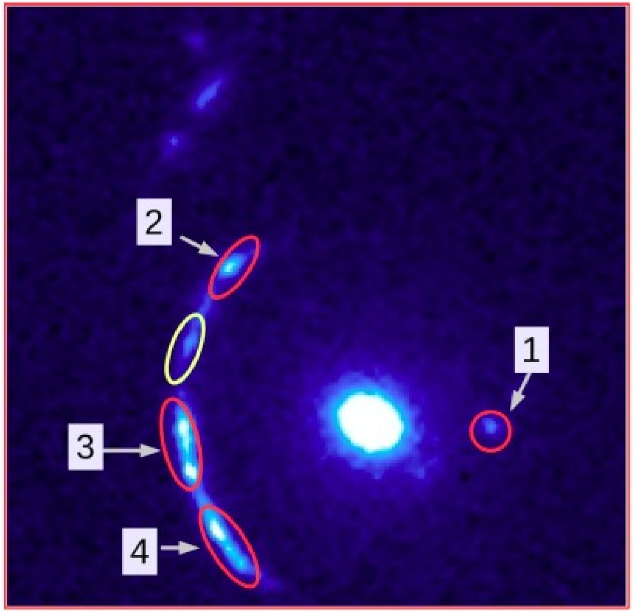
Towards an universal model for strong gravitational lenses

The singular perturbative theory of gravitational lenses

General problems with the modeling of gravitational lenses

The singular background
The perturbative solution
Physical meaning of the perturbative fields
Caustics
Potential iso-contour
Relation to multipole expansion
Some selected applications
Statistical formulation
Future & prospective

Reconstructing strong gravitational lenses



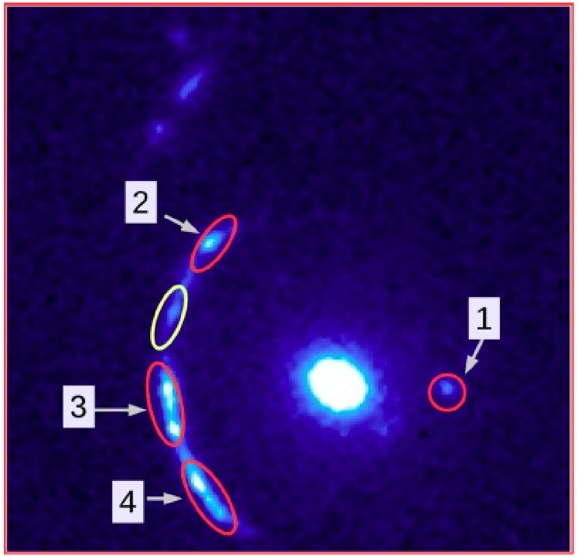
We observe different images of the source
All images must remap to the same source

This gives constraints on the potential: $\vec{r}_s = \vec{r} - \vec{\nabla}\phi$

Main problem: the potential models are degenerates

In the litterature we find NFW, cored-isothermal, power-law models,…,all these models fit the data well

Reconstructing strong gravitational lenses



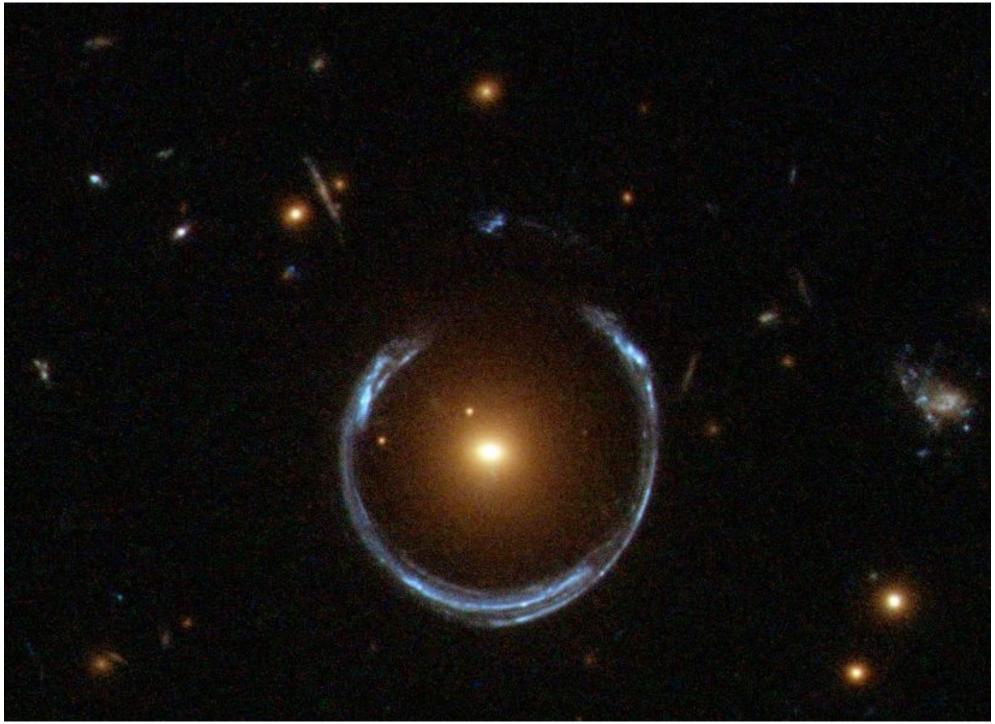
As a consequence

Possible models for a lens belong to large family of models

What are the common properties of all these models ?

What kind of non- degenerate information can we extract ?

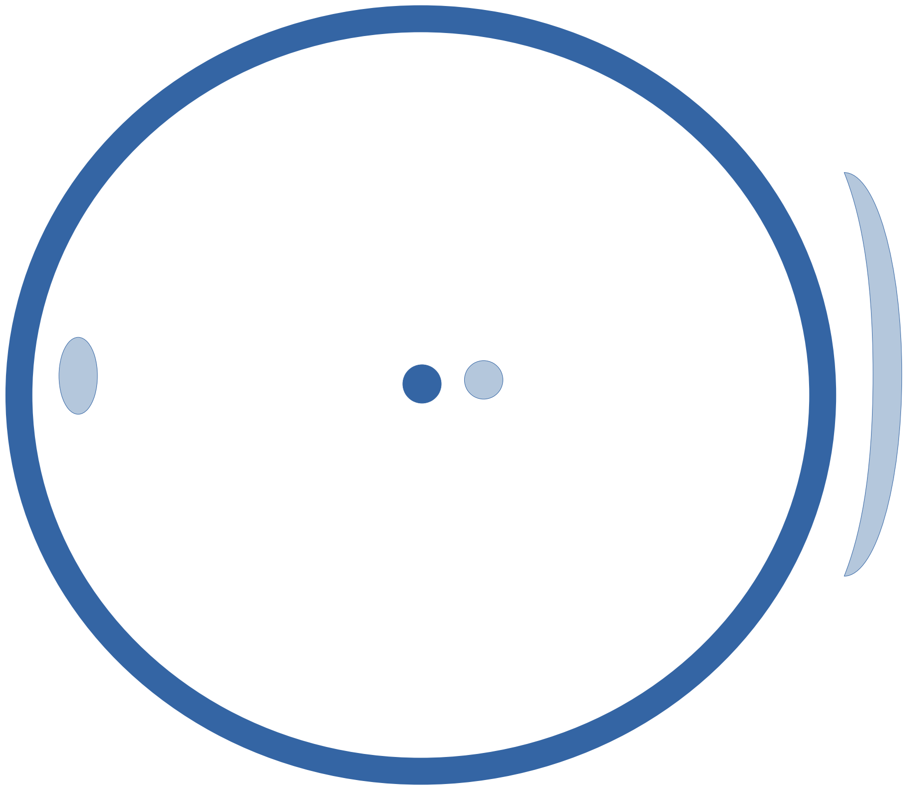
The problem is related to the nature of gravitational arcs
What are gravitational arcs ?



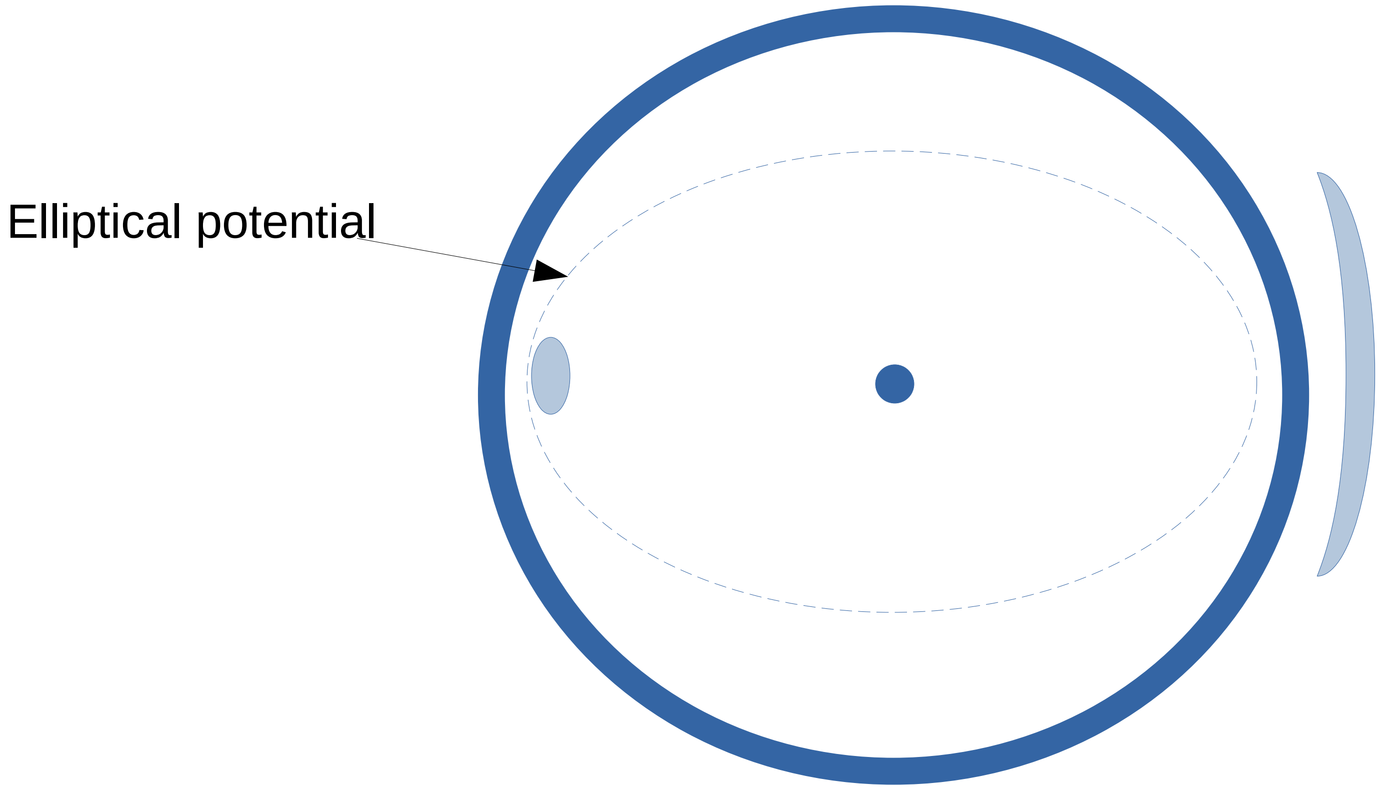
Obviously gravitational arcs are some
Perturbation of the Einstein ring situation

The source is slightly off-centered
The potential deviates from circular symmetry

First perturbation of the perfect ring situation
an off centered source
In a circularly symmetric potential



Second perturbation of the perfect ring situation
a centered source
In a non-circularly symmetric potential



The general situation is a combination of both type of perturbations

I) Of centering of the source

II) Non circular perturbation of the potential

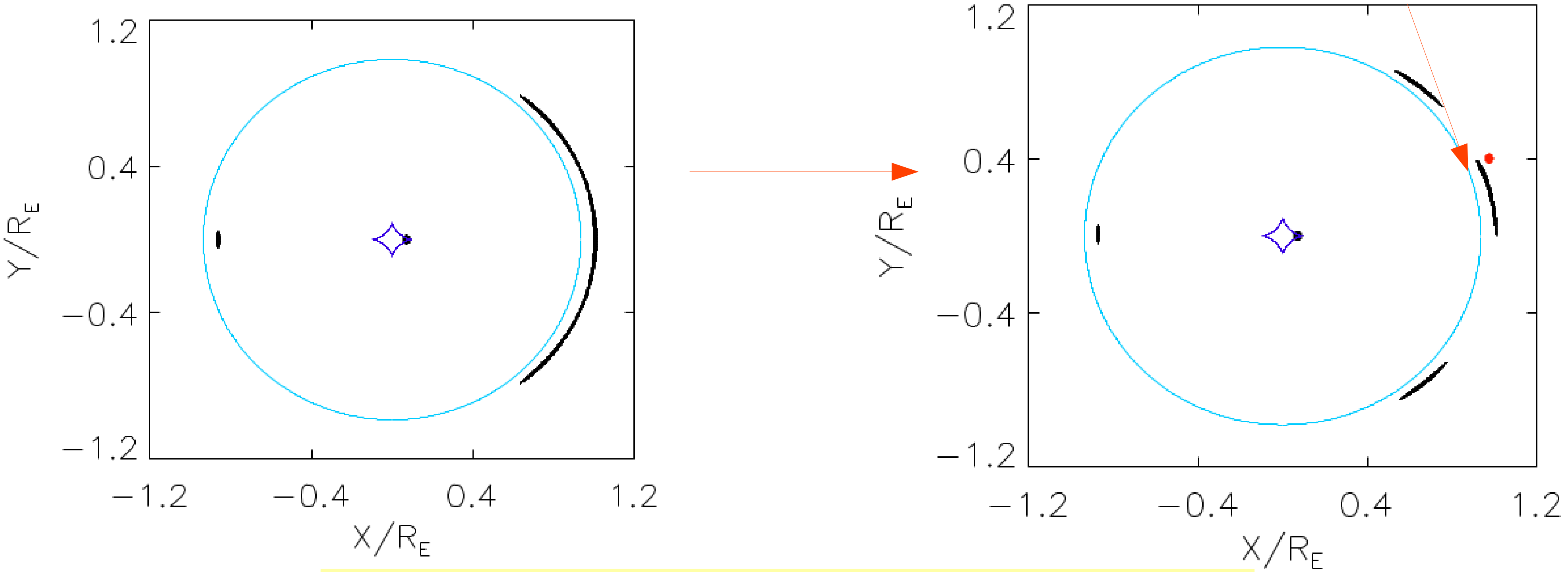
Thus we should write a perturbative theory of strong lensing

The perturbative fields should be the proper non-degenrate quantities

But a perturbative theory of strong lensing looks un-tractable
For a simple reason

Main problem: strong lensing is highly non-linear

1% perturbator



But the non-linearity is in the angular dimension only:
is a perturbative theory possible ?

