THEORY OF GRAVITATIONAL WAVE EMISSION

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PART 1

EINSTEIN FIELD EQUATIONS

AND

QUADRUPOLE MOMENT FORMALISM

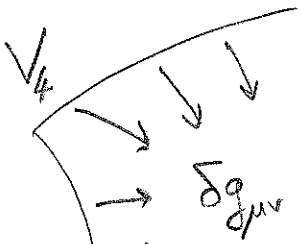
EINSTEIN FIELD EQUATIONS

They derive from the action

$$\boxed{S = \frac{c^3}{16\pi G}\int d^4x \sqrt{-g}\, R \;+\; S_m[g, \psi_m]}$$

Einstein-Hilbert action

matter action (all matter fields <u>universally coupled</u> to the metric $g_{\mu\nu}$)



(with $\delta g_{\mu\nu} = 0$ when $|x^\lambda| \to \infty$)

10 differential equations of second order

$$\underbrace{G^{\mu\nu}[g, \partial g, \partial^2 g]}_{\text{Einstein tensor}} = \frac{8\pi G}{c^4}\underbrace{T^{\mu\nu}}_{\text{stress-energy tensor}}[g]$$

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \qquad\qquad T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\mu\nu}}$$

4 eqs. give the evolution of matter fields

$$\boxed{\nabla_\nu G^{\mu\nu} \equiv 0 \quad \Rightarrow \quad \nabla_\nu T^{\mu\nu} = 0}$$

contracted Bianchi identity (or Einstein identity)

Geometry is governed by 6 eqs., 4 eqs. can be imposed by a choice of coordinates

$$h^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} - \eta^{\mu\nu} \qquad\qquad \eta^{\mu\nu} = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \;\; 1 \end{pmatrix}$$

auxiliary Minkowski metric (signature $-+++$)

<u>Choice of coordinates</u> $\boxed{\partial_\nu h^{\mu\nu} = 0}$

Harmonic or de Donder

$$\boxed{\Box h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu}}$$

ordinary flat d'Alembertian $\Box = \eta^{\rho\sigma}\partial_\rho\partial_\sigma$

stress-energy pseudo tensor (actually a Lorentz tensor) of matter and gravitational fields (in harm. coordinates)

$$\tau^{\mu\nu} = |g|\, T^{\mu\nu} + \underbrace{\frac{c^4}{16\pi G} \Lambda^{\mu\nu}(h, \partial h, \partial^2 h)}_{\text{includes all non-linearities of Einstein's eqs.}}$$

$\Lambda^{\mu\nu} = O(h^2)$

Harmonic coordinate condition is equivalent to matter equation

$$\boxed{\partial_\nu h^{\mu\nu} = 0 \iff \partial_\nu \tau^{\mu\nu} = 0 \iff \nabla_\nu T^{\mu\nu} = 0}$$

NO-INCOMING RADIATION CONDITION

Boundary conditions are imposed at past null infinity (case where $T^{\mu\nu}$ has a spatially compact support)

Spatio-temporal infinities

I^+ = future temporal infinity $\begin{pmatrix} t \to +\infty \\ r = \text{const} \end{pmatrix}$

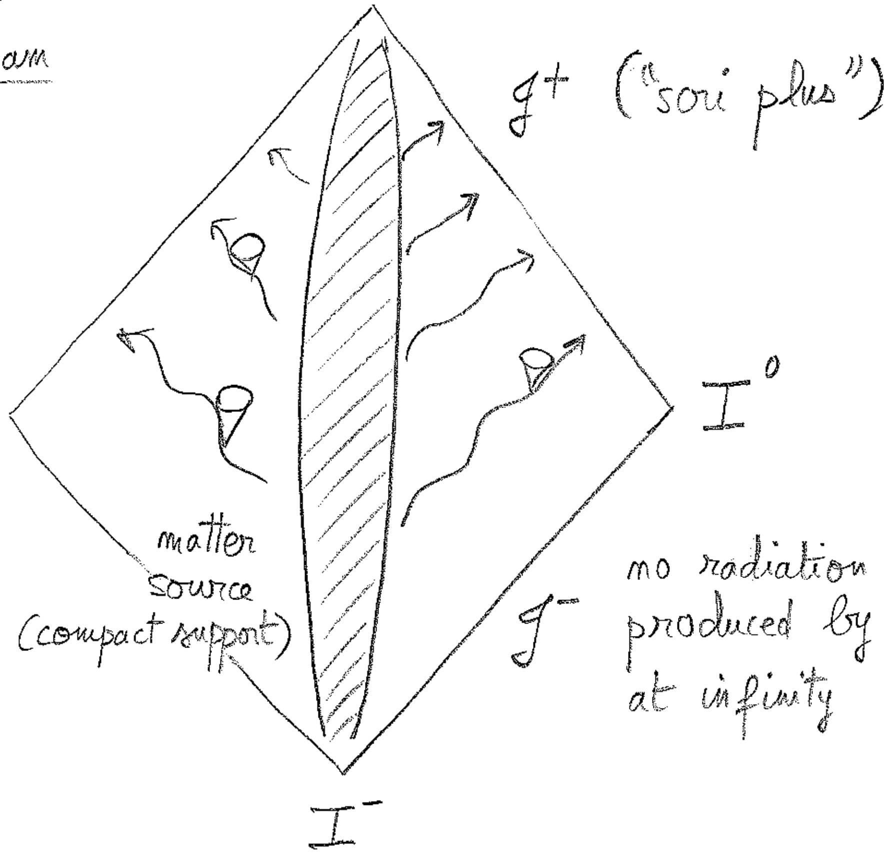
$\mathcal{J}^+$ = future null infinity $\begin{pmatrix} r \to +\infty \\ t - r/c = \text{const} \end{pmatrix}$

I^0 = spatial infinity $\begin{pmatrix} r \to +\infty \\ t = \text{const} \end{pmatrix}$

$\mathcal{J}^-$ = past null infinity $\begin{pmatrix} r \to +\infty \\ t + r/c = \text{const} \end{pmatrix}$

I^- = past temporal infinity $\begin{pmatrix} t \to -\infty \\ r = \text{const} \end{pmatrix}$

Carter-Penrose diagram

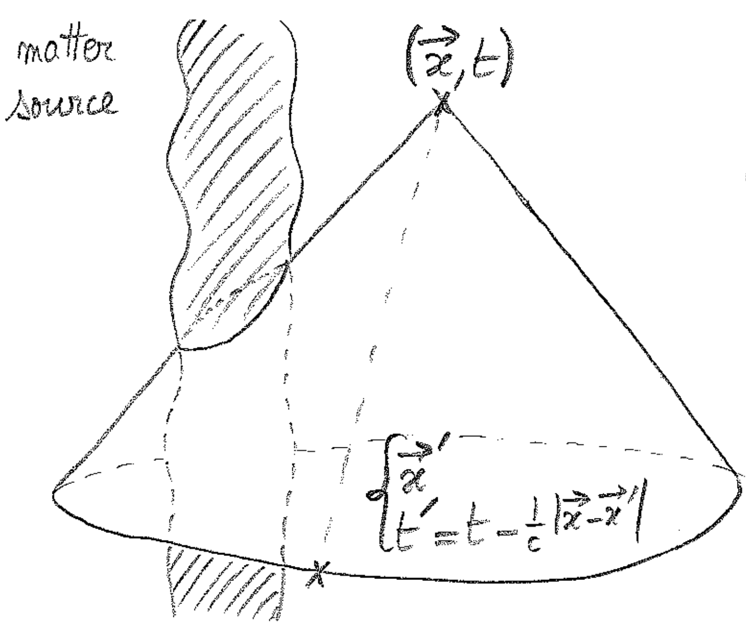


no radiation is produced by sources located at infinity

Kirchhoff's formula for the homogeneous sol. of

$$\Box\, h_{Hom} = 0$$

$$\boxed{h_{Hom}(\vec{x},t) = \lim_{|\vec{x}'| \longrightarrow \infty} \iint \frac{d\Omega'}{4\pi}\left(\frac{\partial}{\partial r} + \frac{1}{c}\frac{\partial}{\partial t}\right)(r\, h_{Hom})\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}$$



$(\vec{x},t)$ = field point

$(\vec{x}',t')$ = source point

No-incoming rad. cond. is

$$\boxed{\lim_{\mathscr{I}^-} \left(\frac{\partial}{\partial r} + \frac{1}{c}\frac{\partial}{\partial t} \right)(r\, h^{\mu\nu}) = 0}$$

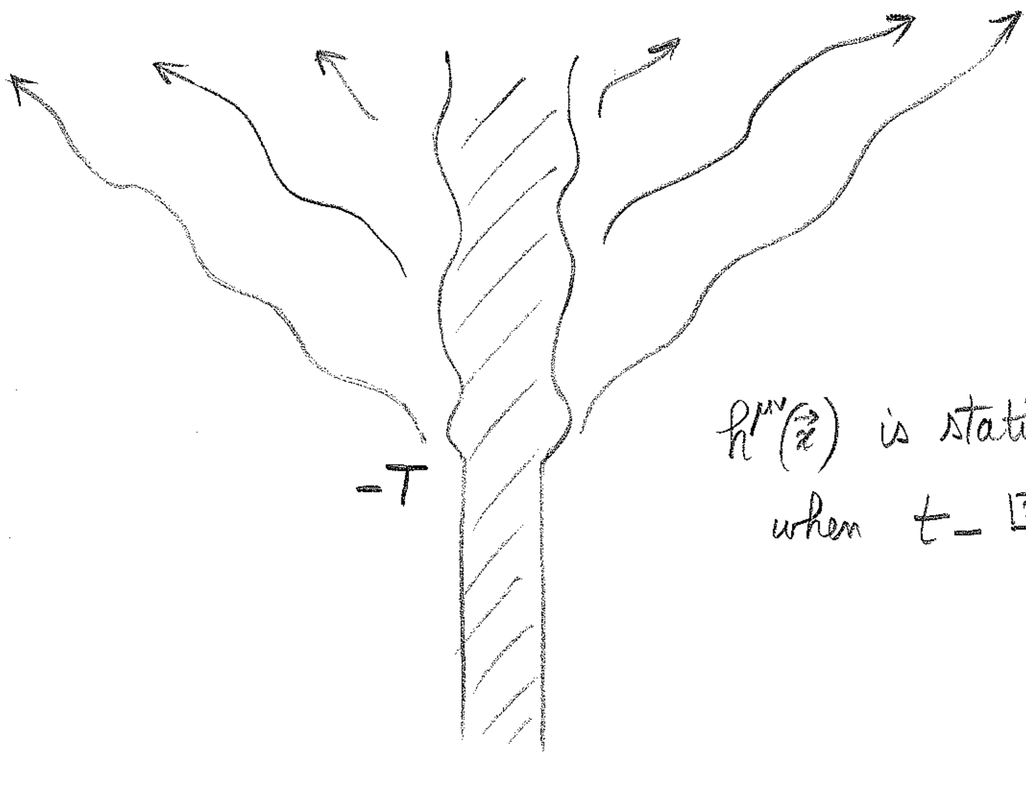
This excludes advanced waves $r\, h_{adv} \sim f(t + r/c)$ at $\mathscr{I}^-$

Einstein field eqs. can be solved (in an iterative way) by means of standard retarded integral in 3+1 dimensions

$$\boxed{h^{\mu\nu}(\vec{x}, t) = -\frac{4G}{c^4} \iiint \frac{d^3 \vec{x}'}{|\vec{x} - \vec{x}'|}\, \tau^{\mu\nu}\left(\vec{x}', t - \frac{1}{c}|\vec{x} - \vec{x}'|\right)}$$

note this is infact an integro-differential equation because $\tau^{\mu\nu}$ depends on $h, \partial h, \partial^2 h$

<u>Stationarity in the past</u> (simple way to implement the no-incoming rad. condition)



$h^{\mu\nu}(\vec{x})$ is stationary (ind. of t)
when $t - \frac{|\vec{x}|}{c} \lesssim -T$

LINEARIZED GRAVITATIONAL WAVES IN VACUUM

$$\begin{cases} \Box h^{\mu\nu} = 0 \\ \partial_\nu h^{\mu\nu} = 0 \end{cases} \qquad \left(\text{we neglect } \mathcal{O}(h^2)\right)$$

<u>Gauge transformation</u> preserving the harmonic cond. $\partial h = 0$

$$h'^{\mu\nu} = h^{\mu\nu} + \partial^\mu \xi^\nu + \partial^\nu \xi^\mu - \eta^{\mu\nu} \partial_\rho \xi^\rho$$

where $\Box \xi^\mu = 0$

Fourier decomposition

$$\boxed{h^{\mu\nu}(x) = \int d^4x \, H^{\mu\nu}(k) \, e^{i k_\lambda x^\lambda}}$$

Fourier amplitude of monochromatic wave $k_\lambda = \begin{pmatrix}\text{wave}\\ \text{vector}\end{pmatrix}$

$$\boxed{\begin{aligned} & k^2 \equiv \eta_{\mu\nu} k^\mu k^\nu = 0 \\ & k_\nu H^{\mu\nu} = 0 \end{aligned}}$$

Can perform a gauge transf.

with <u>any</u> $\xi^\mu(x) = \int d^4x \, \mathcal{E}^\mu(k) e^{i k \cdot x}$

<u>TT coordinates</u> u^μ four-vector constant (independent of x) and not orthogonal to k_μ (i.e. $u_\mu k^\mu \neq 0$) for instance

u^μ = four velocity of an observer (time-like)

There exists a gauge such that (at once)

$$\boxed{\begin{aligned} u_\nu H^{\mu\nu} &= 0 \\ H \equiv \eta_{\mu\nu} H^{\mu\nu} &= 0 \end{aligned}}$$

$\longleftarrow$ transverse (T) condition

$\longleftarrow$ traceless (T) condition

Proof: perform a gauge transf. in Fourier domain

$$H^{\mu\nu} = H_0^{\mu\nu} + i k^\mu \varepsilon^\nu + i k^\nu \varepsilon^\mu - i \eta^{\mu\nu} k_\rho \varepsilon^\rho$$

Then TT conditions are satisfied with gauge vector

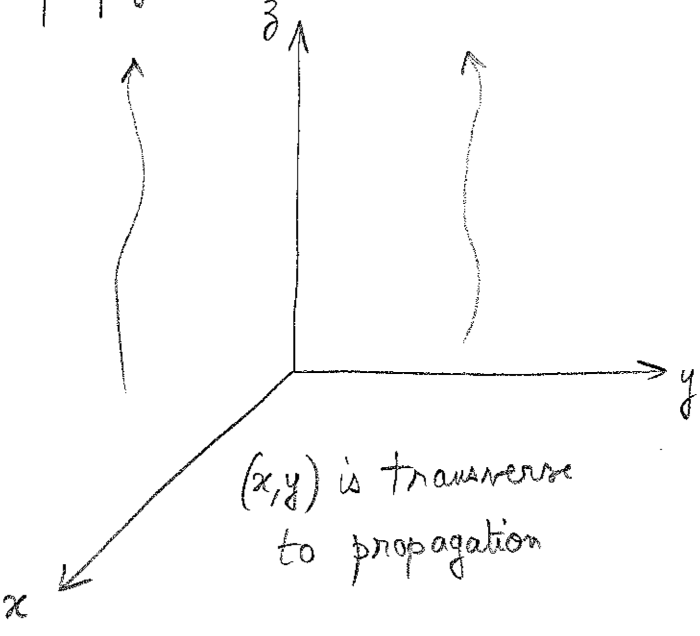
$$\varepsilon^\mu = \frac{i}{(uk)} \left[u_\nu \bar{H}_0^{\mu\nu} - \frac{k^\mu}{2(uk)} u_\rho u_\sigma \bar{H}_0^{\rho\sigma} \right]$$

where $\bar{H}_0^{\mu\nu} = H_0^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} H_0$

$$\boxed{\begin{array}{c} 10 - 4 - (4-1) - 1 = 2 \text{ independent components of } H^{\mu\nu} \\ 2 \text{ polarization states} \end{array}}$$

$u^\mu = (1, \vec{0})$ in rest frame of observer

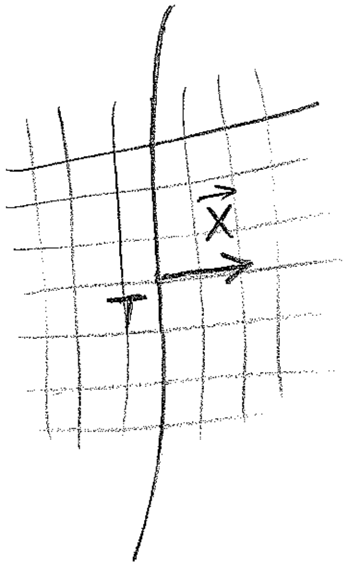
propagation in z-direction



$$h_{\mu\nu}^{TT} = \begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(t-z/c) & h_\times(t-z/c) & 0 \\ 0 & h_\times(t-z/c) & -h_+(t-z/c) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

ACTION OF GRAVITATIONAL WAVES ON MATTER

central geodesics ($X^i=0$)



Fermi coordinates (X^i, T) in the neighborhood of central geodesics

T= proper time along central geodesics

$$\boxed{g_{\mu\nu}(\vec{X},T) = \eta_{\mu\nu} + \underbrace{F_{\mu\nu ij}(T)}_{\text{function of time } T} X^i X^j + \mathcal{O}(|\vec{X}|^3)}$$

Geodesic equ. in vicinity of central geodesic ($|\vec{X}| \ll \lambda^{GW}$)

$$\frac{d^2X^i}{dT^2} = -c^2 \frac{\partial \Gamma^i_{00}}{\partial X^j}(T,\vec{0})X^j = -c^2 R^i_{\cdot 0j0}(T,\vec{0})X^j$$

(to first order in X^i) Riemann in Fermi coord.

