^2}{a^2} + \frac{g}{3!}p_\varphi^3 + \mathcal{O}\left(p_\varphi^4\right)\right] \\ &= \int \mathrm{d}t\mathrm{d}^3x\, a^3\left[\frac{1}{2}p_\varphi^2 + \frac{1}{2}\frac{(\partial_i\varphi)^2}{a^2} + \frac{g}{3!}p_\varphi^3 + \mathcal{O}\left(p_\varphi^4\right)\right]\,. \end{aligned} \tag{5.35}$$

[4]We have inverted the non-linear equation perturbatively

$$\dot\varphi = \frac{p_\varphi}{1-\frac{g}{2}\dot\varphi} \approx p_\varphi\left(1+\frac{g}{2}\dot\varphi\right) \approx p_\varphi + \frac{g}{2}p_\varphi^2\,. \tag{5.34}$$

Note that this only works at this order in perturbation theory!

In Fourier space, this Hamiltonian can be recasted in the form (4.1) and the identification of the tensors (in this case containing a single element) yields

$$\Delta_{\varphi\varphi} = 1\,, \quad M_{\varphi\varphi} = -\frac{k^2}{a^2}\,, \quad D_{\varphi\varphi\varphi} = -\frac{g}{3}\,. \tag{5.36}$$

We would have found the same result if one the linear momentum was considered.

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