Gravitational wave tails and binary star systems

1993 Class. Quantum Grav. 10 2699

(http://iopscience.iop.org/0264-9381/10/12/026)

Class. Quantum Grav. 10 (1993) 2699–2721. Printed in the UK

Gravitational wave tails and binary star systems

Luc Blanchet† and Gerhard Schäfer‡

† Département d'Astrophysique Relativiste et de Cosmologie, Centre National de la Recherche Scientifique (UPR 176), Observatoire de Paris, 92195 Meudon Cedex, France

‡ Max-Planck-Gesellschaft, Arbeitsgruppe Gravitationstheorie an der Universität Jena, Max-Wien-Platz 1, 0-6900 Jena, Federal Republic of Germany

Received 7 May 1993

Abstract. Gravitational wave tails are produced by back-scattering of the outgoing gravitational radiation (emitted by an isolated system) off the curved spacetime associated with the total mass of the system. This paper investigates the spectral (or Fourier) decomposition of gravitational wave tails at large distances from the system, at the 1.5 post-Newtonian order in the wave field. It is shown that the effects of wave tails are (i) to increase the amplitude of the Fourier components of the (linear) waves by a factor linearly depending on the frequency, and (ii) to add to the phase of the waves a supplementary phase depending on the frequency as $\omega \ln \omega$. The latter frequency-dependent phase introduces a new effect which should be observable in any radiation containing more than one frequency, for instance in the radiation emitted by a binary star system orbiting a Keplerian ellipse with non-zero eccentricity, or in the radiation emitted by an inspiralling (compact) binary star system. We propose in this paper to include the tail-induced effects (i) and (ii) in the matched filters of the future data analysis of inspiralling compact binary signals in laser interferometer gravity-wave detectors (at least in future, very sensitive, such detectors). In this way, the filters will be highly correlated with the actual signal, and in particular will remain, as the frequency of the signal increases, in *accurate phase* with it. The contribution of the wave tail in the total gravitational energy emitted by a binary system is also calculated, and a numerical application to the binary pulsar PSR 1913+16 is presented. We find that the tail-induced relative correction in the orbital $\dot{P}_{\rm Th}$ of the pulsar is equal to $+1.65 \times 10^{-7}$ (too small to be observed).

PACS numbers: 0430, 0480

1. Introduction

With the advent, by the end of the century, of a new generation of detectors (laser interferometer detectors like VIRGO [1] and LIGO [2]), we shall probably detect for the first time the gravitational radiation generated by rapidly evolving systems such as coalescing binary systems of neutron stars or black holes. The scientific benefits of this discovery are likely to be extremely important (see [3] for a review). On the other hand, the regular acquisition of timing data from pulsars in relativistic binary systems [4–9] leads to very precise observations of dynamical effects, in reaction to the emission of gravitational radiation, which are, for the moment, in excellent agreement with the predictions of general relativity [10–14] (see [15] for a review).

These two totally different means of observations—direct observation of gravitational radiation and direct observation of its back-reaction on the source—will in fact collect closely related astrophysical information. Indeed, the radiation field at large distances from the source (where the detector is located) in principle contains all the information concerning the back-reaction on the source. For instance, the rate of decrease of the orbital period of the binary pulsar can be (heuristically) deduced by equating the energy flux carried out at infinity by the radiation field with the work done in the source by the reaction forces. Similarly, we expect that the net change of linear momentum of the source in reaction to the emission of radiation (recoil) will follow from the conservation of total linear momentum of the matter and of the gravitational field [16–18].

In this paper, we shall be interested in a particular effect that is present both in the radiation field at large distances from the source, and in the radiation reaction forces within the source. This effect, known as the *tail effect*, is physically due to the back-scattering of the outgoing radiation off the (approximate) Schwarzschild spacetime of the source. The tail effect is essentially a non-linear effect because the wave tails are produced by non-linear interaction of the time-varying multipole moments associated with the radiation field, and of the static mass monopole moment of the source. (Note that the tail effect also arises for electromagnetic radiation propagating on the Schwarzschild background.) A well known consequence of the existence of wave tails is the non-locality (in time) of the gravitational field, namely the fact that the field, at some time t and distance r from the source, depends, in a 'hereditary' way, on the state of the source at all instants in the past that are anterior or equal to the simply retarded time $t - r/c$. Thus, on average, the gravitational field seems to propagate both on and *within* the local light cone, i.e. seems to propagate with all velocities smaller than or equal to the velocity of light.

The tail effect has been extensively studied in the literature. Let us quote some mathematical investigations of the existence and construction of solutions [75–80]; the general investigations, by means of retarded post-Minkowskian approximation schemes, of the non-linear structure of the gravitational radiation field [19–26]; the investigations of the propagation of integer spin fields on the Schwarzschild background [27–34, 81]; more specific studies of the formation of gravitational wave tails and backscattering effects [35–39]; and more specific studies of electromagnetic wave tails [40–43]. Recently, the contribution of the tail effect in the far-zone wave field (at large distances from the source), has been derived within a post-Newtonian wave-generation formalism valid for slowly moving, but possibly strongly self-gravitating, systems of bodies [44–46].

In the far-zone wave field (neglecting terms that die out at large distances from the source like the inverse square of the distance), the non-local tail contribution appears at a relatively low post-Newtonian order of magnitude given by ε^3, where ε is the ratio of a typical internal velocity in the source and of the velocity of light. Using the post-Newtonian terminology, this corresponds to the three-halves post-Newtonian (1.5 PN) order beyond the 'Newtonian' order, by which we refer the '0 PN' order where the wave form can be computed using the Newtonian dynamics of the system. The wave tail at the 1.5 PN order arises from the interaction of the varying mass quadrupole moment of the source with its static mass monopole moment, or ADM mass. (Note that the latter non-local tail contribution must be distinguished from another non-local contribution that formally arises at the order ε^5 or 2.5 PN in the wave form, and that is due to the re-radiation of the stress–energy tensor of the field itself [47–50].) It has been shown [51, 52] that, associated with the tail term in the far zone field, there is also a tail term in the equations of motion of the source, which modifies the usual Burke and Thorne [53–55] radiation reaction potential by a non-local correction of relative order ε^3 or 1.5 PN—this corresponds to the fourth post-Newtonian order in the inner metric of the system. (Note that the same effect also exists for electromagnetic radiation reaction in an exterior gravitational field [56–57].) Furthermore, the tail term in the radiation reaction potential is perfectly consistent, as concerns energy conservation, with the tail term in the far-zone wave field [38, 46].

The aim of the present paper is first to investigate in general terms the spectral (or Fourier) decomposition of the wave tail in the far zone, and then to apply the result of this investigation to the emission of waves by binary star systems. We have in view both the future detection by VIRGO and LIGO [1, 2] of inspiralling compact binary systems, and the current timing of relativistic binary pulsars [4–9]. (However, the former application seems to be much more important.)

The gravitational wave tail depends on the dynamics of the source at all instants in the past (before the simply retarded time $t - r/c$), but, as shown in [51], it is in fact only slightly sensitive to the detailed dynamics of the source at very early times, as soon as a weak assumption of 'moderation' of the wave emission in the past is satisfied. Subject to this assumption, we determine in this paper that the effect of the non-local wave tail, at the 1.5 PN level, is (i) to modify each Fourier component of the (local-in-time) wave by a multiplicative factor in the amplitude, which is linear in the frequency and, perhaps more importantly, (ii) to add to the phase of the wave a supplementary *frequency-dependent* phase, depending on the frequency as $\omega \ln \omega$.

The latter frequency-dependent phase introduces a new differential effect because different Fourier components of the wave, corresponding to different frequencies, will undergo different phase shifts (which are not simply proportional to the differences of frequencies). This can be more clearly viewed if we think in terms of wave packets. Different wave packets, centred around different frequencies (but belonging to the 'same' wave), will propagate with the same group velocity, but will have the position of their maxima of amplitude (in space) slightly spread out along the line of sight, with relative positions fixed by the differences of the *logarithms* of their central frequencies. This effect is in principle observable in all radiation which contains more than one frequency in Fourier space. For instance, this is the case for the radiation emitted by a binary star system orbiting a Keplerian ellipse with non-zero eccentricity (like PSR 1913+16). This is also the case for the radiation emitted by a binary star system on a circular orbit whose radius and orbital frequency are changing with time, e.g. because of radiation reaction effects. Such inspiralling binary star systems, whose dynamics is driven by radiation reaction, constitute in their late stages of evolution the most promising known source of gravitational radiation. Their detection by VIRGO and LIGO will rely on data analysis techniques such as matched filtering (see e.g. [3, 58, 59]).

In this paper, we propose that the frequency-dependent corrections, brought about by the tail effect, in the waves emitted by an inspiralling binary system, be included in the matched filters of the future data analysis of inspiralling signals, at least for future very sensitive generations of detectors. Indeed, in order to have a good detection, it is important to correlate the output of the detector with a filter which is as precise as possible, and in particular which remains in *accurate phase* with the signal (see e.g. [64] for a discussion). A filter that is matched merely on a Newtonian (or even first post-Newtonian) theoretical expectation of the signal, will acquire with respect to the actual signal a phase difference growing, as the frequency of the signal increases, like $\omega \ln \omega$. To what extent exactly the use of an accurate filter incorporating the tail effect, i.e. a filter matched on (4.12)–(4.13) below, can improve the data analysis of laser interferometer detectors will be examined in future work.

Finally, the contribution of the tail effect in the total gravitational luminosity emitted by a binary pulsar orbiting a Keplerian ellipse (with arbitrary eccentricity) is computed. The same contribution also appears (by the argument based on energy conservation) in the rate of decrease of the orbital period of the pulsar as due to the emission of gravitational radiation. We find that the result is enhanced by a dimensionless function of the eccentricity which is somewhat analogous to the enhancement function of Peters and Mathews [10]. In the case of the binary pulsar PSR 1913+16, the numerical value of the tail-induced relative contribution in $\dot{P}_{\mathrm{Th}}$ is equal to $+1.65 \times 10^{-7}$. This is too small to be measured, and is also negligible as compared to many other effects and uncertainties (see Damour and Taylor [7]). However, the computation may become significant for other, not yet discovered, binary pulsars. It also completes a work on the first-order corrections in $\dot{P}_{\rm Th}$ [60], and it permits the recovery, in the circular orbit case, of a result obtained by previous authors working with a different method [61–64].

The plan of this paper is as follows. In the first section, we review the 1.5 post-Newtonian wave-generation formalism of [44–46]. In the second section, we Fourier analyse the wave field and we obtain the tail-induced corrections in the wave amplitude and in the wave phase. An alternate derivation is presented in an appendix. In the third section, we apply these results to the computation of the field generated by an inspiralling binary star system (we neglect in the applications all post-Newtonian corrections which do not belong to the tail). The validity of the computation in the case of an inspiralling (decaying) orbit is proved in an appendix. Finally, in the fourth section, we calculate the influence of the wave tail in the gravitational energy emitted by a binary system, and we make a numerical application to the binary pulsar PSR 1913+16.