($-c^2 R_{i0j0}$ is a relativistic version of the tidal tensor $\partial_i\partial_j U$)

$$R^i_{\cdot 0j0} = \frac{\partial X^i}{\partial x^\lambda}\frac{\partial x^\mu}{\partial X^0}\cdots \overset{TT}{R}{}^\lambda_{,\mu\nu\rho} \approx \overset{TT}{R}{}^i_{\cdot 0j0} \approx -\frac{1}{2c^2}\frac{\partial^2 h^{TT}_{ij}}{\partial t^2}$$

Riemann in TT coordinates

$$\boxed{\frac{d^2X^i}{dT^2} = \frac{1}{2}\frac{\partial^2 h^{TT}_{ij}}{\partial t^2}(T,\vec{0})X^j}$$

acceleration in Fermi coord. wave form in TT coord. evaluated on central geodesic

$$\boxed{X^i(T) = X^i(0) + \frac{1}{2}h^{TT}_{ij}(T,\vec{0})X^j(0)}$$

(to first order in h)

position before passage of GW

QUADRUPOLE MOMENT FORMALISM

Matter source is

- isolated ($T^{\mu\nu}$ has a compact support)
- post-Newtonian $\boxed{\varepsilon \approx \frac{v}{c} \ll 1}$
- self-gravitating : internal motion is due to gravitational forces

$$\gamma \sim \frac{v^2}{a} \sim \frac{GM}{a^2}$$

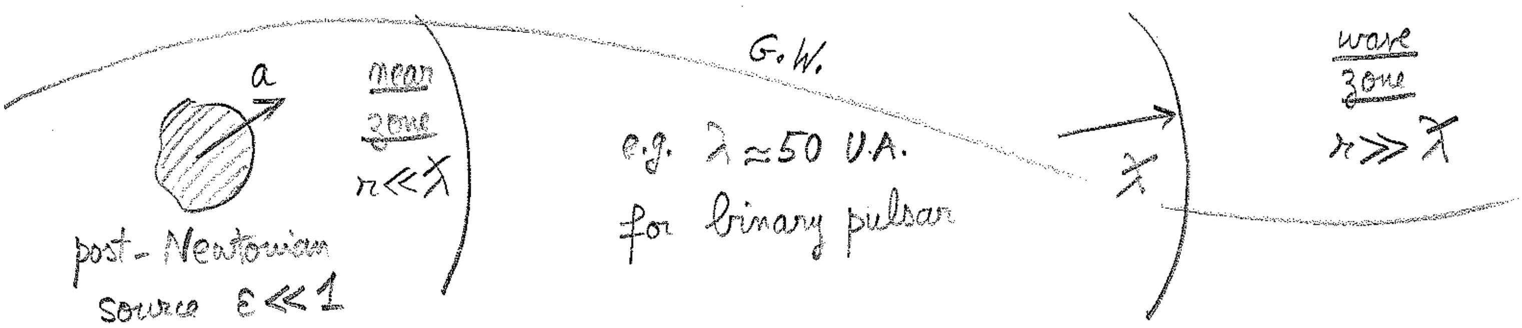
a = size of source
M = its mass

Period of motion $P \sim \frac{2\pi a}{v}$

Gravitational wave length $\lambda = cP$

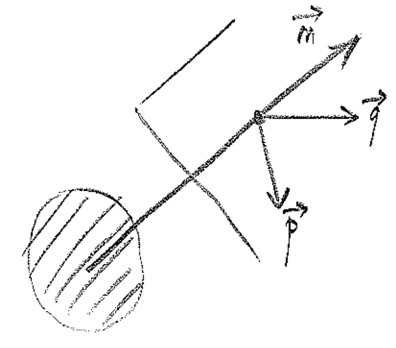
$$ƛ = \frac{\lambda}{2\pi}$$

$$\boxed{\frac{R}{ƛ} \sim \frac{v}{c} \approx \varepsilon}$$



The near zone ($r \ll ƛ$) covers entirely the post-Newtonian source

$$Q_{ij}(t) = \int_{\text{source}} d^3x \, \rho(\vec{x},t)\left(x_i x_j - \frac{1}{3}\delta_{ij}\vec{x}^2\right)$$



$$\boxed{h_{ij}^{TT} = \frac{2G}{c^4 r} P_{ijkl}(\vec{n}) \left\{ \ddot{Q}_{kl}\left(t - \frac{r}{c}\right) + \mathcal{O}(\varepsilon) \right\} + \mathcal{O}\left(\frac{1}{r^2}\right)}$$

TT projection operator $P_{ijkl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$ where $P_{ij} = \delta_{ij} - n_i n_j$

Polarization states w.r.t. $\vec{p}, \vec{q}$

$$h_+ = \frac{p_i p_j - q_i q_j}{2} h_{ij}^{TT}$$

$$h_\times = \frac{p_i q_j + p_j q_i}{2} h_{ij}^{TT}$$

$\vec{p}, \vec{q}$ polarization vectors

$$\boxed{\mathcal{F}^{GW} \equiv \left(\frac{dE}{dt}\right)^{GW} = \frac{G}{5c^5}\left\{\dddot{Q}_{ij}\,\dddot{Q}_{ij} + \mathcal{O}(\varepsilon^2)\right\}}$$

Einstein quadrupole formula

order of magnitude of radiation reaction

$\mathcal{O}(\varepsilon^5)$ called also 2.5PN

Typically $\quad Q \sim M a^2 \qquad \dddot{Q} \sim M a^2 \omega^3 \qquad \omega = \frac{2\pi}{P}$

Self-gravitating source $\quad \omega^2 \sim \frac{GM}{a^3}$

$$\boxed{\mathcal{F}^{GW} \sim \left(\frac{c^5}{G}\right)\left(\frac{GM\omega}{c^3}\right)^{10/3}}$$

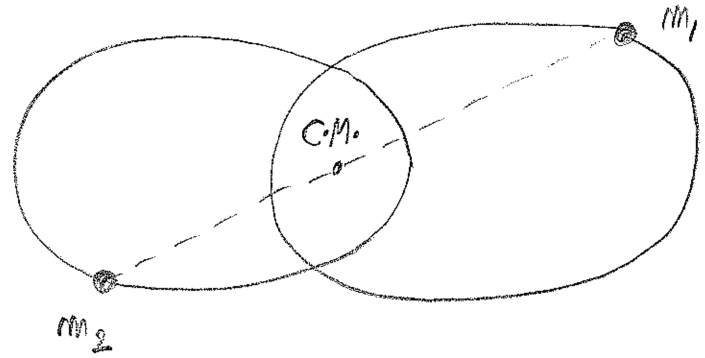
Ultra-relativistic source $\quad v \sim c \quad$ or $\quad \frac{GM\omega}{c^3} \sim 1$

$$\underbrace{\mathcal{F}^{GW}\Big|_{\substack{\text{ultra}\\ \text{relativistic}}} \sim \frac{c^5}{G} = 3.63\ 10^{52}\,W}_{\text{value independent of source}}$$

GW has typically the frequency $\quad \omega \sim \frac{c^3}{GM}$

$M \sim 1\, M_\odot$	$\omega \sim 10^3\, Hz$	bandwidth of LIGO/VIRGO
$M \sim 10^6\, M_\odot$	$\omega \sim 10^{-3}\, Hz$	bandwidth of LISA

PETERS & MATHEWS FORMULA



Two compact objects (without spin) on a Keplerian ellipse

$a =$ semi-major axis

$e =$ eccentricity

$$M = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{M}$$

$$\boxed{\nu = \frac{\mu}{M} \text{ such that } 0 < \nu \leq \frac{1}{4}}$$

test-mass limit (0) — equal masses ($\frac{1}{4}$)

$$\boxed{\langle \mathcal{F}^{GW} \rangle \equiv \frac{1}{P}\int_0^P dt\, \mathcal{F}^{GW}(t) = \frac{32}{5}\frac{c^5}{G}\nu^2\left(\frac{GM}{ac^2}\right)^5 \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}}}$$

eccentricity dependent "enhancement" factor $f(e)$

Energy balance argument

$$\frac{dE}{dt} = -\langle \mathcal{F}^{GW} \rangle \qquad \text{with} \qquad E = -\frac{GM^2\nu}{2a}$$

$$GM = \omega^2 a^3$$

$$\boxed{\dot{P} = -\frac{192\pi}{5c^5}\left(\frac{2\pi GM}{P}\right)^{5/3}\nu\, f(e) = -2.4\; 10^{-12}\ s/s}$$

Binary pulsar PSR 1913+16

in agreement with observations (Taylor et al).

INSPIRALLING COMPACT BINARIES

Evolution of eccentricity $e(t)$

Orbit's energy and angular momentum

$$\boxed{\begin{aligned} \frac{E}{\nu} &= -\frac{GM^2}{2a} \\ \frac{J}{\nu} &= \sqrt{GM^3 a(1-e^2)} \end{aligned}} \qquad \nu \equiv \mu/M$$

Apply quadrupole formulas for both E and J

$$\dot{E} = -\left\langle \frac{G}{5c^5} \dddot{Q}_{ij} \dddot{Q}_{ij} \right\rangle$$

$$\dot{J}^i = -\left\langle \frac{2G}{5c^5} \varepsilon_{ijk} \ddot{Q}_{jl} \dddot{Q}_{kl} \right\rangle$$

$$\boxed{\frac{e^2}{(1-e^2)^{19/6}}\left(1+\frac{121}{304}e^2\right)^{\frac{145}{121}} = \left(\frac{\omega}{\omega_0}\right)^{-\frac{19}{9}}}$$

gives $e(t)$ as a function of $\omega(t)$ during the inspiral (ω_0 is determined from initial conditions) ($e^2 \sim P^{19/9}$ for small e)

For the binary pulsar $\quad e_{now} = 0.617$

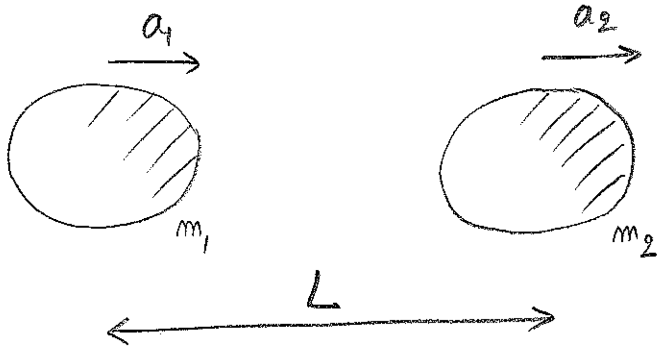
$\omega_{now} = 2.24\ 10^{-4}\,\mathrm{Hz}$

hence GWs are visible by VIRGO/LIGO when

$$\boxed{\omega \sim 30\ \mathrm{Hz} \quad \Longrightarrow \quad e \sim 5\ 10^{-6}}$$

eccentricity is negligible in general.

Finite size effects



Look for influence of quadrupole moments Q_1 and Q_2 induced by <u>tidal interactions</u> between <u>non-spinning compact objects</u>

$$\boxed{Q_1 = k_1 m_2 \frac{a_1^5}{L^3} \qquad Q_2 = k_2 m_1 \frac{a_2^5}{L^3}}$$

$k_{1,2}$ = Love numbers (depend on internal structure)

$Q_{1,2}$ scale like L^{-3} because of tidal field $\partial_{ij} U \sim \frac{1}{L^3}$

Introduce the compacity parameters

$$K_1 = \frac{2Gm_1}{a_1 c^2} \qquad K_2 = \frac{2Gm_2}{a_2 c^2}$$

The quadrupoles modify the energy and GW flux and the orbital frequency ω and phase $\phi = \int \omega dt$

$$\dot{E} = -\mathcal{F}^{GW} \quad \Rightarrow \quad \phi = -\int \frac{\omega dE}{\mathcal{F}^{GW}}$$

Effect of quadrupoles is

depends on internal structure

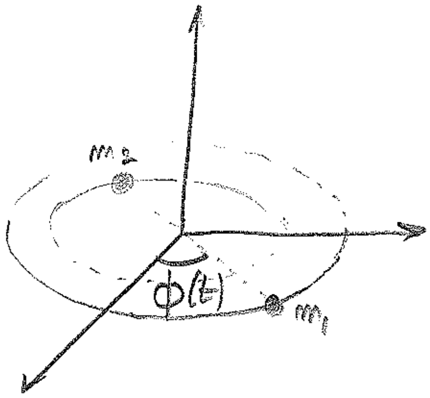
$$\boxed{\phi^{\text{finite-size}} = \underbrace{\phi_0 - \frac{1}{8x^{5/2}}\Big\{1}_{\text{point-mass result}} + (\text{const})\left(\frac{x}{K}\right)^5\Big\}}$$

$x \equiv \left(\frac{GM\omega}{c^3}\right)^{2/3}$ Since $K \sim 1$ for compact objects the formal order of magnitude of the <u>finite-size effect is 5PN</u> (namely $x^5 \sim \frac{1}{c^{10}}$)

Orbital phase evolution $\phi(t)$

(same as for binary pulsar, i.e. based on

$$\frac{dE}{dt} = -\mathcal{F}^{GW})$$



where $\quad \frac{E}{M} = -\frac{c^2}{2}\nu x$

$$\mathcal{F}^{GW} = \frac{32}{5}\frac{c^5}{G}\nu^2 x^5$$

$$\boxed{x = \left(\frac{GM\omega}{c^3}\right)^{2/3}} = \text{PN parameter } \mathcal{O}(\varepsilon^2)$$

$$\dot{E} = -\mathcal{F}^{GW} \Rightarrow \dot{x} = \frac{64}{5}\frac{c^3}{G}\frac{\nu}{M}x^5 \Rightarrow x(t) = \left[\frac{256}{5}\frac{c^3}{G}\frac{\nu}{M}(t_c - t)\right]^{-1/4}$$

t_c = instant of coalescence

$$\phi(t) = \int \omega\, dt = \frac{5}{64\nu}\int x^{-7/2} dx \Rightarrow \boxed{\phi(t) = \phi_c - \frac{x(t)^{-5/2}}{32\nu}}$$

Number of orbital cycles left till coalescence from time t

$$\mathcal{N} = \frac{\phi_0 - \phi(t)}{\pi} = \frac{1}{32\pi\nu}\left(\frac{GM\omega}{c^3}\right)^{-5/3} \underbrace{= \mathcal{O}(\varepsilon^{-5})}_{}$$

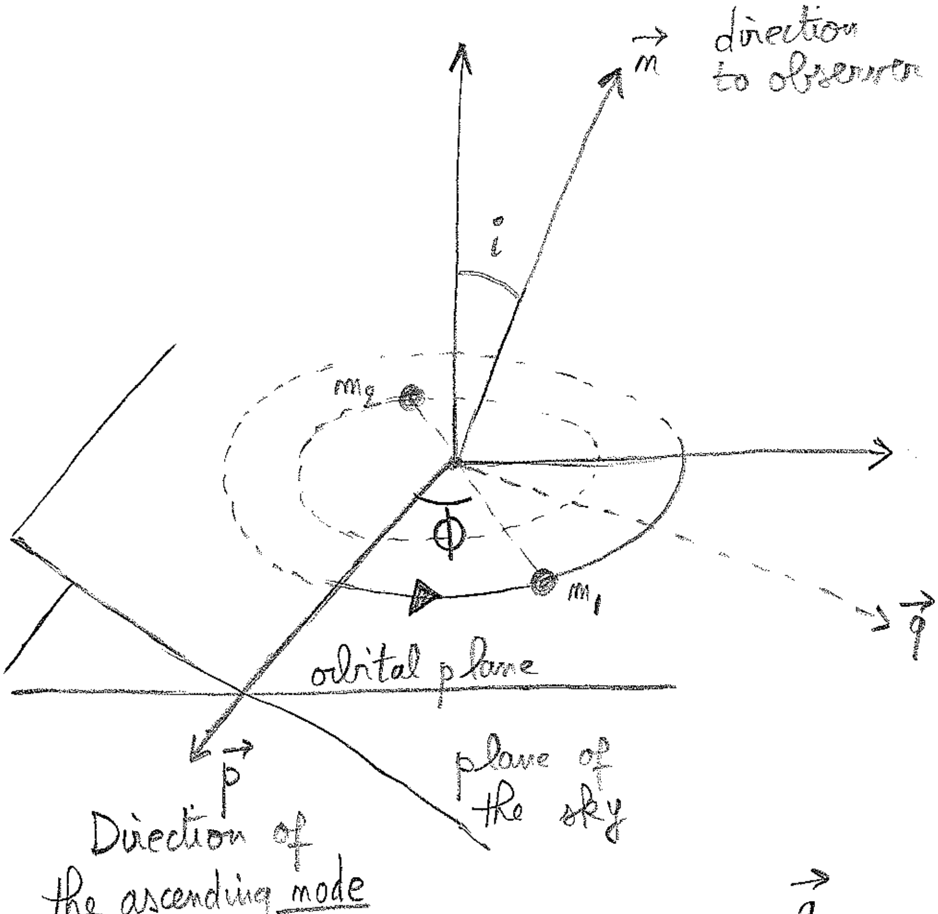
inverse of order of radiation reaction $\varepsilon^{-5} \sim \left(\frac{c}{v}\right)^5$

But $\mathcal{N}$ should be monitored in LIGO/VIRGO with precision

$$\delta\mathcal{N} \sim 1$$

so it is evident that PN corrections in the phase will play a crucial role up to at least the 2.5PN order. Detailed analysis show that good templates for inspiralling compact binaries should have 3PN accuracy. Current theoretical prediction is 3.5PN.

Wave form of inspiralling compact binaries (ICBs)

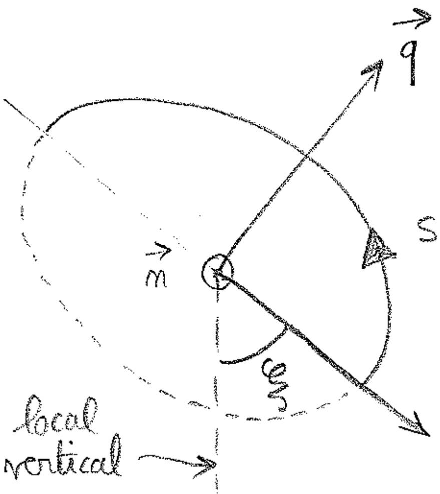


$\vec{p}, \vec{q}$ = polarization vectors (in the plane of sky)

i = inclination angle

$\phi(t)$ = orbital phase

As seen from observer:



sens of motion (bodies move toward observer when they cross $\vec{p}$)

ξ = polarization angle (between $\vec{p}$ and local vertical of observer)

Response of detector $\boxed{h \equiv \frac{2\delta L}{L} = F_+ h_+ + F_\times h_\times}$

$F_{+,\times}$ = detector's pattern functions

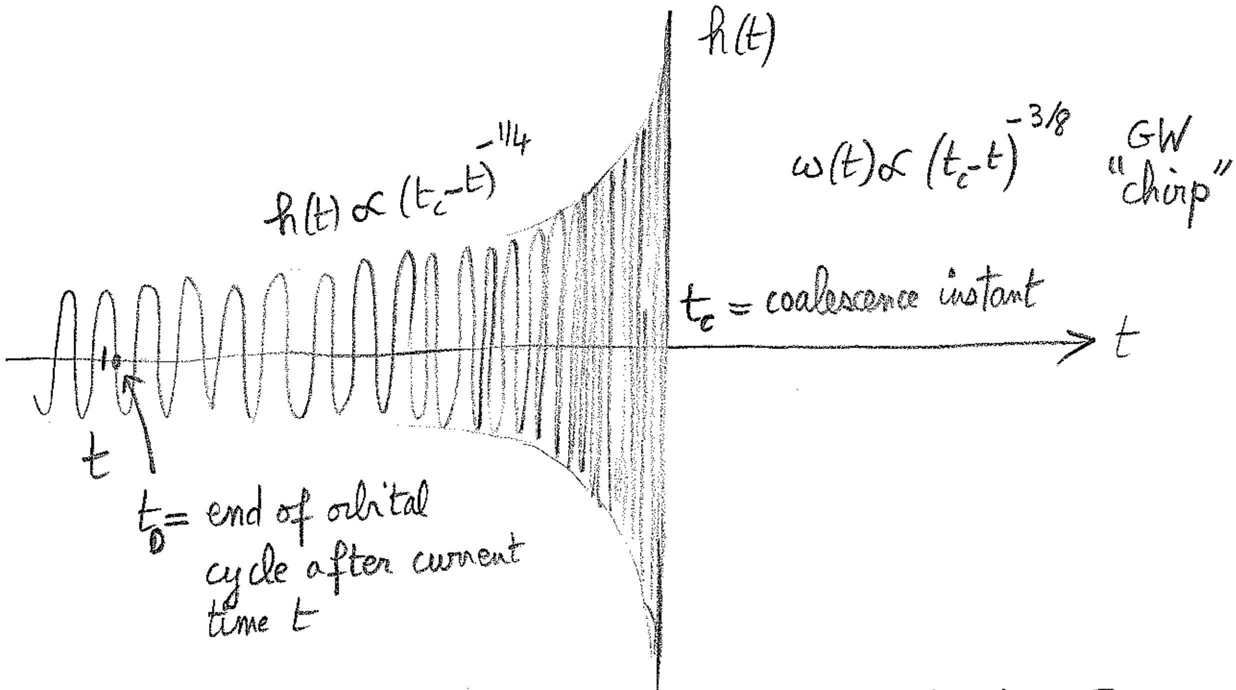
depend on $-\vec{n}$ (direction of source) and ξ

In quadrupole approximation

$$\boxed{\begin{aligned} h_+ &= \frac{2G\mu}{c^2 D}\left(\frac{GM\omega}{c^3}\right)^{2/3}(1+\cos^2 i)\cos(2\phi) \\ h_\times &= \underline{\qquad\qquad\qquad} \;(2\cos i)\sin(2\phi) \end{aligned}}$$

D = distance of source = luminosity distance in cosmology

where
$$\boxed{\begin{aligned} \phi(t) &= \phi_c - \frac{1}{\nu}\left(\frac{\nu c^3}{5GM}(t_c - t)\right)^{5/8} \\ \omega(t) &= \frac{c^3}{8GM}\left(\frac{\nu c^3}{5GM}(t_c-t)\right)^{-3/8} \end{aligned}}$$



