MULTIPOLAR RADIATION REACTION IN GENERAL RELATIVITY

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Received 4 January 1984
Revised manuscript received 6 June 1984

We point out that several previous derivations of quadrupolar radiation reaction based on "matched asymptotic expansions" were incomplete because of the lack of consideration of non-linear effects in the exterior gravitational field. Using a post minkowskian algorithm together with multipolar expansions we show how to take into account the non-linear effects. We derive an explicit expression for the radiation reaction force density when the first radiating multipoles are of order $l \geqslant 2$. When $l = 2$ we recover the Burke–Thorne reaction potential.

The announcement of the observation of secular kinematical effects in the binary pulsar [1] has motivated recently several new calculations of gravitational radiation reaction in slowly moving gravitationally bound systems. Many different lines of attack have been used, among them: post newtonian methods [2], asymptotic matching methods [3], balance equations approaches [4], initial value approaches [5], and post minkowskian methods [6]. Up to now the only complete calculation of the secular kinematical effects happening in gravitationally bound systems, as well as the only one applicable to the binary pulsar, has been performed by using a post minkowskian method [7]. However, the latter calculation applies only to a particular system (a two compact body system) and at the lowest order where irreversible effects appear ("quadrupolar damping"). On the other hand the existence of systems where the quadrupolar gravitational radiation is strongly suppressed [8], and the astrophysical importance of gravitational radiation recoil [9] where the coupling between quadrupole and higher multipoles is essential, makes it important to investigate the radiation reaction due to higher multipoles in more general systems. The purpose of this letter is: (i) to point out that several previous derivations of *quadrupolar* radiation reaction, based on asymptotic matching method [10,11,3], were incomplete because they did not take into account the non-linear effects in the "exterior region", and (ii) to complete the latter "quadrupole" derivations, to generalize this approach to the case where all the multipoles of order $< l$ are constant and to compute the radiation reaction caused by the first radiating multipoles (either of "mass" or "current" type). Concerning the criticism (i) above, let us emphasize, firstly that it does not apply to the calculations of refs. [12] and [13] whose validity is however restricted to sources having a spherical or axial symmetry (this symmetry allowing one to meaningfully consider linearized perturbations on an exact (non-linearized) background), and, secondly that this criticism goes beyond the one of ref. [14] which pointed out the need to take into account the non-linearities in the "resistive" *interior* field (or "near-zone" field). Here we show the need to consider also the non-linearities in the exterior field (see e.g. eq. (8d) below: this is a non-linear term in the exterior field which has not been taken into account in ref. [3] which however meets the criticism [14] for the near-zone field).

The basic idea of "matching methods" is to determine the gravitational field generated by a bounded system both in an "interior region" (or "near zone") which covers the system and extends up to a radius r_1 away from it, and in an "exterior region" which starts at a radius $r_0 < r_1$ outside the system, and then to match the two metrics in the overlapping region $r_0 < r < r_1$. Let us make clear that in our approach r_0 is only restricted to be greater than the radius of the system and not to be a large multiple of the gravitational wave length λ. This is necessary for allowing one to match the exterior metric (determined below) to a post newtonian expansion (or "near zone" expansion), valid only for $r_1 \ll \lambda$, of the interior metric. Correspondingly we shall make no use of "rescaled" variables (e.g. $r^* = r/\lambda$), the use of which can dangerously hide the necessity of considering non-linear effects in the exterior region.