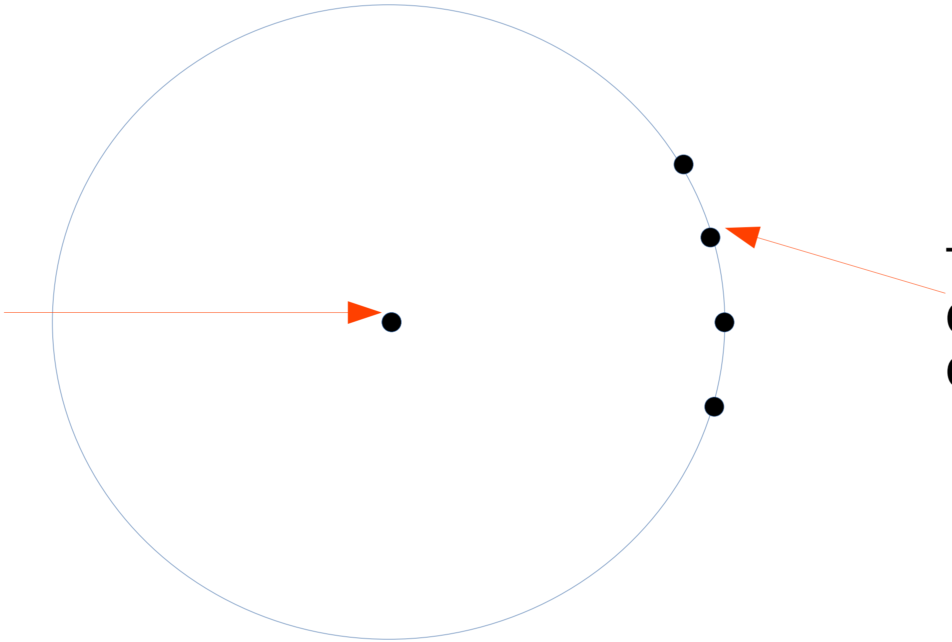
Solving the problem

An effective perurbative theory of strong gravitational lensing

The singular perturbative solution

A perturbative approach is possible
if the un-perturbed situation is a singularity

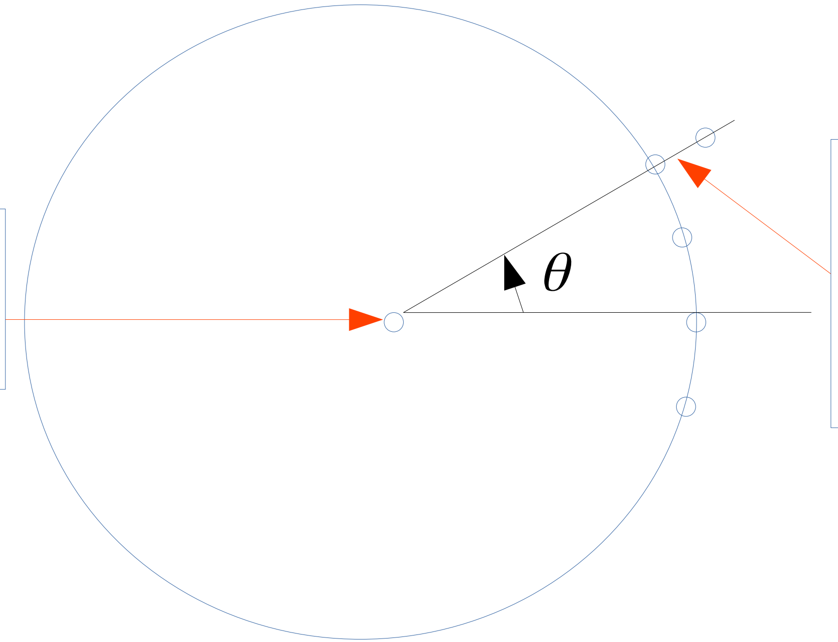
A point is at the center
Of a circularly symmetric
potential



There is an infinity
Of image of the point
On Einstein circle

The perturbative situation

The point is not at the center
The potential is not
circularly symmetric



There is always
An un-perturbed point
On the circle
Close to the perturbed point
For any θ

This solution is the singular perturbative solution

We can find an un-perturbed point for any θ
The perturbation is only in the radial dimension

$$\begin{cases} \phi(r,\theta)=\phi_0(r)+\epsilon\,\psi(r,\theta) \\ r=1+\epsilon\,dr \\ \vec{r}_s=\epsilon\,\vec{r}_S \end{cases}$$

→ We expand only in dr
from the unit Einstein circle

For convenience the un-perturbed Einstein circle has radius unity

$$\left.\begin{array}{l}\phi(r,\theta)=\phi_0(r)+\epsilon\,\psi(r,\theta)\\ r=1+\epsilon\,dr\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\phi_0(r)\simeq\phi_0(1)+\phi_0^{'}(1)\,\epsilon dr+\frac{1}{2}\,\phi_0^{''}(1)(\epsilon dr)^2\\ \psi(r,\theta)\simeq\epsilon\left(f_0(\theta)+f_1(\theta)\,\epsilon dr\right)\end{array}\right.$$

$$\phi(r,\theta)\simeq\phi_0(1)+\phi_0^{'}(1)\,\epsilon dr+\frac{1}{2}\,\phi_0^{''}(1)(\epsilon dr)^2+\epsilon\left(f_0(\theta)+f_1(\theta)\,\epsilon dr\right)$$

$$\vec{r}_s=\vec{r}-\vec{\nabla}\,\phi=\left(r-\frac{\partial\phi}{\partial r}\right)\vec{u}_r-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\vec{u}_\theta$$

With: $\partial r\equiv\partial\,\epsilon dr$

$$\vec{r}_s=\left(1-\phi_0^{'}(1)\right)\vec{u}_r+\left(\left(1-\phi_0^{''}(1)\right)dr-f_1(\theta)\right)\vec{u}_r-\frac{df_0}{d\theta}\vec{u}_\theta$$

$$\vec{r}_s = (1 - \phi_0^{'}(1))\vec{u}_r + ((1 - \phi_0^{''}(1))dr - f_1(\theta))\vec{u}_r - \frac{d f_0}{d\theta}\vec{u}_\theta$$

=0
Unperturbed unit
Einstein circle

$$\vec{r}_s = ((1 - \phi_0^{''}(1))dr - f_1(\theta))\vec{u}_r - \frac{d f_0}{d\theta}\vec{u}_\theta$$

$$\kappa_2 = 1 - \left[\frac{d^2\phi_0}{dr^2}\right]_{r=1}$$

$$\vec{r}_S = (\kappa_2 dr - f_1)\vec{u}_r - \frac{df_0}{d\theta}\vec{u}_\theta$$

The singular perturbative theory

$$\phi(r,\theta)=\phi_0(r)+\epsilon\,\psi(r,\theta) \qquad r=1+\epsilon\,dr \qquad \vec{r}_s=\epsilon\,\vec{r}_S$$

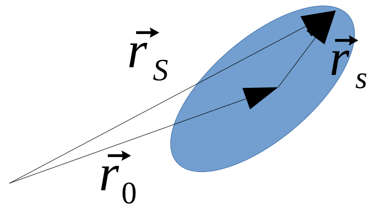
$$\vec{r}_s=\vec{r}-\vec{\nabla}\phi \quad \longrightarrow \quad \vec{r}_S=(\kappa_2\,dr-f_1)\,\vec{u}_r-\frac{df_0}{d\theta}\vec{u}_\theta$$

$$f_1=\left[\frac{d\psi}{dr}\right] \quad ; \quad f_0=\psi(1,\theta) \quad ; \quad \kappa_2=1-\left[\frac{d^2\phi_0}{dr^2}\right]_{r=1}$$

Alard (2007)

κ_2 ⟷ Mass-sheet degeneracy

Let consider a source with an impact parameter $\vec{r}_0$



$$\vec{r}_S = \vec{r}_0 + \vec{r}_s$$

$$\vec{r}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\vec{r}_S = (\kappa_2 dr - f_1)\vec{u}_r - \frac{df_0}{d\theta}\vec{u}_\theta$$

$$\vec{r}_s = (\kappa_2 dr - \tilde{f}_1)\vec{u}_r - \frac{d\tilde{f}_0}{d\theta}\vec{u}_\theta$$

With: $\tilde{f}_i = f_i + x_0 \cos(\theta) + y_0 \sin(\theta)$

For a circular source

$$\vec{r}_s = (\kappa_2 dr - \tilde{f}_1)\vec{u}_r - \frac{d\tilde{f}_0}{d\theta}\vec{u}_\theta \quad ; \quad |r_s|^2 = r_0^2$$

↓

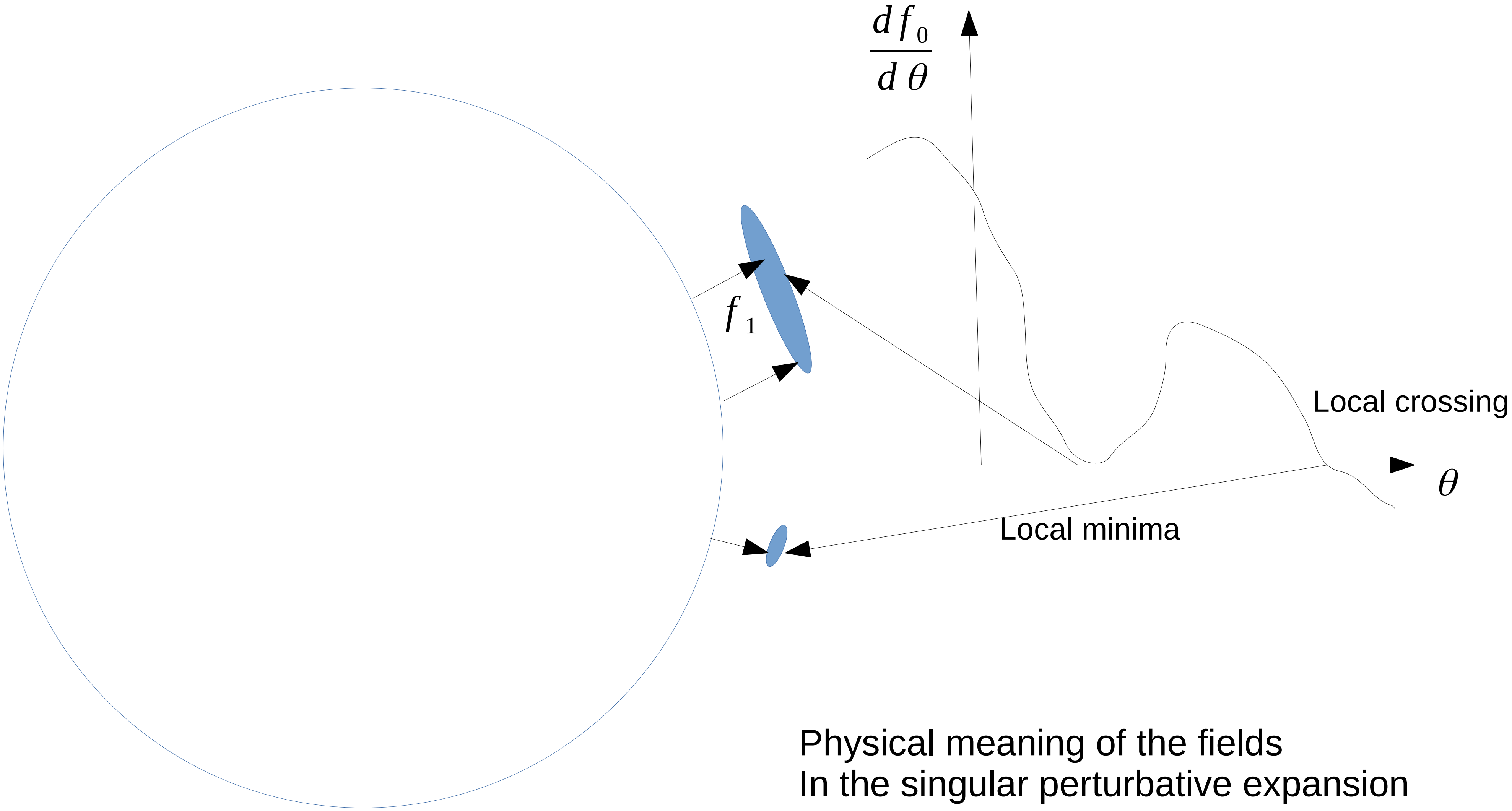
$$\kappa_2 dr = \tilde{f}_1 \pm \sqrt{r_0^2 - \frac{d\tilde{f}_0}{d\theta}^2}$$

The 2 perturbative fields have strong physical meaning

$\tilde{f}_1$ Images positions (deviation from the circle)

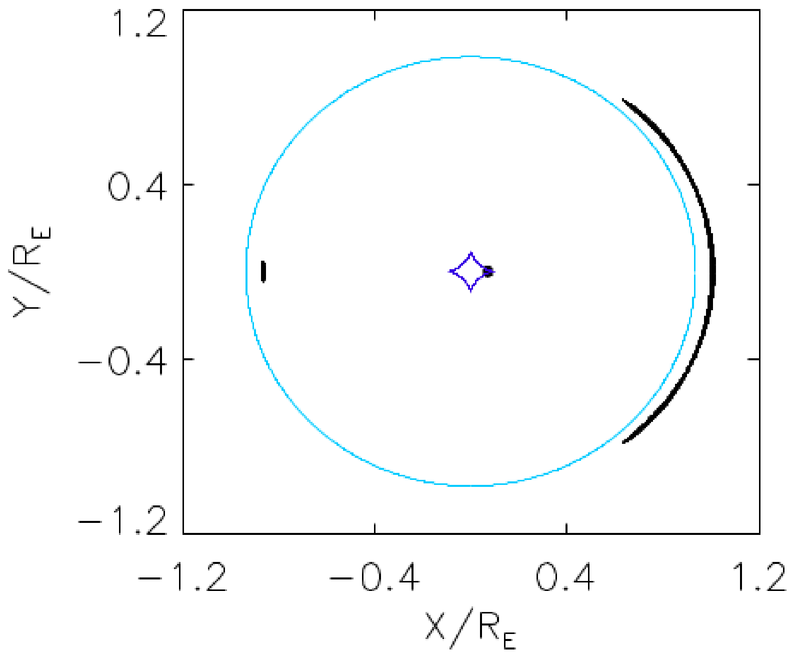
$\frac{d\tilde{f}_0}{d\theta}$ Where the images forms (small values of the field)

Alard (2007)