Suppose current time t is such that $t_c - t \gg P$ (non-relativistic limit, two bodies are well-separated)

$$t_c - t = (t_c - t_0)\left[1 + \frac{t_0 - t}{t_c - t_0}\right] \qquad \text{with} \quad \frac{t_0 - t}{t_c - t_0} \ll 1$$

$$\begin{aligned} \phi(t) &\approx \phi_c - \frac{1}{\nu}\left(\frac{\nu c^3}{5GM}(t_c - t_0)\right)^{5/8}\left[1 + \frac{5}{8}\,\frac{t_0 - t}{t_c - t_0} + \cdots\right] \\ &\approx \phi_0 + \frac{5}{8\nu}\left(\frac{\nu c^3}{5GM}\right)^{5/8}(t_c - t_0)^{-3/8}\, t + \cdots \end{aligned}$$

thus $\boxed{\phi(t) \approx \phi_0 + \omega_0 t + \cdots}$ constant orbital motion in the non relativistic limit

Orders of magnitude

$$h \sim \frac{GM\nu}{c^2 D}\left(\frac{GM\omega}{c^3}\right)^{2/3}$$

Number of cycles around frequency ω

$$n = \frac{\omega^2}{\dot{\omega}} \sim \frac{1}{\nu}\left(\frac{GM\omega}{c^3}\right)^{-5/3} = \mathcal{O}(\varepsilon^{-5})$$ inverse of rad. reaction order

Effective amplitude after matched filtering

$$\boxed{h_{\text{eff}} = h\sqrt{n} \sim \frac{GM\sqrt{\nu}}{c^2 D}\left(\frac{GM\omega}{c^3}\right)^{-1/6}}$$

Example: coalescence of two supermassive BHs in LISA

Characteristic frequency $\omega_c \sim \omega_{I.C.O.}$

innermost circular orbit (defined by the minimum of the energy function)

$$\frac{GM\omega_c}{c^3} \sim 0.1 \quad \Rightarrow \quad f_c \sim 10^4\,\text{Hz}\left(\frac{M_\odot}{M}\right)$$

(from 3PN theory) For LISA $f_c \in [10^{-4}\,\text{Hz}, 10^{-1}\,\text{Hz}]$

Hence LISA should observe

$$\boxed{10^5\,M_\odot \lesssim M \lesssim 10^8\,M_\odot}$$

$$\boxed{h_{eff} \sim 10^{-14} \left(\frac{1\,\mathrm{Gpc}}{D}\right)\left(\frac{\nu}{0.25}\right)^{1/2}\left(\frac{M}{10^7 M_\odot}\right)^{-5/6}\left(\frac{f}{10^{-4}\,\mathrm{Hz}}\right)^{-1/6}}$$

Separation of BHs ($M \sim 10^7 M_\odot$) at entry frequency of LISA

$$r = \left(\frac{GM}{\omega^2}\right)^{1/3} \sim 1\ \mathrm{A.U.}$$

Time left till coalescence

$$T = \frac{5GM}{\nu c^3}\left(\frac{8GM\omega}{c^3}\right)^{-8/3} \sim 10\ \text{days}$$

The signal-to-noise of the supermassive BH coalescence in LISA is enormous

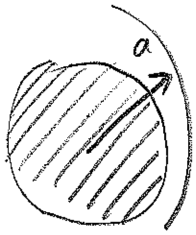
$$\boxed{\frac{S}{N} = \left(\int_{-\infty}^{+\infty} d\omega \frac{|\tilde{h}(\omega)|^2}{S_n(\omega)}\right)^{1/2} \sim \frac{h_{eff}}{\sqrt{\omega S_n(\omega)}} \sim 10^4}$$

$S_n(\omega) \sim 10^{-34}\,\mathrm{Hz}^{-1}$ for LISA

PART 2

EXTERNAL FIELD OF AN ISOLATED SOURCE

NON-LINEARITY (POST MINKOWSKIAN) EXPANSION



isolated source (radius a)

In exterior region ($r > a$)

of order $\mathcal{O}(h_{ext}^2)$

$$\begin{cases} \Box h^{\mu\nu}_{ext} = \Lambda^{\mu\nu}(h_{ext}) \\ \partial_\nu h^{\mu\nu}_{ext} = 0 \quad \text{harmonic coordinate condition} \end{cases}$$

We solve these equations by means of <u>post-Minkowskian</u> (PM) or <u>non-linearity</u> expansion

$$\boxed{h^{\mu\nu}_{ext} = \sum_{n=1}^{+\infty} G^n h^{\mu\nu}_{(n)}}$$

G = Newton's constant
(viewed here as a "bookeeping" parameter to label the successive PM orders)

Insert PM expansion into <u>vacuum</u> Einstein field eqs.

$$\Box\left(G h^{\mu\nu}_{(1)} + G^2 h^{\mu\nu}_{(2)} + \dots\right) = G^2 \Lambda^{\mu\nu}_{(2)}(h_{(1)}) + G^3 \Lambda^{\mu\nu}_{(3)}(h_{(1)}, h_{(2)}) + \dots$$

$$\partial_\nu\left(\text{———}\right) = 0$$

where $\Lambda_{(2)} \sim h_{(1)} \partial^2 h_{(1)} + \partial h_{(1)} \partial h_{(1)}$

$\Lambda_{(3)} \sim h_{(1)} \partial h_{(1)} \partial h_{(1)} + h_{(1)} \partial^2 h_{(2)} + h_{(2)} \partial^2 h_{(1)} + \partial h_{(1)} \partial h_{(2)}$

...

Hierarchy of PM equations equivalent to Einstein eqs.

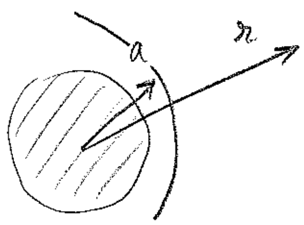
$\forall n \geq 1$

$$\boxed{\begin{aligned} \Box h^{\mu\nu}_{(n)} &= \Lambda^{\mu\nu}_{(n)}(h_{(1)} h_{(2)} \cdots h_{(n-1)}) \\ \partial_\nu h^{\mu\nu}_{(n)} &= 0 \end{aligned}}$$

The source term $\Lambda_{(n)}$ is known from previous iterations

LINEARIZED SOLUTION

Solve $\Box h_{(1)} = 0$ by means of multipole expansion (valid in exterior $r > a$)



"Monopolar" general solution

$$h^{\text{Mono.}}_{(1)}(\vec{x}, t) = \frac{R(t - r/c) + A(t + r/c)}{r}$$

Impose no incoming rad. cond.

$$0 = \lim_{\substack{t \to -\infty \\ t + \frac{r}{c} = \text{const}}} \left[\partial_r (r h_{(1)}) + \partial_t (r h_{(1)}) \right] = 2 A'(t + \tfrac{r}{c})$$ hence $A(u)$ is constant and can be included into definition of $R(t - \frac{r}{c})$.

$$h^{\text{Mono.}}_{(1)} = \frac{R(t - r/c)}{r}$$

$(i = 1,2,3)$

"Dipolar" solution is obtained by applying $\partial_i \equiv \dfrac{\partial}{\partial x^i}$

hence $h^{\text{Dip.}}_{(1)} = \partial_i \left(\frac{R_i(t - r/c)}{r} \right)$. General multipolar solution is obtained by applying l spatial derivatives

$$\boxed{h^{\mu\nu}_{(1)}(\vec{x}, t) = \sum_{l=0}^{+\infty} \partial_L \left(\frac{R^{\mu\nu}_L(u)}{r} \right)} \quad \left(u \equiv t - \frac{r}{c} \right)$$

$L = i_1 i_2 \cdots i_l$ a multi-index with l spatial indices

$$\partial_L \equiv \partial_{i_1 i_2 \cdots i_l} = \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_l}}$$

Without loss of generality we can assume that R_L is <u>symmetric and trace-free</u> (STF)

STF tensors

$$R_L = \hat{R}_L + \sum_{j \leq l-1} \varepsilon \underbrace{\delta \delta \cdots \delta}_{1 \text{ to } [\frac{l}{2}]} \hat{U}_J$$

0 or 1 Levi-Cinita symbol

Kronecker symbol

where the $\hat{U}_J$'s are linear in the $\varepsilon \delta \ldots \delta R_K$'s.

For example:

$$\begin{cases} R_{ij} = \hat{R}_{ij} + \varepsilon_{ijk} \hat{U}_k + \delta_{ij} \hat{U} \\ \hat{U}_k = \frac{1}{2} \varepsilon_{kab} R_{ab} \\ \hat{U} = \frac{1}{3} R_{kk} \end{cases}$$

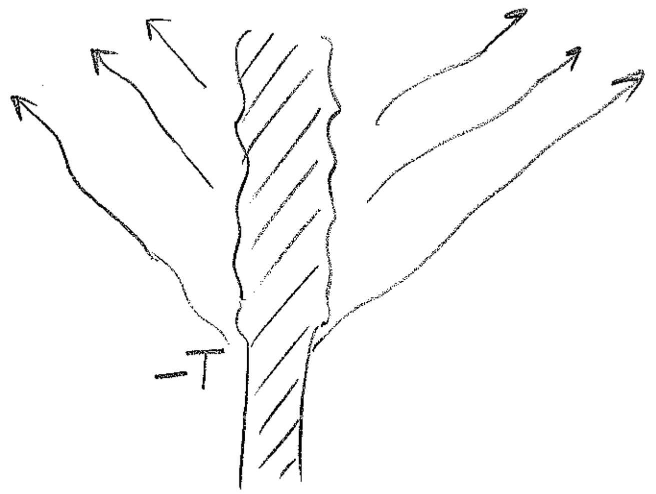
$\hat{R}_{ij} = \frac{R_{ij} + R_{ji}}{2} - \frac{1}{3} \delta_{ij} R_{kk}$ is the STF part of R_{ij}.

$$\partial_L\left(\frac{1}{r}R_L\right) = \partial_L\left(\frac{1}{r}\hat{R}_L\right) + \sum_{k\geq 1} \Delta^k \partial_{L-2K}\left(\frac{1}{r}\hat{U}_{L-2K}\right)$$

because of k Kronecker δs
(terms with one ε cancelled by symmetry of ∂_L)

$$\Delta^k \partial\left(\frac{1}{r}\hat{U}(u)\right) = \partial\left(\frac{1}{r}\frac{d^{2k}\hat{U}}{c^{2k}du^{2k}}(u)\right)$$ takes same structure

For simplicity assume that source emits GWs only from some finite instant $-T$ in the past (<u>stationarity in the past</u>)



$h^{\mu\nu}_{ext}(\vec{x})$ is independent of time when $t \leqq -T$

(and even when
$$\underbrace{t - \frac{r}{c} - \frac{2GM}{c^3}\ln\left(\frac{r}{r_0}\right) + \cdots}_{\text{"light cone" in coordinates } (t,r)} \leqq -T)$$

There are 10 independent functions $R_L^{\mu\nu}(u)$ (for each multi-index L) at this stage.

We impose now the harmonicity condition $\partial_\nu h^{\mu\nu}_{(1)} = 0$ which gives 4 differential relations between the R_L's. Hence we end up with <u>6 independent functions</u> (6 types of "source" multipole moments).

Most general solution of $\Box h_{(1)} = 0 = \partial h_{(1)}$ is (Thorne 1980)

$$\boxed{h^{\mu\nu}_{(1)} = k^{\mu\nu}_{(1)} + \underbrace{\partial^\mu \varphi^\nu_{(1)} + \partial^\nu \varphi^\mu_{(1)} - \eta^{\mu\nu} \partial_\rho \varphi^\rho_{(1)}}_{\text{linearized gauge transformation}}}$$

where $k^{\mu\nu}_{(1)}$ depends on two sets of <u>STF multipole moments</u>

$$\boxed{I_L(u) \quad \text{and} \quad J_L(u)}$$

I_L: <u>mass-moment of order l</u> $\qquad$ J_L: <u>current-moment of order l</u>

and $\varphi^\mu_{(1)}$ depends on four sets of moments (for its four components $\mu = 0,1,2,3$)

$$W_L(u) \quad X_L(u) \quad Y_L(u) \quad \text{and} \quad Z_L(u)$$

$$\boxed{\begin{aligned}
k^{00}_{(1)} &= -\frac{4}{c^2} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left(\frac{1}{r} I_L(u) \right) \\
k^{0i}_{(1)} &= \frac{4}{c^3} \sum_{l=1}^{+\infty} \frac{(-)^l}{l!} \left\{ \partial_{L-1} \left(\frac{1}{r} \dot{I}_{iL-1}(u) \right) + \frac{l}{l+1} \varepsilon_{iab} \partial_{aL-1} \left(\frac{1}{r} J_{bL-1}(u) \right) \right\} \\
k^{ij}_{(1)} &= -\frac{4}{c^4} \sum_{l=2}^{+\infty} \frac{(-)^l}{l!} \left\{ \partial_{L-2} \left(\frac{1}{r} \ddot{I}_{ijL-2} \right) + \frac{2l}{l+1} \partial_{aL-2} \left(\frac{1}{r} \varepsilon_{ab(i} \dot{J}_{j)L-2} \right) \right\}
\end{aligned}}$$

Dots mean derivative w.r.t. time $u = t - r/c$

$I_L(u)$ and $J_L(u)$ are arbitrary functions of time u except for the <u>conservation laws</u> (directly issued from the harmonicity condition $\partial h_{(1)} = 0$)

$$\boxed{\begin{array}{lll} M \equiv I = \text{const} & & \text{total mass} \\ X_i \equiv \dfrac{I_i}{I} = \text{const} & & \text{center-of-mass position} \\ P_i \equiv \dot{I}_i = 0 & & \text{linear momentum} \\ S_i \equiv J_i = \text{const} & & \text{angular momentum} \end{array}}$$

These conservation laws are exact (by definition of the moments) and refer to the total quantities associated with the source and <u>including the contributions of GWs emitted by the source</u>

They describe the <u>initial</u> state of the source before emisson of GWs.

In particular <u>$M = I$ is the total ADM mass of source</u>

Finally $h_{(1)}$ (and hence $h_{ext} = \sum G^n h_{(n)}$) will be described by

$$\boxed{\underbrace{I_L(u) \quad J_L(u)}_{\substack{\text{main moments} \\ \text{(encode all properties of} \\ \text{source at linear order)}}} \quad \underbrace{W_L(u) \;\dots\; Z_L(u)}_{\substack{\text{gauge moments} \\ \text{(will play a role} \\ \text{at non-linear order)}}} = \begin{array}{c}\text{six source} \\ \text{multipole} \\ \text{moments}\end{array}}$$

NON-LINEAR VACUUM SOLUTION

When $r \to 0$ $\quad h_{(1)} \sim \partial\left(\frac{R(t-r)}{r}\right)$ diverges. This is because $h_{(1)}$ is valid only in the exterior $r > a$. Inserting $h_{(1)}$ into $\Lambda_{(2)}$ we get

$$\Lambda_{(2)} \sim \partial\left(\frac{R(t-r)}{r}\right)\partial\left(\frac{S(t-r)}{r}\right)$$

$$\sim \sum_{k\geq 2} \frac{\hat{n}_L}{r^k} F(t-r)$$

STF product of unit vectors n_i $\boxed{\hat{n}_L = n_{\langle i_1} \cdots n_{i_\ell \rangle}}$
is equivalent to spherical harmonics $Y_{\ell m}(\theta, \varphi)$

$$\begin{cases} \hat{n}_L(\theta,\varphi) = \sum_{m=-\ell}^{\ell} \alpha_L^m \, Y_{\ell m}(\theta,\varphi) \\ \alpha_L^m = \int d\Omega \, \hat{n}_L \, Y_{\ell m}^* \end{cases} \longleftarrow \text{constant STF tensor}$$

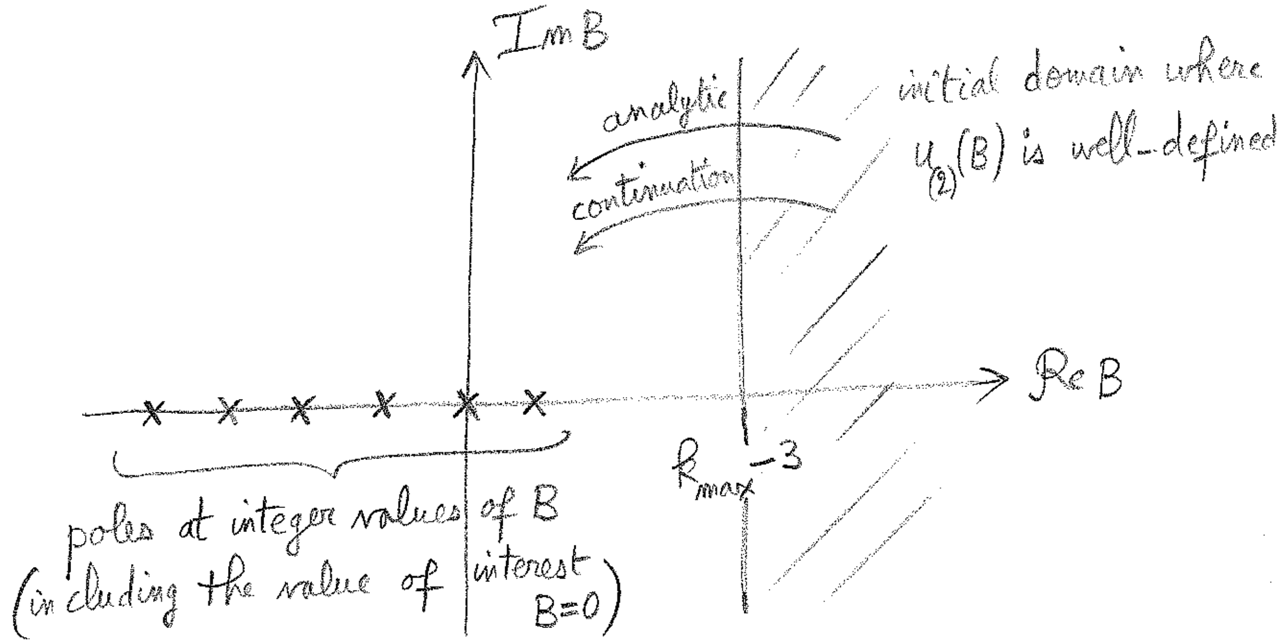
Because of divergence when $r \to 0$ one cannot apply the standard retarded integral.

If we assume $h_{(1)}$ is made of a finite set of moments, say $\ell \leq \ell_{max}$, there is a maximal order of divergencies in $\Lambda_{(2)}$, $k \leq k_{max}$. We can <u>regularize</u> $\Lambda_{(2)}$ by multiplying by some factor r^B (where $B \in \mathbb{C}$).