As a basis for our work we use the multipole moment formalism of ref. [15] together with a procedure of complex analytic continuation. We wish to construct the most general solution of the vacuum Einstein equations which admit a post minkowskian expansion (G := Newton's constant, $f^{\alpha\beta}$:= flat metric):

$$|g|^{1/2} g^{\alpha\beta} = f^{\alpha\beta} + G h^{\alpha\beta}_{(1)} + \ldots + G^n h^{\alpha\beta}_{(n)} + \ldots . \quad (1)$$

Replacing the expansion (1) into Einstein's vacuum equations, and, using harmonic coordinates, lead to the following equations ($\Box := \Delta - c^{-2}\partial_t^2$):

$$\Box h^{\alpha\beta}_{(n)} = N^{\alpha\beta}_{(n)}(h_{(m)}) , \qquad \partial_\beta h^{\alpha\beta}_{(n)} = 0 , \quad (2,3)$$

where $N_{(n)}$ is a polynomial in $h_{(m)}$ and its first two derivatives ($m < n$, $N_1 \equiv 0$). We shall take for $h^{\alpha\beta}_{(1)}$ the explicit expression (8.12) of ref. [15] given as a sum of terms of the type $\partial_{i_1 i_2 \ldots i_l}[r^{-1}F(t - r/c)]$, where the functions F are the "mass" or "current" multipole moments or their derivatives. We shall denote these moments respectively by M_L and $\epsilon_{abc}S_{cL}$ where $L := (i_1, i_2 \ldots i_l)$ is a multi-index of order l. M_L and S_L are symmetric trace-free tensors, M and S_i are constant and $M_i = 0$. In order to have a well-defined iteration at any order which incorporates Fock's "no incoming radiation" condition we shall assume here that all the multipole moments are stationary before a fixed instant in the past. Starting from $h^{\alpha\beta}_{(1)}[M_L, S_L]$ we define recursively $h^{\alpha\beta}_{(n)}$ from $N^{\alpha\beta}_{(n)}(h_{(m)})$ ($m < n$) as:

$$h^{\alpha\beta}_{(n)} := p^{\alpha\beta}_{(n)} + q^{\alpha\beta}_{(n)} , \quad (4)$$

$$p^{\alpha\beta}_{(n)} := \text{Finite part}_{B=0}\, \Box^{-1}_{\text{ret}}[(r/c)^B N^{\alpha\beta}_{(n)}(h_{(m)})] , \quad (5)$$

$$\partial_\beta q^{\alpha\beta}_{(n)} = -\text{Residue}_{B=0}\, \Box^{-1}_{\text{ret}}[(r/c)^B r^{-1} n_i N^{i\alpha}_{(n)}(h_{(m)})] , \quad (6)$$

where $\Box^{-1}_{\text{ret}}$ denotes the usual retarded integral (all over space, including the vicinity of the "center" $r := (x^2 + y^2 + z^2)^{1/2} = 0$), B is a complex number, $n^i := x^i/r$, c is the velocity of light (introduced here only for convenience), and where the explicit expression of $q_{(n)}$ in terms of the r.h.s. of eq. (6) will be found in ref. [16]. The meaning of the r.h.s. of eq. (5) and (6) is the following: It can be shown that each term of the multipolar expansion of $N_{(n)}$ in the exterior region $r > r_0 > 0$, is a function of r which can be (analytically) continued for $0 < r \leqslant r_0$ and has only a "tempered" singularity in $r = 0$ (less than some r^{-k}). Therefore, because of our assumption of stationarity of the multipole moments in the past, each integral appearing in (5), (6) will be well defined as a usual integral all over space for B pertaining to some strip in the complex plane, and then can be uniquely analytically continued all over the complex plane except for multiple poles at integer values of B. The Laurent expansion of these functions near $B = 0$ then defines (5) (the B^0 term) and (6) (the B^{-1} term). Because $h^{\alpha\beta}_{(1)}$ represents the most general linearized solution which is stationary in the remote past (when using "arbitrary" sequences of multipole moments which are constant in the past) it can be proven formally that our construction: $\Sigma\, G^n h^{\alpha\beta}_{(n)}[M_L, S_L]$, which is uniquely defined in terms of the preceding sequences of multipole moments, yields, modulo an arbitrary coordinate transformation $\Sigma\, G^n \xi^\alpha_{(n)}$, the most general formal power series solution of the vacuum Einstein equations which is stationary in the past (taking for granted the convergence of all the multipole expansions).