2. Three-halves post-Newtonian gravitational wave generation

We consider in this paper an isolated system, which is the source of gravitational radiation, and which is at once *slowly moving* and *weakly stressed*. This means that we have a post-Newtonian dimensionless parameter, say

$$\varepsilon = \sup\left\{\left|\frac{T^{0i}}{T^{00}}\right|, \left|\frac{T^{ij}}{T^{00}}\right|^{1/2}\right\} \tag{2.1}$$

which is required to be small compared with 1. In (2.1) we denote by $T^{\mu\nu}(\boldsymbol{x},t)$ the components of the stress–energy tensor of the gravitating source in some coordinate system $\boldsymbol{x}, t$ covering the source. The small parameter ε is equal to the ratio between a typical internal velocity in the source and the velocity of light, or equivalently to the ratio between the radius a of the source and a typical reduced wavelength $\lambda/2\pi$ of the emitted radiation. In terms of the (angular) frequency $\omega = 2\pi c/\lambda$ of the radiation, this reads as

$$\frac{a\omega}{c} \approx \varepsilon \ll 1\,. \tag{2.2}$$

Note that for a bounded self-gravitating system we have typically $\omega^2 a^3 \approx GM$, where M is the total mass of the system, and thus (2.2) implies

$$\frac{GM\omega}{c^3} \approx \varepsilon^3 \ll 1\,. \tag{2.3}$$

We shall *not* require that the system be everywhere weakly self-gravitating. For instance we could have a system made of several strongly self-gravitating bodies like neutron stars orbiting each other with typical orbital frequencies satisfying (2.3).

The far-zone gravitational field generated by the system, at large distances from the system, has been computed in a recent sequence of papers [44–46] with a relative precision equal to ε^3 (equation (2.3)). This precision corresponds to what can be called the three-halves post-Newtonian approximation level (in short the 1.5 PN level) beyond the dominant 'Newtonian' level at which the Newtonian dynamics of the source is sufficient to compute the wave form. We refer to [44] for discussion of the link, at the 1 PN order, of the present formalism and of the earlier formalism of Epstein and Wagoner [65], and of Thorne [23]. Let us denote by

$X^\mu = (cT, \boldsymbol{X})$ a *far-zone* coordinate system covering the regions at large distances from the source, and such that the metric admits in these coordinates an expansion in simple inverse powers of the distance $R = |\boldsymbol{X}|$ (without logarithms of R). We assume that this coordinate system is transverse and trace-free (TT) and denote by

$$\mathcal{P}_{abij}(\boldsymbol{N}) = (\delta_{ai} - N_a N_i)(\delta_{bj} - N_b N_j) - \tfrac{1}{2}(\delta_{ab} - N_a N_b)(\delta_{ij} - N_i N_j) \quad (2.4)$$

the TT projection operator onto the plane orthogonal to the radial direction $\boldsymbol{N} = \boldsymbol{X}/R$ from the source (the indices $a, b, i, j, \ldots$ range from 1 to 3). The asymptotic wave field in the TT coordinates, $h_{ab}^{\mathrm{TT}} = (g_{ab} - \delta_{ab})^{\mathrm{TT}}$, valid at first order in the inverse of the distance and at 1.5 PN relative order (i.e. accurate to within ε^3), then reads as [46]

$$h_{ab}^{\mathrm{TT}}(T, \boldsymbol{X}) = \frac{2G}{Rc^4}\mathcal{P}_{abij}(\boldsymbol{N})\left\{K_{ij}^{(2)}(U, \boldsymbol{N}) + \frac{2GM}{c^3} \times \int_{-\infty}^{U} \mathrm{d}V \left[\ln\left(\frac{U-V}{2b}\right) + \frac{11}{12}\right] I_{ij}^{(4)}(V) + \mathrm{O}(\varepsilon^4)\right\} + \mathrm{O}\left(\frac{1}{R^2}\right). \quad (2.5)$$

In this expression, we denote by $U = T - R/c$ the retarded time of the far-zone coordinates, and by $K^{(n)}(U)$ the time-derivative $\mathrm{d}^n K(U)/\mathrm{d}U^n$. The first term in (2.5) involves Newtonian and post-Newtonian terms of order 0 PN, 0.5 PN, 1 PN and 1.5 PN. The tensor $K_{ij}(U, \boldsymbol{N})$ is given by the irreducible multipole decomposition

$$\begin{aligned} K_{ij}(U, \boldsymbol{N}) = I_{ij}(U) &+ \frac{1}{c}\left[\frac{1}{3}N_k I_{ijk}^{(1)}(U) + \frac{4}{3}\varepsilon_{kl(i}J_{j)k}(U)N_l\right] \\ &+ \frac{1}{c^2}\left[\frac{1}{12}N_k N_l I_{ijkl}^{(2)}(U) + \frac{1}{2}\varepsilon_{kl(i}J_{j)km}^{(1)}(U)N_l N_m\right] \\ &+ \frac{1}{c^3}\left[\frac{1}{60}N_k N_l N_m I_{ijklm}^{(3)}(U) + \frac{2}{15}\varepsilon_{kl(i}J_{j)kmn}^{(2)}(U)N_l N_m N_n\right] \end{aligned} \quad (2.6)$$

where I_{ij}, I_{ijk}, I_{ijkl} and I_{ijklm} are the mass-type multipole moments of the source (quadrupole, octupole, ...), and where J_{ij}, J_{ijk} and J_{ijkl} are the corresponding spin-type multipole moments. All these moments are symmetric and trace-free (STF) in their indices (we use the notation $T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$). They are given as explicit integrals extending over the compact-supported stress–energy distribution of matter in the source. Their precision must be consistent with the 1.5 PN approximation, which means that the quadrupole and octupole mass moments I_{ij} and I_{ijk}, and the quadrupole spin moment J_{ij}, must involve post-Newtonian correcting terms of order $\varepsilon^2 \sim 1/c^2$, while all higher moments can take their usual Newtonian form. We shall quote here only the post-Newtonian expression of the mass quadrupole moment I_{ij}. It reads [44]

$$\begin{aligned} I_{ij}(U) = \int \mathrm{d}^3\boldsymbol{x}\,\hat{x}_{ij}\frac{T^{00}(\boldsymbol{x}, U) + T^{ss}(\boldsymbol{x}, U)}{c^2} &+ \frac{1}{14c^2}\frac{\mathrm{d}^2}{\mathrm{d}U^2}\int \mathrm{d}^3\boldsymbol{x}\,\hat{x}_{ij}\boldsymbol{x}^2\frac{T^{00}(\boldsymbol{x}, U) + T^{ss}(\boldsymbol{x}, U)}{c^2} \\ &- \frac{20}{21c^2}\frac{\mathrm{d}}{\mathrm{d}U}\int \mathrm{d}^3\boldsymbol{x}\,\hat{x}_{ijk}\frac{T^{0k}(\boldsymbol{x}, U)}{c} \end{aligned} \quad (2.7)$$

where T^{00}, T^{0k} and $T^{ss} = \Sigma_i T^{ii}$ denote the contravariant components of the stress–energy tensor of the source in the (harmonic) coordinate system $\boldsymbol{x}, t$, and where $\hat{x}_{ij}$ and $\hat{x}_{ijk}$ denote the trace-free parts of $x^i x^j$ and $x^i x^j x^k$. We refer to [44] for the similar expression of the mass octupole moment I_{ijk}, and to [45] for the (more complicated) expression of the spin quadrupole moment J_{ij}. (Note that the Newtonian precision of the quadrupole moment (2.7) is sufficient in the second term of (2.5).)

All the source moments $I_{ij}(U)$, $J_{ij}(U)$, ... in (2.6), and the tensor $K_{ij}(U, \boldsymbol{N})$ itself, are local-in-time (or 'instantaneous') functionals of the source in the sense that they depend at time U on the components of the stress–energy tensor of the source at the same time U (see (2.7)). By contrast, we see that the second term in the asymptotic wave field (2.5), which involves the fourth derivative of the source quadrupole moment (2.7) and the total mass M of the source, depends on the state of the source at all times $V \leqslant U$ in the past. This term, which we shall qualify as *non-local-in-time* (or 'hereditary'), represents a component of the radiation field which propagates on average inside the light cone. Physically, this term can be viewed as the wave tail produced by the continuous backscattering of the linear quadrupolar wave off the curved spacetime generated by the mass M of the source. It has been shown that to the non-local wave tail in (2.5) corresponds a non-local modification of the radiation reaction force in the source [51, 52], and that the energy carried out at infinity by the wave tail is exactly balanced by the work done in the source by this non-local force [46].

The wave tail in (2.5) involves two constants, besides the mass M of the source: b and $\frac{11}{12}$. The constant $\frac{11}{12}$ was computed in [46]. Here we have chosen to include this constant in the tail even though it is in front of a term which is proportional to the third derivative of $I_{ij}(U)$, and which is thus purely local. (Note that the same $\frac{11}{12}$ appears also in the radiation reaction force acting within the source [52].) The constant b is an arbitrary constant which is strictly positive and has the dimension of time. It parametrizes the coordinate transformation between the far-zone coordinate system T, $\boldsymbol{X}$ and the (harmonic) source coordinate system t, $\boldsymbol{x}$ [46]. More precisely, the constant b relates the origin of the far-zone coordinate time $U = T - R/c$ (at some fixed large distance of the source) to the origin of the harmonic coordinate time t in the source. The relation between both times reads as

$$U = t - \frac{r}{c} - \frac{2GM}{c^3} \ln\left(\frac{r}{cb}\right) + O(\varepsilon^5) \tag{2.8}$$

where $r = |\boldsymbol{x}|$ is the distance of the source in harmonic coordinates. By inserting (2.8) into (2.5) we easily see that the far-zone field does not depend (within the accuracy with which it is derived) on the constant b when it is expressed in source-rooted coordinates t, $\boldsymbol{x}$ [46].

The integrand of the wave tail in (2.5) contains a logarithmic kernel which blows up when the span interval $U - V$ between the 'current' time U and the actual time V of dependence of the field on the source goes to infinity. We must therefore supplement the study of the wave tail by some assumption concerning the behaviour of the source at very early times $V \to -\infty$. In this paper we shall assume that the *second* time derivative of the quadrupole moment I_{ij} (equation (2.7)) becomes asymptotically constant when $V \to -\infty$, i.e.

$$I_{ij}^{(2)}(V) = A_{ij} + o(1) \tag{2.9}$$

where A_{ij} is some constant tensor. Furthermore we shall assume that the o symbol in (2.9) satisfies $\partial^n o(1)/\partial V^n = o(1/V^n)$. (By $f(V) = o(g(V))$ we mean that $f(V)/g(V)$ tends to zero when $V \to -\infty$.) The assumption (2.9) precludes the emission of a strong burst of gravitational radiation in the past (see e.g. [51]), and it is satisfied for instance in the case of an initial scattering situation where the system is formed by accretion of bodies moving on an initially hyperbolic orbit. (The tensor A_{ij} is in this case equal to $\Sigma_A m_A v_A^i v_A^j$ where m_A and v_A^i are the mass and initial velocities of the bodies.) Subject to the assumption (2.9), the integrand of the wave tail behaves as $\ln(-V)o(1/V^2)$ when $V \to -\infty$, and the integral is perfectly convergent. (The wave tail (2.5) was derived in [46] under the more restrictive assumption of stationarity in the past—before some fixed date in the past. We shall admit here that (2.5) still holds under the weaker assumption (2.9).)