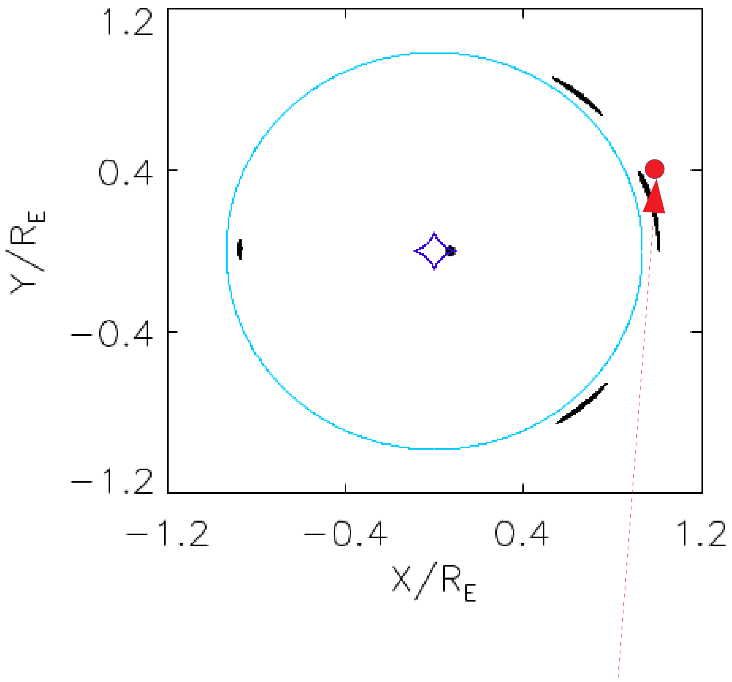


Physical meaning of the fields
In the singular perturbative expansion

Exemple of reconstruction using the singular perturbative method
Presentation of the of the lens systems

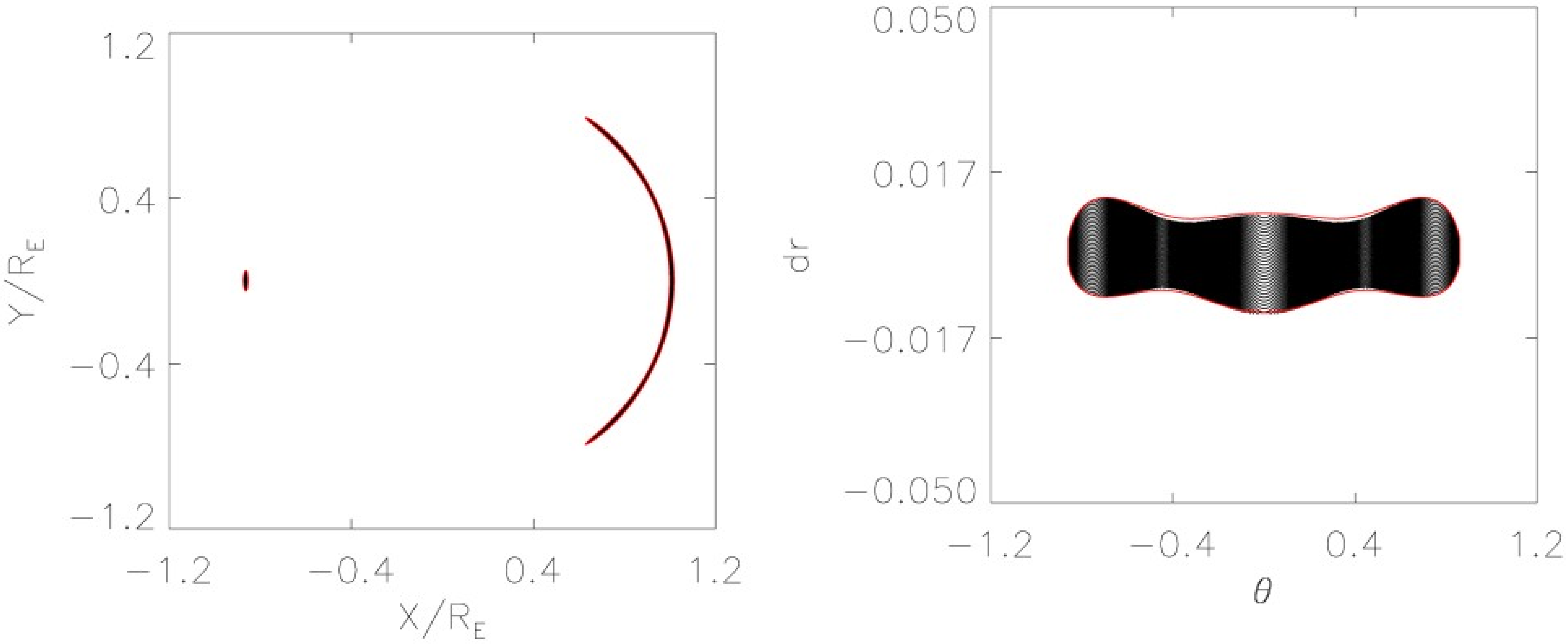


Isothermal lens source in sub-critical regime

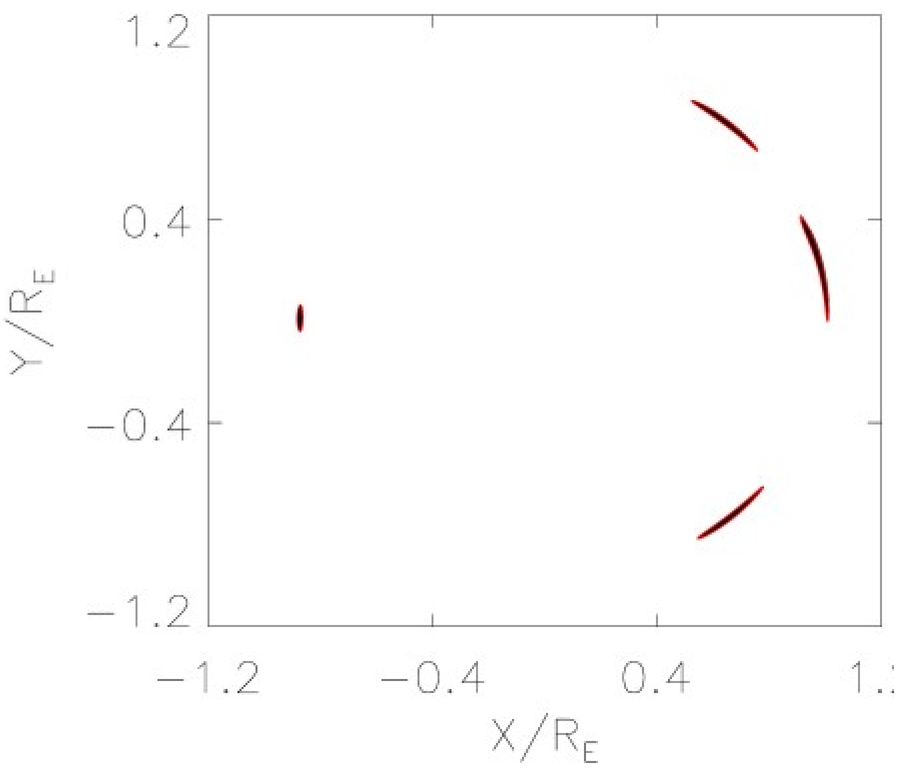


Same lens perturbed by 1% point mass

Reconstruction for the isothermal potential



Same lens perturbed by 1% point mass



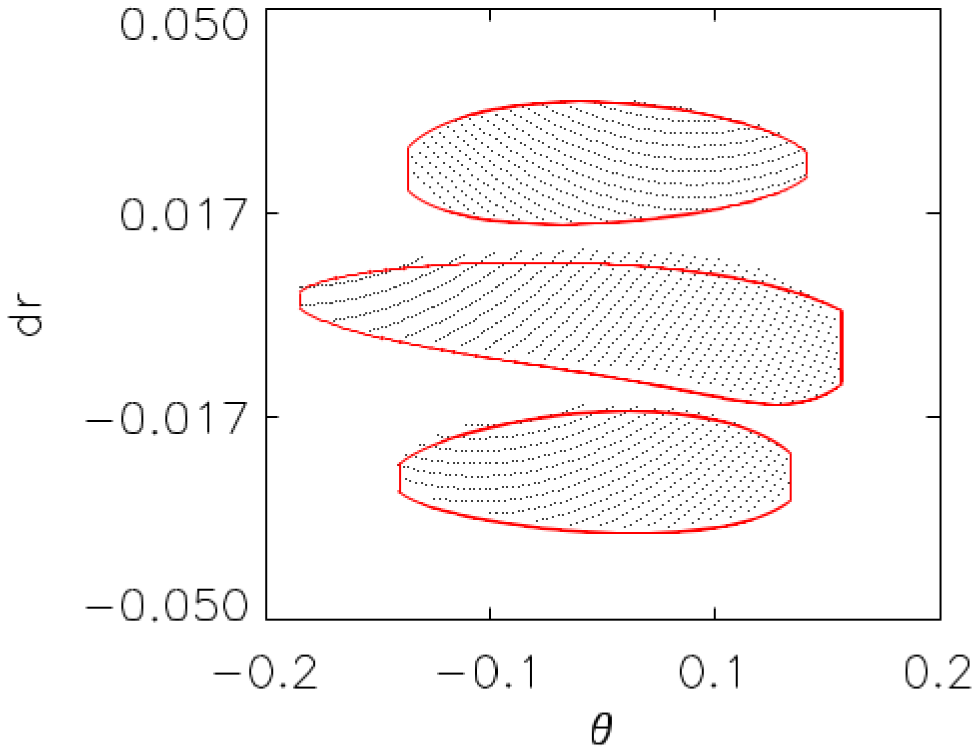
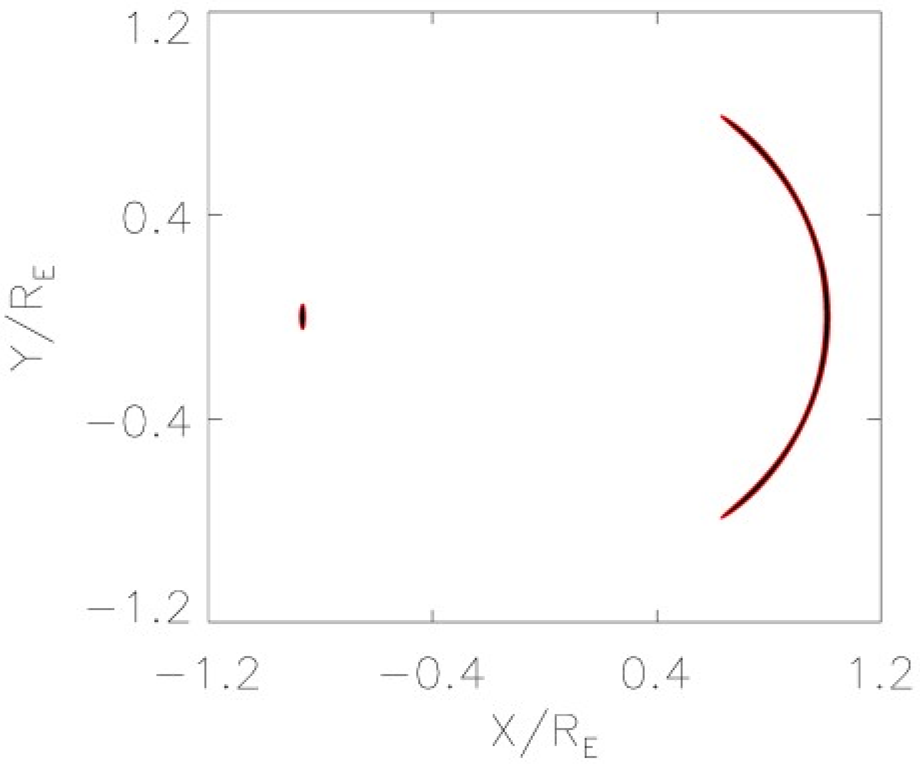