If we endow the multipole moments with their usual physical dimensions (in terms of length, mass and velocity units) each $h^{\alpha\beta}_{(n)}$ becomes a function of t, $\boldsymbol{x}$ and c. We have proven [16] that each $h^{\alpha\beta}_{(n)}(t, \boldsymbol{x}, c)$ admits an asymptotic expansion for $c \to \infty$ on the scale functions $(\log c)^p/c^q$ ($p, q \in \mathbb{N}$). The resulting expansion can be called the "near zone expansion of the exterior field" or the "post newtonian expansion of the post minkowskian field" and the pith of the matching approach is to match this expansion to the "near zone" (or "post newtonian") expansion of the "near zone field". We have derived simple formulae for computing the former expansion without having to compute explicitly the post minkowskian $h_{(n)}$. If we consider the case where the low order multipole moments are always stationary and where the first "radiating" moments are $M_{i_1 \ldots i_l}$ and/or $\epsilon_{i_1 i_2 c} S_{c i_3 \ldots i_l}$ (usually $l = 2$), we found that, for any iteration order n, the "near zone" expansion of the non-linear exterior term $p^{\alpha\beta}_{(n)}$ is given by (i being the number of spatial indices among α, β):

$$\bar{p}^{\alpha\beta}_{(n)} = \text{Finite part}_{B=0} \sum_{k \geqslant 0} \frac{1}{c^{2k}} \frac{\partial^{2k}}{\partial t^{2k}} \Delta^{-k-1} [(r/c)^B \bar{N}^{\alpha\beta}_{(n)}] + \mathrm{O}[(\log c)^{n-1}/c^{3n+2l-i}] , \tag{7}$$

where the bars over $p^{\alpha\beta}$ and $N^{\alpha\beta}$ denote the near zone expansion, and where the iterated inverse laplacians of eq. (7) are defined by analytic continuation in B. Because of the remarkable absence of explicit odd powers of c^{-1} in (7) it is especially simple to compute the "odd" terms in $\bar{p}^{\alpha\beta}$: we mean by "odd" terms the coefficients of c^{-q} with $i + q$ odd, i.e. the terms that change sign upon time reversal, or the "resistive" terms (there are no "log c" in these terms at lowest order).

Simple dimensional arguments show beforehand that in order to get the complete radiation reaction we need to compute among $\bar{h}^{\alpha\beta} := \sum_{n=1}^{\infty} G^n \bar{h}^{\alpha\beta}_{(n)}$, the $c^{-(2l+3)}$ parts of $\bar{h}^{00}$ and $\bar{h}^{ss}$ (denoted ${}_{2l+3}h^{00}$ and ${}_{2l+3}h^{ss}$), as well as ${}_{2l+2}h^{0j}$ and ${}_{2l+1}h^{jk}$. Now the point is that $G^n \bar{h}^{\alpha\beta}_{(n)}$ starts at order c^{-2n} ($\sim (GM/rc^2)^n$), therefore it is indispensable to control not only the linearized $\bar{h}^{\alpha\beta}_{(1)}$, but also the non-linear $\bar{h}^{\alpha\beta}_{(2)}, \bar{h}^{\alpha\beta}_{(3)}, \ldots, \bar{h}^{\alpha\beta}_{(l+1)}$ (with $l \geqslant 2$) before being sure to have included all the contributions to ${}_{2l+3}h^{00}$ for instance. This has been possible thanks to the previous formula (7) for $\bar{p}^{\alpha\beta}_{(n)}$, together with a related formula for $\bar{q}^{\alpha\beta}_{(n)}$. We find (all lowest order odd terms being zero):

$$\tfrac{1}{2}({}_{2l+3}h^{00}_{(1)} + {}_{2l+3}h^{ss}_{(1)}) = [(-1)^l 2^{l+1}/(2l+1)!]\, x^L M_L^{(2l+1)}(t) , \tag{8a}$$