Next we define:

$$\boxed{u^{\mu\nu}_{(2)}(B) \equiv \Box^{-1}_{\mathrm{Ret}}\left[\left(\frac{r}{r_0}\right)^B \Lambda^{\mu\nu}_{(2)}\right]}$$

The retarded integral is convergent when $\mathcal{R}e\, B > k_{max} - 3$



$$u_{(2)}(B) = \sum_{p=p_0}^{+\infty} \lambda_p B^p$$

Laurent expansion when $B \to 0$ $(p \in \mathbb{Z})$

Applying $\Box$ we get $\left(\frac{r}{r_0}\right)^B \Lambda_{(2)} = \sum (\Box \lambda_p) B^p$

$$p_0 \leq p \leq -1 \Rightarrow \Box \lambda_p = 0$$

$$p \geq 0 \Rightarrow \Box \lambda_p = \frac{(\ln(r/r_0))^p}{p!} \Lambda_{(2)}$$

In particular when $p=0$ we obtain a solution of the eq. we want $(\Box u_{(2)} = \Lambda_{(2)})$. Pose $u^{\mu\nu}_{(2)} \equiv \lambda^{\mu\nu}_0$

$$\boxed{u^{\mu\nu}_{(2)} = \underset{B\to 0}{\text{Finite Part}}\ \Box^{-1}_{\text{Ret}}\left[r^B \Lambda^{\mu\nu}_{(2)}\right]} \qquad (r_0 = 1)$$

Thus $\Box u_{(2)} = \Lambda_{(2)}$ is satisfied and $u_{(2)}$ has the same structure $\sim \sum \frac{\hat{n}_L}{r^k} G(t-r)$ as $\Lambda_{(2)}$ but $\partial_\nu u^{\mu\nu}_{(2)} \neq 0$ in general.

$$\boxed{w^{\mu}_{(2)} \equiv \partial_\nu u^{\mu\nu}_{(2)} = \underset{B\to 0}{FP}\ \Box^{-1}_{\text{Ret}}\left[B\, n_i\, r^{B-1} \Lambda^{\mu i}_{(2)}\right]}$$

computed from the fact that $\partial_\nu \Lambda^{\mu\nu}_{(2)} = 0$

Because of factor B (coming from $\partial_i r^B = B r^{B-1} n_i$) $w^\mu_{(2)}$ is non zero when the integral <u>develops a pole</u> $\propto \frac{1}{B}$. The structure of the pole is that of a source-free (retarded) solution of d'Alembert's eq.

$$\boxed{w^{\mu}_{(2)} = \sum_{l=0}^{\infty} \partial_L\left(\frac{S^\mu_L(u)}{r}\right)}$$

Indeed $\Box w_{(2)} = \underset{B\to 0}{FP}\left(B n r^{B-1} \Lambda\right) = 0$. From that structure one can construct "<u>algorithmically</u>"

$$v^{\mu\nu}_{(2)} = \mathcal{H}^{\mu\nu}(w_{(2)})$$

an algorithm which gives a unique $v^{\mu\nu}_{(2)}$ starting from any $w^\mu_{(2)}$ (source-free solution) such that (at once) $\Box v_{(2)} = 0$ and $\partial v_{(2)} = - u_{(2)}$

$$\boxed{v_{(2)}^{\mu\nu} = \sum_{l=0}^{\infty} \partial_L \left(\frac{T_L^{\mu\nu}(u)}{r} \right)}$$

where the $T_L^{\mu\nu}$'s are given in terms of the S_L^μ's by the algorithm $\mathcal{H}$. Solution is thus

$$\boxed{h_{(2)}^{\mu\nu} = u_{(2)}^{\mu\nu} + v_{(2)}^{\mu\nu}}$$

Same method applies by induction to any n (Blanchet & Damour 1986)

$$\boxed{\begin{aligned} u_{(n)}^{\mu\nu} &= \underset{B\to 0}{\text{Finite Part}}\ \Box^{-1}_{\text{Ret}} \left[\left(\frac{r}{r_0}\right)^B \Lambda_{(n)} (h_{(1)} \cdots h_{(n-1)}) \right] \\ v_{(n)}^{\mu\nu} &= \mathcal{H}^{\mu\nu}(\partial u_{(n)}) \\ h_{(n)}^{\mu\nu} &= u_{(n)}^{\mu\nu} + v_{(n)}^{\mu\nu} \end{aligned}}$$

To $h_{(n)}$ one can still add a <u>homogeneous</u> solution (such that $\Box h_{(n)}^{\text{Hom}} = 0 = \partial h_{(n)}^{\text{Hom}}$) but $h_{(n)}^{\text{Hom}}$ is necessarily of the form $h_{(1)}[\text{some moments}]$. Hence

$$h^{gen}_{(n)} = h_{(n)}[I_L \cdots Z_L] + \underbrace{h_{(1)}[\delta I_L \cdots \delta Z_L]}_{\text{can be re-absorbed into } h_{(1)}[I_L \cdots Z_L] \text{ by posing}}$$

$$\begin{cases} I_L^{new} = I_L + G^{n-1}\delta I_L \\ \vdots \\ Z_L^{new} = Z_L + G^{n-1}\delta Z_L \end{cases}$$

Hence the previous construction represents the <u>most general</u> solution of Einstein's field eqs. outside the source

Resulting metric $\boxed{g^{ext}_{\mu\nu}(x;\, \underbrace{I_L\, J_L\, \overbrace{W_L\, X_L\, Y_L\, Z_L}^{4 \text{ gauge moments}}}_{6 \text{ source moments}})}$

One can define by coord. transformation $x \longrightarrow x'$ a "canonical" metric which depends only on 2 moments $M_L\ S_L$.

Thus $\boxed{g^{can}_{\mu\nu}(x';\, \underbrace{M_L\, S_L}_{2 \text{ canonical moments}})}$

is <u>isometric</u> to $g^{ext}_{\mu\nu}$ i.e. $g^{can}_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g^{ext}_{\rho\sigma}(x)$ where

$$\boxed{x'^\mu = x^\mu + G\underbrace{\varphi^\mu_{(1)}(x;\, W_L X_L Y_L Z_L)}_{\text{gauge vector in the general linear solution}} + \underbrace{O(G^2)}_{\text{crucial non-linear corrections}}}$$

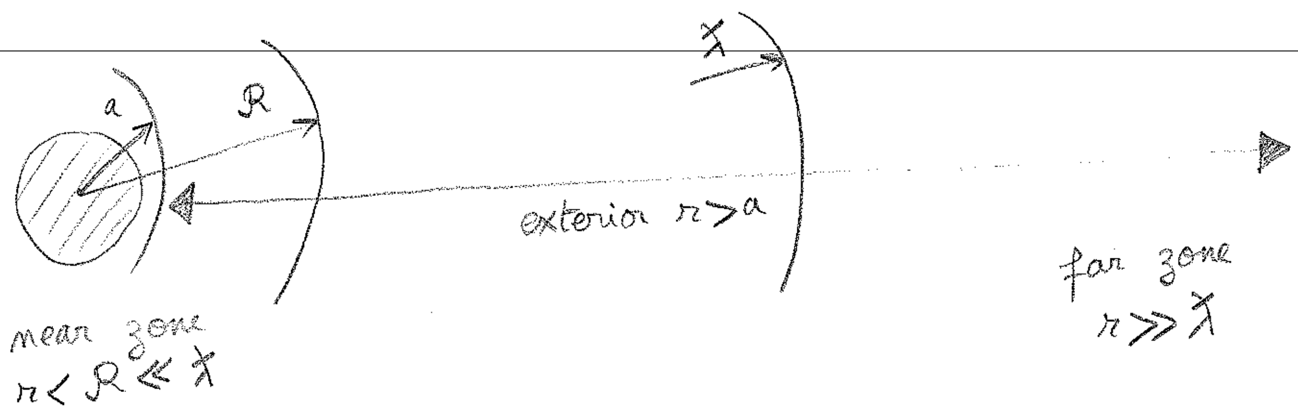
Hence any isolated system can be described by 2 sets of moments

$$M_L(u) \qquad S_L(u)$$
mass-type   current-type

$$\boxed{\begin{aligned} M_L &= I_L + O(G) \\ S_L &= J_L + O(G) \end{aligned}}$$

complicated non-linear functionals of $I_L\, J_L\, X_L \ldots Z_L$

GENERAL STRUCTURE OF THE SOLUTION



The solution $h_{ext} = \sum G^n h_{(n)}$ is physically valid in the exterior $r > a$ but is defined for any $r > 0$. When $r \to 0$

$$\boxed{h_{(n)} = \sum_{p \leq N} \hat{n}_L(\theta,\varphi)\, r^p (\ln r)^q F(t) + O(r^N)}$$

(proved by induction on n in the construction of $h_{(n)}$).
Note appearance of powers of $\ln r$ with $q \leq n-2$.

Since $r \to 0$ means $\frac{r}{c} \to 0$ or $c \to \infty$ we have the general structure of the post-Newtonian (PN) expansion

$$\boxed{h_{(n)}(c) = \sum_{p \leq N} \frac{(\ln c)^q}{c^p} + \mathcal{O}\left(\frac{1}{c^N}\right)}$$

When $r \to \infty$ (wave zone) we find also a "poly-logarithmic" structure

$$h_{(n)} = \sum_{k \leq N} \hat{n}_L \frac{(\ln r)^q}{r^k} G(u) + o\left(\frac{1}{r^N}\right) \qquad \text{where } u = t - r/c$$

(expansion at $\mathcal{J}^+$)

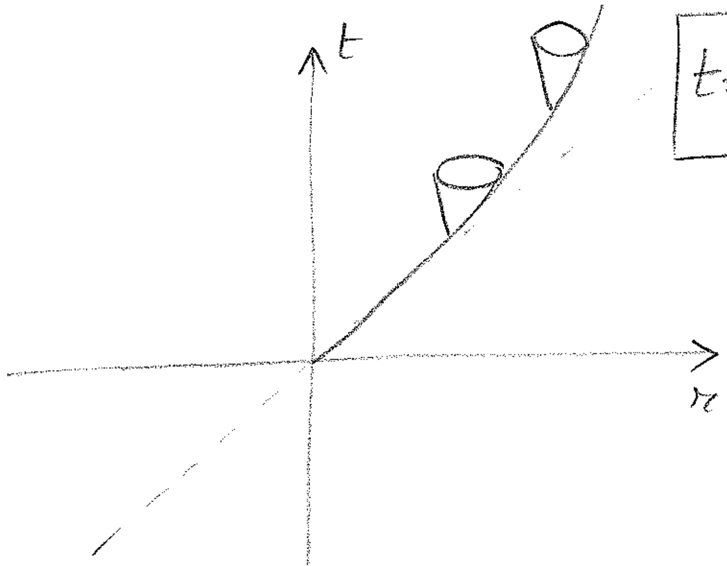
The logs here come from the well-known deviation of light rays in harmonic coordinates.

Schwarzschild in harmonic coord.

$$ds^2 = -\left(\frac{r-M}{r+M}\right)dt^2 + \left(\frac{r+M}{r-M}\right)dr^2 + (r+M)^2 d\Omega^2$$

For an <u>outgoing radial</u> ($\theta = $ const $\varphi = $ const) photon

$$dt = \frac{r+M}{r-M}\,dr \quad \Rightarrow \quad t = r + 2M \ln\left(\frac{r-M}{\text{const}}\right)$$



$$\boxed{t = \frac{r}{c} + \frac{2GM}{c^3} \ln\left(\frac{r}{r_0}\right) + \mathcal{O}(G^2)}$$

We shall see that all these logs (in the FZ) can be removed by a coord. transformation

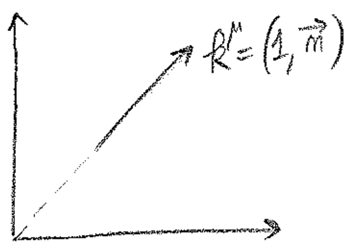
STRUCTURE OF THE QUADRATIC SOLUTION

$$h_{(1)} = \sum \partial_L \left(\frac{1}{r} R(t-r) \right) = \sum \frac{(-)^l n_L}{r} \overset{(l)}{R}(u) + \mathcal{O}(r^{-2})$$

Pose $$\boxed{h^{\mu\nu}_{(1)} = \frac{1}{r} z^{\mu\nu}(\vec{n}, u) + \mathcal{O}(r^{-2})}$$

Inserting $h_{(1)}$ into $\Lambda_{(2)}(h_{(1)}) \sim h_{(1)} \partial^2 h_{(1)} + \partial h_{(1)} \partial h_{(1)}$ obtain

$$\boxed{\Lambda^{\mu\nu}_{(2)} = \frac{1}{r^2} \left[4M \overset{(2)}{z}{}^{\mu\nu} + k^\mu k^\nu \sigma \right] + \mathcal{O}(r^{-3})}$$



where $\sigma(\vec{n}, u) = \frac{1}{2} \overset{(1)}{z}{}^{\mu\nu} \overset{(1)}{z}_{\mu\nu} - \frac{1}{4} \overset{(1)}{z}{}^{\mu}{}_{\mu} \overset{(1)}{z}{}^{\nu}{}_{\nu}$

$$\boxed{\sigma \propto \left(\frac{dE}{du\, d\Omega} \right)^{GW}}$$

σ is generated by distribution of energy of linearized GWs.

Structure of the remainder can be proved to be

$$\mathcal{O}(r^{-3}) = \sum_{3 \leq k \leq l+2} \partial_P \left(\frac{\hat{n}_L}{r^k} H(u) \right)$$

We need the retarded integrals of source terms of the type $\frac{\hat{n}_L}{r^2} F(u)$ or $\frac{\hat{n}_L}{r^k} H(u)$ with $3 \leq k \leq l+2$

$$\Box^{-1}_{Ret}\left(\frac{\hat{n}_L}{r^2}F(u)\right) = -\frac{\hat{n}_L}{r}\int_r^{+\infty} dz\, F(t-z)\, Q_l\left(\frac{z}{r}\right)$$

Legendre function of second kind

When $r \longrightarrow +\infty$

$$\boxed{\Box^{-1}_{Ret}\left(\frac{\hat{n}_L}{r^2}F(u)\right) = \frac{\hat{n}_L}{2r}\int_0^{+\infty} dy\, F(u-y)\left[\ln\left(\frac{y}{2r}\right) + 2\sum_{i=1}^{l}\frac{1}{i}\right] + O\left(\frac{\ln r}{r^2}\right)}$$

integral over the entire "past" of the source (so called <u>hereditary</u> terms)

appearance of $\ln r$ (linked with deviation of light cones)

When $3 \leq k \leq l+2$ the result is simple

$$\boxed{\text{F.P. } \Box^{-1}_{Ret}\left(\frac{\hat{n}_L}{r^k}F(u)\right) = \hat{n}_L\sum_{j=0}^{k-3} C_{j,l,k}\,\frac{F^{(k-3-j)}(u)}{r^{j+1}}}$$

composed only of <u>instantaneous</u> terms

$$\boxed{u^{\mu\nu}_{(2)} = \text{F.P. } \Box^{-1}_{Ret}\Lambda^{\mu\nu}_{(2)} = \underbrace{\Box^{-1}_{Ret}\left[\frac{4M}{r^2}z^{(2)\mu\nu}\right]}_{\text{produces the so-called tails}} + \underbrace{\Box^{-1}_{Ret}\left[\frac{k^\mu k^\nu}{r^2}\sigma\right]}_{\text{responsible for non-linear memory integral}} + \left(\begin{matrix}\text{instantaneous}\\ \text{terms}\end{matrix}\right)}$$

produces the so-called <u>tails</u> (Blanchet & Damour 1992)

responsible for <u>non-linear memory</u> integral (Thorne 91 Christodoulou 91 Will & Wiseman 92 BD92)

SHOW STRUCTURE OF TAILS AND MEMORY

We have also the other piece

$$v_{(2)}^{\mu\nu} = \mathcal{F}\mathcal{H}^{\mu\nu}(w_{(2)}) \quad \text{where} \quad w_{(2)}^{\mu} = \partial_\nu u_{(2)}^{\mu\nu}$$

If
$$w_{(2)}^{0} = \frac{1}{r}A(u) + \partial_i\left(\frac{1}{r}A_i(u)\right) + \begin{pmatrix}\text{contributions}\\ l \geq 2\end{pmatrix}$$
$$w_{(2)}^{i} = \frac{1}{r}C_i(u) + \varepsilon_{iab}\,\partial_a\left(\frac{1}{r}D_b(u)\right) + \begin{pmatrix}\text{other}\\ \text{contributions}\end{pmatrix}$$

$$\boxed{\begin{gathered} v_{(2)}^{00} = \underbrace{-\frac{1}{r}\int_{-\infty}^{u} dv\, A(v)}_{\text{"hereditary" modification of the mass}} \underbrace{- \partial_i\left[\frac{1}{r}\int_{-\infty}^{u} dv \int_{-\infty}^{v} dw\, A_i(w)\right]}_{\text{hered. modif. of mass dipole}} + \begin{pmatrix}\text{instantaneous}\\ \text{terms}\end{pmatrix} \\ v_{(2)}^{0i} = \underbrace{-\frac{1}{r}\int_{-\infty}^{u} dv\, C_i(v)}_{\text{hered. modif. linear momentum}} \underbrace{- \varepsilon_{iab}\,\partial_a\left[\frac{1}{r}\int_{-\infty}^{u} dv\, D_b(u)\right]}_{\text{hered. modif. angular momentum (spin)}} + \begin{pmatrix}\text{inst.}\\ \text{terms}\end{pmatrix} \\ v_{(2)}^{ij} = \begin{pmatrix}\text{inst.}\\ \text{terms}\end{pmatrix} \end{gathered}}$$

These hereditary modifications account for the losses of mass, etc... by GW emission.

$$h_{ext}^{00} = G\,h_{(1)}^{00} + G^2 h_{(2)}^{00} + \dots = \frac{4\,M_{Bondi}}{r} + \begin{pmatrix}\text{other}\\ \text{moments}\end{pmatrix}$$

where M_{Bondi} is the mass measured at $\mathcal{J}^+$

$$\boxed{M_{Bondi}(u) = M_{ADM} - \frac{1}{5}\int_{-\infty}^{u} dv\, \dddot{I}_{ij}(v)\,\dddot{I}_{ij}(v) + \begin{pmatrix}\text{other } l \geq 3\\ \text{and}\\ \text{higher PM}\end{pmatrix}}$$

in agreement with quadrupole formula

RADIATIVE MULTIPOLE MOMENTS

From $h_{ext} = \sum G^n h_{(n)}$ (in harmonic coordinates) we can eliminate all the log terms at infinity $r \gg \lambda\!\!\!^{-}$

$$(t, \vec{x}) \longrightarrow (T, \vec{X})$$

harmonic coordinates — radiative coordinates

$$\boxed{U \equiv T - \frac{R}{c} \text{ is null in rad. coordinates } \quad g^{\mu\nu} \partial_\mu U \partial_\nu U = 0}$$

At each PM order we correct from the "logarithmic" deviation of light cones. At linearized order

$$\boxed{H_{(1)}^{\mu\nu} = h_{(1)}^{\mu\nu} + \underbrace{\partial^\mu \xi_{(1)}^\nu + \partial^\nu \xi_{(1)}^\mu - \eta^{\mu\nu} \partial_\rho \xi_{(1)}^\rho}_{\text{gauge transformation at linear order } \mathcal{O}(G)}}$$

where $\xi_{(1)}^\mu = 2M \, \eta^{\mu 0} \ln\left(\frac{r}{r_0}\right)$

This gauge transformation is non-harmonic

$$\partial_\nu H_{(1)}^{\mu\nu} = \Box \xi_{(1)}^\mu = \frac{2M}{r^2} \eta^{\mu 0}$$

Need to control the term r^{-2} in $\Lambda_{(2)}$

$$\boxed{\Lambda_{(2)}^{\mu\nu}(H_{(1)}) = \frac{k^\mu k^\nu}{r^2} \sigma(\vec{n}, u) + \mathcal{O}\left(\frac{1}{r^3}\right)}$$

has the structure of the energy-momentum tensor of massless particles (gravitons)

Apply same "algorithm" as in harm. coord.

$$U^{\mu\nu}_{(2)} = \mathrm{FP}\, \Box^{-1}_{R}\, \Lambda^{\mu\nu}_{(2)}$$

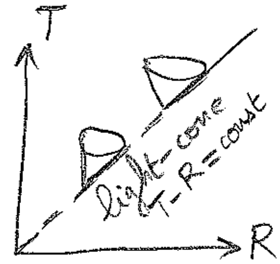
$$V^{\mu\nu}_{(2)} = \mathcal{V}^{\mu\nu}\left(W_{(2)} \equiv \partial U_{(2)}\right)$$

$$\boxed{H^{\mu\nu}_{(2)} = U^{\mu\nu}_{(2)} + V^{\mu\nu}_{(2)} + \underbrace{\partial^{\mu}\xi^{\nu}_{(2)} + \partial^{\nu}\xi^{\mu}_{(2)} - \eta^{\mu\nu}\partial_{\rho}\xi^{\rho}_{(2)}}_{\text{gauge transformation at quadratic order } \mathcal{O}(G^2)}}$$

where $$\boxed{\xi^{\mu}_{(2)} = \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}}\left[\frac{k^{\mu}}{2r^2}\int_{-\infty}^{u} dv\, \sigma(\vec{n}, v)\right]}$$