Image formation
Isothermal case

$$\frac{d f_0}{d \theta}$$



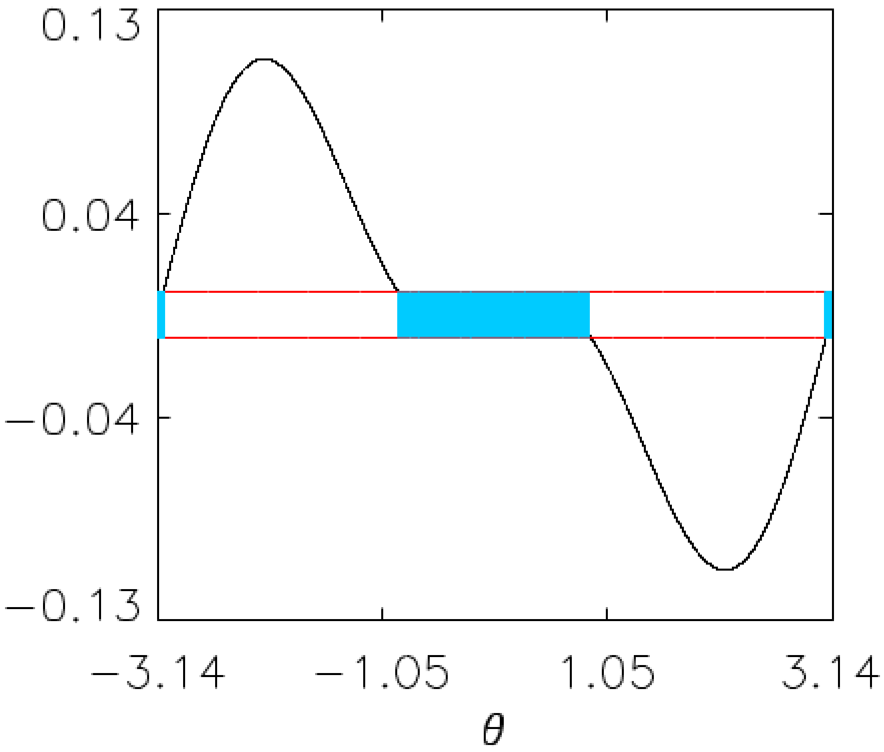
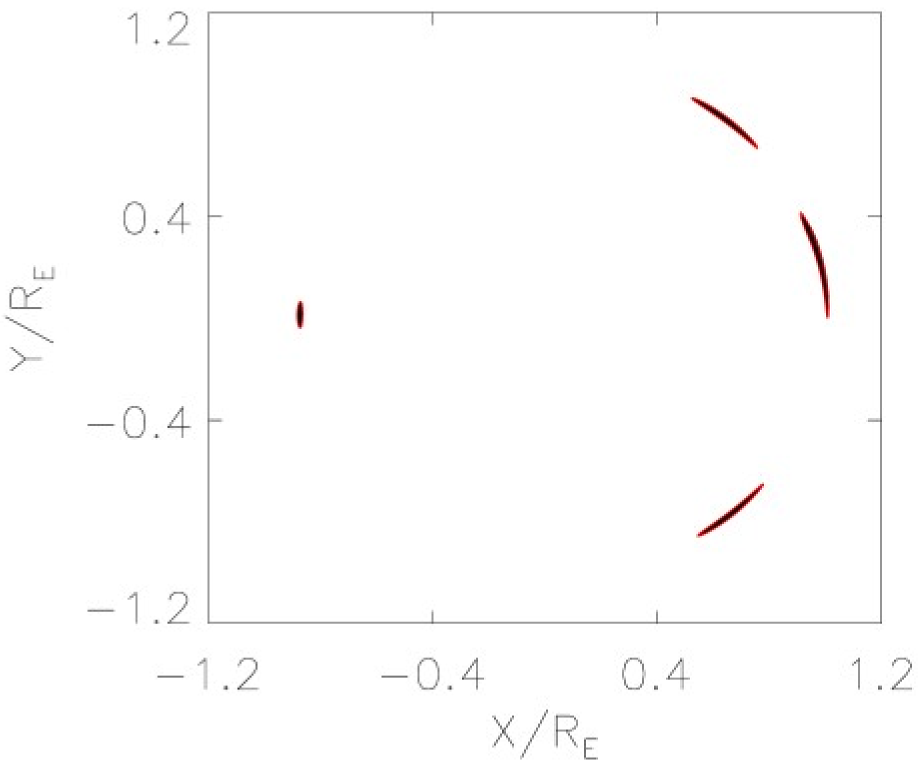
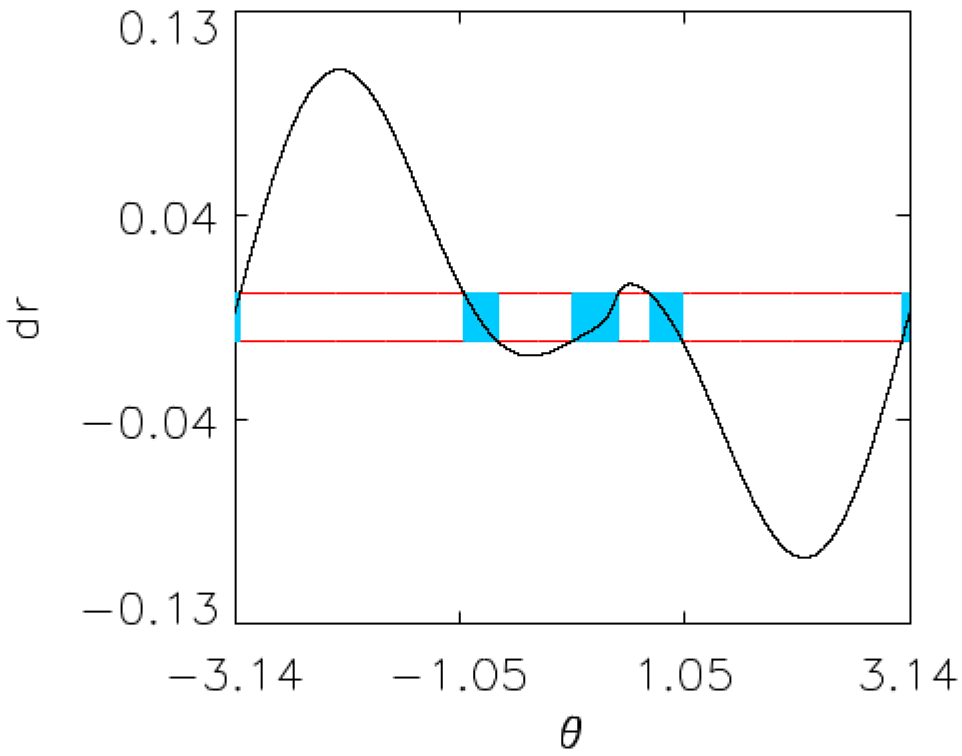


Image formation
Perturbed

$$\frac{d f_0}{d\theta}$$





Equation for caustics

$$\vec{r}_s = (\kappa_2 dr - \tilde{f}_1)\vec{u}_r - \frac{d\tilde{f}_0}{d\theta}\vec{u}_\theta$$

$$J \propto \frac{\partial x_s}{\partial r}\frac{\partial y_s}{\partial \theta} - \frac{\partial x_s}{\partial \theta}\frac{\partial y_s}{\partial r} = 0$$

Critical lines: $$dr = \frac{1}{\kappa_2}\left[f_1 + \frac{d^2 f_0}{d\theta^2}\right]$$

Caustics lines:
$$\begin{cases} x_S = \dfrac{d^2 f_0}{d\theta^2}\cos\theta + \dfrac{d f_0}{d\theta}\sin\theta \\ y_S = \dfrac{d^2 f_0}{d\theta^2}\sin\theta - \dfrac{d f_0}{d\theta}\cos\theta \end{cases}$$

Potential iso-contours

$$\phi(r,\theta)=\phi_0(r)+\epsilon f_0(\theta)+\epsilon f_1(\theta)(r-1)=C$$

Potential iso-contour near unit Einstein circle $r_i=1+\epsilon\, dr_i$

To first order leads to: $dr_i=-f_0$

The Fourier series expansion of the fields
And the multipole expansion:
Inner and outer contribution can be separated

$$\psi=-\left(\sum_n \frac{a_n}{r^n}\cos n\,\theta+\frac{b_n}{r^n}\sin n\,\theta+c_n r^n\cos n\,\theta+d_n r^n\sin n\,\theta\right)$$

$$\begin{cases} a_n = \frac{1}{2\pi n}\int_0^{2\pi}\int_0^{r=1}\rho(u,v)\cos nv\; u^{n+1}\,\mathrm{d}u\,\mathrm{d}v, \\ b_n = \frac{1}{2\pi n}\int_0^{2\pi}\int_0^{r=1}\rho(u,v)\sin nv\; u^{n+1}\,\mathrm{d}u\,\mathrm{d}v, \\ c_n = \frac{1}{2\pi n}\int_0^{2\pi}\int_{r=1}^{\infty}\rho(u,v)\cos nv\; u^{1-n}\,\mathrm{d}u\,\mathrm{d}v, \\ d_n = \frac{1}{2\pi n}\int_0^{2\pi}\int_{r=1}^{\infty}\rho(u,v)\sin nv\; u^{1-n}\,\mathrm{d}u\,\mathrm{d}v. \end{cases}$$

Multipole expansion

$$\begin{cases} f_1 = \left(\frac{\partial\psi}{\partial r}\right)_{(r=1)} = \sum_n n(a_n - c_n)\cos n\theta + n(b_n - d_n)\sin n\theta, \\ \frac{\mathrm{d}f_0}{\mathrm{d}\theta} = \left(\frac{\partial\psi}{\partial\theta}\right)_{(r=1)} = \sum_n -n(b_n + d_n)\cos n\theta + n(a_n + c_n)\sin n\theta. \end{cases}$$

Knowing the perturbative field the multipole expansion
Can be reconstructed

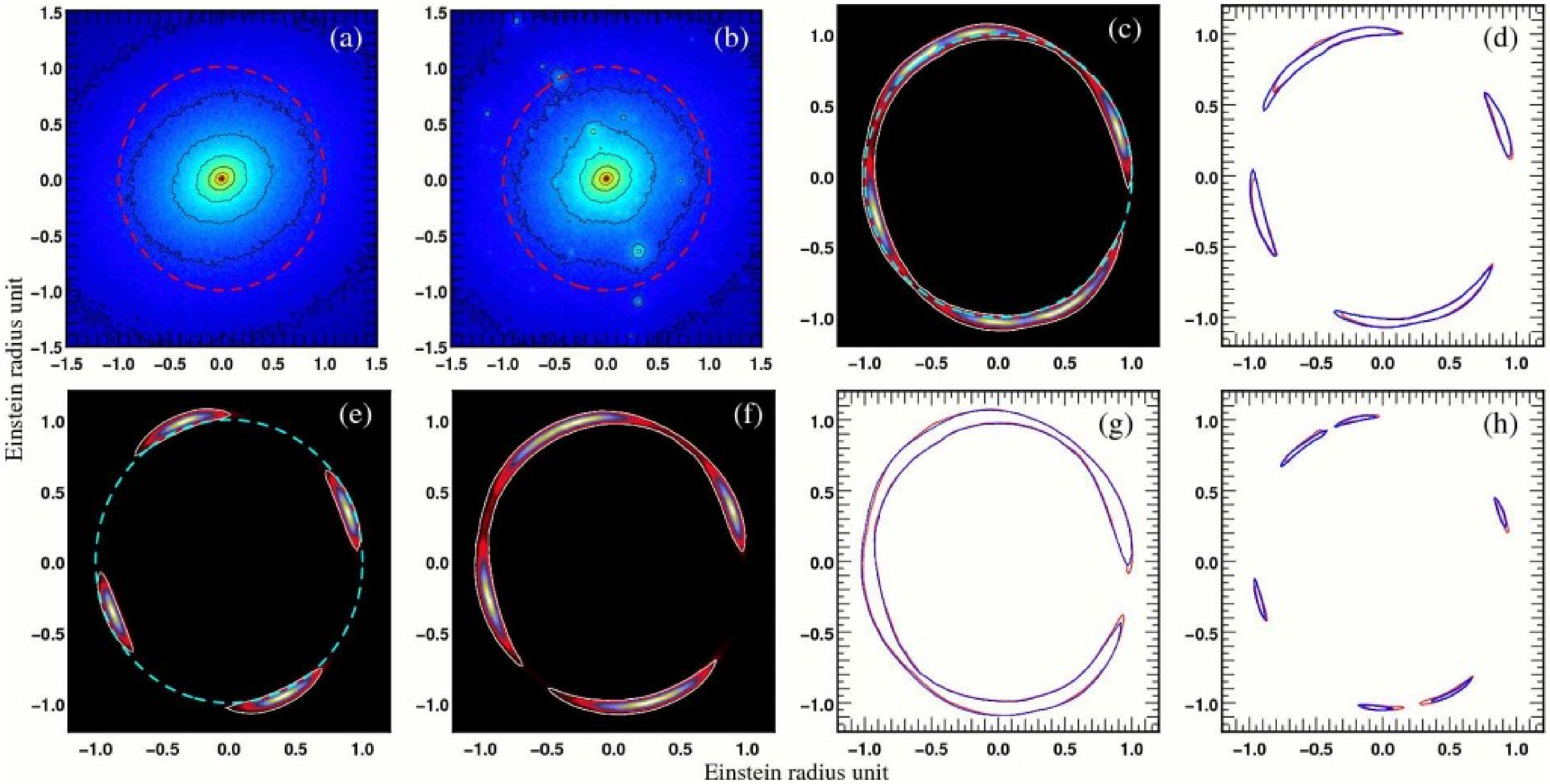
It allows to separate the inner terms a_n, b_n

And the outer terms c_n, d_n

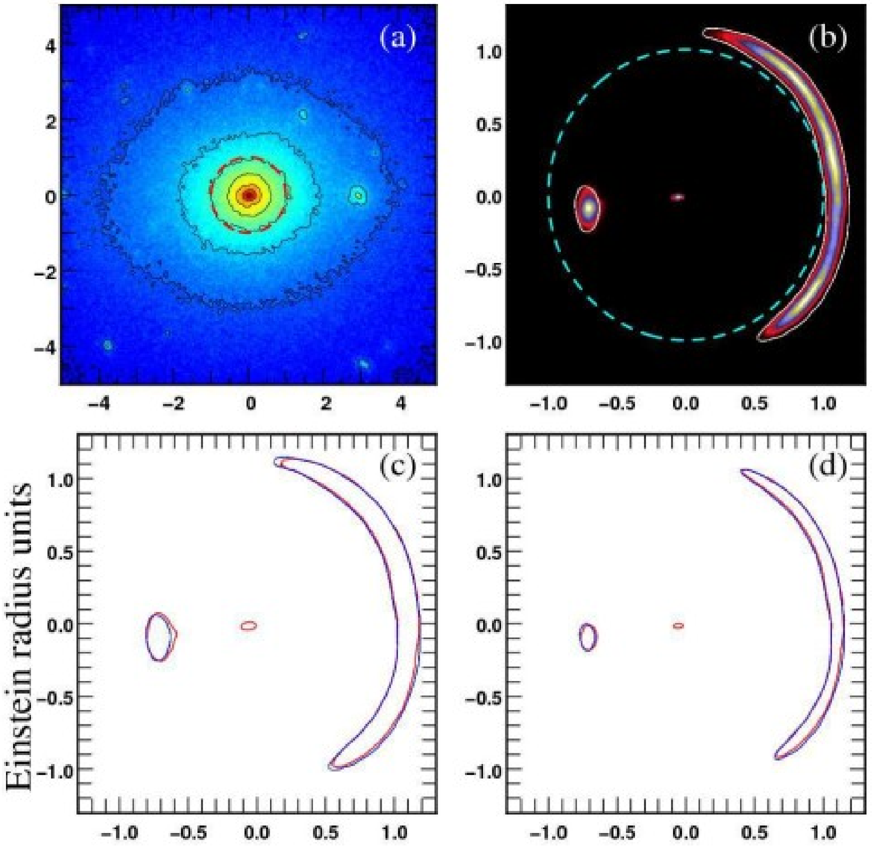
How does the perturbative fields expansion works with real halo’s ?