Finally it will be useful in this paper to express the wave tail in an equivalent form, but containing a better behaving kernel when $V \to -\infty$. To this end, we split the integral in (2.5) into two integrals, one corresponding to the 'recent past' of the source and extending from $V = U$ to $V = U - \mathcal{T}$, where $\mathcal{T}$ is some constant > 0, and one corresponding to the 'remote past' of the source and extending from $V = U - \mathcal{T}$ to $V = -\infty$. Then we integrate by parts the remote past integral and use (2.9) to cancel out the all-integrated term at the limit $V = -\infty$. As a result we obtain

$$h^{\mathrm{TT}}_{ab}(U) = \frac{2G}{Rc^4}\mathcal{P}_{abij}(\mathbf{N})\left\{K^{(2)}_{ij}(U,\mathbf{N}) + \frac{2GM}{c^3}\left[\left(\ln\left(\frac{\mathcal{T}}{2b}\right) + \frac{11}{12}\right)I^{(3)}_{ij}(U) + \mathcal{T}\int_0^1 \mathrm{d}x\,\ln x\, I^{(4)}_{ij}(U - \mathcal{T}x) + \int_1^{+\infty}\frac{\mathrm{d}x}{x}I^{(3)}_{ij}(U - \mathcal{T}x)\right]\right\} \tag{2.10}$$

in which we use the variable $x = (U - V)/\mathcal{T}$. (We no longer mention the neglected terms $O(\varepsilon^4)$ and $O(1/R^2)$.)

3. Spectral decomposition of the gravitational wave field

We investigate in this section the continuous Fourier decomposition of the wave field in the form derived in (2.10). Note that since the wave field satisfies the post-Newtonian assumption $\varepsilon \ll 1$, where ε is the small parameter (2.1), its Fourier decomposition will be valid only for low enough frequencies ω satisfying (2.2) or (2.3).

Thanks to the fall-off property $\sim 1/x$ of the kernel in the 'remote past' contribution of the tail in (2.10), we see that each separate Fourier component of the wave field (2.10) will yield a convergent integral. (Note that this does not mean that the Fourier decomposition of the field in the original form (2.5) is divergent, but simply that one cannot compute it by inverting the summations in V and in ω.) Let us denote by $\widetilde{K}_{ij}(\omega,\mathbf{N})$ and $\widetilde{I}_{ij}(\omega)$ the Fourier transforms of the source moments $K_{ij}(U,\mathbf{N})$ and $I_{ij}(U)$ given by (2.6) and (2.7). We have

$$K_{ij}(U,\mathbf{N}) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\widetilde{K}_{ij}(\omega,\mathbf{N})\,\mathrm{e}^{-\mathrm{i}\omega U} \tag{3.1a}$$

$$I_{ij}(U) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\widetilde{I}_{ij}(\omega)\,\mathrm{e}^{-\mathrm{i}\omega U}\,. \tag{3.1b}$$

Since the moments (2.6) and (2.7) are real, their Fourier transforms satisfy $\widetilde{K}_{ij}(-\omega,\mathbf{N}) = \widetilde{K}^*_{ij}(\omega,\mathbf{N})$ and $\widetilde{I}_{ij}(-\omega) = \widetilde{I}^*_{ij}(\omega)$ where $*$ denotes the complex conjugation. We insert (3.1) into the field (2.10), and we use the following identity, valid for any non-zero real number λ:

$$\lambda\int_0^1 \mathrm{d}x\,\ln x\,\mathrm{e}^{\mathrm{i}\lambda x} + \mathrm{i}\int_1^{+\infty}\frac{\mathrm{d}x}{x}\,\mathrm{e}^{\mathrm{i}\lambda x} = -\frac{\pi}{2}\mathrm{sign}(\lambda) - \mathrm{i}(\ln|\lambda| + C) \tag{3.2}$$

where $\mathrm{sign}(\lambda)$ and $|\lambda|$ denote the sign and absolute value of λ, and where $C = 0.577\ldots$ is the Euler constant (see e.g. [66] pp 583 and 928). As a result, we obtain the following expression of the Fourier decomposition of the wave field (2.10):

$$h_{ab}^{\mathrm{TT}}(U) = \frac{2G}{Rc^4}\mathcal{P}_{abij}(N)\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}(-\omega^2)\,\mathrm{e}^{-\mathrm{i}\omega U}\left\{\widetilde{K}_{ij}(\omega, N) + \frac{2GM}{c^3}\left[\frac{\pi}{2}|\omega| + \mathrm{i}\omega\left(\ln(2|\omega|b) + C - \frac{11}{12}\right)\right]\widetilde{I}_{ij}(\omega)\right\}. \tag{3.3}$$

This expression does not depend on the constant $\mathcal{T}$ we introduced in (2.10) to separate the wave tail into 'recent past' and 'remote past' contributions. Note that the separation into recent past and remote past contributions is not the only way to derive (3.3). In appendix A we proceed in another way which is based on the introduction in the original form of the wave tail in (2.5) of an 'adiabatic damping' factor $\mathrm{e}^{\alpha V}$, where α is some positive constant. The Fourier decomposition (3.3) is then recovered in the limit $\alpha \to 0$.

We shall now express the Fourier decomposition (3.3) in a more convenient form for our purpose. We note first that in the second term of (3.3), which has in front of it the small factor $2GM/c^3$, we can replace consistently with the 1.5 PN approximation the quadrupole moment $\widetilde{I}_{ij}(\omega)$ by the post-Newtonian moment $\widetilde{K}_{ij}(\omega, N)$, because $\widetilde{I}_{ij}$ and $\widetilde{K}_{ij}$ differ from each other by small post-Newtonian corrections (see (2.6)). Then, the moment $\widetilde{K}_{ij}$ is in factor of a complex-valued expression which can be written as a real amplitude times a complex exponential. Neglecting higher-order post-Newtonian terms, we finally obtain

$$h_{ab}^{\mathrm{TT}}(U) = \frac{2G}{Rc^4}\mathcal{P}_{abij}(N)\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}(-\omega^2)\widetilde{K}_{ij}(\omega, N)\left[1 + \frac{\pi GM}{c^3}|\omega|\right]\mathrm{e}^{-\mathrm{i}\omega U + \mathrm{i}\theta(\omega)} \tag{3.4}$$

where we have posed

$$\theta(\omega) = \frac{2GM\omega}{c^3}\left[\ln(2|\omega|b) + C - \frac{11}{12}\right]. \tag{3.5}$$

The final form (3.4), (3.5) of the Fourier decomposition of the far-zone gravitational field constitutes a central result of this paper. It shows that the effect of the backscattering of the linear waves off the monopolar spacetime associated with the mass of the source (tail effect) is:

(i) to *increase* the amplitude of each Fourier component of the linear waves by a multiplicative factor $1 + \pi GM|\omega|/c^3$ which is linear in the absolute value of the frequency;

(ii) to add to the phase of the linear waves the supplementary *frequency-dependent* phase $\theta(\omega)$ given by (3.5).

Note that the tail-induced modifications (i) and (ii) of the linear waves apply identically on all the components of the waves, and in particular on the two independent polarization states of the waves. Recall also that the modifications (i) and (ii) are valid only for frequencies ω satisfying the post-Newtonian assumption (2.3); thus we expect that they will be of very small numerical order of magnitude. However, the frequency-dependent modification of the phase could be important in the case of the radiation emitted during the late stages of inspiralling compact binary systems.

If we assume for simplicity that the components of the tensor $\widetilde{K}_{ij}(\omega, N)$ are either real or purely imaginary (see e.g. (5.5)), we can write the total phase of the gravitational wave (3.4), (3.5) in the form

$$\varphi(\omega, T, R) = \omega\,(T - R/c) - \frac{2GM\omega}{c^3}\left[\ln(2|\omega|b) + C - \frac{11}{12}\right] \tag{3.6a}$$

(modulo the possible addition of $\pm\pi/2$), where T and R are the far-zone coordinate time and radial distance to the source. In these coordinates, the wave vector along the radial direction is $K = -\partial\varphi/\partial R = \omega/c$. We can also write the total phase (3.6a) in terms of the source-rooted (harmonic) coordinates t and $r = |\boldsymbol{x}|$. By inserting (2.8) into (3.6a) (and neglecting higher-order terms) we obtain

$$\varphi(\omega, t, r) = \omega\,(t - r/c) - \frac{2GM\omega}{c^3}\left[\ln\left(2|\omega|r/c\right) + C - \frac{11}{12}\right]. \tag{3.6b}$$

This phase does not depend on the constant b. The associated wave vector is $k = -\partial\varphi/\partial r = (\omega/c)[1 + 2GM/(rc^2)]$.

A case of interest is when the gravitational wave (3.4) is in the form of one or several *wave packets*, centred around one or several frequencies ω_0, $\omega_1, \ldots$, i.e. when the components of the tensor $\widetilde{K}_{ij}(\omega, \boldsymbol{N})$ are all sharply picked around ω_0, $\omega_1, \ldots$. The velocity of propagation of the wave packets (or group velocity) is the same for all wave packets; it is given by $V_{\mathrm{g}} = \partial\omega/\partial K = c$ in the far-zone coordinates (T, R), and by $v_{\mathrm{g}} = \partial\omega/\partial k = c[1-2GM/(rc^2)]$ in the source coordinates (t, r). (These velocities are the same as for high-frequency electromagnetic waves propagating on a Schwarzschild background.) The position of the maxima of amplitude in space of the wave packets along the radial distance (or line of sight) is determined by solving the equation $(\partial\varphi/\partial\omega)(\omega_0, T, R_{\max}(T)) = 0$ in far-zone coordinates, or the equation $(\partial\varphi/\partial\omega)(\omega_0, t, r_{\max}(t)) = 0$ in source coordinates. From (3.6) we find

$$R_{\max}(T) = cT - \frac{2GM}{c^2}\left[\ln(2\omega_0 b) + C + \frac{1}{12}\right] \tag{3.7a}$$

and

$$r_{\max}(t) = ct - \frac{2GM}{c^2}\left[\ln(2\omega_0 t) + C + \frac{1}{12}\right]. \tag{3.7b}$$

This shows that the maxima of amplitude of different wave packets corresponding to different central frequencies (but belonging to the same wave (3.4)) are at some given time distributed along the line of sight with slightly different relative positions fixed by the ratios of frequencies. Equivalently, the maxima of two wave packets corresponding to two different central frequencies ω_0 and ω_1 arrive at some given distance with a slight relative delay given by

$$T_0 - T_1 = t_0 - t_1 = \frac{2GM}{c^3}\ln\left(\frac{\omega_0}{\omega_1}\right). \tag{3.8}$$

The wave packet with the higher frequency is slightly delayed with respect to the wave packet with the lower frequency.