Thanks to the structure of the r^{-2} term in $\Lambda_{(2)}$ $(\propto k^{\mu}k^{\nu}\sigma)$ this term is cancelled by the gauge transformation

$$U^{\mu\nu}_{(2)} + \partial\xi^{\mu\nu}_{(2)} = \underbrace{\Box^{-1}\left(\frac{k^{\mu}k^{\nu}\sigma}{r^2}\right) + \partial^{(\mu}\Box^{-1}\left(\frac{k^{\nu)}}{r^2}\int\sigma\right)}_{\substack{\text{cancel at leading order } r^{-2} \\ \text{since } \partial^{\mu} = -k^{\mu}\partial_u}} + \cdots$$



Hence no logs are produced

$$\xi^{\mu} = G\,\xi^{\mu}_{(1)} + G^2\,\xi^{\mu}_{(2)} + \cdots$$

gives the (full non-linear) coordinate transformation between harmonic and radiative coordinates $(\vec{x}, t) \longrightarrow (\vec{X}, T)$ (Blanchet 1987)

We find

$$U_{(2)}^{\mu\nu} + \partial \xi_{(2)}^{\mu\nu} = \frac{1}{r} \underbrace{\int_{-\infty}^{u} dv\, K^{\mu\nu}(v, \vec{n})}_{\text{non-linear memory integral}} \quad \text{where} \quad K \approx \sum \hat{n}_L \overset{(p)}{I}_{L_1} \overset{(p)}{I}_{L_2}$$

In rad. coord. $(T, \vec{X})$ the metric admits a Bondi-type expansion $R \longrightarrow +\infty$ $U = T - R/c = \text{const}$ $(\mathcal{J}^+)$

$$\boxed{H_{(n)}(T, \vec{X}) = \sum_{k \leq N} \frac{\hat{N}_L}{R^k} K(U) + \mathcal{O}\left(\frac{1}{R^N}\right)}$$

One can then prove the "asymptotic simplicity" (Penrose 1963, 1965) i.e. existence of a conformal transformation such that

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad \text{is } C^\infty \text{ at } \mathcal{J}^+$$

The radiative moments are then defined by (Thorne 1980)

$$\boxed{H_{ij}^{TT} = \frac{4G}{c^2 R} P_{ijkl}(\vec{N}) \sum_{l=2}^{\infty} \frac{1}{c^l\, l!} \Big\{ \underbrace{N_{L-2}\, U_{klL-2}(T-R)}_{\text{mass-type}} + \underbrace{\frac{l}{c} N_{aL-2}\, \varepsilon_{ab(k}\, V_{l)bL-2}(T-R)}_{\text{current-type}} \Big\}} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

Purely an "algebraic" definition from the $1/R$ term of the metric in radiative coordinates

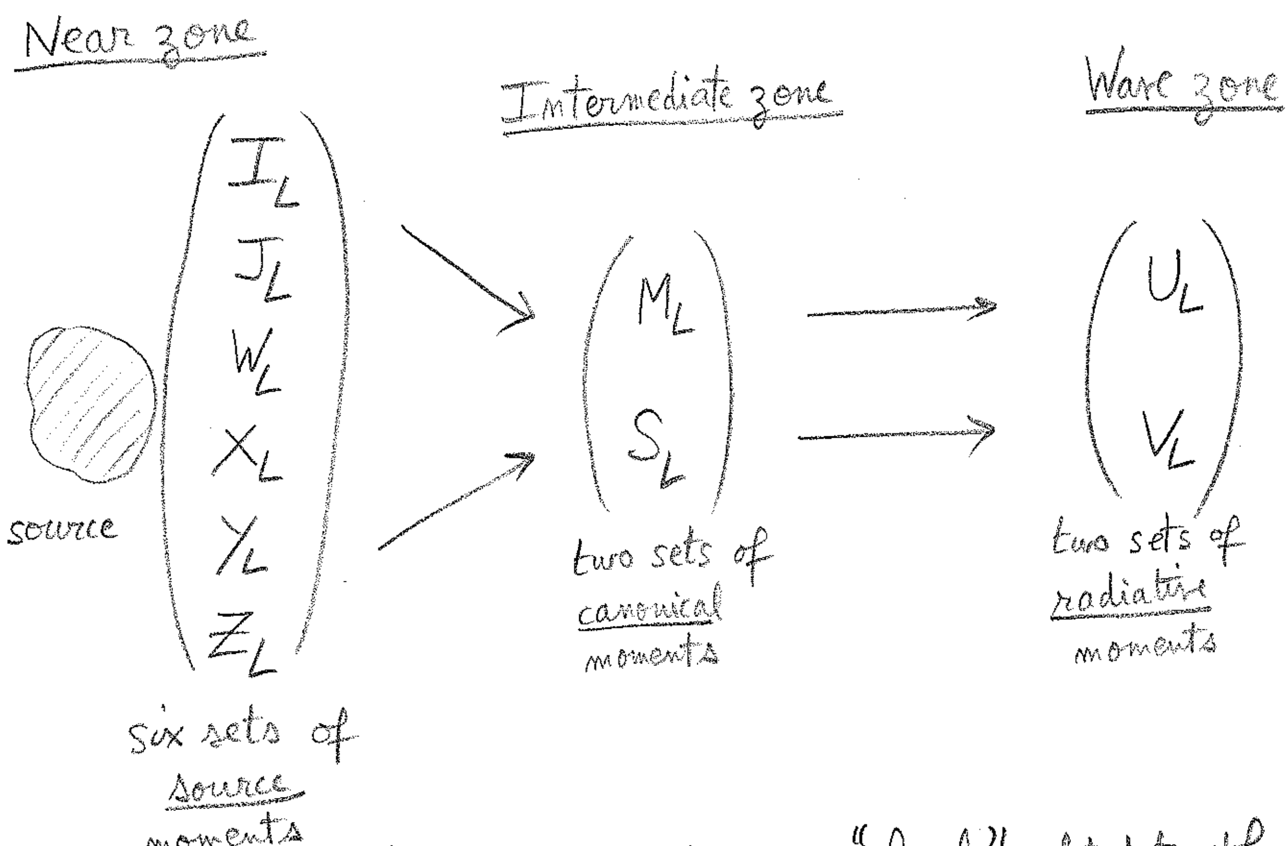
Energy flux in GWs is

$$\boxed{\mathcal{F} \equiv \left(\frac{dE}{dT}\right)^{GW} = \sum_{l=2}^{\infty} \frac{1}{c^{2l+1}} \left\{ a_l \overset{(1)}{U}_L \overset{(1)}{U}_L + \frac{b_l}{c^2} \overset{(1)}{V}_L \overset{(1)}{V}_L \right\}}$$

The rad. moments agree with the canonical ones at leading PM order

$$\boxed{\begin{aligned} U_L &= \overset{(l)}{M}_L + \mathcal{O}(G) \\ V_L &= \overset{(l)}{S}_L + \mathcal{O}(G) \end{aligned}}$$

non-linear corrections including tails, tails-of-tails non-linear memory etc...



We shall see that the source moments are "closely" related to the source in the sense that they admit closed form expressions in terms of the source's parameters.

PART 3

MATCHING TO THE FIELD OF A POST-NEWTONIAN SOURCE

THE MATCHING EQUATION

We have contructed the exterior field (physically valid when $r > a$) of <u>any isolated source</u>

$$h_{ext} = \sum_{n=1}^{+\infty} G^n h_{(n)} [\underbrace{I_L \, J_L \, W_L \dots Z_L}_{\text{source moments (for the moment arbitrary)}}]$$

We suppose that h_{ext} comes from the <u>multipole expansion</u> of h defined everywhere inside and outside the source (for any r)

$$\boxed{h_{ext} = \mathcal{M}(h)}$$

operation of taking the multipole expansion

Note that $\mathcal{M}(h)$ is defined of any $r > 0$ but agrees with the "true" field h only when $r > a$

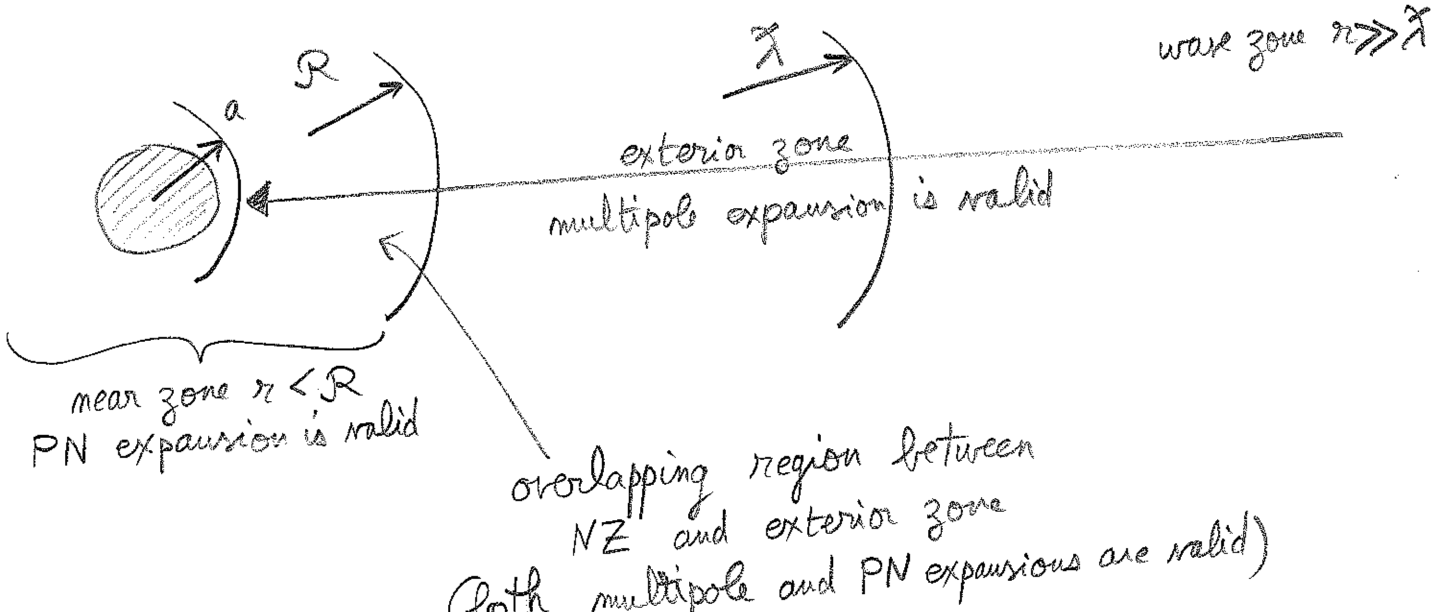
$$\boxed{r > a \quad \Rightarrow \quad \mathcal{M}(h) = h \quad \text{(numerically)}}$$

But when $r \to 0$ $\mathcal{M}(h)$ diverges while h is a perfectly smooth solution Einstein field eqs. inside the matter (of the extended source).

Suppose the source is <u>post-Newtonian</u> (existence of the PN parameter $\varepsilon = \frac{v}{c} \ll 1$). We know that the near zone $r < \mathcal{R}$ where $\mathcal{R} \ll \lambda\!\!\!^{-}$ encloses totally the PN source ($\mathcal{R} > a$).

In the NZ the field h can be expanded as a PN expansion ($\bar{h} = \sum c^{-p} (\ln c)^q$)

$$\boxed{r < \mathcal{R} \quad \Rightarrow \quad h = \bar{h} \quad \text{(numerically)}}$$



$$\boxed{a < r < \mathcal{R} \quad \Rightarrow \quad \mathcal{M}(h) = \bar{h} \quad \text{(numerically)}}$$

The <u>matching equation</u> follows from transforming the latter numerical equality in a <u>functional identity</u> (valid $\forall (\vec{x}, t)$ in $\mathbb{R}^3_* \times \mathbb{R}$) between two <u>formal asymptotic series</u>

Matching equation: $\boxed{\overline{\mathcal{M}(h)} \equiv \mathcal{M}(\overline{h})}$

NZ expansion $\left(\frac{r}{c} \to 0\right)$ of each multipolar coeff. of $\mathcal{M}(h)$

multipole expansion of each PN coefficient of $\overline{h}$

We assume (as part of our fundamental assumptions) that the matching eq. is correct (in the sense of formal series)

$$\boxed{\underset{\frac{r}{c} \to 0}{\text{NZ expansion}} \left(\underset{\frac{a}{r} \to 0}{\text{multipolar expansion}} \right) \equiv \underset{r \to \infty}{\text{FZ expansion}} \left(\underset{c \to \infty}{\text{PN series}} \right)}$$

The NZ expansion $\frac{r}{c} \to 0$ is "equivalent" to the PN expansion $c \to +\infty$ for fixed r

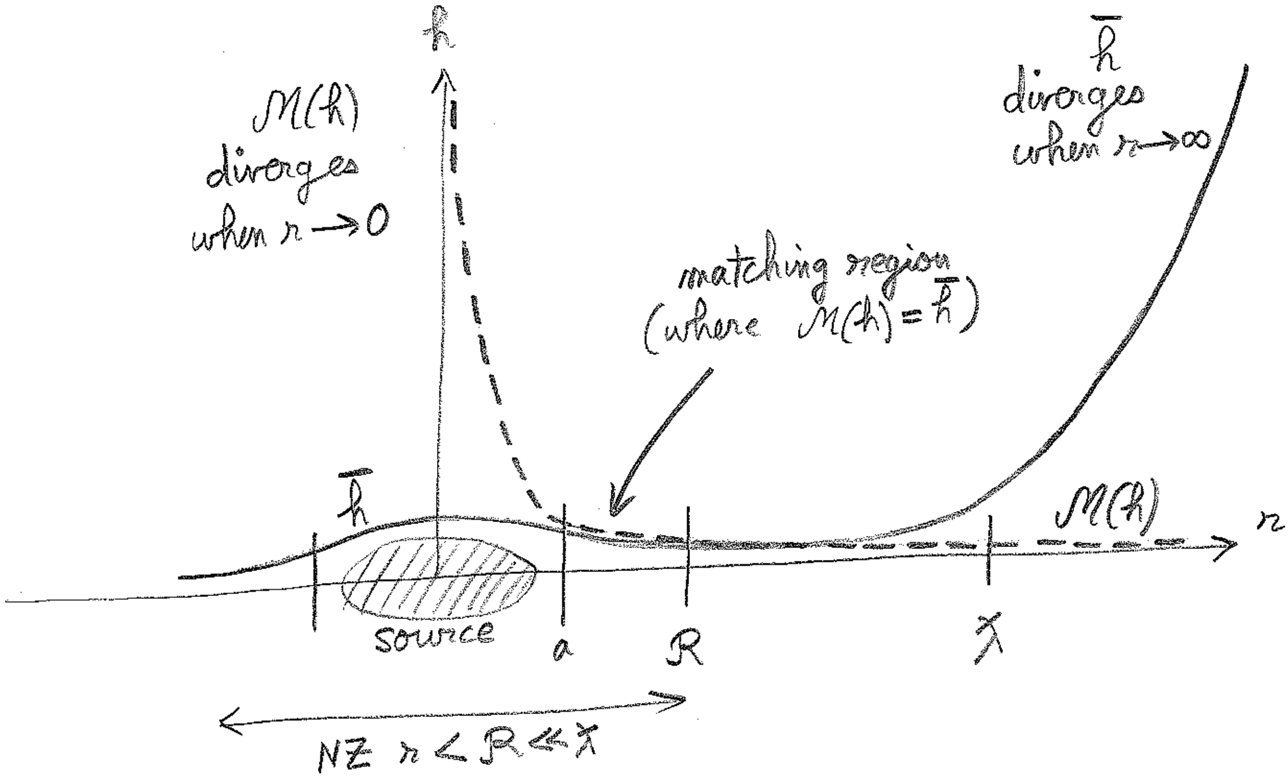
The multipole expansion $\frac{a}{r} \to 0$ is equivalent to the FZ expansion $r \to +\infty$ for a given source (fixed a)

The matching equation says basically the NZ and multipole expansions can be commuted.

Thus there is a common structure for the formal NZ and FZ expansions

$$\overline{\mathcal{M}(h)} \equiv \underbrace{\sum \hat{n}_L\, r^p (\ln r)^q F(t)} \equiv \mathcal{M}(\bar{h})$$

can be interpreted either as
- NZ singular expansion when $r \to 0$
- FZ ——————— $r \to \infty$



GENERAL EXPRESSION OF THE MULTIPOLE MOMENTS

h is the sol. of Einstein eqs (in harmonic coord. $\partial h = 0$) valid everywhere inside and outside the source

$$h = \frac{16\pi G}{c^4} \Box^{-1}_{\mathrm{Ret}} \tau \qquad \left(\begin{array}{c}\text{suppress} \\ \text{indices } \mu\nu\end{array}\right)$$

where $\tau = |g|\, T + \underbrace{\frac{c^4}{16\pi G}\Lambda}_{\substack{\text{gravitational source-term} \\ (\text{non-linearities in } h)}}$

Define $$\boxed{\Delta \equiv h - \mathrm{FP}\,\Box^{-1}_{\mathrm{Ret}}\,\mathcal{M}(\Lambda)}$$

where $\mathcal{M}(\Lambda) = \Lambda[\mathcal{M}(\Lambda)] = \Lambda_{ext}$ and FP is the finite part when $B \to 0$ (plays a crucial role because Λ_{ext} diverges when $r \to 0$)

$$\Delta = \underbrace{\frac{16\pi G}{c^4}\Box^{-1}_{\mathrm{Ret}}\tau}_{\substack{\text{no FP here}\\ \text{since } \tau \text{ is regular } (C^\infty)}} - \mathrm{FP}\,\Box^{-1}_{\mathrm{Ret}}\,\mathcal{M}(\Lambda)$$

However we can add FP on the first term (do not change the value because it converges). Using also $\mathcal{M}(\tau) = 0$ since τ has a compact support

$$\Delta = \frac{16\pi G}{c^4}\,\mathrm{FP}\,\Box^{-1}_{\mathrm{Ret}}\left[\tau - \mathcal{M}(\tau)\right]$$

Hence Δ appears as the retarded integral of a source <u>with compact support</u>. Indeed

$$\tau = \mathcal{M}(\tau) \quad \text{when } r > a$$

$$\boxed{\mathcal{M}(\Delta) = -\frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\,\partial_L\left(\frac{1}{r}\mathcal{H}_L(u)\right)}$$

This is standard expression of multipolar expansion outside a <u>compact-support</u> source. Here the moments are

$$\mathcal{H}_L = \mathrm{FP} \int d^3x \, x_L \underbrace{\left[\tau - \mathcal{M}(\tau)\right]}$$

since this has compact support ($r < a$, inside the NZ) we can replace by the NZ or PN expansion

$$\mathcal{H}_L = \mathrm{FP} \int d^3x \, x_L \left[\overline{\tau} - \overline{\mathcal{M}(\tau)}\right]$$

But we know the structure $\overline{\mathcal{M}(\tau)} = \sum \hat{n}_L r^p (\ln r)^q F(t)$ which is sufficient to prove that the second term is <u>zero</u> by analytic continuation

$$\mathrm{FP} \int d^3x \, x_L \, \overline{\mathcal{M}(\tau)} = \sum \mathrm{FP} \int d^3x \, x_L \, \hat{n}_Q \, r^p (\ln r)^q$$

$$= \sum \underset{B \to 0}{\text{Finite Part}} \int dr \, r^{B+S} (\ln r)^p$$

integrate over angles

$$= \sum \underset{B \to 0}{\mathrm{FP}} \left(\frac{d}{dB}\right)^p \int_0^{+\infty} dr \, r^{B+S}$$

$$\int_0^{+\infty} dr \, r^{B+S} = \underbrace{\int_0^{R} dr \, r^{B+S}}_{\substack{\text{computed} \\ \text{when } \mathrm{Re}\,B > -S-1}} + \underbrace{\int_R^{+\infty} dr \, r^{B+S}}_{\substack{\text{computed} \\ \text{when } \mathrm{Re}\,B < -S-1}}$$

$$= \underbrace{\frac{R^{B+S+1}}{B+S+1}}_{\text{by analytic continuation}} \qquad = \underbrace{-\frac{R^{B+S+1}}{B+S+1}}_{\text{by analytic continuation}}$$

$$\boxed{\text{Analytic Continuation} \quad \int_0^{+\infty} dr\, r^{B+s} (\ln r)^p = 0 \qquad \forall B \in \mathbb{C}}$$