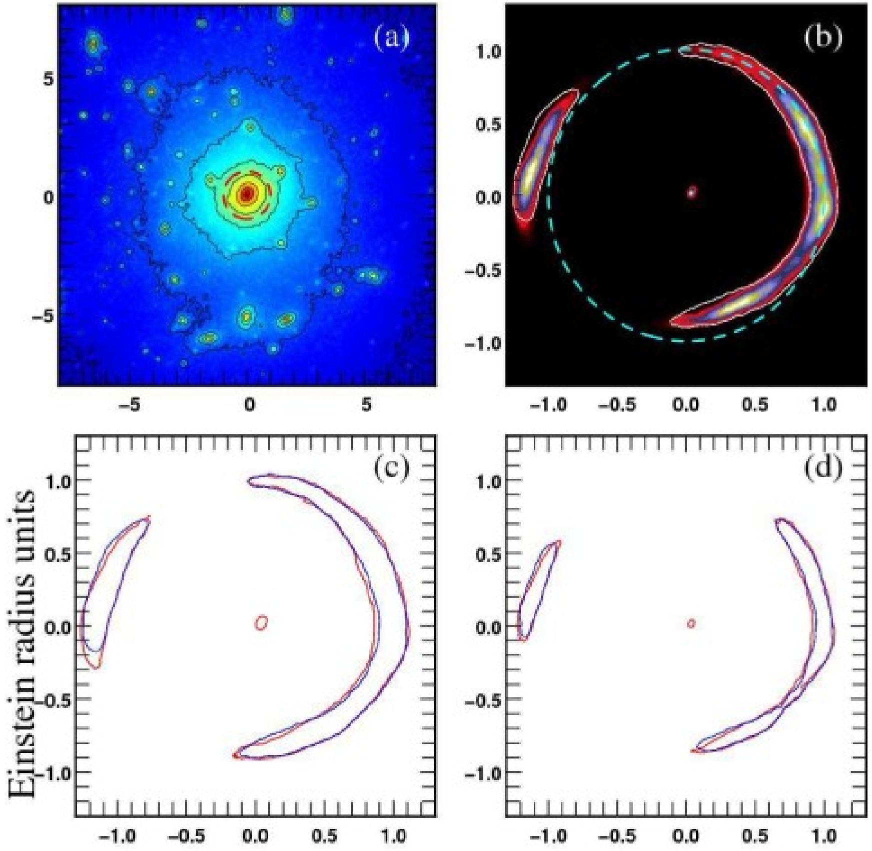
Here we present some comparison between the contours
Reconstructed for the perturbative method and real ray tracing

The perturbative expansion compared to ray tracing in numerical simulations (Peirani etal. 2008)



Some more comparisons





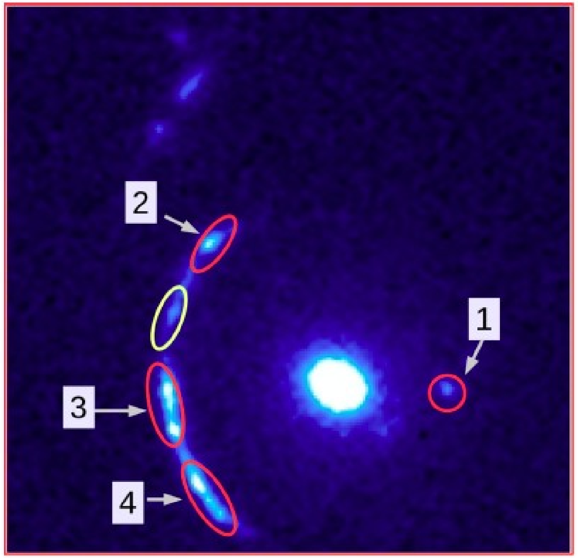
Some example of reconstruction
With the singular perturbative method

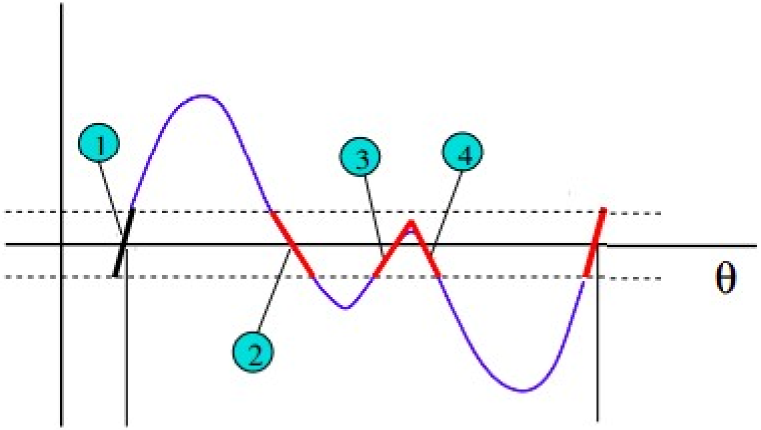
1) single galaxy in perturbed environment

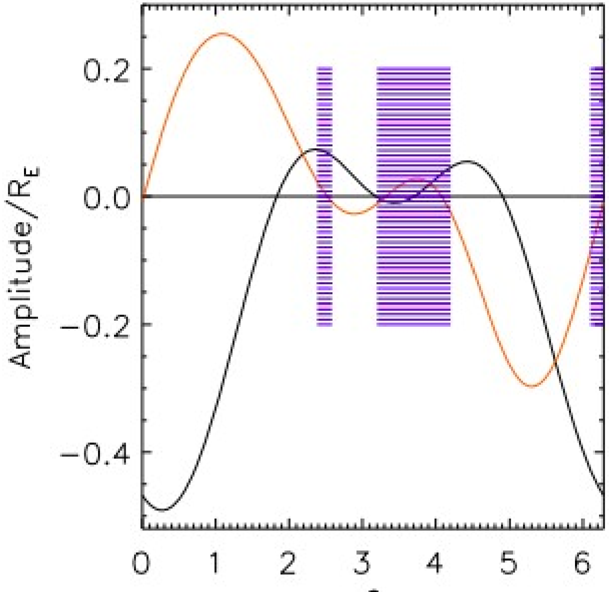
2) small group of galaxies

3) The cosmic horseshoe lens

Alard (2010)

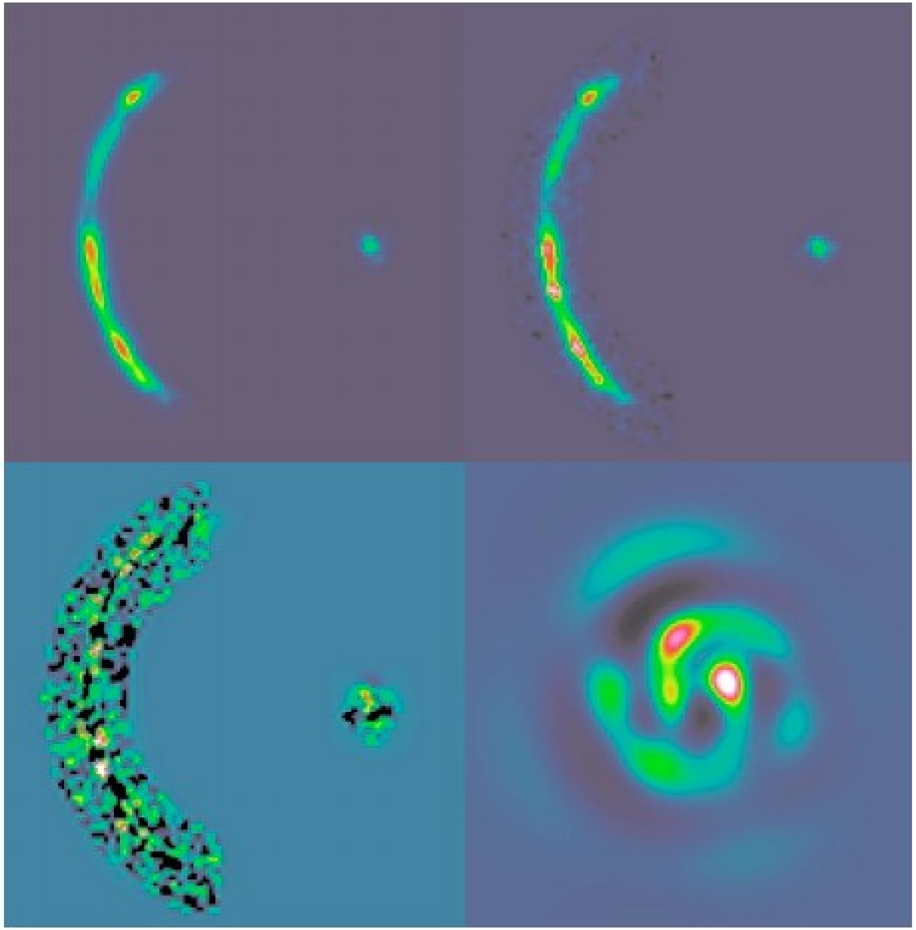


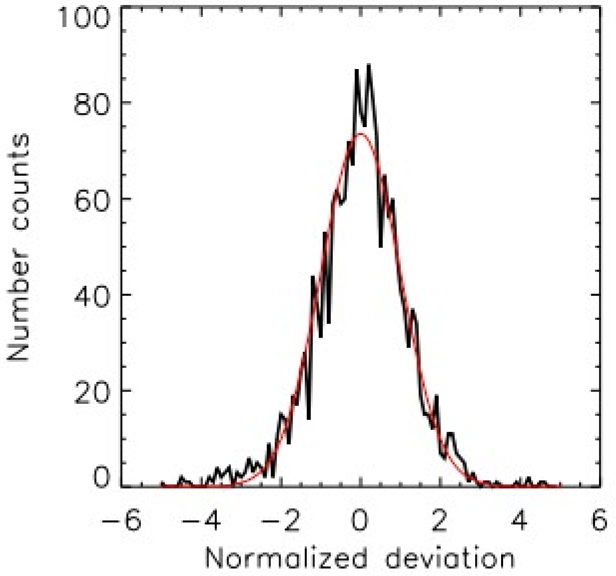




The lens system and the reconstruction
Of the 2 fields

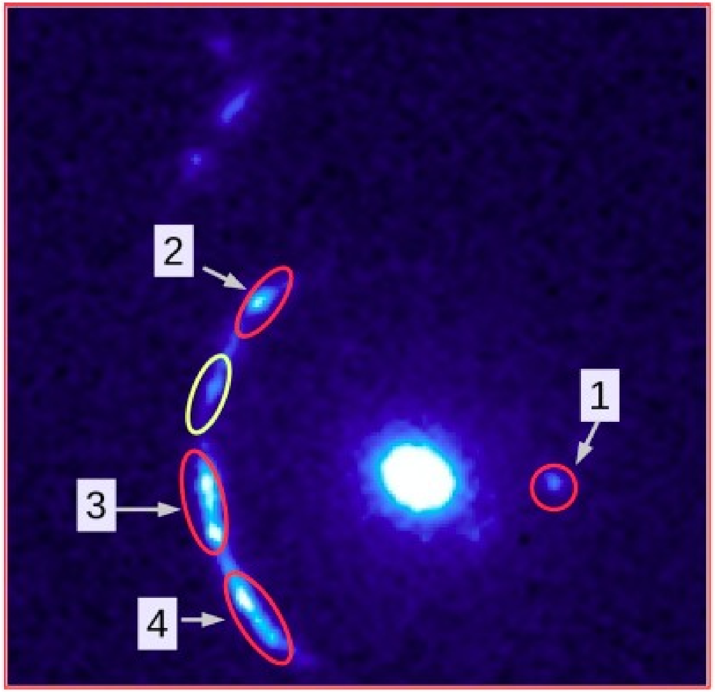
Image and source reconstruction

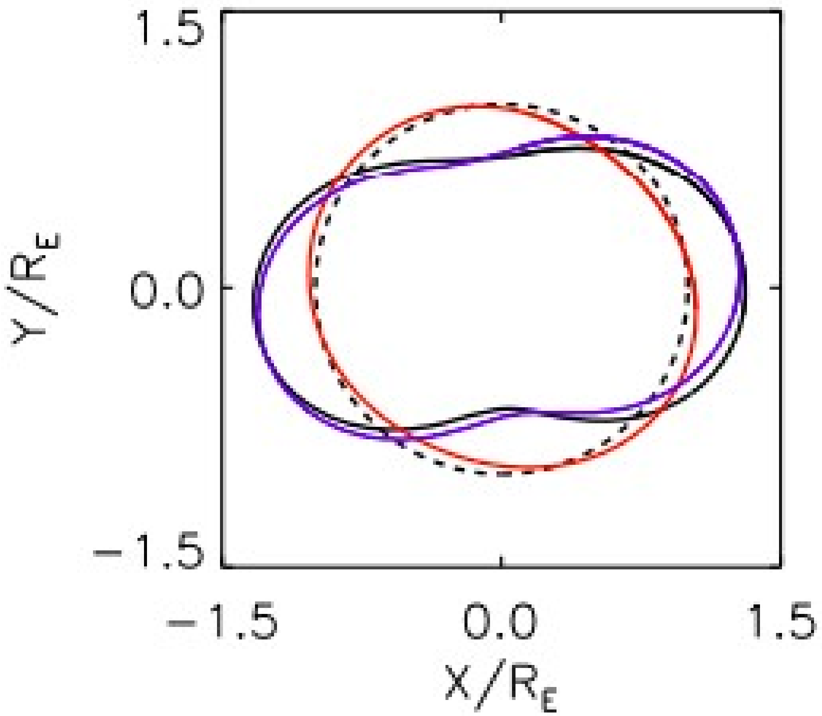




Alard (2010)

The reconstruction of the potential iso-contours



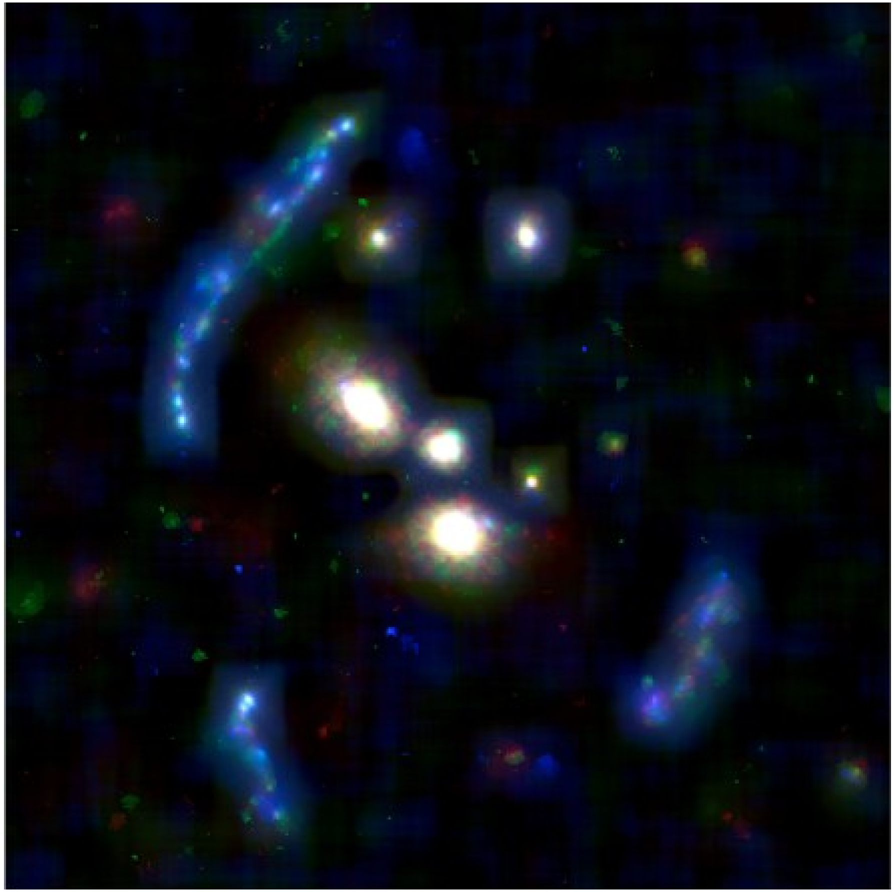


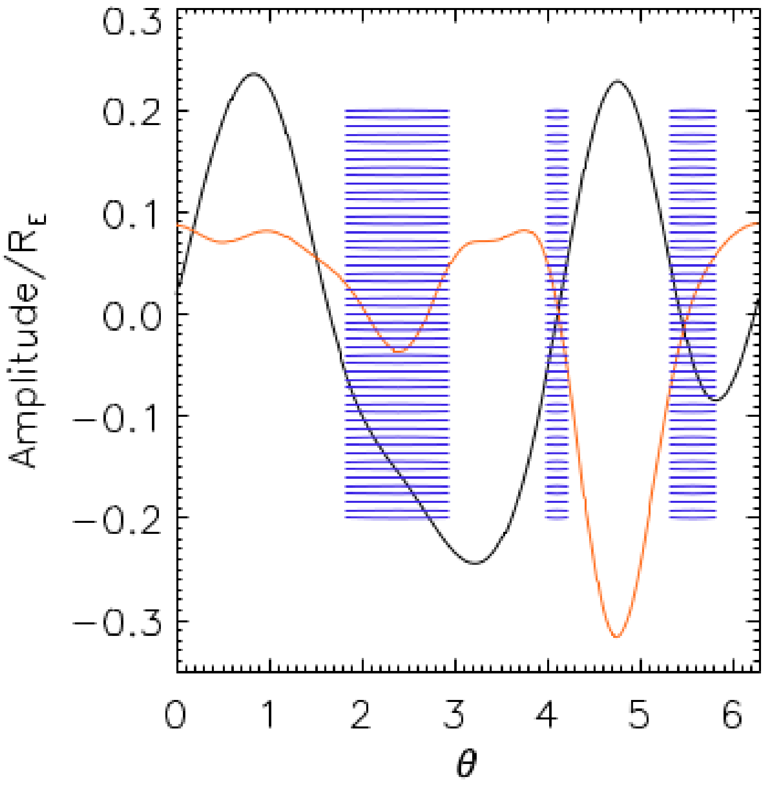
Inner iso-contour

outer iso-contour

Alard (2010)