4. Application to the radiation of an inspiralling binary star system

As we recalled in the introduction, inspiralling (compact) binary star systems constitute probably, in their late stages of evolution, the most promising sources of gravitational radiation. They seem also to provide the most interesting application of the tail-modified expression

(3.4), (3.5) of the Fourier decomposition of the far-zone wave field. (Recall that the wave-generation formalism summarized in section 2 is valid for systems containing strongly self-gravitating compact bodies like neutron stars [46].)

For simplicity we shall neglect in the applications (this section and the next section) all post-Newtonian corrections in the wave form (3.4), (3.5) except the one which is associated with the tail (including its associated coefficient $\frac{11}{12}$). Namely, we shall replace in (3.4) the post-Newtonian tensor $K_{ij}(U, \boldsymbol{N})$ of (2.6) by the usual *Newtonian* quadrupole moment of the system. This will permit us to separate more clearly the effects which are specifically due to the wave tail. On the other hand, post-Newtonian corrections other than the one associated with the wave tail are well known, especially for binary systems [67–71], and thus can straightforwardly be added if necessary. We denote the Newtonian quadrupole moment of the binary system (assimilated to a system of two point-masses) by

$$Q_{ij} = m_1 r_1^i r_1^j + m_2 r_2^i r_2^j = \mu r^i r^j \tag{4.1}$$

(in a mass-centred frame), where $r^i = r_2^i - r_1^i$ is the relative position of the two point-masses m_1 and m_2, and where $\mu = m_1 m_2 / M$ is the reduced mass of the system (with $M = m_1 + m_2$ its total mass). For later convenience, we assume in (4.1) that Q_{ij} is not trace-free, so that the trace-free moment K_{ij} of (2.6) is equal, in the Newtonian limit, to $Q_{ij} - \frac{1}{3}\delta_{ij} Q_{kk}$.

In the case of a two-point-masses binary system moving on a Keplerian orbit, we have instead of the continuous Fourier decomposition (3.1) a decomposition into a discrete series of frequencies. Let us write in this case

$$Q_{ij}(U) = \sum_{n=-\infty}^{+\infty} {}_n\widetilde{Q}_{ij}\, \mathrm{e}^{-\mathrm{i}n\Omega_0 U} \tag{4.2a}$$

where Ω_0 is the orbital frequency of the binary ($\Omega_0 = 2\pi / P_0$, where P_0 is the orbital period), and where the Fourier discrete coefficients ${}_n\widetilde{Q}_{ij}$ satisfy ${}_n\widetilde{Q}_{ij} = {}_{-n}\widetilde{Q}^*_{ij}$. The coefficients ${}_n\widetilde{Q}_{ij}$ are related to the continuous Fourier transform $\widetilde{Q}_{ij}(\omega)$ by

$$\widetilde{Q}_{ij}(\omega) = \int_{-\infty}^{+\infty} \mathrm{d}U\, Q_{ij}(U)\, \mathrm{e}^{\mathrm{i}\omega U} = 2\pi \sum_{n=-\infty}^{+\infty} {}_n\widetilde{Q}_{ij}\, \delta(\omega - n\Omega_0)\,. \tag{4.2b}$$

Thus we replace in (3.4) the tensor $\widetilde{K}_{ij}(\omega, \boldsymbol{N})$ by the tensor $\widetilde{Q}_{ij}(\omega)$ (the trace of $\widetilde{Q}_{ij}$ will be cancelled out by the TT operator $\mathcal{P}_{abij}$) and we make use of (4.2b) to go from continuous to discrete frequencies. This yields the wave field

$$h_{ab}^{\mathrm{TT}}(U) = \frac{2G}{Rc^4} \mathcal{P}_{abij}(\boldsymbol{N}) \sum_{n=-\infty}^{+\infty} (-n^2\Omega_0^2)\, {}_n\widetilde{Q}_{ij} \left[1 + \frac{\pi G M}{c^3} |n| \Omega_0\right] \mathrm{e}^{-\mathrm{i}n\Omega_0 U + \mathrm{i}\theta(n\Omega_0)} \tag{4.3}$$

(where we can use $M = m_1 + m_2$). Let us also write the two independent polarizations $h_+(U)$ and $h_\times(U)$, associated with the wave field (4.3), with respect to two perpendicular unit directions $\boldsymbol{P}$ and $\boldsymbol{Q}$ in the plane orthogonal to $\boldsymbol{N}$ (so that $\boldsymbol{N}$, $\boldsymbol{P}$, $\boldsymbol{Q}$ forms an oriented triad). They are

$$h_{+,\times}(U) = \frac{2G}{Rc^4} \sum_{n=-\infty}^{+\infty} (-n^2\Omega_0^2)\, {}_n\widetilde{Q}_{+,\times} \left[1 + \frac{\pi G M}{c^3} |n| \Omega_0\right] \mathrm{e}^{-\mathrm{i}n\Omega_0 U + \mathrm{i}\theta(n\Omega_0)} \tag{4.4}$$

where we have posed

$$ {}_n\widetilde{Q}_+ = \frac{P_iP_j - Q_iQ_j}{2}\, {}_n\widetilde{Q}_{ij} \tag{4.5a} $$

$$ {}_n\widetilde{Q}_\times = \frac{P_iQ_j + P_jQ_i}{2}\, {}_n\widetilde{Q}_{ij}\,. \tag{4.5b} $$

We shall now restrict our attention to a binary star system orbiting a *circular* (Newtonian) orbit. This is a safe restriction during the late stages of the inspiralling of a binary system because the orbit of the system, having evolved through the action of radiation reaction, will have circularized earlier (see e.g. [3]). In the circular case we have only one frequency in the radiation, namely twice the orbital frequency $\omega_0 = 2\Omega_0$, and the corresponding $n = 2$ components of the quadrupole moment are given by

$$ {}_2\widetilde{Q}_{xx} = -{}_2\widetilde{Q}_{yy} = \frac{\mu a_0^2}{4}\, \mathrm{e}^{-\mathrm{i}\varphi_0} \tag{4.6a} $$

$$ {}_2\widetilde{Q}_{xy} = \mathrm{i}\frac{\mu a_0^2}{4}\, \mathrm{e}^{-\mathrm{i}\varphi_0} \tag{4.6b} $$

where a_0 is the radius of the orbit (satisfying $\omega_0^2 a_0^3 = 4GM$), and where φ_0 is some constant phase. Let us denote by ι the angle between the line of sight from the source to the observer and the normal to the orbital plane (we assume $0 \leqslant \iota \leqslant \pi/2$), and let us orient the polarization axes $\boldsymbol{P}$ and $\boldsymbol{Q}$ along the major and minor axes of the projection of the orbital plane on the sky, respectively. Then the two wave polarizations (4.4) are easily obtained in the form

$$ h_+(U) = \frac{2G\mu}{Rc^4}(1+\cos^2\iota)\left(\frac{1}{2}GM\omega_0\right)^{2/3}\left[1 + \frac{\pi GM\omega_0}{c^3}\right]\cos[\omega_0 U + \varphi_0 - \theta(\omega_0)] \tag{4.7a} $$

$$ h_\times(U) = \pm\frac{2G\mu}{Rc^4}(2\cos\iota)\left(\frac{1}{2}GM\omega_0\right)^{2/3}\left[1 + \frac{\pi GM\omega_0}{c^3}\right]\sin[\omega_0 U + \varphi_0 - \theta(\omega_0)] \tag{4.7b} $$

(where $\omega_0 = 2\Omega_0$). The expressions (4.7) have also been derived by Poisson [61] (see his equations (6.1)–(6.3)), albeit in another form and in the limiting case $\mu \ll M$. (The coefficient $\frac{17}{12}$ found by Poisson differs from our coefficient $\frac{11}{12}$ because he uses Schwarzschild coordinates instead of harmonic coordinates in (2.8).) Note that these expressions are *a priori* valid only in the case of a fixed, non-inspiralling (non-decaying) orbit. However, as we shall prove, the expressions valid in the case of a *decaying* orbit, and in an appropriate regime, can be simply obtained by replacing in (4.7) the constant frequency ω_0 by the varying frequency $\omega(U)$ of the decaying orbit.

The dynamics of the decaying circular orbit, driven by radiation reaction, is known from the work of Peters [72]. It is as follows. The radius of the orbit, as a function of the far-zone retarded time U (say), reads as

$$ a(U) = \alpha[\tau_c(U)]^{1/4} \tag{4.8a} $$

where we denote by $\tau_c(U) = t_c - U$ the coalescing time left till coalescence starting from time U (t_c is the instant of coalescence), and where the constant α depends on the masses and is given by

$$ \alpha = 4\left(\frac{G^3\mu M^2}{5c^5}\right)^{1/4}. \tag{4.8b} $$

As the orbit decays, the frequency increases, and by Kepler's law $\omega^2 a^3 = 4GM$ we get

$$\omega(U) = \beta^{-1}[\tau_c(U)]^{-3/8} \tag{4.9a}$$

where the constant β is given by

$$\beta = 4\left(\frac{G^{5/3}\mu M^{2/3}}{5c^5}\right)^{3/8}. \tag{4.9b}$$

The phase associated with the time-varying frequency (4.9) reads as

$$\varphi(U) = \int_{t_c}^{U} \omega(t)\,\mathrm{d}t + \varphi_c = -\frac{8}{5\beta}[\tau_c(U)]^{5/8} + \varphi_c \tag{4.10}$$

where φ_c denotes a constant final phase at the instant of coalescence. Finally we also introduce the dimensionless ratio between (half) the period of the binary and the coalescing time, i.e.

$$\xi(U) = \frac{2\pi}{\omega(U)\tau_c(U)} = 2\pi\beta[\tau_c(U)]^{-5/8}. \tag{4.11}$$

The condition $\xi \ll 1$ defines the regime of the decay of the orbit in which the relative changes of the frequency and the radius of the binary in *one* period of the binary are small (and of order ξ). The regime $\xi \ll 1$ is the one during which most observations of inspiralling binaries will take place [3]. Note that the parameter ξ is of order $\xi \approx \varepsilon^5$ where ε is the post-Newtonian parameter (2.1).