The <u>general multipole expansion</u> outside the domain of a <u>PN isolated source</u> reads (Blanchet 1995, 1998)

$$\boxed{\mathcal{M}(h) = \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \mathcal{M}(\Lambda) - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left(\frac{1}{r} \mathcal{H}_L(u) \right)}$$

where

$$\boxed{\mathcal{H}_L(u) = \mathrm{FP} \int d^3x \; x_L \, \overline{\tau}(\vec{x}, u)}$$

PN expansion crucial here
(this is where the formalism applies only to PN sources)

Same result but in STF guise

$$\mathcal{M}(h) = \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \mathcal{M}(\Lambda) - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left(\frac{1}{r} \mathcal{F}_L(u) \right)$$

where

$$\mathcal{F}_L(u) = \mathrm{FP} \int d^3x \; \hat{x}_L \int_{-1}^{1} dz\, \delta_l(z)\, \overline{\tau}(\vec{x}, u + z|\vec{x}|/c)$$

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1}\, l!} (1-z^2)^l \quad \text{such that} \quad \int_{-1}^{1} dz\, \delta_l(z) = 1, \qquad \lim_{l \to +\infty} \delta_l(z) = \delta(z)$$

Practical way to implement the STF multipole expansion is to use the PN series

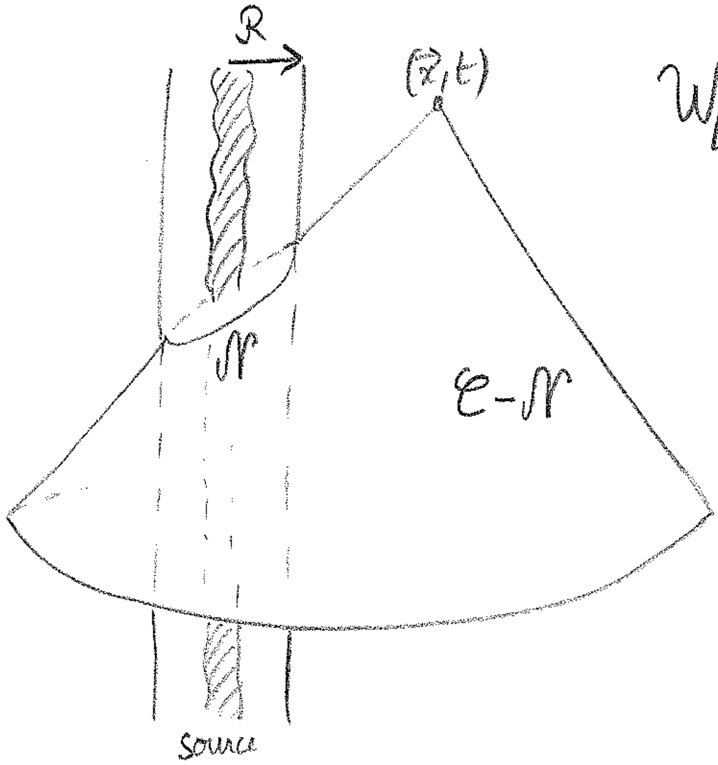
$$\boxed{\int_{-1}^{1} dz\, \delta_\ell(z)\, \overline{\tau}\left(\vec{x}, u + z|\vec{x}|/c\right) = \sum_{k=0}^{+\infty} \underbrace{\alpha_k^\ell}_{\frac{(2\ell+1)!!}{(2k)!!\,(2\ell+2k+1)!!}} \left(\frac{|\vec{x}|}{c}\frac{\partial}{\partial u}\right)^{2k} \overline{\tau}(\vec{x}, u)}$$

There is an alternative formalism for writing the general multipole expansion (Will & Wiseman 1996)

$$\mathcal{M}(h) = \underbrace{\Box^{-1}_{Ret}\, \mathcal{M}(\Lambda)\Big|_{\mathcal{C}-\mathcal{N}}}_{\text{the retarded integral excludes the NZ of source}} - \frac{4G}{c^4}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}\mathcal{W}_L(t-r)\right)$$

where

$$\mathcal{W}_L(u) = \underbrace{\int_{r<\mathcal{R}} d^3x\; x_L\, \overline{\tau}(\vec{x}, u)}_{\text{volume integral limited to the NZ of the source }(\mathcal{N})}$$



<u>The two formalisms are equivalent</u>

Next we identify $h_{ext} = \mathcal{M}(h)$ which means

$$G\, h_{(1)}[I_L J_L W_L \cdots Z_L] + G^2 h_{(2)} + \cdots + G^n h_{(n)} + \cdots$$

$$= \underbrace{-\frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_L\left(\frac{1}{r} \mathcal{F}_L^1(u)\right)} + \underbrace{\mathrm{FP}\, \Box^{-1}_{Ret} \mathcal{M}(\Lambda)}$$

has the form of the linear metric $Gh_{(1)}$ where the $\mathcal{F}_L^1$'s are "equivalent" to $I_L \cdots Z_L$

represents the non-linear corrections $G^2 h_{(2)} + \cdots + G^n h_{(n)} + \cdots$

Note that for the identification to work the $\mathcal{F}_L^1$'s in the right-hand-side should be considered as of <u>zero-th order in G</u>

Then we obtain $I_L \cdots Z_L$ in terms of the components of $\mathcal{F}_L^{\mu\nu}$ and hence of the source's pseudo-tensor $\overline{\tau}^{\mu\nu}$.

Decompose the $\mathcal{F}_L^{\mu\nu}$'s into ten irreducible STF tensors $R_L\ T^{(+)}_{L+1} \cdots U^{(-2)}_{L-2}\ V_L$

$$\begin{cases} \mathcal{F}_L^{00} = R_L \\ \mathcal{F}_L^{0i} = T^{(+)}_{iL} + \varepsilon_{ai\langle i_\ell} T^{(0)}_{L-1\rangle a} + \delta_{i\langle i_\ell} T^{(-)}_{L-1\rangle} \\ \mathcal{F}_L^{ij} = U^{(+2)}_{ijL} + \underset{L}{\mathrm{STF}}\, \underset{ij}{\mathrm{STF}} \Big[\varepsilon_{aii_\ell} U^{(+1)}_{ajL-1} + \delta_{ii_\ell} U^{(0)}_{jL-1} \\ \qquad + \delta_{ii_\ell} \varepsilon_{aji_{\ell-1}} U^{(-1)}_{aL-2} + \delta_{ii_\ell} \delta_{ji_{\ell-1}} U^{(-2)}_{L-2} \Big] + \delta_{ij} V_L \end{cases}$$

The final result is

$$\boxed{\begin{aligned}
I_L &= \mathrm{FP}\int d^3x \int_{-1}^{1} dz \Big\{ \delta_\ell(z)\,\hat{x}_L\,\Sigma - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)}\,\delta_{\ell+1}\,\hat{x}_{iL}\,\Sigma_i^{(1)} \\
&\qquad + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)}\,\delta_{\ell+2}\,\hat{x}_{ijL}\,\Sigma_{ij}^{(2)} \Big\}(\vec{x}, u+z|\vec{x}|/c) \\[2ex]
J_L &= \mathrm{FP}\int d^3x \int_{-1}^{1} dz\; \varepsilon_{ab\langle i_\ell} \Big\{ \delta_\ell\,\hat{x}_{L-1\rangle a}\,\Sigma_b \\
&\qquad - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)}\,\delta_{\ell+1}\,\hat{x}_{L-1\rangle ac}\,\Sigma_{bc}^{(1)} \Big\}(\vec{x}, u+z|\vec{x}|/c)
\end{aligned}}$$

where
$$\begin{cases} \Sigma = \dfrac{\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2} \\ \Sigma_i = \dfrac{\overline{\tau}^{0i}}{c} \\ \Sigma_{ij} = \overline{\tau}^{ij} \end{cases}$$

There are similar expressions for $W_L \dots Z_L$

These expressions give the source moments of any isolated PN source, <u>up to any PN order</u> (formally).

<u>POST-NEWTONIAN EXPANSION IN THE NEAR ZONE</u>

Consider the PN expansion of the field in the NZ ($r < \mathcal{R}$)

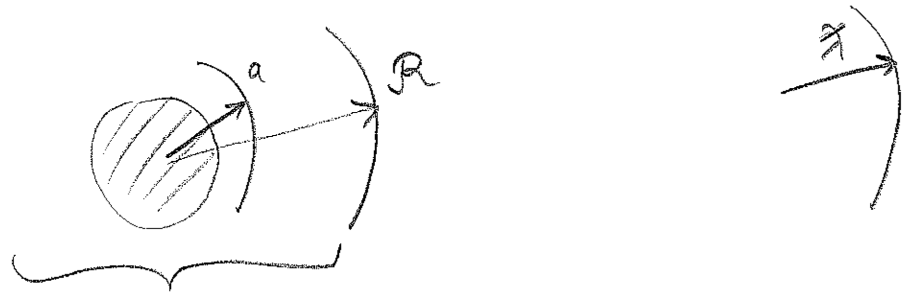
$$\boxed{\overline{h}(\vec{x}, t, c) = \sum_{p=2}^{+\infty} \frac{1}{c^p}\,\overline{h}_p(\vec{x}, t, \ln c)}$$

formal PN series (appearance of $\ln c$'s at 4PN order)

<u>Note</u>: $\overline{h}_p$ denotes the PN coefficient of $\frac{1}{c^p}$ while $h_{(n)}$ denotes the PM coefficient of G^n

To compute iteratively the $\bar{h}_n$'s we meet two problems

① <u>Problem of NZ limitation</u>



$\bar{h}$ is valid only in NZ
(and diverges in the FZ, when $r \to \infty$)

How to incorporate into the PN series the information about boundary conditions at infinity (notably the <u>no-incoming radiation condition</u> which is imposed at $\mathcal{J}^-$) ?

② <u>Problem of divergencies</u>

$$\Delta \bar{h}_p = \begin{pmatrix} \text{source term} \\ \text{with } \underline{\text{non-compact}} \\ \text{support} \\ \text{which blows up when } r \to +\infty \end{pmatrix}$$

Then the usual Poisson integral is divergent

$$\boxed{\bar{h}_p = \int \frac{d^3\vec{x}'}{|\vec{x}-\vec{x}'|} (\text{source term})}$$

<u>diverges</u> at the bound $|\vec{x}'| = +\infty$
(for high p)

Problem ① will be solved by matching: $\overline{\mathcal{M}(h)} = \mathcal{M}(\bar{h})$

Problem ② will be solved by finding a suitable solution of the Poisson equation (different from the Poisson integral)

Insert $\bar{h} = \sum \frac{1}{c^p} \bar{h}_p$ into $\begin{cases} \Box \bar{h} = \frac{16\pi G}{c^4} \bar{\tau} \\ \partial \bar{h} = 0 \end{cases}$

Hierarchy of PN equations ($\forall n \geqslant 2$)

$$\boxed{\begin{aligned} &\Delta \bar{h}_p^{\mu\nu} = 16\pi G \, \bar{\tau}_{p-4}^{\mu\nu} + \partial_t^2 \bar{h}_{p-2}^{\mu\nu} \\ &\partial_\nu \bar{h}_p^{\mu\nu} = 0 \end{aligned}}$$

At any given p the right-hand-side is known from previous iteration (using recursive treatment).

Construct first a particular solution of these equations using the generalized Poisson integral (Poujade & Blanchet 2002)

$$\mathrm{FP}\Delta^{-1}\left[\bar{\tau}_p\right] \equiv \underset{B\to 0}{\text{Finite Part}} \; \underbrace{\frac{-1}{4\pi} \int \frac{d^3\vec{x}' \, |\vec{x}'|^B}{|\vec{x}-\vec{x}'|} \bar{\tau}_p(\vec{x}', t)}_{\text{defined by analytic continuation}}$$

Then we add the general homogeneous solution of Laplace's equation which is regular in the source ($r \to 0$)

$$\Delta\Big[\underset{\substack{\uparrow \\ \text{solution regular} \\ \text{when } r\to 0}}{a\, \hat{x}_L} + \underset{\substack{\uparrow \\ \text{solution regular} \\ \text{when } r\to\infty}}{b\, \hat{\partial}_L \frac{1}{r}} \Big] = 0$$

Most general solution is

$$\boxed{\overline{h}'^{\mu\nu}_p = \underbrace{\mathrm{FP}\,\Delta^{-1}\left\{16\pi G\,\overline{\tau}'^{\mu\nu}_{p-4} + \partial_t^2\,\overline{h}'^{\mu\nu}_{p-2}\right\}}_{\substack{\text{particular solution} \\ \text{(well-defined thanks to} \\ \text{the Finite Part)}}} + \underbrace{\sum_{l=0}^{+\infty} B'^{\mu\nu}_{pL}(t)\,\hat{x}_L}_{\substack{\text{homogeneous solution} \\ \text{(unknown for the} \\ \text{moment)}}}}$$

To compute the homogenous solution we require that it matches the external field in the sense

$$\mathcal{M}\left(\sum \frac{1}{c^p}\,\overline{h}'^{\mu\nu}_p\right) = \overline{\mathcal{M}(h)} = \overline{\sum G^n h_{(n)}}$$

where $\mathcal{M}(h) = h_{ext} = \sum G^n h_{(n)}$. This fixes uniquely the homogeneous solution which is associated with radiation reaction forces inside the source, appropriate to an isolated system emitting GWs but not receiving GWs from $\mathcal{I}^-$.

Summing up $\overline{h} = \sum \frac{1}{c^p}\,\overline{h}_p$ we get

$$\boxed{\overline{h}'^{\mu\nu} = \underbrace{\frac{16\pi G}{c^4}\left\{\sum_{k=0}^{\infty}\left(\frac{\partial}{c\,\partial t}\right)^{2k}\mathrm{FP}\,\Delta^{-k-1}\,\overline{\tau}^{\mu\nu}\right\}}_{\substack{\text{particular solution} \\ \text{of d'Alembert eq.} \\ \text{denoted } \mathrm{FP}\,\square^{-1}\overline{\tau}^{\mu\nu}}} - \underbrace{\frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\,\hat{\partial}_L\left\{\frac{A^{\mu\nu}_L(t-r) - A^{\mu\nu}_L(t+r)}{2r}\right\}}_{\substack{\text{homogeneous solution} \\ \text{of d'Alembert eq.} \\ \text{which is regular when } r\to 0}}}$$

It's an anti-symmetric wave (retarded)−(advanced)

Result of the matching is (Poujade & Blanchet 2002)

$$\mathcal{A}_L^{\mu\nu}(u) = \mathcal{F}_L^{\mu\nu}(u) + \mathcal{R}_L^{\mu\nu}(u)$$

where $\mathcal{F}_L^{\mu\nu}$ is the source's multipole moment (computed previously)

$$\mathcal{F}_L^{\mu\nu}(u) = \mathrm{FP}\int d^3x\, \hat{x}_L \int_{-1}^{1} dz\, \delta_\ell(z)\, \overline{\tau}^{\mu\nu}(\vec{x}, u + z|\vec{x}|/c)$$

PN expansion of τ

and where $\mathcal{R}_L^{\mu\nu}(u)$ is a new type of moment which turns out to parametrize non-linear radiation reaction effects in the source (Blanchet 1993)

$$\boxed{\mathcal{R}_L^{\mu\nu}(u) = \mathrm{FP}\int d^3x\, \hat{x}_L \int_{1}^{+\infty} dz\, \gamma_\ell(z)\, \mathcal{M}(\tau^{\mu\nu})(\vec{x}, u - z|\vec{x}|/c)}$$

multipole expansion of τ

where $\gamma_\ell(z) = -2\,\delta_\ell(z)$ satisfies (by analytic continuation in ℓ)

$$\int_1^{+\infty} dz\, \gamma_\ell(z) = 1 \qquad \gamma_\ell(z) = (-)^{\ell+1}\frac{(2\ell+1)!!}{2^\ell \ell!}(z^2-1)^\ell$$

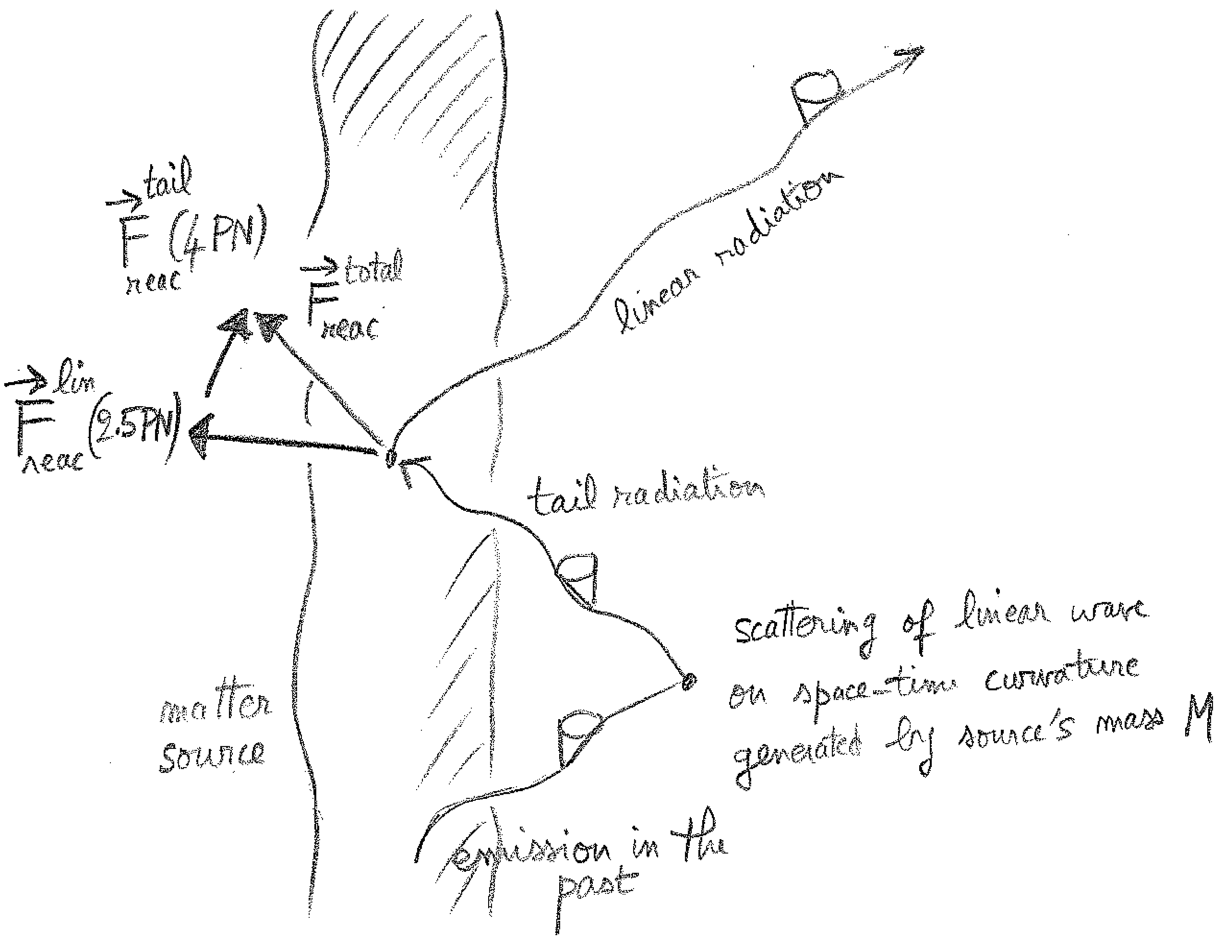
This comes from

$$\underbrace{0 = \int_{-\infty}^{+\infty} dz\, \delta_\ell(z)}_{\text{by analytic continuation in } \ell\in\mathbb{C}} = 2\int_1^{\infty} dz\, \delta_\ell(z) + \int_{-1}^{1} dz\, \delta_\ell(z) = -\int_1^{\infty} dz\, \gamma_\ell(z) + 1$$

Note that the PN expansion in the NZ ($r < \mathcal{R}$) depends on the multipole exp. $\mathcal{M}(\tau^{\mu\nu})$ and therefore on the properties of the field in the FZ ($r \gg \lambda\!\!\!^{-}$).