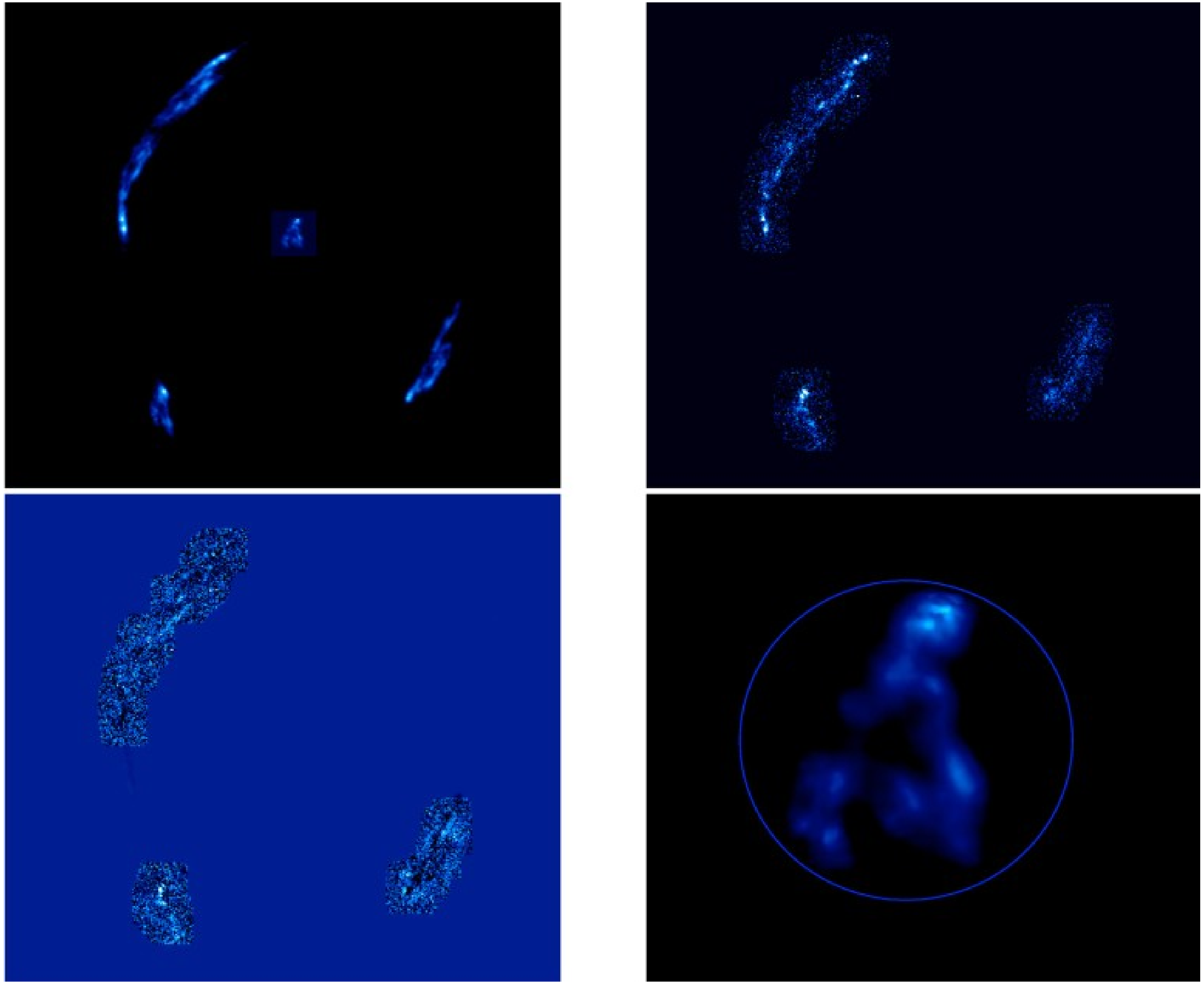
Alard (2009)

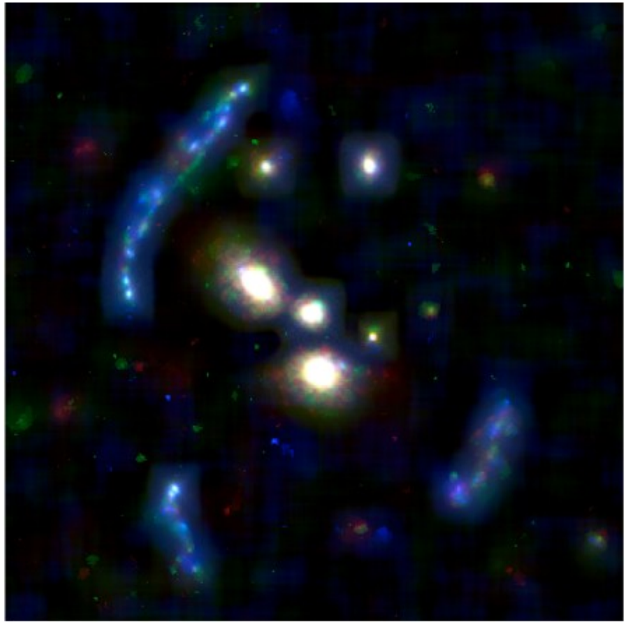




Fields reconstruction for the lens

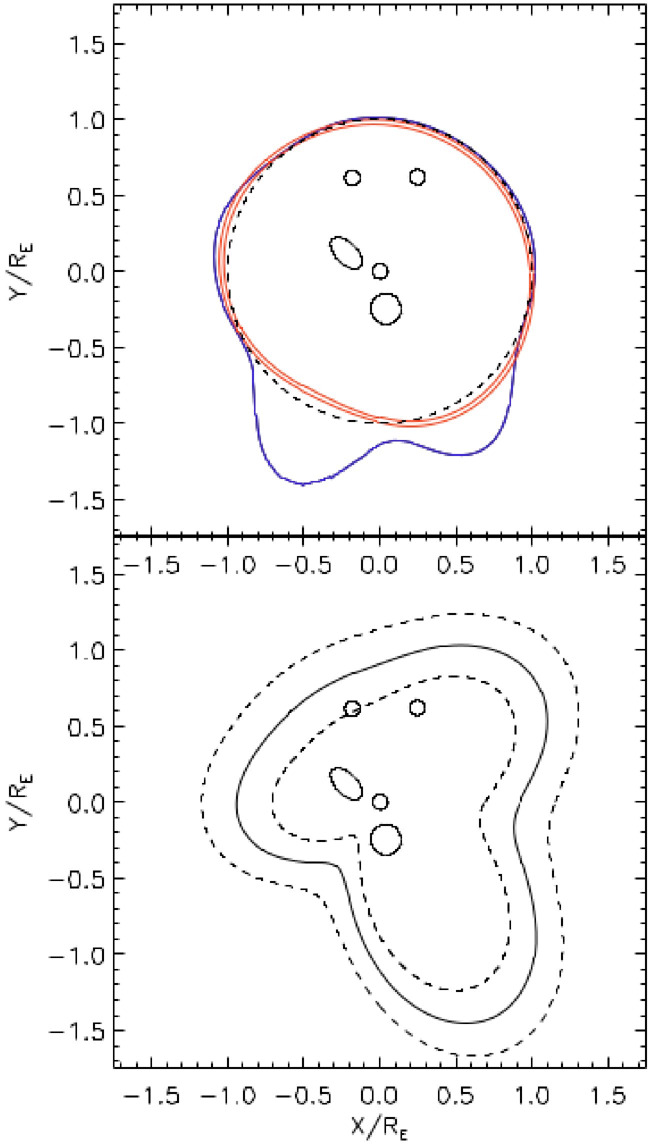
Image and source reconstruction





Potential reconstruction

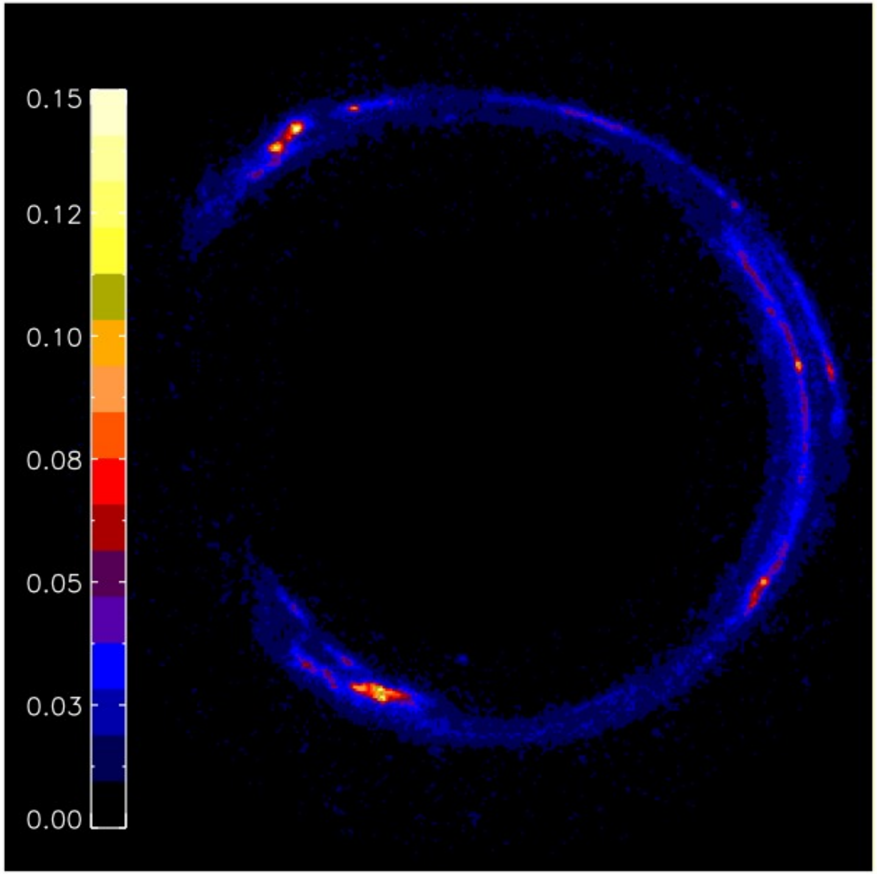
Density reconstruction



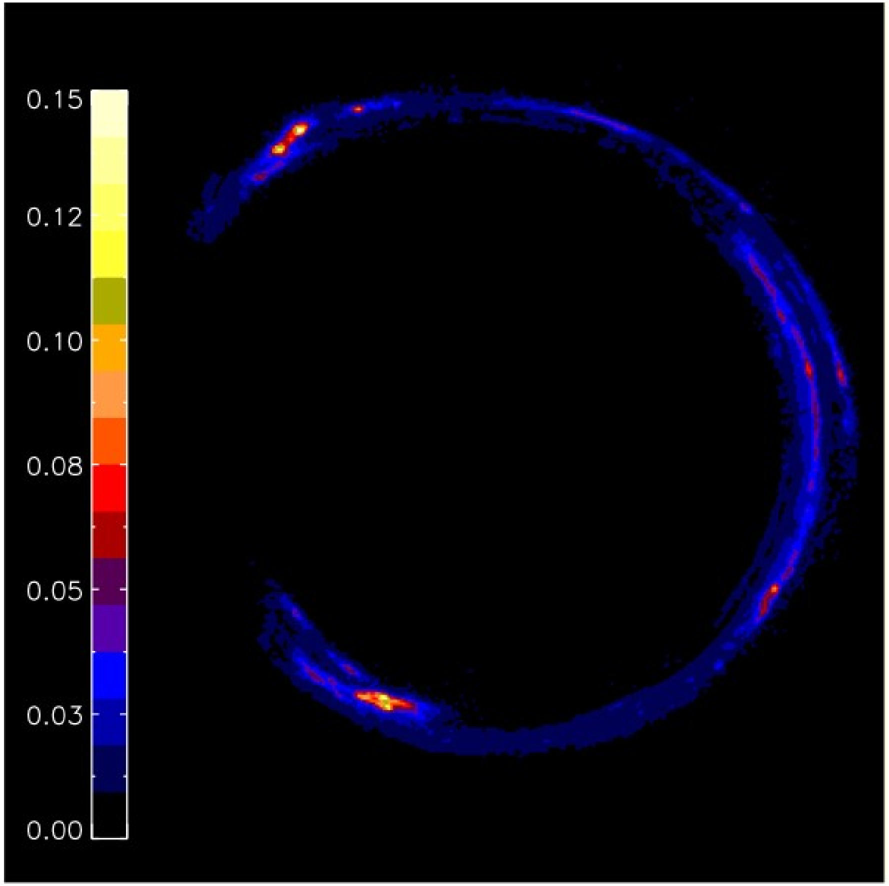
In this small cluster mass does
Not follow light

Alard (2009)