At the level of the *first* post-Newtonian approximation in the wave form (i.e. including terms of relative order 1, ε and ε^2), it is legitimate, so long as the parameter ξ given by (4.11) remains small, to derive the wave form in the case of the decaying orbit by a simple replacement in (4.7) of the constant frequency ω_0 of the fixed orbit by the varying frequency (4.9) of the decaying orbit. Indeed, the wave form at the 1 PN level is a local-in-time functional of the source, i.e. it depends on the source's parameters at the 'current' time U only, and thus in such a replacement $\omega_0 \to \omega(U)$ we shall make errors of order $\xi(U)$ at most. However, the tail contribution we are interested in, which arises at the relative 1.5 PN order ε^3, is not a local-in-time functional of the source's parameters. This means that the tail contribution involves for instance, besides the current value $\omega(U)$ of the frequency, all arbitrarily early values $\omega(V)$ of the frequency (with $V \leqslant U$), which differ from $\omega(U)$ by relative terms tending to one in the limit $V \to -\infty$, and thus *not* being of order $\xi(U)$. Therefore, it seems to be necessary to perform another computation of the wave tail in the case of a decaying orbit satisfying the laws (4.8)–(4.10). A possible computation is to go back to the continuous Fourier decomposition (3.4), (3.5), substitute in it the Fourier transform of the decaying orbit obtained by means of the stationary-phase approximation method, and then obtain the inverse Fourier transform. However, we have chosen to perform, in appendix B, a direct computation in the time series which shows that, most satisfactorily, the replacement $\omega_0 \to \omega(U)$ (and also $\omega_0 U + \varphi_0 \to \varphi(U)$) is correct even for the wave tail, at first order in the *current* value $\xi(U)$ of the parameter (4.11) or, more exactly, at first order in $\xi(U)\ln\xi(U)$. The reason why this works is that the 'remote past' contribution of the wave tail is itself of small numerical order $\xi(U)$, when we choose the constant $\mathcal{T}$ in (2.10) to be the current value of the coalescing time $\tau_c(U)$ (see appendix B), and thus that only remains the 'recent past' contribution of the wave tail, in which the frequency of the orbit takes essentially its current value $\omega(U)$. (Note that since

$\xi(U)$ is of order $(\varepsilon(U))^5$, the errors made in the replacement $\omega_0 \to \omega(U)$ in the 'Newtonian' wave form are formally smaller than the wave tail, which is of order $(\varepsilon(U))^3$ only.)

In conclusion, we can now write down from (4.7) and appendix B the expressions of the wave polarizations generated in the far zone by an inspiralling (compact) binary system,

$$h_+(U) = \frac{2G\mu}{Rc^4}(1+\cos^2\iota)\left(\frac{1}{2}GM\omega(U)\right)^{2/3}\left[1+\frac{\pi GM\omega(U)}{c^3}\right]\cos[\varphi(U)-\theta(\omega(U))] + \mathrm{O}[\xi(U)\ln\xi(U)] \tag{4.12a}$$

$$h_\times(U) = \pm\frac{2G\mu}{Rc^4}(2\cos\iota)\left(\frac{1}{2}GM\omega(U)\right)^{2/3}\left[1+\frac{\pi GM\omega(U)}{c^3}\right]\sin[\varphi(U)-\theta(\omega(U))] + \mathrm{O}[\xi(U)\ln\xi(U)] \tag{4.12b}$$

where the frequency $\omega(U)$ and the phase $\varphi(U)$ are given by (4.9) and (4.10), where the neglected terms are of order $\xi(U)\ln\xi(U)$, $\xi(U)$ being the parameter given by (4.11) (see appendix B), and where the tail-induced phase $\theta(\omega(U))$ is given by

$$\theta(\omega(U)) = \frac{2GM\omega(U)}{c^3}\left[\ln(2\omega(U)b) + C - \frac{11}{12}\right]. \tag{4.13}$$

The latter tail-induced phase, with its $\omega(U)\ln\omega(U)$ dependence on frequency, could be important in the search of inspiralling compact binary signals in the raw output of future laser interferometer detectors. Indeed, as the frequency of the signal increases with time, (4.13) will induce a phase difference, growing as $\omega\ln\omega$, between the actual signal, which is composed of a front-wave and of a tail-wave, and a 'Newtonian' signal (or even a 'first-post-Newtonian' signal), which is composed only of a front-wave. The data analysis technique for extracting the signal out of the noise is to correlate the output of the detector with a filter that is matched on the expected signal itself (see e.g. [3, 58, 59]). Thus, we think that it is important to include the tail-induced phase (4.13), and in fact all the tail-induced corrections appearing in (4.12), in the construction of matched filters, so that signal and filters remain as long as possible correlated.

5. Application to the timing of a relativistic binary pulsar

The binary pulsar PSR 1913+16 has been timed since 1974 with increasing precision, and the observed change $\dot{P}_{\mathrm{Obs}}$ of its orbital period now agrees to within 0.5% with the theoretical prediction $\dot{P}_{\mathrm{Th}}$ [9]. In this section, we wish to compute the value of the relative correction in the $\dot{P}_{\mathrm{Th}}$ of the binary pulsar (or of another pulsar), which is due to the wave tail arising at the 1.5 PN approximation in the wave form. Our aim is also to show how the computation proceeds, to recover in a particular case a result of previous authors [61–64] working with different methods, and to complete a work [60] on the relative correction in $\dot{P}_{\mathrm{Th}}$ which is due to the 1 PN (local) approximation in the wave form.

We shall need the expression of the 'gravitational luminosity' $\mathcal{L}(U,\boldsymbol{N}) = (\mathrm{d}E^{\mathrm{grav}}/\mathrm{d}U\,\mathrm{d}\Omega)(U,\boldsymbol{N})$ of the binary system, i.e. the gravitational energy emitted by the system in one unit of time U and in the solid angle $\mathrm{d}\Omega$ around the direction $\boldsymbol{N}$. The luminosity $\mathcal{L}(U,\boldsymbol{N})$, including the tail effect, can be straightforwardly computed, using standard formulae, by differentiating and squaring the gravitational field (4.3). Then, as $\mathcal{L}(U,\boldsymbol{N})$ is a periodic function of time with the orbital period $P_0 = 2\pi/\Omega_0$, we can perform its time-average. As a result, we obtain (neglecting higher-order terms)

$$\langle \mathcal{L}(N) \rangle = \frac{1}{P_0} \int_0^{P_0} \mathrm{d}U\, \mathcal{L}(U, N) = 2 \sum_{n=1}^{+\infty} {}_n\langle \mathcal{L}_0(N) \rangle \left[1 + \frac{2\pi G M}{c^3} (n\Omega_0) \right] \tag{5.1}$$

where ${}_n\langle \mathcal{L}_0(N) \rangle$ denotes the nth Fourier component of the 'Newtonian' averaged luminosity,

$${}_n\langle \mathcal{L}_0(N) \rangle = \frac{G}{8\pi c^5} (n\Omega_0)^6\, \mathcal{P}_{abij}(N)\, {}_n\widetilde{Q}_{ab}\, {}_n\widetilde{Q}^*_{ij}\,. \tag{5.2}$$

(Recall that we neglect all post-Newtonian approximations except the one which is associated with the wave tail.) By averaging (5.1) over all directions of emission N we get

$$\langle \mathcal{L} \rangle = \int \mathrm{d}\Omega \langle \mathcal{L}(N) \rangle = 2 \sum_{n=1}^{\infty} {}_n\langle \mathcal{L}_0 \rangle \left[1 + \frac{2\pi G M}{c^3} (n\Omega_0) \right] \tag{5.3}$$

where ${}_n\langle \mathcal{L}_0 \rangle$ is the nth Fourier component of the standard 'Newtonian' quadrupole formula

$${}_n\langle \mathcal{L}_0 \rangle = \frac{G}{5c^5} (n\Omega_0)^6 \left[{}_n\widetilde{Q}_{ij}\, {}_n\widetilde{Q}^*_{ij} - \frac{1}{3}\, {}_n\widetilde{Q}_{ii}\, {}_n\widetilde{Q}^*_{jj} \right]. \tag{5.4}$$

As we see in (5.1) and (5.3), the effect of the wave tail is to modify each nth Fourier component of the luminosity by a multiplicative factor linear in the frequency of the nth harmonics. (This is valid so long as the post-Newtonian condition (2.3) remains satisfied.)

The *Newtonian* averaged luminosity $\langle \mathcal{L}_0 \rangle = 2 \sum_{n=1}^{\infty} {}_n\langle \mathcal{L}_0 \rangle$ has been computed for a system of two point-masses orbiting a Keplerian ellipse by Peters and Mathews [10]. These authors obtained the expression of $\langle \mathcal{L}_0 \rangle$ by two methods. The first method was to take directly the average in time of the usual quadrupole formula computed for the Keplerian ellipse, and the second method consisted of summing up the series of the ${}_n\langle \mathcal{L}_0 \rangle$s obtained by Fourier analysis of the Keplerian motion. Let us consider in more detail the second method. The Fourier coefficients of the quadrupole moment of a system of two point-masses orbiting a Keplerian ellipse are given (for $n \neq 0$) by

$${}_n\widetilde{Q}_{xx} = 2\mu a_0^2 \left[\frac{1-e^2}{e} \frac{1}{n} J'_n(ne) - \frac{1}{n^2 e^2} J_n(ne) \right] \tag{5.5a}$$

$${}_n\widetilde{Q}_{xy} = \frac{2\mu a_0^2 \sqrt{1-e^2}}{\mathrm{i}n} \left[\frac{1}{ne} J'_n(ne) + \left(1 - \frac{1}{e^2} \right) J_n(ne) \right] \tag{5.5b}$$

$${}_n\widetilde{Q}_{yy} = 2\mu a_0^2 (1-e^2) \left[-\frac{1}{ne} J'_n(ne) + \frac{1}{n^2 e^2} J_n(ne) \right] \tag{5.5c}$$

where a_0 is the semi-major axis of the (relative) orbit and e is the eccentricity. We denote by $J_n(x)$ the usual Bessel function, and by $J'_n(x)$ its derivative $\mathrm{d}J_n(x)/\mathrm{d}x$. When $e \to 0$, the Fourier coefficients (5.5) are all zero except the ones corresponding to $n = 2$, and we recover (4.6) above (with $\varphi_0 = 0$). By inserting (5.5) into (5.4), we obtain

$$2\, {}_n\langle \mathcal{L}_0 \rangle = \frac{32 G^4}{5c^5} \frac{\mu^2 M^3}{a_0^5} g(n, e) \tag{5.6}$$

where $g(n,e)$ is a quadratic product of Bessel functions J_n and J'_n which is given by

$$g(n,e) = \frac{n^2}{2}\left\{(J_n(ne))^2\left[\frac{1}{e^4} - \frac{1}{e^2} + \frac{1}{3} + n^2\left(\frac{1}{e^4} - \frac{3}{e^2} + 3 - e^2\right)\right] + nJ_n(ne)J'_n(ne)\right.$$
$$\left.\times\left[-\frac{4}{e^3} + \frac{7}{e} - 3e\right] + (J'_n(ne))^2\left[\frac{1}{e^2} - 1 + n^2\left(\frac{1}{e^2} - 2 + e^2\right)\right]\right\}. \tag{5.7}$$

(See equation (A1) of [10] in which the fifth term should read $(1-e^2)(J'_n/n)^2(4/e)^2$ instead of $(1-e^2)(J'_n/n)^2(4/e^2)^2$.) Then, the result obtained by Peters and Mathews (which was found to be identical by the two methods) reads as

$$\langle\mathcal{L}_0\rangle = 2\sum_{n=1}^{+\infty} {}_n\langle\mathcal{L}_0\rangle = \frac{32G^4}{5c^5}\frac{\mu^2M^3}{a_0^5}\,f(e) \tag{5.8}$$

where $f(e)$ is the famous 'enhancement' dimensionless factor

$$f(e) = \sum_{n=1}^{+\infty} g(n,e) = \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}}. \tag{5.9}$$

The proof that the series of the $g(n,e)$s can be summed up to yield (5.9) is given in the appendix of [10].