Indeed the PN exp. includes the radiation reaction terms appropriate to an isolated system, satisfying the correct boundary conditions at infinity (notably $\mathcal{I}^-$).

$$\mathcal{A}_L^{\mu\nu} = \underbrace{\mathcal{F}_L^{\mu\nu}}_{\substack{\text{describes "linear"} \\ \text{radiation reaction terms} \\ \text{and \underline{starts at 2.5PN}}}} + \underbrace{\mathcal{R}_L^{\mu\nu}}_{\substack{\text{describes "non-linear"} \\ \text{effects (tails) in the} \\ \text{radiation reaction and} \\ \text{\underline{starts at 4PN}}}}$$



The linear rad. reac. (parametrized by $\mathcal{F}_L^{\mu\nu}$) can be recombined with the particular solution

$$\mathrm{FP}\,\mathcal{I}^{-1}\,\overline{\tau}^{\mu\nu} = \sum_{k=0}^{+\infty}\left(\frac{\partial}{c\partial t}\right)^{2k} \mathrm{FP}\,\Delta^{-k-1}\,\overline{\tau}^{\mu\nu}$$

to give simply the retarded integral

$$\mathrm{FP}\,\Box^{-1}_{\mathrm{Ret}}\,\overline{\tau}^{\mu\nu} = -\frac{1}{4\pi}\underbrace{\sum_{p=0}^{+\infty}\frac{(-)^p}{p!}\left(\frac{\partial}{c\partial t}\right)^p \mathrm{FP}\int d^3x'\,|x-x'|^{p-1}\,\overline{\tau}^{\mu\nu}(\vec{x}',t)}_{\substack{\text{formal expansion } c\to+\infty \\ \text{of the retardation } t-\frac{1}{c}|\vec{x}-\vec{x}'| \\ \text{(well-defined thanks to the FP)}}}$$

The sol. $\mathrm{FP}\,\mathcal{I}^{-1}$ corresponds to the even-parity part $p=2k$.
The odd-parity $p=2k+1$ is exactly given by the terms with $\mathcal{F}_L^{\mu\nu}$

Final result is thus (Blanchet, Faye & Nissanke 2005)

$$\boxed{\overline{h}^{\mu\nu} = \underbrace{\frac{16\pi G}{c^4}\,\mathrm{FP}\,\Box^{-1}_{\mathrm{Ret}}\,\overline{\tau}^{\mu\nu}}_{\substack{\text{corresponds to the old} \\ \text{way of performing the} \\ \text{PN expansion (Anderson \& DeCanio} \\ \text{1975)}}} - \underbrace{\frac{4G}{c^4}\sum_{l=0}^{+\infty}\partial_L\left\{\frac{\mathcal{R}_L^{\mu\nu}(t-r)-\mathcal{R}_L^{\mu\nu}(t+r)}{2r}\right\}}_{\text{starts at 4PN}}}$$

PART 4

APPLICATION TO COMPACT BINARIES

THE 3PN METRIC

Detailed calculations at 3PN use explicit expressions of the near-zone metric coefficients (in harm. coord.)

$$\boxed{\begin{aligned} g_{00} &= -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_i V_i + \frac{V^3}{6}\right) \\ &\quad + \frac{32}{c^8}\left(\hat{T} + \dots\right) + \mathcal{O}\left(\frac{1}{c^{10}}\right) \\ g_{0i} &= -\frac{4}{c^3}V_i - \frac{8}{c^5}\hat{R}_i - \frac{16}{c^7}\left(\hat{Y}_i + \dots\right) + \mathcal{O}\left(\frac{1}{c^9}\right) \\ g_{ij} &= \delta_{ij}\left[1 + \frac{2}{c^2}V + \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + \dots\right)\right] \\ &\quad + \frac{4}{c^4}\hat{W}_{ij} + \frac{16}{c^6}\left(\hat{Z}_{ij} + \dots\right) + \mathcal{O}\left(\frac{1}{c^8}\right) \end{aligned}}$$

The potentials are generated by $T^{\mu\nu}$

$$\boxed{\begin{aligned} \sigma &= \frac{T^{00} + T^{ii}}{c^2} \\ \sigma_i &= \frac{T^{0i}}{c} \\ \sigma_{ij} &= T^{ij} \end{aligned}}$$

$\sigma = \rho + \mathcal{O}\left(\frac{1}{c^2}\right)$

where ρ is source's Newtonian density

V and V_i represent some retarded versions of the Newtonian and "gravitomagnetic" potentials

$$\boxed{\begin{aligned} V &= \Box^{-1}_{Ret}\left(-4\pi G\sigma\right) \\ V_i &= \Box^{-1}_{Ret}\left(-4\pi G\sigma_i\right) \end{aligned}}$$

$\hat{W}_{ij}$ is generated by matter + gravitational "stresses"

$$\boxed{\hat{W}_{ij} = \Box^{-1}_{Ret}\left[-4\pi\left(\sigma_{ij} - \delta_{ij}\,\sigma_{kk}\right) - \underbrace{\partial_i V \partial_j V}_{\text{quadratic non-linearity}}\right]}$$

$\hat{X}$, $\hat{R}_i$, $\hat{Z}_{ij}$, $\hat{T}$ are higer-order PN potentials

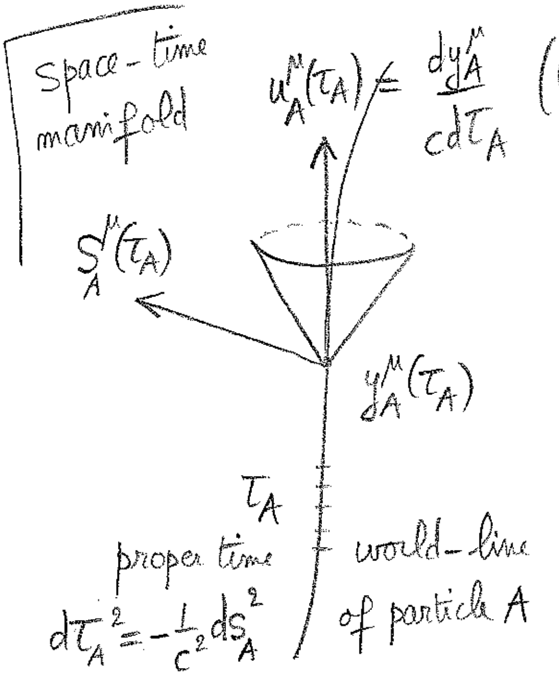
$$\hat{X} = \Box^{-1}_{Ret}\left[-4\pi G V \sigma_{ii} + \underbrace{\hat{W}_{ij}\,\partial_{ij} V}_{\text{cubic term}} + \cdots\right]$$

$$\hat{T} = \Box^{-1}_{Ret}\left[-4\pi G\left(\frac{1}{4}\sigma_{ij}\hat{W}_{ij} + \cdots\right) + \hat{Z}_{ij}\,\partial_{ij} V + \cdots\right]$$

and so on. The 3PN metric parametrized by these potentials is very useful in practice (permits to separate out different problems associated with quadratic, cubic, etc... non-linearities). At Newtonian order

$V = U + O\left(\frac{1}{c^2}\right)$ where $\underbrace{U = \Delta^{-1}\left(-4\pi G\rho\right)}_{\text{Newtonian potential}}$ is the usual

STRESS-ENERGY TENSOR OF POINT PARTICLES



$$\boxed{T^{\mu\nu}(x) = \sum_A \int_{-\infty}^{+\infty} d\tau_A \, p_A^{(\mu} u_A^{\nu)} \frac{\delta_4(x-y_A)}{\sqrt{-g_A}}}$$

where $p^\mu_A = m\, u^\mu_A$ (without spin)

In PN calculations we "split" space & time

$y^\mu_A = (ct, \vec{y}_A)$ $\qquad v^\mu_A = (c, \vec{v}_A)$ $\qquad$ where $\boxed{v^i_A = \frac{dy^i_A}{dt} = c\,\frac{u^i_A}{u^0_A}}$

ordinary (coordinate) velocity

$$\boxed{T^{\mu\nu}(\vec{x},t) = \sum_A \frac{m_A\, v^\mu_A\, v^\nu_A}{\sqrt{-g^A_{\rho\sigma}\, v^\rho_A\, v^\sigma_A}} \, \frac{\delta(\vec{x}-\vec{y}_A)}{\sqrt{-g_A}}}$$

$\delta(\vec{x}-\vec{y}_A)$ is Dirac's 3-dim function

For <u>spinning</u> particles we can add

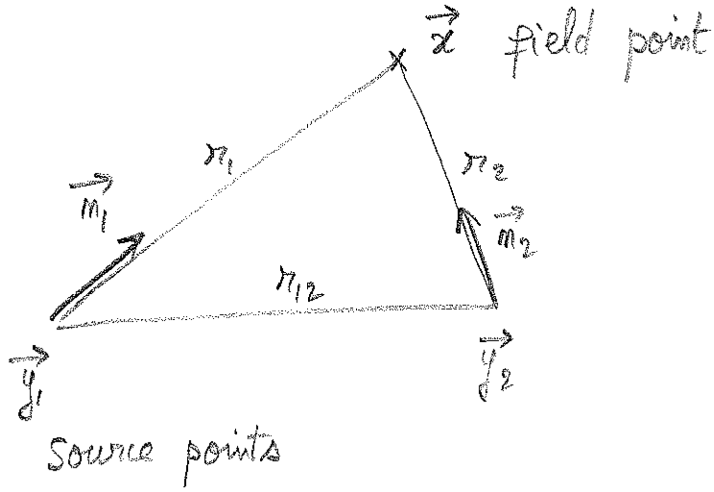
$$\boxed{T^{\mu\nu}_{\text{spin}}(x) = -\sum_A \nabla_\rho \left[\int_{-\infty}^{+\infty} d\tau_A \, S_A^{\rho(\mu}\, u_A^{\nu)} \frac{\delta_4(x-y_A)}{\sqrt{-g_A}} \right]}$$

(Dixon 1970, Bailey & Israel 1980)

where $S^{\mu\nu}_A$ is the spin anti-symmetric tensor

PROBLEM OF POINT PARTICLES

Two (say) point-like particles (masses m_1 and m_2)



$$r_A = |\vec{x} - \vec{y}_A| \qquad \vec{n}_A = \frac{\vec{x} - \vec{y}_A}{r_A}$$

$$r_{12} = |\vec{y}_1 - \vec{y}_2|$$

Newtonian potential U generated by the point masses

$$\Delta U = -4\pi G \rho = -4\pi G \left[m_1 \, \delta(\vec{x} - \vec{y}_1) + m_2 \, \delta(\vec{x} - \vec{y}_2) \right]$$

Using $\Delta \frac{1}{r} = -4\pi \, \delta(\vec{x})$ $\qquad U(\vec{x}) = \frac{G m_1}{r_1} + \frac{G m_2}{r_2}$

$$\frac{d\vec{v}_1}{dt} = \left(\vec{\nabla} U\right)(\vec{y}_1) = \Big(\underbrace{-\frac{G m_1}{r_1^2} \vec{n}_1}_{\substack{\text{self-force on the} \\ \text{point-particle is divergent}}} - \frac{G m_2}{r_2^2} \vec{n}_2 \Big)(\vec{y}_1)$$

<u>Problem 1</u> If $F(\vec{x})$ is divergent at $\vec{y}_1$ (say, with a power-like singular expansion around $\vec{y}_1$) what is the meaning of $F(\vec{y}_1)$?

Stress-energy tensor of point-particles

$$T^{\mu\nu} = \sum_A m_A \int_{-\infty}^{+\infty} d\tau_A \, u_A^\mu u_A^\nu \frac{\delta_4(x-y_A)}{\sqrt{-g}} = \sum_A \frac{m_A v_A^\mu v_A^\nu}{\sqrt{-g^A_{\rho\sigma} v_A^\rho v_A^\sigma}} \frac{\delta(\vec{x}-\vec{y}_A)}{\sqrt{-g}}$$

But $g \simeq -1 + \frac{U}{c^2} + \cdots$ where $U(\vec{x})$ is singular at $\vec{x} = \vec{y}_A$

<u>Problem 2</u> What is the meaning of $F(\vec{x})\,\delta(\vec{x}-\vec{y}_1)$?

Non-linear source of Einstein-field eqs

$$\Lambda_2^{00} \simeq h^{ij} \partial_i \partial_j h^{00} + \partial_i h^{00} \partial_j h^{00} + \cdots$$

with $h^{00} \simeq \frac{U}{c^2}$ Need to differentiate U

<u>Problem 3</u> How to differentiate singular functions

$$\partial_i \partial_j F \,?$$

For instance should we use standard distribution theory

$$\partial_i \partial_j \frac{1}{r_1} = \frac{3 n_1^i n_1^j - \delta^{ij}}{r_1^3} \underbrace{- \frac{4\pi}{3} \delta^{ij} \delta(\vec{x}-\vec{y}_1)}_{\text{distributional term}} \quad ?$$

<u>Problem 4</u> What is the meaning of the divergent integral

$$\int d^3\vec{x}\, F(\vec{x}) \;?$$

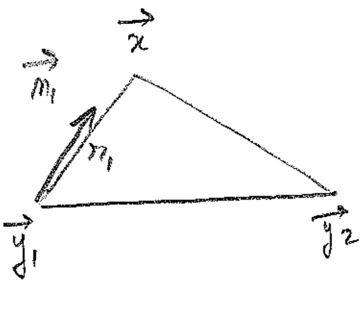
We must suplement the calculation of point particles by some <u>self-field regularization</u> to remove the formally infinite "self-field" of point particles.

- Hadamard's regularization (Hadamard 1932, Schwartz 1957) which is very efficient in practical calculations but yields some <u>ambiguity parameters</u> (coefficients which cannot be computed) at high PN orders ($\geqslant 3PN$)

- Dimensional regularization ('t Hooft and Veltman 1972), extremely powerful and <u>free of ambiguities</u> but cannot be implemented at present for general d (only $d = 3 + \varepsilon$ where $\varepsilon \to 0$)

HADAMARD SELF-FIELD REGULARIZATION

$F(\vec{x})$ is smooth except at $\vec{y}_1$ and $\vec{y}_2$. When $r_1 \to 0$

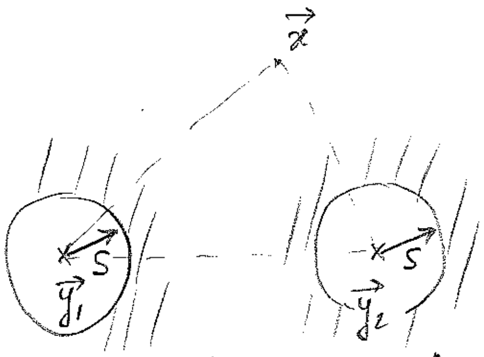


$$F(\vec{x}) = \sum_{a_0 \leq a \leq N} r_1^a \underset{1}{f_a}(\vec{n}_1) + o(r_1^N) \qquad a_0 \in \mathbb{Z}$$

Hadamard's partie finie of F at singular point $\vec{y}_1$

$$\boxed{(F)_1 \equiv \int \frac{d\Omega_1}{4\pi} \underset{1}{f_0}(\vec{n}_1)}$$

Hadamard's partie finie (Pf) of the divergent integral $\int d^3\vec{x}\, F(\vec{x})$



Two balls (radius s) are excised

$$\boxed{\mathrm{Pf}\int d^3x\, F(\vec{x}) = \lim_{s\to 0}\left\{ \int_{\substack{r_1>s \\ r_2>s}} d^3x\, F(\vec{x}) + \sum_{a+3<0} \frac{s^{a+3}}{a+3}\int d\Omega_1\, \underset{1}{f_a} + \ln\left(\frac{s}{s_1}\right)\int d\Omega_1\, \underset{1}{f_{-3}} + 1 \longleftrightarrow 2 \right\}}$$

These terms cancel out the divergencies of the integral over the "exterior"

Note the log terms depending on <u>two arbitrary constants</u> s_1 s_2 (one for each particle)

Hadamard Pf is equivalent to an analytic continuation

$$\boxed{\mathop{\mathrm{Pf}}_{s_1 s_2}\int d^3x\, F = \underbrace{\mathop{\mathrm{FP}}_{\alpha\to 0}\ \mathop{\mathrm{FP}}_{\beta\to 0}}_{\text{operations in whatever order}} \int d^3x \left(\frac{r_1}{s_1}\right)^{\alpha}\left(\frac{r_2}{s_2}\right)^{\beta} F}$$

Note the integral of a gradient is not zero (because of the singularities)

$$\boxed{\mathrm{Pf}\int d^3x\, \partial_i F = -4\pi\left(n_1^i r_1^2 F\right)_1 - 4\pi\left(n_2^i r_2^2 F\right)_2}$$

Partie finie pseudo-functions (Blanchet & Faye 2000)

$\mathcal{F}$ is the set of all such $F(\vec{x})$

$\mathrm{Pf}\,F$ is a linear form on $\mathcal{F}$

$$\forall G \in \mathcal{F} \qquad \boxed{\underbrace{\langle \mathrm{Pf}\,F, G \rangle}_{\substack{\text{result of action}\\ \text{of Pf F on } G\in\mathcal{F}}} = \underbrace{\mathrm{Pf}\int d^3x\, FG}_{\substack{\text{a real number}\\ \text{given by Hadamard's Pf}}}}$$

Define the partie finie δ-function $\mathrm{Pf}\,\delta_1$ (where $\delta_1 = \delta(\vec{x}-\vec{y}_1)$)

$$\forall F \qquad \langle \mathrm{Pf}\,\delta_1, F \rangle = (F)_1$$

We can give also a meaning to the product of F with δ_1

$$\forall G \qquad \langle \mathrm{Pf}(F\delta_1), G \rangle = (FG)_1$$

Derivative of a pseudo-function is defined by the "rule of integration by parts"

$$\boxed{\forall F, G \qquad \langle \partial_i(\mathrm{Pf}\,F), G \rangle = -\langle \partial_i(\mathrm{Pf}\,G), F \rangle}$$

This constitutes a generalization of Schwartz distributional derivative.

This "extended Hadamard regularization" gives a set of rules to compute all terms at say 3PN. However because of the constants s_1 s_2 it leaves finally some unknown "ambiguity parameters" at 3PN order (Jaranowski & Schäfer 1999).

Hadamard's regularization works well up to 2PN but fails to provide a complete answer at 3PN. One reason is that from the definition of $(F)_1$ we have

$$(FG)_1 \neq (F)_1 (G)_1 \quad \text{in general.}$$

Hence basic symmetries of GR such as diffeomorphism invariance at not respected (at PN orders $\geq$ 3PN)

DIMENSIONAL SELF-FIELD REGULARIZATION

Work in a space with d dimensions (so space-time has $D = d+1$ dimensions).

Idea of the regularization is to apply complex analytic continuation in the dimension $d \in \mathbb{C}$.