$${}_{2l+2}h^{0j}_{(1)} = [(-1)^{l+1} 2^{l+1}/l(2l-1)!]\, [x^{L-1} M^{(2l)}_{j\,L-1}(t) + (l-1)\epsilon_{jab} x^{b\,L-2} S^{(2l-1)}_{a\,L-2}(t)] , \tag{8b}$$

$${}_{2l+1}h^{jk}_{(1)} = [(-1)^l 2^l/l(l-1)(2l-3)!]\, [x^{L-2} M^{(2l-1)}_{jk\,L-2}(t) + 2(l-1) x^{b\,L-2} \epsilon_{ab(j} S^{(2l-2)}_{k)a\,L-3}(t)] , \tag{8c}$$

$$\tfrac{1}{2}({}_{2l+3}h^{00}_{(2)} + {}_{2l+3}h^{ss}_{(2)}) = [(-1)^{l+1} 2^{l-1}/l(l-1)^2(2l-3)!]\, [x^{L-1} M^{(2l-1)}_{a\,L-1}(t) + \frac{2(l-1)^2}{l-2} \epsilon_{abc} x^{c\,L-2} S^{(2l-2)}_{b\,L-2}(t)]$$
$$\times \sum_{l'=0}^{\infty} \frac{2(-1)^{l'+1}}{l'!} (\partial_{aL'} r^{-1}) M_{L'}(t) + \sum_{l'=1}^{\infty} \frac{2(-1)^{l'+1}}{l'!} (\partial_{L'} r^{-1}) Q_{L'}(t) , \tag{8d}$$

$${}_{2l+2}h^{0j}_{(2)} = 0 ; \qquad {}_{2l+1}h^{jk}_{(2)} = 0 ; \qquad {}_{2l+3-i}h^{\alpha\beta}_{(n)} = 0 \quad \text{when} \quad n \geqslant 3 . \tag{8e}$$

In eqs. (8) $L-1$ denotes $(i_1, i_2, \ldots, i_{l-1})$, $M^{(p)} = \mathrm{d}^p M/\mathrm{d}t^p$, $n^L = x^L/r^l$ is the trace free part of $n^{i_1} n^{i_2} \ldots n^{i_l}$, and $Q_{L'}$ is a contracted tensor product of $M^{(2l-1)}_L$, $S^{(2l-2)}_{L-1}$ or $S^{(2l-3)}_{L-1}$ with $M_{L''}$ or $S_{L''}$. In order to stress the necessity of considering non-linear terms even if one intends only to do a partial matching (looking up the highest powers of r for instance) let us point out that the intermediate calculations of $h^{\alpha\beta}_{(n)}$ ($n \geqslant 2$) lead in addition to the type of terms appearing in (8d) to terms of the type $\sim P_{L'} x^{L'}$ similar to the terms, coming from $h_{(1)}$, which give rise to radiation reaction effects (see eqs. (8a) and (11)). Therefore in absence of a proof (contained in our detailed calculations leading to eqs. (8)) that these terms, of non-linear origin, do not appear, and thence do not modify the corresponding terms coming from $h_{(1)}$, all the derivations of radiation reaction based on the sole linearized approximation of the exterior field (like in refs. [10,11,3]) are certainly inconclusive.