We now use these results to express the relative correction in the averaged luminosity $\langle\mathcal{L}\rangle$ (equation (5.3)) that is due to the tail effect, by means of the functions $g(n,e)$. By inserting (5.8) and (5.9) into (5.3) we readily obtain

$$\langle\mathcal{L}\rangle = \langle\mathcal{L}_0\rangle\left\{1 + 4\pi\left(\frac{GM}{c^2a_0}\right)^{3/2}\varphi(e)\right\} \tag{5.10}$$

where we have used $\Omega_0^2a_0^3 = GM$, and where $\varphi(e)$ is the dimensionless factor

$$\varphi(e) = \frac{1}{f(e)}\sum_{n=1}^{+\infty}\frac{n}{2}\,g(n,e). \tag{5.11}$$

In the case of a circular orbit ($e=0$), the only non-zero $g(n,e)$ is $g(2,e=0)=1$ and thus we see that the function (5.11) satisfies $\varphi(e=0)=1$. In this case the relative correction in $\langle\mathcal{L}\rangle$ as due to the tail is equal to 4π times the parameter $(GM/c^2a_0)^{3/2}$ (which is $\approx\varepsilon^3$ where ε is the post-Newtonian parameter (2.1)). This coefficient 4π was derived first in a study of the circular motion of a small mass around a large one [61], and an alternative derivation, valid for arbitrary mass ratios, was subsequently given [62] (see also [63,64]). In the general case of a non-zero eccentricity, the coefficient 4π has to be multiplied by the function $\varphi(e)$ of (5.11). Note that we tried to sum up the infinite series (5.11), as Peters and Mathews did to obtain their function $f(e)$, but we did not succeed. Apparently, no closed-form expression for the function $\varphi(e)$ can be found in this way. We have therefore resorted to a numerical computation, the result of which is shown in figure 1. As we see, the function $\varphi(e)$ increases from 1 to $+\infty$ when e goes from 0 to 1 (when $e=1$ the series diverges). Thus, the function $\varphi(e)$ plays the role of an 'enhancement' factor, when the eccentricity is large, for the tail-induced relative correction in the total luminosity (5.10), exactly as does the function $f(e)$ of Peters and Mathews for the

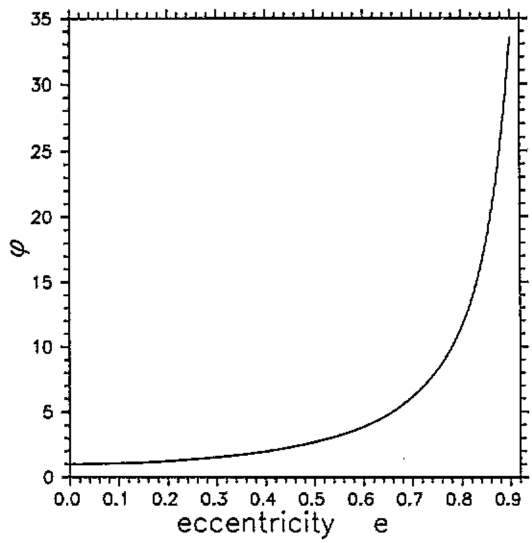

Figure 1. Function φ plotted against the eccentricity e.

Newtonian luminosity (leading term in (5.10)). Note, however, that the function $\varphi(e)$ increases less rapidly than the function $f(e)$. For instance we have $\varphi(0.9) \sim 30$ while $f(0.9) \sim 1000$.

The eccentricity of the orbit of the binary pulsar PSR 1913+16 is $e = 0.617$ and, in this case, we find $\varphi(e) = 4.16$. Inserting this value into (5.10), together with the numerical values of the (relative) semi-major axis $a_0 = 1.95 \times 10^9$m and of the total mass $M = 2.83\, M_\odot$, we get the numerical contribution of the tail effect in the gravitational luminosity of the binary pulsar system:

$$\left(\frac{\langle\mathcal{L}\rangle - \langle\mathcal{L}_0\rangle}{\langle\mathcal{L}_0\rangle}\right)_{\text{PSR 1913+16}} = +1.65 \times 10^{-7}\,. \tag{5.12}$$

We can then apply the usual (heuristic) energy balance argument for the computation of the orbital $\dot{P}_{\text{Th}}$ of the pulsar. Since in the equations of motion the first half-integer ('odd') post-Newtonian approximation is a higher-order approximation (namely the 2.5 PN approximation), and since (as it is easy to check) (5.12) represents in fact the whole contribution of the 1.5 PN approximation in the luminosity, we see that (5.12) will give also the whole 1.5 PN relative contribution in the theoretical formula for $\dot{P}_{\text{Th}}$. Hence, extending our previous work [60], we can write

$$\dot{P}_{\text{Th}} = -\frac{192\pi}{5c^5}\frac{\mu}{M}(GMn)^{5/3} f(e_{\text{T}}) \left\{1 + \frac{1}{c^2}X_{1\ \text{PN}} + \frac{1}{c^3}X_{1.5\ \text{PN}}\right\} \tag{5.13}$$

where n is the orbital frequency of the pulsar ($n \equiv \Omega_0$), where e_{T} is some post-Newtonian eccentricity which is associated with the object which is timed [73,74], and where the relative 1 PN and 1.5 PN corrections have the numerical values

$$\frac{1}{c^2}X_{1\ \text{PN}} = +2.15 \times 10^{-5} \tag{5.14}$$

$$\frac{1}{c^3}X_{1.5\ \text{PN}} = +1.65 \times 10^{-7}\,. \tag{5.15}$$

The value of $c^{-2}X_{1\ \mathrm{PN}}$ follows from equation (4.27) of [60], and the value of $c^{-3}X_{1.5\ \mathrm{PN}}$ follows from (5.12) above. These values are far smaller than the present accuracy in the measurement of $\dot{P}_{\mathrm{Obs}}$, which is presently 0.5% [9], and they are also negligible as compared with many other effects and uncertainties (see table 1 of [7]). We hope, however, that the expression (5.10), together with the graph of the function $\varphi(e)$ in figure 1, will become useful in future observations of other relativistic binary pulsars.

Acknowledgments

We are particularly grateful to Norbert Wex for his numerical computation of the function $\varphi(e)$ in figure 1, and to B S Sathyaprakash for informative discussions on matched filtering. We also thank Bruno Bertotti and Brandon Carter for reading the manuscript, and Eric Gourgoulhon, Kostas Kokkotas and Horacio Sivak for numerical checks of some formulae and discussions. One of us (LB) wishes to thank the Max-Planck-Arbeitsgruppe Gravitationstheorie at the University of Jena for an invitation during which this work was initiated.

Appendix A. Adiabatic damping computation of the wave tail

The method used in the text to compute the Fourier decomposition of the field is to express the field in the form (2.10), where the wave tail is split into two contributions, one extending over the 'recent past' of the source, and one extending over its 'remote past'. The latter remote past contribution involves a kernel whose behaviour at very early times allows the computation of each separate Fourier components.

In this appendix we present a simpler method which is based on an 'adiabatic damping' regularization procedure, which can be seen as an adiabatic switching of the total mass of the source in the remote past. Namely, we introduce in the integrand of the wave tail in (2.5) an exponential damping factor $e^{\alpha V}$ where α is some positive constant. We thus consider the following α-depending family of fields:

$$h^{\mathrm{TT}}_{ab}[\alpha] = \frac{2G}{Rc^4}\mathcal{P}_{abij}\left\{K^{(2)}_{ij}(U,N) + \frac{2GM}{c^3}\int_{-\infty}^{U} \mathrm{d}V\,e^{\alpha V}\left[\ln\left(\frac{U-V}{2b}\right)+\frac{11}{12}\right]I^{(4)}_{ij}(V)\right\}. \tag{A.1}$$

The Fourier decomposition of the fields (A.1) can be straightforwardly computed (when $\alpha > 0$) by means of the identity, valid for any complex number Z having $\mathrm{Re}(Z) > 0$,

$$\int_0^{+\infty} \mathrm{d}x\ \ln x\,e^{-Zx} = -\frac{1}{Z}(\ln Z + C) \tag{A.2}$$

where C is Euler's constant (see e.g. [66] p 573). The result is

$$\begin{aligned} h^{\mathrm{TT}}_{ab}[\alpha] = \frac{2G}{Rc^4}\mathcal{P}_{abij}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}(-\omega^2)e^{-\mathrm{i}\omega U}\Big\{\widetilde{K}_{ij}(\omega,N) \\ +\frac{2GM}{c^3}\frac{\omega^2 e^{\alpha U}}{\alpha-\mathrm{i}\omega}\left[\ln(2b[\alpha-\mathrm{i}\omega])+C-\frac{11}{12}\right]\widetilde{I}_{ij}(\omega)\Big\}. \end{aligned} \tag{A.3}$$

In the limit $\alpha \to 0$, (A.3) yields (3.3) which we obtained in the text. (Indeed we have $\ln(-\mathrm{i}\omega) = \ln|\omega| - \mathrm{i}\frac{\pi}{2}\,\mathrm{sign}\,(\omega)$.)

Note that the identity (3.2) used in the method of the text can be derived from the identity (A.2) used in the method of this appendix by analytic continuation in Z to purely imaginary values $Z = -\mathrm{i}\lambda$. This shows the equivalence (in an analytic continuation sense) between the two methods.

Appendix B. The wave tail in the case of a fixed or decaying circular orbit

We shall use the expression (2.10) of the wave form, which depends on an arbitrary constant $\mathcal{T}$ used to separate the remote past and the recent past contributions of the wave tail, and in which we replace the moments $K_{ij}(U, N)$ and $I_{ij}(U)$ by the usual Newtonian quadrupole moment $Q_{ij}(U)$ of the binary system (equation (4.1)). Thus, we write

$$h_{ab}^{\mathrm{TT}}(U) = \frac{2G}{Rc^4}\mathcal{P}_{abij}(N)\left\{Q_{ij}^{(2)}(U) + \frac{2GM}{c^3}T_{ij}(U)\right\} \tag{B.1}$$

where $T_{ij}(U)$ denotes the wave tail

$$T_{ij}(U) = \left[\ln\left(\frac{\mathcal{T}}{2b}\right) + \frac{11}{12}\right]Q_{ij}^{(3)}(U) + \mathcal{T}\int_0^1 \mathrm{d}x\ \ln x\, Q_{ij}^{(4)}(U-\mathcal{T}x) + \int_1^{+\infty}\frac{\mathrm{d}x}{x}Q_{ij}^{(3)}(U-\mathcal{T}x)\,. \tag{B.2}$$

In this appendix we shall make two computations of the wave tail by direct insertion of the quadrupole moment of the binary system into (B.2). First, we shall redo the computation in the case of a fixed circular orbit (the result has already been obtained in the text by means of a Fourier analysis), and then we shall do the computation in the case of a *decaying* orbit, whose dynamics is driven by radiation reaction. We shall see exactly the differences between the two cases, and how at the end the results are the same (in an appropriate regime) if we identify the constant frequency of the fixed orbit with the current value of the frequency of the decaying orbit.