Reconstruction of the cosmic horseshoe

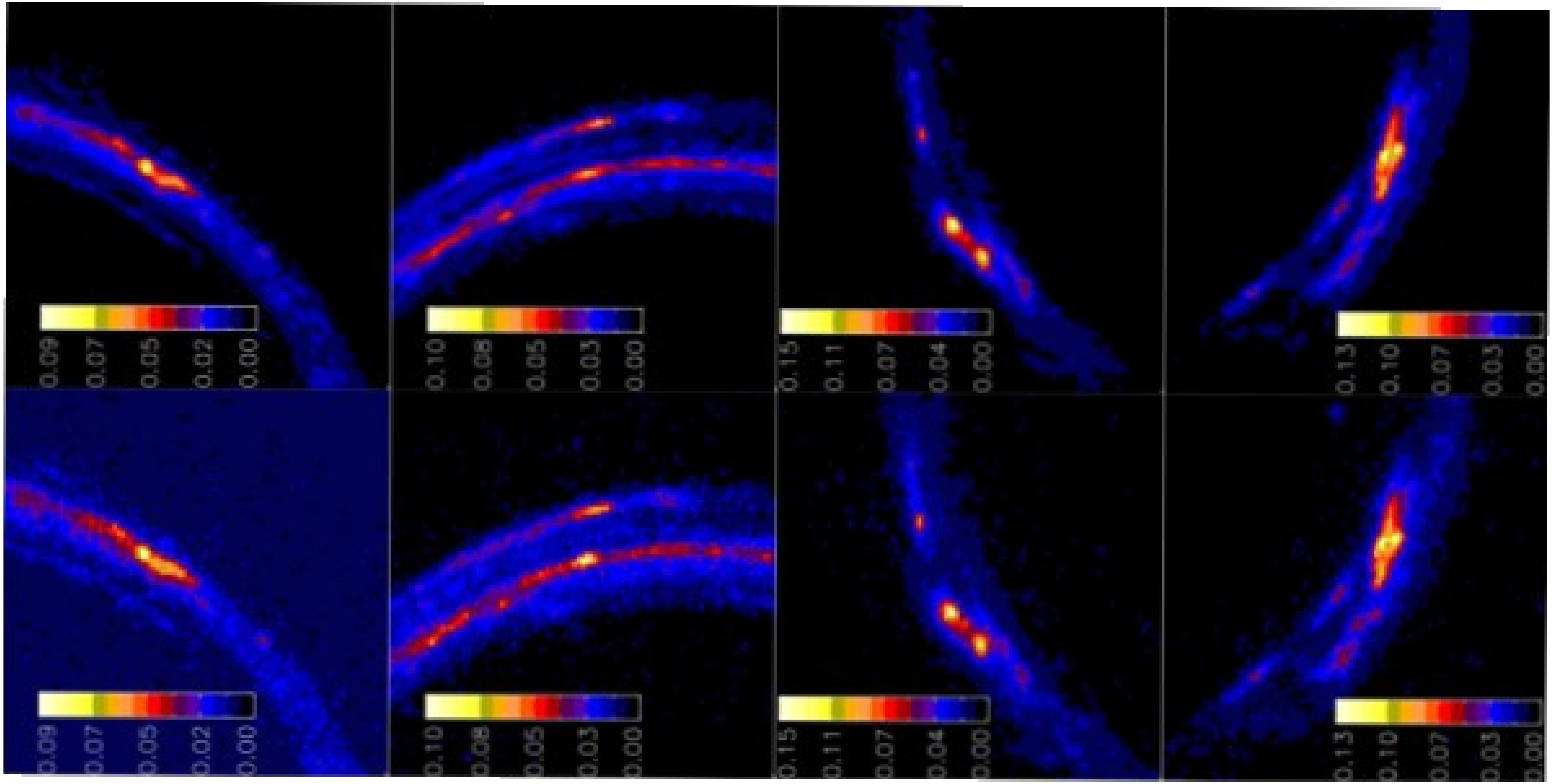


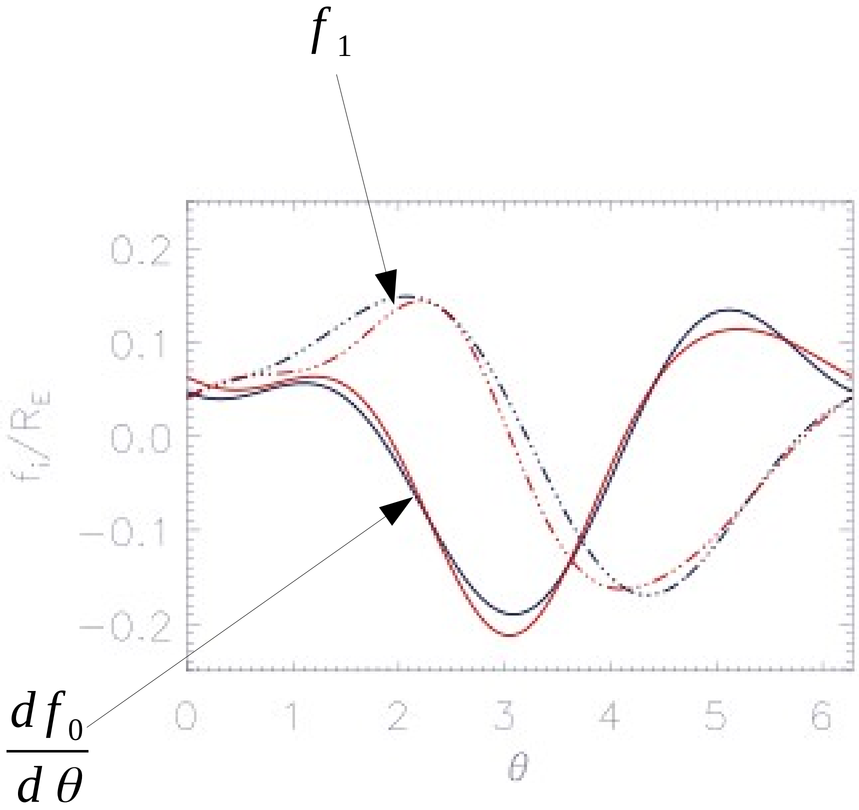
Original (HST data)



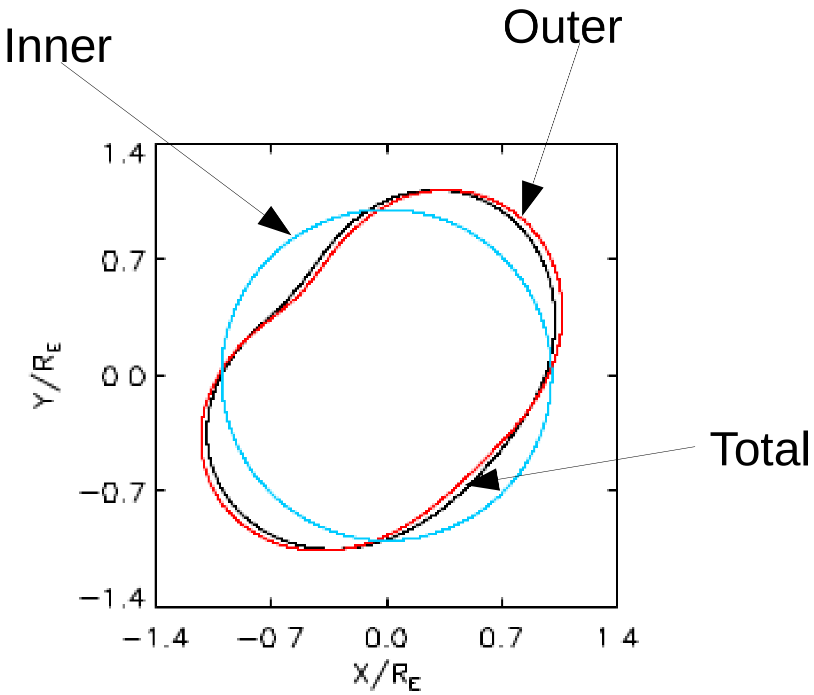
Reconstructed

Comparison of details original/reconstruction

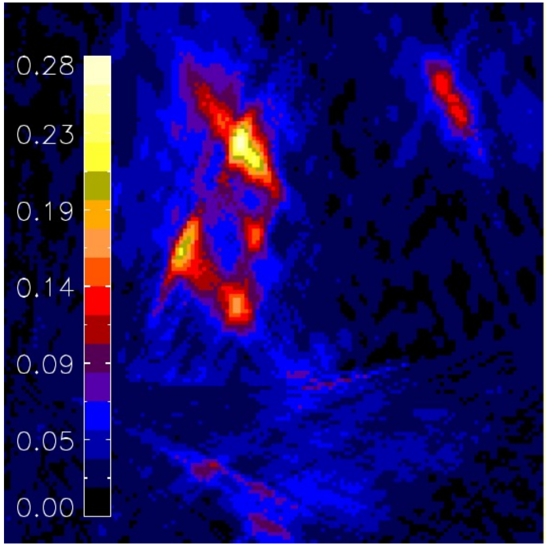




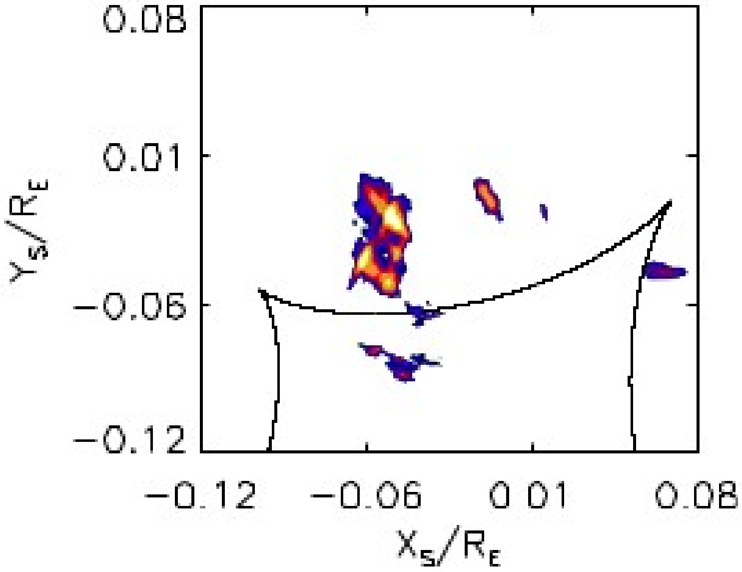
Solution for the fields



Potential iso-contours



Source reconstruction



Source/caustic configuration

Very important assets of the perturbative analysis

Universal approach for all lenses

Universal modeling and parameters

Consequence:

It makes statistical analysis possible

The singular perturbative method
A statistical approach

As an illustration: the statistical signature of substructures

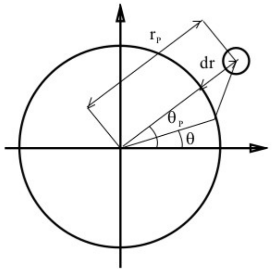
The presence of substructure in the lens near the Einstein ring produce local perturbations

These local perturbations have specific statistical signature in the singular perturbative theory

In particular they stand up as higher order terms in the Fourier expansion of the fields.

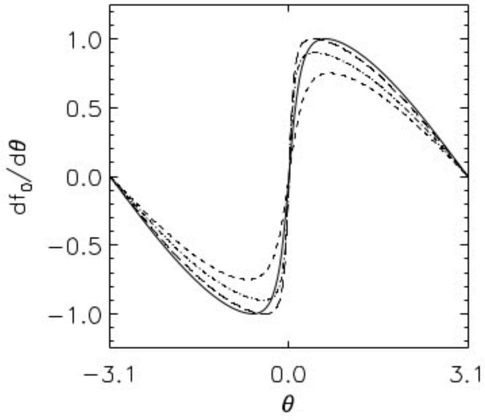
The singular perturbative method
A statistical approach

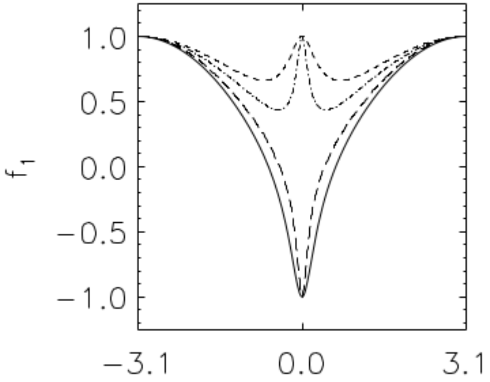
Analytical calculations of the perturbation due to a point mass



$$\begin{cases} f_1 = \dfrac{m_p[1 - r_p \cos(\theta - \theta_p)]}{\sqrt{1 - 2r_p \cos(\theta - \theta_p) + r_p^2}}, \\ \dfrac{\mathrm{d}f_0}{\mathrm{d}\theta} = \dfrac{m_p[r_p \sin(\theta - \theta_p)]}{\sqrt{1 - 2r_p \cos(\theta - \theta_p) + r_p^2}}. \end{cases}$$

Perturbation fields due to a substructure

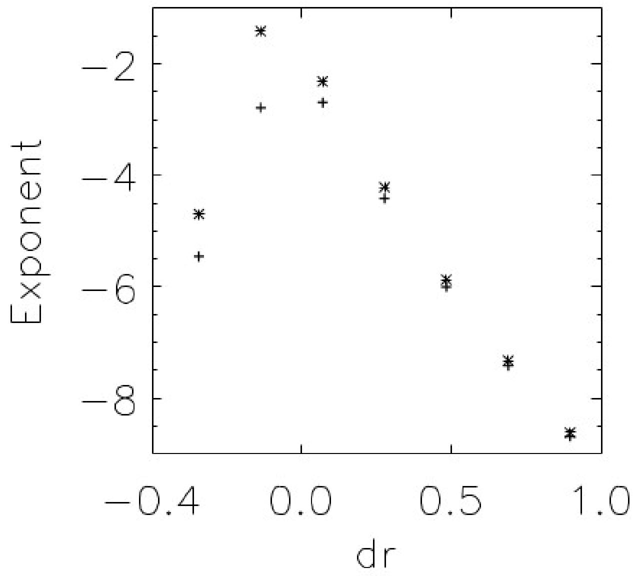




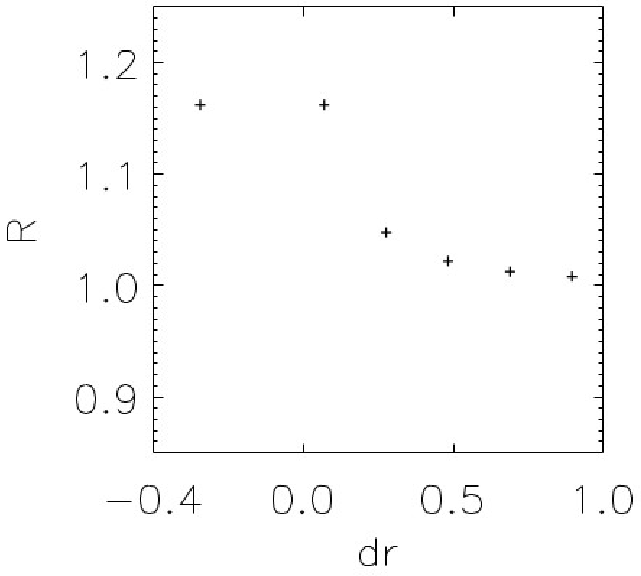
The effect on the fields as a function of the distance of the substructure

Alard (2008)

The statistical signature of substructure
Alard (2008)



Power-law modelling of the Fourier expansion
Coefficients as function of the substructure position



Mean ratio of the 2 fields Fourier coefficients

The substructure signature is a long tail at higher order in the Fourier expansion
With distinct nature between the 2 fields.