Let us now perform the following coordinate transformation in the overlapping region between the interior and the exterior region ($r_0 < r < r_1 \ll \lambda$): $x'^\mu = x^\mu + \xi^\mu$ with:

$$\xi_0 = -\xi^0 = (1/c^{2l+2})[(-1)^l 2^l/l^2(l-1)(2l-1)!]\, x^L M_L^{(2l)}(t) , \tag{9a}$$

$$\xi_i = (1/c^{2l+1})[(-1)^{l+1} 2^{l-1}/l(l-1)^2(2l-3)!]\, \{x^{L-1} M^{(2l-1)}_{i\,L-1}(t) + [2(l-1)^2/(l-2)]\, \epsilon_{iab} x^{b\,L-2} S^{(2l-2)}_{a\,L-2}(t)\} . \tag{9b}$$

We find for the new "near-zone-expanded" exterior metric (going back to $g^{\alpha\beta}$ instead of $h^{\alpha\beta}$), for $r_0 < r < r_1$:

$$ {}_{2l+3}g'_{00} = -2V_{\rm R} + \sum_{l'=1}^{\infty} \frac{2(-1)^{l'}}{l'!}(\partial_{L'}r^{-1})Q_{L'}(t)\,, \quad {}_{2l+2}g'_{0j} = A^j_{\rm R}\,, \quad {}_{2l+1}g'_{jk} = 0\,, \tag{10} $$

with:

$$ V_{\rm R} = \frac{(-1)^l 2^l (l+1)(l+2)}{l(l-1)(2l+1)!}\, x^L M_L^{(2l+1)}(t)\,, \quad A^j_{\rm R} = \frac{(-1)^l 2^{l+1}(l+1)(l-1)}{l(l-2)(2l-1)!}\, \epsilon_{jab} x^{b\,L-2} S_{a\,L-2}^{(2l-1)}(t)\,. \tag{11} $$

Let us now assume that one has constructed a "near zone expanded" interior metric (for example for a slowly moving self-gravitating system of fluid balls) containing only "even" terms which matches our general near-zone-expanded exterior metric up to order $1/c^{2l+2-i}$ (this construction will be discussed in a forthcoming paper). Then, as it is easy to check that the expressions: $\delta g_{00} = c^{-2l-3}(-2V_{\rm R})$, $\delta g_{0j} = c^{-2l-2}A^j_{\rm R}$, $\delta g_{jk} = 0$, considered all over space, satisfy everywhere the linearized Einstein's vacuum equations (modulo terms of higher order in c^{-1}), it is clear that in order to match the interior and exterior fields we must: (1) add to the previous "even" interior metric the "odd" terms: δg_{00}, δg_{0j}, δg_{jk} and, (2) link the multipole moments, which, up to now, were only arbitrary parameters in the exterior metric, to the matter density ρ by equations of the type:

$$ M_{L'} + c^{-(2l'+1)}Q_{L'} = \int {\rm d}^3x\, \rho x^{L'} + \text{"even" corrections}\,. $$

Finally the "odd" terms δg_{00}, δg_{0j} in the interior metric imply a small "odd" correction to the local equations of motion ($T^{\alpha\beta}{}_{;\beta} = 0$) which can be expressed in the usual way as a reaction force density. We find it convenient to express it as an electromagnetic-like force:

$$ \boldsymbol{F}_{\rm R} = (1/c^{2l+1})\rho(\boldsymbol{E}_{\rm R} + \boldsymbol{v} \times \boldsymbol{B}_{\rm R})\,, \tag{12a} $$

with:

$$ \boldsymbol{E}_{\rm R} = -\nabla V_{\rm R} - \partial_t \boldsymbol{A}_{\rm R}\,, \quad \boldsymbol{B}_{\rm R} = \nabla \times \boldsymbol{A}_{\rm R}\,. \tag{12b} $$

The scalar and vector "reaction potentials" being given by eq. (11). When $l = 2$ (quadrupolar damping) the "vector reaction potential" vanishes (constant spin) and we recover the usual Burke–Thorne scalar potential. For mass moments $l \geqslant 3$ our $V_{\rm R}$ agrees with the related proposals of refs. [10,12]. We have shown that the reaction force $\boldsymbol{F}_{\rm R}$ causes the expected [15] secular decreases of energy and angular momentum of the system. The details of this work will be published elsewhere.

One of us (T.D.) would like to acknowledge stimulating discussions with J. Ehlers and B. Schmidt.

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