B1. Case of a fixed circular orbit

The components of the quadrupole moment of the binary system are given by

$$Q_{xx}(U) = \tfrac{1}{2}\,\mu a_0^2[1+\cos(\omega_0 U+\varphi_0)] \tag{B.3a}$$

$$Q_{xy}(U) = \tfrac{1}{2}\,\mu a_0^2\sin(\omega_0 U+\varphi_0) \tag{B.3b}$$

$$Q_{yy}(U) = \tfrac{1}{2}\,\mu a_0^2[1-\cos(\omega_0 U+\varphi_0)] \tag{B.3c}$$

where a_0 is the radius of the orbit, where ω_0 is twice the orbital frequency ($a_0^3\omega_0^2 = 4GM$) and where φ_0 is some constant phase. The components of the wave tail (B.2) can then be written in the form

$$\begin{pmatrix} T_{xx}(U) \\ T_{xy}(U) \\ T_{yy}(U)\end{pmatrix} = \frac{2G\mu M\omega_0}{a_0}\begin{pmatrix} -\mathrm{Re}[A\,\mathrm{e}^{-\mathrm{i}(\omega_0 U+\varphi_0)}] \\ \mathrm{Im}[A\,\mathrm{e}^{-\mathrm{i}(\omega_0 U+\varphi_0)}] \\ \mathrm{Re}[A\,\mathrm{e}^{-\mathrm{i}(\omega_0 U+\varphi_0)}]\end{pmatrix} \tag{B.4}$$

where Re and Im denote the real and imaginary parts, and where A is a constant complex amplitude given by

$$A = -\mathrm{i}\left[\ln\left(\frac{\mathcal{T}}{2b}\right) + \frac{11}{12}\right] - \omega_0\mathcal{T}\int_0^1 \mathrm{d}x\ \ln x\,\mathrm{e}^{\mathrm{i}\omega_0\mathcal{T}x} - \mathrm{i}\int_1^{+\infty}\frac{\mathrm{d}x}{x}\mathrm{e}^{\mathrm{i}\omega_0\mathcal{T}x}\,. \tag{B.5}$$

The value of A readily follows from the identity (3.2) in the text. However, having in view a better comparison with the case of the decaying orbit below, we introduce, before using (3.2), a dimensionless parameter $\xi = 2\pi/(\omega_0 \mathcal{T})$, and we compute the value of A in the limit $\xi \to 0$. This will give the correct result since A does not depend on $\mathcal{T}$ nor on ξ. We have

$$A = -\mathrm{i}\left[\ln\left(\frac{\pi}{\xi\omega_0 b}\right) + \frac{11}{12}\right] - \frac{2\pi}{\xi}\int_0^1 \mathrm{d}x\ \ln x\, \mathrm{e}^{2\pi \mathrm{i}x/\xi} - \mathrm{i}\int_1^{+\infty} \frac{\mathrm{d}x}{x}\, \mathrm{e}^{2\pi \mathrm{i}x/\xi}\,. \tag{B.6}$$

The last integral is, when $\xi \to 0$, of order $\mathrm{O}(\xi)$, as can be shown for instance by integrating the integral by parts. Hence we write

$$A = -\mathrm{i}\left[\ln\left(\frac{\pi}{\xi\omega_0 b}\right) + \frac{11}{12}\right] - \frac{2\pi}{\xi}\int_0^1 \mathrm{d}x\ \ln x\, \mathrm{e}^{2\pi \mathrm{i}x/\xi} + \mathrm{O}(\xi) \tag{B.7}$$

and, by using finally the identity (3.2) with $\lambda = 2\pi/\xi \to +\infty$, we obtain

$$A = \frac{\pi}{2} + \mathrm{i}\left[\ln(2\omega_0 b) + C - \frac{11}{12}\right]. \tag{B.8}$$

We have suppressed the terms of order $\mathrm{O}(\xi)$ since these terms are zero (A does not depend on ξ). With (B.8), we recover the expression (4.7) computed in the text.

B2. Case of a decaying circular orbit

We shall admit that the instantaneous components of the quadrupole moment are given by

$$Q_{xx}(U) = \tfrac{1}{2}\mu a^2(U)[1 + \cos\varphi(U)] \tag{B.9a}$$

$$Q_{xy}(U) = \tfrac{1}{2}\mu a^2(U)\sin\varphi(U) \tag{B.9b}$$

$$Q_{yy}(U) = \tfrac{1}{2}\mu a^2(U)[1 - \cos\varphi(U)]\,. \tag{B.9c}$$

We denote by $a(U)$, $\omega(U)$ and $\varphi(U)$ the radius, twice-frequency and associated phase of the decaying orbit. They are given by (4.8)–(4.10) in the text, namely

$$a(U) = \alpha[\tau_c(U)]^{1/4} \tag{B.10a}$$

$$\omega(U) = \beta^{-1}[\tau_c(U)]^{-3/8} \tag{B.10b}$$

$$\varphi(U) = -\frac{8}{5\beta}[\tau_c(U)]^{5/8} + \varphi_c \tag{B.10c}$$

where $\tau_c(U) = t_c - U$ is the coalescing time. We introduce also, as in (4.11), the parameter

$$\xi(U) = \frac{2\pi}{\omega(U)\tau_c(U)} = 2\pi\beta[\tau_c(U)]^{-5/8} \tag{B.11}$$

and we do the computation in the case where the parameter $\xi(U)$ is small. In this case, we can neglect the variations of the radius $a(U)$ and of the frequency $\omega(U)$, and we can consider only the variation of the phase $\varphi(U)$. The errors made in this way are of order $\xi(U)$. Thus, by inserting the components of the quadrupole moment (B.9) into the wave tail (B.2), we obtain error terms of order $\xi(U)$ coming from the first term in (B.2), and also terms involving $\xi(U - \mathcal{T}x)$ where x can take any value from 0 to $+\infty$. The absolute values of the latter terms can, however, be majored by $\xi(U)$ times an absolutely convergent integral (using

$\xi(U - \mathcal{T}x) \leqslant \xi(U)$ for any $x \geqslant 0$), and thus these terms are also of order $\xi(U)$. Hence we can write, similarly to (B.4),

$$\begin{pmatrix} T_{xx}(U) \\ T_{xy}(U) \\ T_{yy}(U) \end{pmatrix} = \frac{2G\mu M\omega(U)}{a(U)} \begin{pmatrix} -\operatorname{Re}[A(U)\,\mathrm{e}^{-\mathrm{i}\varphi(U)}] \\ \operatorname{Im}[A(U)\,\mathrm{e}^{-\mathrm{i}\varphi(U)}] \\ \operatorname{Re}[A(U)\,\mathrm{e}^{-\mathrm{i}\varphi(U)}] \end{pmatrix} + \mathrm{O}[\xi(U)] \tag{B.12}$$

where the complex amplitude $A(U)$ is now a function of time, and is given by

$$A(U) = -\mathrm{i}\left[\ln\left(\frac{\mathcal{T}}{2b}\right) + \frac{11}{12}\right] - \mathcal{T}\frac{a(U)}{\omega(U)}\,\mathrm{e}^{\mathrm{i}\varphi(U)} \int_0^1 \mathrm{d}x\, \ln x\, \frac{\omega^2(U - x\mathcal{T})}{a(U - x\mathcal{T})}\,\mathrm{e}^{-\mathrm{i}\varphi(U - x\mathcal{T})}$$
$$- \mathrm{i}\frac{a(U)}{\omega(U)}\,\mathrm{e}^{\mathrm{i}\varphi(U)} \int_1^{+\infty} \frac{\mathrm{d}x}{x} \frac{\omega(U - x\mathcal{T})}{a(U - x\mathcal{T})}\,\mathrm{e}^{-\mathrm{i}\varphi(U - x\mathcal{T})}. \tag{B.13}$$

At this stage, it is convenient to choose the constant $\mathcal{T}$ in (B.13) to be the coalescing time $\tau_c(U)$ left till coalescence from time U:

$$\mathcal{T} = \tau_c(U) = t_c - U. \tag{B.14}$$

With this choice, we can easily express the early values of the radius and of the frequency (at times $U - \tau_c(U)x$) in terms of their current values (at time U) and of some powers of $1 + x$. We find

$$a(U - \tau_c(U)x) = a(U)(1 + x)^{1/4} \tag{B.15a}$$

$$\omega(U - \tau_c(U)x) = \omega(U)(1 + x)^{-3/8}. \tag{B.15b}$$

Similarly, using also the expression of the parameter $\xi(U)$ of (B.11), we can express the phase as

$$\varphi(U - \tau_c(U)x) = \varphi(U) - \frac{2\pi}{\xi(U)}\frac{8}{5}[(1 + x)^{5/8} - 1]. \tag{B.15c}$$

Substituting (B.15) into (B.13), we obtain the amplitude $A(U)$ in the form

$$A(U) = -\mathrm{i}\left[\ln\left(\frac{\pi}{\xi(U)\omega(U)b}\right) + \frac{11}{12}\right] - \frac{2\pi}{\xi(U)} \int_0^1 \frac{\mathrm{d}x\, \ln x}{1 + x}\,\mathrm{e}^{2\pi\mathrm{i}\frac{8}{5}[(1+x)^{5/8} - 1]/\xi(U)}$$
$$- \mathrm{i}\int_1^{+\infty} \frac{\mathrm{d}x}{x(1 + x)^{5/8}}\,\mathrm{e}^{2\pi\mathrm{i}\frac{8}{5}[(1+x)^{5/8} - 1]/\xi(U)}. \tag{B.16}$$

This expression should be compared to the expression (B.6) we obtained in the case of a fixed orbit. Now, when $\xi(U) \ll 1$, we easily find that the last integral in (B.16) is of order $\mathrm{O}[\xi(U)]$. This can be shown by integrating by parts. Hence we have

$$A(U) = -\mathrm{i}\left[\ln\left(\frac{\pi}{\xi(U)\omega(U)b}\right) + \frac{11}{12}\right] - \frac{2\pi}{\xi(U)} \int_0^1 \frac{\mathrm{d}x\, \ln x}{1 + x}\,\mathrm{e}^{2\pi\mathrm{i}\frac{8}{5}[(1+x)^{5/8} - 1]/\xi(U)} + O[\xi(U)]. \tag{B.17}$$

Thus, it remains to prove that the integral in (B.17) is in fact equivalent, when $\xi(U) \to 0$, to the simpler integral appearing in (B.7). Indeed, when $\xi(U) \to 0$, the phase of the integrand in