The Fourier series expansion of the fields
Is rich in statistical information

$$\begin{cases} \frac{\mathrm{d}f_0}{\mathrm{d}\theta} = \sum_n \alpha_{0,n}\cos(n\theta) + \beta_{0,n}\sin(n\theta), \\ f_1 = \sum_n \alpha_{1,n}\cos(n\theta) + \beta_{1,n}\sin(n\theta), \\ P_i(n) = \alpha_{i,n}^2 + \beta_{i,n}^2, \quad i = 0, 1. \end{cases}$$

$$\psi = -\left(\sum_n \frac{a_n}{r^n}\cos n\theta + \frac{b_n}{r^n}\sin n\theta + c_n r^n \cos n\theta + d_n r^n \sin n\theta\right)$$

$$\begin{cases} a_n = \frac{1}{2\pi n}\int_0^{2\pi}\int_0^{r=1} \rho(u,v)\cos nv\; u^{n+1}\,\mathrm{d}u\,\mathrm{d}v, \\ b_n = \frac{1}{2\pi n}\int_0^{2\pi}\int_0^{r=1} \rho(u,v)\sin nv\; u^{n+1}\,\mathrm{d}u\,\mathrm{d}v, \\ c_n = \frac{1}{2\pi n}\int_0^{2\pi}\int_{r=1}^{\infty} \rho(u,v)\cos nv\; u^{1-n}\,\mathrm{d}u\,\mathrm{d}v, \\ d_n = \frac{1}{2\pi n}\int_0^{2\pi}\int_{r=1}^{\infty} \rho(u,v)\sin nv\; u^{1-n}\,\mathrm{d}u\,\mathrm{d}v. \end{cases}$$

Multipole expansion

$$\begin{cases} f_1 = \left(\frac{\partial\psi}{\partial r}\right)_{(r=1)} = \sum_n n(a_n - c_n)\cos n\theta + n(b_n - d_n)\sin n\theta, \\ \frac{\mathrm{d}f_0}{\mathrm{d}\theta} = \left(\frac{\partial\psi}{\partial\theta}\right)_{(r=1)} = \sum_n -n(b_n + d_n)\cos n\theta + n(a_n + c_n)\sin n\theta. \end{cases}$$

The Fourier expansion of the fields contains all the details
Of the multipole expansion on the Einstein circle

The statistical analysis of a large number of lenses (EUCLID)

Reconstruction of the 2 fields for many lenses

Fourier decomposition of the fields

Full statistic of the multipole expansion

Signature from complex halo geometry

Substructures

Light-mass offsets

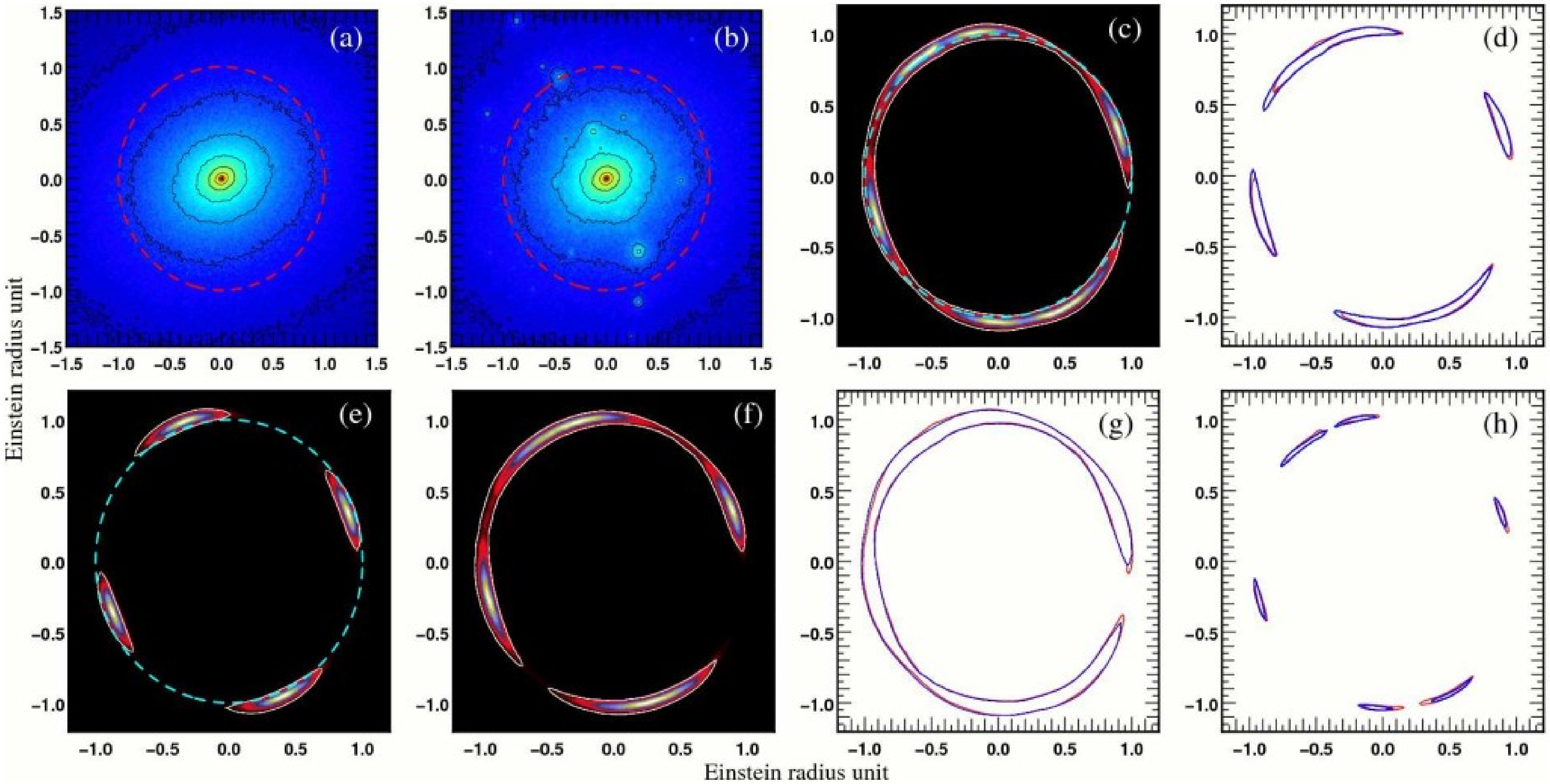
Mass without light counterparts

New results (rings, caustics, filaments, holes,...)

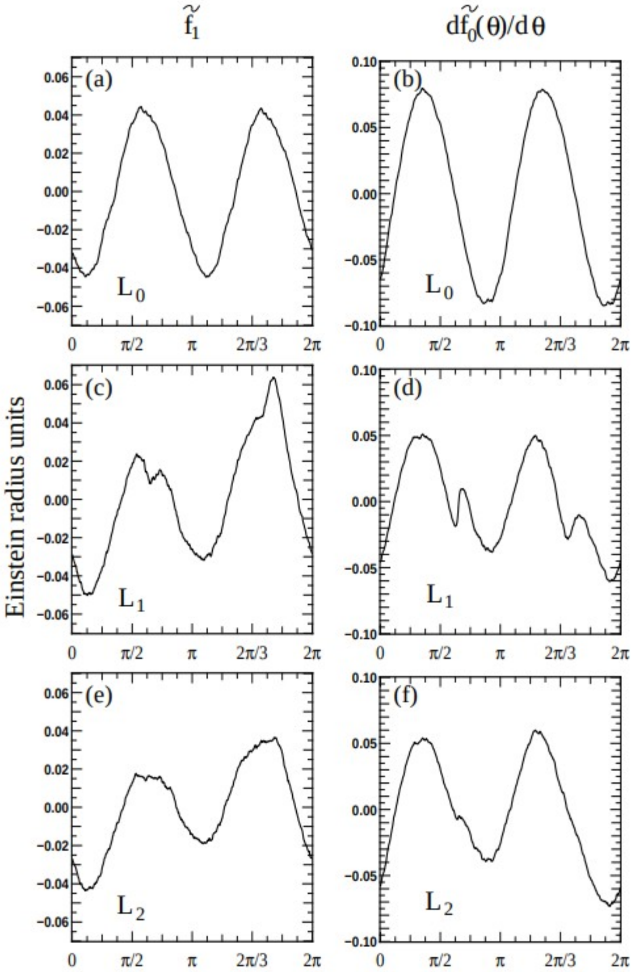
Some practical example of the statistical information
Available in the perturbative fields expansion

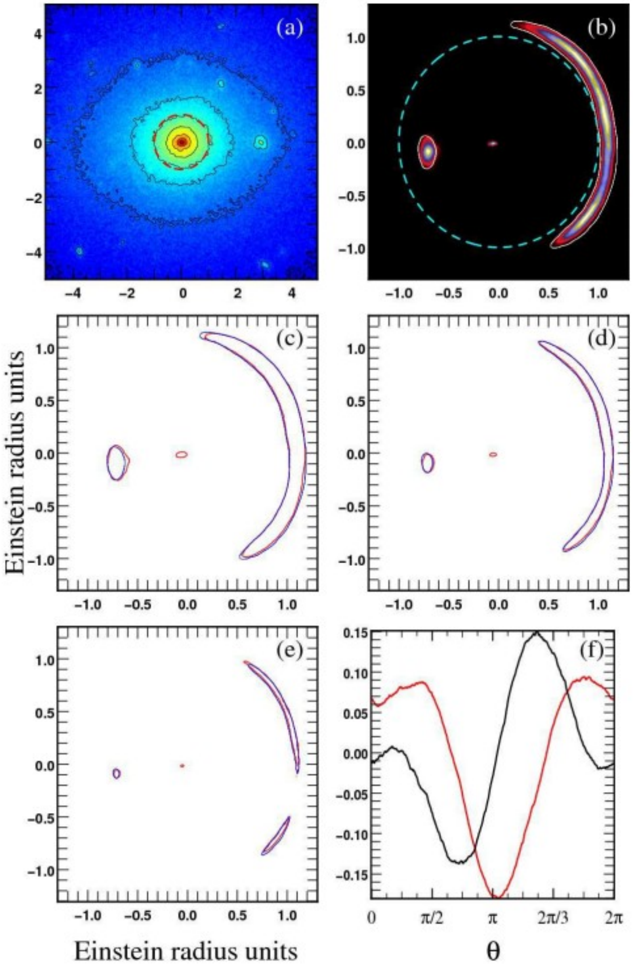
3 halo's from Peirani etal. (2008)
analyzed in detail

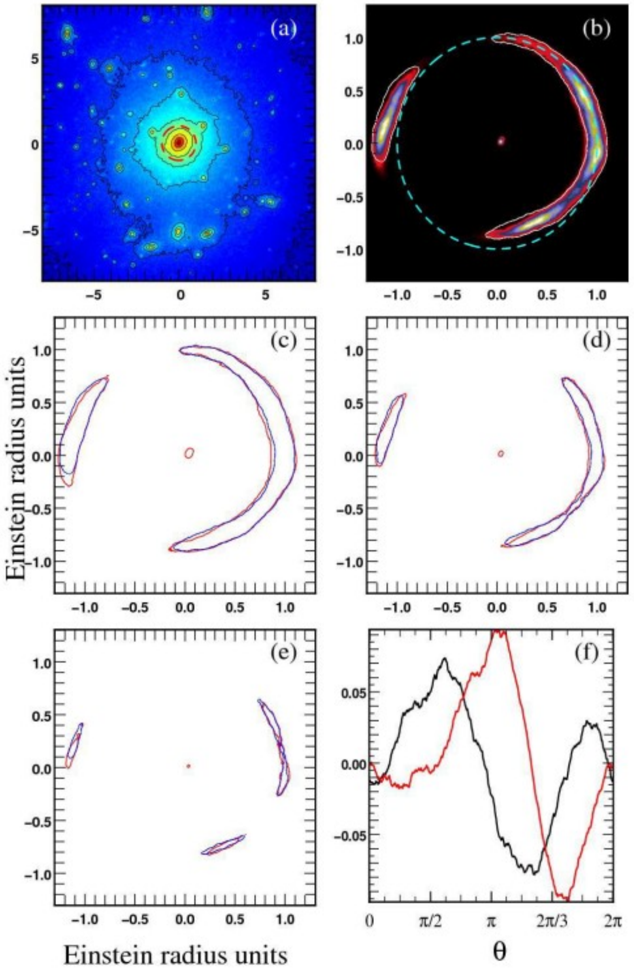
The perturbative expansion compared to ray tracing in numerical simulations (Peirani etal. 2008)



The perturbative expansion compared to ray tracing in numerical simulations: the shape of the perturbative fields







Lens	1	2	3	4	5	6	7
L_0	0.07	4.21	0.02	0.20	0.04	0.07	0.03
L_1	1.62	3.80	0.42	0.18	0.29	0.20	0.33
L_2	1.38	2.86	0.18	0.20	0.10	0.11	0.11

Table 2. Power spectra of $\widetilde{f}_1(\theta)$ shown in the first column of Figure 3 .

Lens	1	2	3	4	5	6	7
L_0	0.08	8.17	0.04	0.39	0.02	0.08	0.03
L_1	1.14	4.12	0.32	1.50	0.28	0.59	0.24
L_2	1.54	5.36	0.20	0.74	0.07	0.14	0.18

Table 3. Power spectra of $\mathrm{d}\widetilde{f}_0(\theta)/\mathrm{d}\theta$ shown in the second column of Figure 3 .

The power spectrum of
the perturbative fields expansion

For various halo’s

When a large set of lens is available It will be possible to build a statistical analysis of the perturbative fields

The statistics of higher order terms will be a direct measure of DM substructure

The whole geometry of the halo‘s will be accessible

Allowing to probe the DM/matter offsets, difference in distribution

Presence of DM in unexpected places….

EUCLID is soon to be launched