(B.17) oscillates very rapidly, except when the variable x tends to zero. Thus, we expect that the main contribution of the integral will come from the neighbourhood of the limit $x = 0$, where the two integrands in (B.17) and (B.7) are equivalent. A direct proof that (B.17) reduces to (B.7) when $\xi(U) \to 0$ is as follows. We denote the integral in (B.17) by

$$I[\xi(U)] = \frac{2\pi}{\xi(U)} \int_0^1 \frac{\mathrm{d}x \, \ln x}{1+x} \, \mathrm{e}^{2\pi \mathrm{i} \frac{8}{5}[(1+x)^{5/8}-1]/\xi(U)} \tag{B.18}$$

and we introduce the new variable $y = \frac{8}{5}[(1+x)^{\frac{5}{8}} - 1]$. We get

$$I[\xi(U)] = \frac{2\pi}{\xi(U)} \int_0^\alpha \frac{\mathrm{d}y}{1+\frac{5}{8}y} \ln\left[(1+5y/8)^{\frac{8}{5}} - 1\right] \mathrm{e}^{2\pi \mathrm{i} y/\xi(U)} \tag{B.19}$$

where $\alpha = \frac{8}{5}[2^{\frac{5}{8}} - 1]$. The integral (B.19) is then reduced in several steps which can all be proved by integrations by parts and by neglecting small terms of order $\mathrm{O}[\xi(U)]$ and $\mathrm{O}[\xi(U) \ln \xi(U)]$. We successively obtain

$$\begin{aligned}
I[\xi(U)] &= \frac{2\pi}{\xi(U)} \int_0^\alpha \frac{\mathrm{d}y \ln y}{1+\frac{5}{8}y} \, \mathrm{e}^{2\pi \mathrm{i} y/\xi(U)} + \mathrm{i} \frac{\ln \alpha}{1+\frac{5}{8}\alpha} \, \mathrm{e}^{2\pi \mathrm{i}\alpha/\xi(U)} + \mathrm{O}[\xi(U)] \\
&= \frac{2\pi}{\xi(U)} \int_0^1 \frac{\mathrm{d}y \ln y}{1+\frac{5}{8}y} \, \mathrm{e}^{2\pi \mathrm{i} y/\xi(U)} + \mathrm{O}[\xi(U)] \\
&= \frac{2\pi}{\xi(U)} \int_0^1 \mathrm{d}y \, (1-5y/8) \ln y \, \mathrm{e}^{2\pi \mathrm{i} y/\xi(U)} + \mathrm{O}[\xi(U)] \\
&= \frac{2\pi}{\xi(U)} \int_0^1 \mathrm{d}y \ln y \, \mathrm{e}^{2\pi \mathrm{i} y/\xi(U)} + \mathrm{O}[\xi(U) \ln \xi(U)]
\end{aligned} \tag{B.20}$$

which shows that the function $A(U)$ of (B.17) can be rewritten as

$$A(U) = -\mathrm{i}\left[\ln\left(\frac{\pi}{\xi(U)\omega(U)b}\right) + \frac{11}{12}\right] - \frac{2\pi}{\xi(U)} \int_0^1 \mathrm{d}x \, \ln x \, \mathrm{e}^{2\pi \mathrm{i} x/\xi(U)} + \mathrm{O}[\xi(U) \ln \xi(U)] \tag{B.21}$$

and is thus given in the limit $\xi(U) \to 0$ by

$$A(U) = \frac{\pi}{2} + \mathrm{i}\left[\ln(2\omega(U)b) + C - \frac{11}{12}\right] + \mathrm{O}[\xi(U) \ln \xi(U)] \, . \tag{B.22}$$

This establishes the fact that the wave field (4.12) in the case of the decaying orbit can be obtained, in the limit $\xi(U) \to 0$, by formal replacement in the wave field (4.7) of the constant frequency ω_0 by the varying frequency $\omega(U)$, and by formal replacement of the linear phase $\omega_0 U + \varphi_0$ by the integral $\varphi(U)$ of the frequency $\omega(U)$.

References

[1] Bradaschia C *et al* 1991 *Proc. Banff Summer Inst. on Gravitation (Banff 1990)* ed R Mann and P Wesson (Singapore: World Scientific)
[2] Abramovici A *et al* 1992 *Science* **256** 325
[3] Thorne K S 1987 *300 Years of Gravitation* ed S W Hawking and W Israel (Cambridge: Cambridge University Press)
[4] Taylor J H and Weisberg J M 1982 *Astrophys. J.* **253** 908
[5] Taylor J H 1989 *Timing Neutron Stars* ed H Ögelman and E P J van den Heuvel (Dordrecht: Reidel)
[6] Taylor J H and Weisberg J M 1989 *Astrophys. J.* **345** 434
[7] Damour T and Taylor J H 1991 *Astrophys. J.* **366** 501
[8] Wolszczan A 1991 *Nature* **350** 688
[9] Taylor J H, Wolszczan A, Damour T and Weisberg J M 1992 *Nature* **355** 132
[10] Peters P C and Mathews J 1963 *Phys. Rev.* **131** 435
[11] Esposito L W and Harrison E R 1975 *Astrophys. J.* **196** L1
[12] Wagoner R V 1975 *Astrophys. J.* **196** L63
[13] Damour T 1983 *Gravitational Radiation* ed N Deruelle and T Piran (Amsterdam: North-Holland) p 59
[14] Damour T 1983 *Phys. Rev. Lett.* **51** 1019
[15] Will C M 1993 *Theory and Experiment in Gravitational Physics* (revised edition) (Cambridge: Cambridge University Press)
[16] Bonnor W B and Rotenberg M A 1961 *Proc. R. Soc.* A **265** 109
[17] Papapetrou A 1971 *Ann. Inst. H. Poincaré* XIV 79
[18] Bekenstein J D 1973 *Astrophys. J.* **183** 657
[19] Bertotti B and Plebanski J 1960 *Ann. Phys., NY* **11** 169
[20] Bonnor W B 1959 *Phil. Trans. R. Soc.* A **251** 233
[21] Bonnor W B and Rotenberg M A 1966 *Proc. R. Soc.* A **289** 247
[22] Thorne K S and Kovács S J 1975 *Astrophys. J.* **200** 245
[23] Thorne K S 1980 *Rev. Mod. Phys.* **52** 299
[24] Bel L, Damour T, Deruelle N, Ibañez J and Martin J 1981 *Gen. Rel. Grav.* **13** 963
[25] Westpfahl K 1985 *Forschr. Phys.* **33** 417
[26] Blanchet L and Damour T 1986 *Phil. Trans. R. Soc.* A **320** 379
[27] Peters P C 1966 *Phys. Rev.* **146** 938
[28] Kundt W and Newman E T 1968 *J. Math. Phys.* **9** 2193
[29] Price R H 1972 *Phys. Rev.* D **5** 2419
[30] Price R H 1972 *Phys. Rev.* D **5** 2439
[31] Bardeen J M and Press W H 1973 *J. Math. Phys.* **14** 7
[32] Schmidt B G and Stewart J M 1979 *Proc. R. Soc.* A **367** 503
[33] Porrill J and Stewart J M 1981 *Proc. R. Soc.* A **376** 451
[34] Leaver E W 1986 *Phys. Rev.* **34** 384
[35] Couch W E, Torrence R J, Janis A I and Newman E T 1968 *J. Math. Phys.* **9** 484
[36] Hunter A J and Rotenberg M A 1969 *J. Phys. A: Math. Gen.* **2** 34
[37] Bonnor W B 1974 *Ondes et Radiations Gravitationnelles* (Paris: CNRS) p 73
[38] Blanchet L and Damour T 1990 Tail effects in the generation of gravitation radiation *Thèse d'habilitation* L Blanchet, Université P et M Curie, p 195 (unpublished)
[39] Schäfer G 1990 *Astron. Nachr.* **311** 213
[40] Rotenberg M A 1971 *J. Phys. A: Math. Gen.* **4** 617
[41] Couch W E and Hallidy W H 1971 *J. Math. Phys.* **12** 2170
[42] Papapetrou A 1975 *J. Phys. A: Math. Gen.* **8** 313
[43] Linet B 1975 *J. Phys. A: Math. Gen.* **8** 328
[44] Blanchet L and Damour T 1989 *Ann. Inst. H. Poincaré (Physique Théorique)* **50** 377
[45] Damour T and Iyer B R 1991 *Ann. Inst. H. Poincaré (Physique Théorique)* **54** 115
[46] Blanchet L and Damour T 1992 *Phys. Rev.* D **46** 4304
[47] Payne P N 1983 *Phys. Rev.* D **28** 1894
[48] Christodoulou D 1991 *Phys. Rev. Lett.* **67** 1486
[49] Thorne K S 1992 *Phys. Rev.* D **45** 520
[50] Wiseman A G and Will C M 1991 *Phys. Rev.* D **44** R2945
[51] Blanchet L and Damour T 1988 *Phys. Rev.* D **37** 1410
[52] Blanchet L 1993 *Phys. Rev.* D **47** 4392
[53] Burke W L 1969 *PhD Thesis* California Institute of Technology (unpublished)

[54] Thorne K S 1969 *Astrophys. J.* **158** 997
[55] Burke W L 1971 *J. Math. Phys.* **12** 401
[56] De Witt B S and Brehme R W 1960 *Ann. Phys.* **9** 220
[57] Rudolph E 1975 *Ann. Inst. H. Poincaré* XXIII 113
[58] Schutz B F 1989 *The Detection of Gravitational Radiation* ed D Blair (Cambridge: Cambridge University Press)
[59] Sathyaprakash B S and Dhurandhar S V 1991 *Phys. Rev.* D **44** 3819
[60] Blanchet L and Schäfer G 1989 *Mon. Not. R. Astron. Soc.* **239** 845
[61] Poisson E 1993 *Phys. Rev.* D **47** 1497
[62] Wiseman A G 1992 *PhD Thesis* Washington University (unpublished)
[63] Cutler C, Finn L S, Poisson E and Sussmann G J 1993 *Phys. Rev.* D **47** 1511
[64] Cutler C *et al* 1993 *Phys. Rev. Lett.* **70** 2984
[65] Epstein R and Wagoner R V 1975 *Astrophys. J.* **197** 717
[66] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series and Products* (New York: Academic)
[67] Wagoner R V and Will C M 1976 *Astrophys. J.* **210** 764
[68] Krolak A and Schutz B F 1987 *Gen. Rel. Grav.* **19** 1163
[69] Lincoln C W and Will C M 1990 *Phys. Rev.* D **42** 1123
[70] Wiseman A G 1992 *Phys. Rev.* D **46** 1517
[71] Junker W and Schäfer G 1992 *Mon. Not. R. Astron. Soc.* **254** 146
[72] Peters P C 1964 *Phys. Rev.* **136** B1224
[73] Damour T and Deruelle N 1985 *Ann. Inst. H. Poincaré (Physique Théorique)* **43** 107
[74] Damour T and Deruelle N 1986 *Ann. Inst. H. Poincaré (Physique Théorique)* **44** 263
[75] Fourès-Bruhat Y 1952 *Acta Math.* **88** 141
[76] Bruhat Y 1964 *Ann. Mat. Pura Appl.* **64** 191
[77] Friedlander F G 1975 *The Wave Equation on a Curved Space-time* (Cambridge: Cambridge University Press)
[78] Waylen P C 1978 *Proc. R. Soc.* A **362** 233
[79] Choquet-Bruhat Y, Christodoulou D and Francaviglia M 1979 *Ann. Inst. H. Poincaré* A **31** 399
[80] Carminati J and McLenaghan R G 1986 *Ann. Inst. H. Poincaré (Physique Théorique)* **44** 115
[81] McLenaghan R G 1969 *Proc. Camb. Phil. Soc.* **65** 139