Volume element $\boxed{d^d x = r^{d-1}\, dr\, d\Omega_{d-1}}$ $\quad r = |\vec{x}|$

Volume of $(d-1)$ dimensional sphere $\Omega_{d-1} = \int d\Omega_{d-1}$

From the Gaussian integral $\int d^d x\, e^{-r^2} = \left(\int dx\, e^{-x^2}\right)^d = \pi^{d/2}$

$$= \Omega_{d-1} \int_0^\infty dr\, r^{d-1} e^{-r^2} = \frac{\Omega_{d-1}}{2} \Gamma\left(\frac{d}{2}\right)$$

$$\boxed{\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}}$$

For instance $\Omega_2 = 4\pi$ and $\Omega_1 = 2\pi$
and $\Omega_0 = 2$ (sphere with 0 dimension is made of 2 points!)

Green's function of Laplace operator:

$$\boxed{\begin{array}{ll} \Delta u = -4\pi\, \delta^{(d)}(\vec{x}) & \text{d-dimensional Dirac function} \\ u = \tilde{k}\, r^{2-d} \quad \text{where } \tilde{k} = \dfrac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-1}{2}}} & \end{array}}$$

Riesz (1949) Euclidean kernels (generalize $\delta^{(d)}$ and u)

$$\boxed{\begin{array}{l} \delta_\alpha^{(d)}(\vec{x}) = K_\alpha\, r^{\alpha-d} \\ \qquad \text{where } K_\alpha = \dfrac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^\alpha \pi^{d/2}\, \Gamma(d/2)} \\ \text{are such that } \Delta\, \delta_{\alpha+2}^{(d)} = -\delta_\alpha^{(d)} \\ \text{and } \delta_\alpha^{(d)} * \delta_\beta^{(d)} = \delta_{\alpha+\beta}^{(d)} \end{array}}$$

hence
$\delta^{(d)} = \delta_0^{(d)}$
and $u = 4\pi\, \delta_2^{(d)}$

this beautiful convolution property is an elegant formulation of Riesz's formula in d dimensions

$$\boxed{\int d^d x \; r_1^\alpha\, r_2^\beta = \pi^{d/2} \frac{\Gamma\left(\frac{\alpha+d}{2}\right)\Gamma\left(\frac{\beta+d}{2}\right)\Gamma\left(-\frac{\alpha+\beta+d}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\beta}{2}\right)\Gamma\left(\frac{\alpha+\beta+2d}{2}\right)}\, r_{12}^{\alpha+\beta+d}}$$

For instance $\displaystyle \int \frac{d^3x}{r_1^2 r_2^2} = \frac{\pi^3}{r_{12}}$

Einstein field equations in $D=d+1$ dimensions

$$R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R=\frac{8\pi G}{c^4}T^{\mu\nu} \Leftrightarrow R^{\mu\nu}=\frac{8\pi G}{c^4}\Big(T^{\mu\nu}-\underbrace{\frac{1}{d-1}g^{\mu\nu}T}_{\text{dimension appears explicitly here}}\Big)$$

still we have
$$\begin{cases}\Box h^{\mu\nu}=\dfrac{16\pi G}{c^4}\tau^{\mu\nu} \text{ with } \partial_\nu h^{\mu\nu}=0\\ \tau^{\mu\nu}=|g|\,T^{\mu\nu}+\dfrac{c^4}{16\pi G}\Lambda^{\mu\nu}\end{cases}$$

$$\Lambda^{\mu\nu}=-h^{\rho\sigma}\partial_{\rho\sigma}h^{\mu\nu}+\partial_\rho h^{\mu\sigma}\partial h^{\nu\rho}_{\sigma}+\dots+\frac{1}{d-1}g\cdots g\,\partial h\,\partial h$$

$$\boxed{G=\ell_0^{d-3}\,G_N}$$

usual Newtonian gravitational constant

DIFFERENCE BETWEEN HADAMARD AND DIMENSIONAL REGULARIZATIONS

Iterating the field equations in PN form we have to solve Poisson equations $\Delta P=F$ with some source term $F(\vec{x})$ which is singular at $\vec{y}_1$ and $\vec{y}_2$ $(F\in\mathcal{F})$. Then we need to compute the value of P at $\vec{y}_1$ and $\vec{y}_2$.

In Had. reg. we use the Partie finie of a Poisson integral

$$P(\vec{x}')=-\frac{1}{4\pi}\underbrace{\mathop{\mathrm{Pf}}_{s_1 s_2}}_{\text{depends on constants } s_1,\, s_2}\int\frac{d^3x}{|\vec{x}-\vec{x}'|}F(\vec{x})$$

To compute the value when $\vec{x}' \longrightarrow \vec{y}_1$ one applies the Partie finie of a singular function.

$$P(\vec{x}') = \sum_{p \leq N} r_1'^{\,p} \underbrace{\left[g_p(\vec{n}_1') + h_p(\vec{n}_1') \ln r_1' \right]}_{\text{appearance of } \ln r_1' \text{ terms in the Poisson integral}} + o(r_1'^{\,N})$$

$$\boxed{(P)_1 = \int \frac{d\Omega_1'}{4\pi} \left[g_0 + h_0 \ln r_1' \right]}$$

here $\ln r_1'$ is considered as a "constant" (though it is really infinite $\ln 0 = -\infty$)

Explicit calulation shows

$$\boxed{(P)_1 = -\frac{1}{4\pi} \underset{r_1', s_2}{\mathrm{Pf}} \int \frac{d^3x}{r_1} F(x) - (r_1^2 F)_1}$$

depends on r_1' and s_2

(similarly $(P)_2$ depends on r_2' and s_1)

In dim. reg. things are simpler:

$$P^{(d)}(\vec{x}') = -\frac{\tilde{k}}{4\pi} \int \frac{d^dx}{|\vec{x}' - \vec{x}|^{d-2}} F^{(d)}(\vec{x})$$

and value at $\vec{x}' = \vec{y}_1$ is obtained by replacing $\vec{x}' \longrightarrow \vec{y}_1$

$$\boxed{P^{(d)}(\vec{y}_1) = -\frac{\tilde{k}}{4\pi} \int \frac{d^dx}{r_1^{d-2}} F^{(d)}(\vec{x})}$$

Point is that the difference between the two regularization depends on the vicinity of singularities only

$$\boxed{\mathcal{D}P(1) \equiv P^{(d)}(\vec{y}_1) - (P)_1}$$

When $r_1 \longrightarrow 0$ (near $\vec{y}_1$)

$$F(\vec{x}) = \sum_p r_1^p \, {}_1f_p(\vec{n}_1) + o(r_1^N)$$

while the analogue in d dimensions, $F^{(d)}(\vec{x})$ (defined by the same PN iteration of field equations but in d dim) admits

$$F^{(d)}(\vec{x}) = \sum_{p,q} r_1^{p+q\varepsilon} \, {}_1f^{(\varepsilon)}_{p,q}(\vec{n}_1) + o(r_1^N)$$

where $\varepsilon = d-3$.

$$\boxed{\begin{aligned}\mathcal{D}P(1) = &-\frac{1}{\varepsilon(1+\varepsilon)} \sum_q \left(\frac{1}{q} + \varepsilon\left[\ln r_1' - 1\right]\right) \int \frac{d\Omega_1}{4\pi} \, {}_1f^{(\varepsilon)}_{-2,q}(\vec{n}_1) \\ &-\frac{1}{\varepsilon(1+\varepsilon)} \sum_q \left(\frac{1}{q+1} + \varepsilon \ln s_2\right) \\ &\qquad \times \sum_{l=0}^{\infty} \frac{(-)^l}{l!} \partial_L\left(\frac{1}{r_{12}^{1+\varepsilon}}\right) \int \frac{d\Omega_2}{4\pi} n_2^L \, {}_2f^{(\varepsilon)}_{-l-3,q}(\vec{n}_2) \\ &+ \mathcal{O}(\varepsilon)\end{aligned}}$$

can be computed from the knowledge of the expansions of $F^{(d)}$ when $r_1 \to 0$, $r_2 \to 0$

Conclusions The difference between Had reg and Dim reg is made of the contribution of poles

$$\boxed{(\text{Dim reg}) - (\text{Had reg}) = \frac{a_{-1}}{\varepsilon} + a_0 + \mathcal{O}(\varepsilon)}$$

$$\varepsilon = d-3$$

This difference can be computed locally, i.e. depends only on the expansions of $F^{(d)}$ around the singularities ($r_1 \longrightarrow 0$ and $r_2 \longrightarrow 0$)

The two regs. agree in the absence of poles. Since no poles occur up to 2PN order (poles in ε correspond to logarithmic divergences in $d=3$) Had reg can be employed without problem up to 2PN.

At 3PN order poles in ε occur and as a result Had reg is not able to give a complete answer, and becomes "ambiguous" with the appearance of unknown "ambiguity parameters" (λ, ξ, κ and ζ) which cannot be computed.

Technically one of the reasons for the problems with Had reg is the "non-distributivity" of the partie finie

$$(FG)_1 \neq (F)_1 (G)_1 \quad \text{in general}$$

(because of the angular integration in the definition of the p.f.)

However Had. reg. is extremely convenient in practical calculations and permits to compute unambiguously all the terms but a few (those corresponding to poles in ε)

By contrast Dim. reg. cannot be implemented (for the moment) for general d but only in the limit $d \to 3$

Strategy

(1) Compute all the terms using Had reg (in $d=3$)

(2) Obtain the Dim reg result by

$$(\text{Dim reg}) = (\text{Had reg}) + \underbrace{\frac{a_{-1}}{\varepsilon} + a_0 + O(\varepsilon)}_{\text{computed locally } r_{1,2} \to 0}$$

SOME EXAMPLES OF COMPUTATION IN $d=3$

In a PN expansion the metric is

$$\begin{cases} g_{00} = -1 + \dfrac{2U}{c^2} + \cdots + \dfrac{\hat{X}}{c^6} + \cdots \\[2ex] g_{0i} = \dfrac{4V_i}{c^3} + \cdots \\[2ex] g_{ij} = \delta_{ij}\left(1 + \dfrac{2U}{c^2} + \cdots\right) + \dfrac{1}{c^4}\hat{W}_{ij} \end{cases}$$

$U =$ Newtonian potential
$\Delta U = -4\pi G\rho$

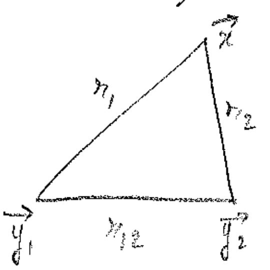
$\hat{X} =$ some higher potential

$V_i =$ gravitomagnetic potential
$\Delta V_i = -4\pi G\rho v^i$

$\hat{W}_{ij} =$ potential generated by gravitational stresses
$\Delta \hat{W}_{ij} = \partial_i U \partial_j U + \cdots$

For 2 particles

$$\rho = m_1\delta_1 + m_2\delta_2 \quad \Rightarrow \quad U = \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}$$



$$V_i = \frac{Gm_1 v_1^i}{r_1} + \frac{Gm_2 v_2^i}{r_2}$$

$$\begin{aligned} \Delta W_{ij} &= \partial_i\left(\frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}\right)\partial_j\left(\frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}\right) \\ &= \partial_i\left(\frac{Gm_1}{r_1}\right)\partial_j\left(\frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}\right) + 1 \longleftrightarrow 2 \\ &= G^2 m_1^2 \frac{n_1^i n_1^j}{r^4} + G^2 m_1 m_2 \frac{\partial^2}{\partial y_1^{(i}\partial y_2^{j)}}\left(\frac{1}{r_1 r_2}\right) + 1 \longleftrightarrow 2 \end{aligned}$$

Can be integrated using

$$\boxed{\begin{aligned} & g = \ln S \qquad S = r_1 + r_2 + r_{12} \\ & \Delta g = \frac{1}{r_1 r_2} \end{aligned}}$$

extremely useful function which permits the 3PN calculation in closed-analytic form

$$\boxed{W_{ij} = \frac{G^2 m_1^2}{8}\left(\partial_{ij} \ln r_1 + \frac{\delta^{ij}}{r^2}\right) + G^2 m_1 m_2 \frac{\partial^2 g}{\partial y_1^{(i} \partial y_2^{j)}} + 1 \longleftrightarrow 2}$$

At higher PN order needs to compute solutions of eqs like

$$\Delta X = W_{ij}\, \partial_{ij} U \qquad \text{where}$$

The closed-form solution can be found using the elementary solutions

$$\boxed{\begin{aligned} \Delta K_1 &= 2\, \partial_{ij} \frac{1}{r_2}\, \partial_{ij} \ln r_1 \\ \Delta H_1 &= 2\, \partial_{ij} \frac{1}{r_1} \frac{\partial^2 g}{\partial y_1^i \partial y_2^j} \end{aligned}}$$

which are known in closed form

$$K_1 = \left(\frac{1}{2}\Delta - \Delta_1 - \Delta_2\right)\left(\frac{\ln r_1}{r_2}\right) + \cdots$$

$$H_1 = \frac{1}{2}\Delta_1\left(\frac{g}{r_1}\right) + \cdots$$

These results permit to derive the metric $g_{\mu\nu}$ at 2PN hence we can deduce the EOM at 2PN (by replacing $g_{\mu\nu}$ into the geodesic equation and applying the regularization)

However at 3PN one cannot derive the metric $g_{\mu\nu}^{3PN}$ in closed form for any field point $\vec{x}$ in the NZ. Only the limit $\vec{x} \to \vec{y}_1$ can be computed (using the regularization) so the 3PN EOM can be obtained (after long and tedious calculations) (Blanchet & Faye 2000)

For the computation of the multipole moments $\underbrace{I_L \ J_L}_{\substack{\text{source-type moments} \\ \text{whose general expression} \\ \text{is known}}}$:

At Newtonian order (quadrupole formula)

$$I_{ij} = \int d^3x \, \rho \, \hat{x}_{ij} = m_1 \, y_1^{<i} y_1^{j>} + m_2 \, y_2^{<i} y_2^{j>} + \cdots$$
$$+ \cdots$$

At higher PN order we have <u>non-compact support</u> terms such as

$$I_{ij}^{(NC)} = \underset{B \to 0}{\text{F.P.}} \int d^3x \, |\vec{x}|^B \, \hat{x}_{ij} \, \partial_k U \, \partial_k U$$

$$= \text{FP} \int d^3x \, |x|^B \, \hat{x}_{ij} \left\{ \underbrace{\frac{G^2 m_1^2}{r_1^4}}_{\substack{\text{gives zero with} \\ \text{Had reg}}} + G^2 m_1 m_2 \frac{\partial^2}{\partial y_1^k \partial y_2^k} \left(\frac{1}{r_1 r_2} \right) + 1 \leftrightarrow 2 \right\}$$

Computation (to this order) is reduced to the computation of

$$\boxed{\begin{aligned} Y_L(\vec{y}_1, \vec{y}_2) &= -\frac{1}{2\pi} \text{F.P.} \int d^3x \, |x|^B \frac{\hat{x}_L}{r_1 r_2} \\ Y_L(y_1, y_2) &= \frac{r_{12}}{l+1} \sum_{p=0}^{l} y_1^{<L-P} \, y_2^{P>} \end{aligned}}$$

To higher PN order more complicated integrals appear (Blanchet, Iyer & Joguet 2002)

Ambiguity parameter λ in 3PN Had. reg. EOM

There are 4 constants which appear (inside logs)

$r'_1 \quad r'_2$ (come from reg. of the potentials)

$s_1 \quad s_2$ (come from reg. of the EOM)

However two of these constants can be removed by a coordinate transformation. It remains only the 2 "constants"

$$\ln\left(\frac{r'_1}{s_1}\right) \quad \text{and} \quad \ln\left(\frac{r'_2}{s_2}\right)$$

We find (Blanchet & Faye 2000) these constants have the form

$$\ln\left(\frac{r'_1}{s_1}\right) = \frac{159}{308} + \lambda \frac{m}{m_1} \qquad (m = m_1 + m_2)$$

and $1 \longleftrightarrow 2$

λ is equivalent to ω_{static} introduced by Jaranowski & Schäfer (1999)

Ambiguity parameters ξ, κ, ζ in 3PN quad. moment

(Blanchet, Iyer & Joguet 2002)

$$\ln\left(\frac{r'_1}{u_1}\right) = \xi + \kappa \frac{m_2}{m_1}$$

(ambiguities in the relation between Had. reg. constants u_1, u_2 similar to $s_1 s_2$ and the EOM-related constants $r'_1\ r'_2$).

In addition ζ reflects the Poincaré invariance of the field (not necessarily satisfied by Had. reg.)

There is complete agreement between all these works (whenever this can be compared) up to 3.5PN.

Final values for the ambiguity parameters are

$$\lambda = -\frac{1987}{3080} \qquad \text{(3PN equations of motion)}$$

$$\begin{cases} \xi = -\dfrac{9871}{9240} \\ \kappa = 0 \\ \zeta = -\dfrac{7}{33} \end{cases} \qquad \text{(3PN radiation field)}$$

All these parameters have been checked by methods <u>independent of the regularization</u>

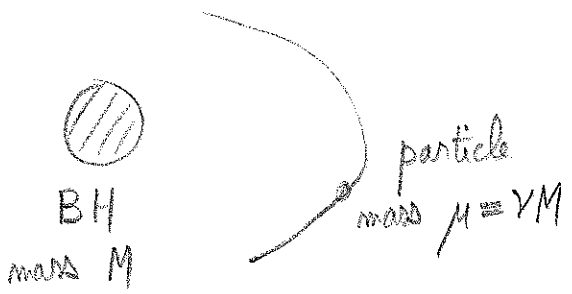
λ by surface-integral method (Itoh & Futamase 2004)

$\xi + \kappa$ by requiring that the binary's mass dipole agrees with the center-of-mass deduced from EOM (BI 04)

κ from argument based on space-time diagrams (BDEI 05)

ζ from a computation of the multipole moments of a boosted Schwarzschild solution (BDI 04)

All results are in agreement with <u>black-hole perturbation theory</u> in the limit $\nu \longrightarrow 0$



Sasaki (1994)
Tagoshi & Sasaki (1994) — 4PN

Tanaka, Tagoshi & Sasaki (1996) — 5.5PN

<u>Spin effects</u> have been added

Barker & O'Connell (1975, 1979)
Kidder, Will & Wiseman (1993)
Kidder (1995)
Porto (2006)
— spin-orbit (1.5PN) and spin-spin (2PN)

Tagoshi, Ohashi & Owen (2000)
Faye, Blanchet & Buonanno (2006)
Blanchet, Buonanno & Faye (2006)
— spin-orbit (2.5PN)

Templates for inspiralling compact binaries (ICBs) are known up to

3.5PN for the phase
2.5PN for the waveform

With spins they are known up to 2.5PN for the phase.

HISTORY OF PN EOM AND RADIATION OF COMPACT BINARIES

PN equations of motion

Lorentz & Droste 1917
Einstein, Infeld & Hoffmann 1938
← surface integral approach

1PN

Damour & Deruelle (1982, 1983) Harm. coord.
Damour & Schäfer (1985) ADM coord.
Kopeikin & Grischuk (1985) extended body approach
Blanchet, Faye & Ponsot (1998) point-particles computation of EOM and metric
Itoh, Futamase & Asada (2001) surface-integral

2.5PN

Jaranowski & Schäfer (1998, 1999) Hadamard reg. in ADM coord. Two ambiguity parameters ω_s, ω_k
Blanchet & Faye (2000, 2001) Had. reg. in harmonic coord. One ambiguity parameter $\lambda \Leftrightarrow \omega_s$
Damour, Jaranowski & Schäfer (2001) Dimensional reg. computation of ω_s
Blanchet, Damour & Esposito-Farèse (2004) Dim reg. computation of $\lambda \Leftrightarrow \omega_s$
Itoh & Futamase (2004) surface-integral method free of ambiguity parameters

3PN

Iyer & Will (1993, 1995) balance equation for computing rad. reaction
Pati & Will (2001) harm. coord.
Königsdörffer, Faye & Schäfer (2003) ADM coord.
Nissanke & Blanchet (2005) harm. coord.

3.5 PN

PN radiation field

Landau & Lifchitz (1941)
Peters & Mathews (1963)

Newtonian (quadrupole order)

Wagoner & Will (1976) using Epstein–Wagoner–Thorne moments
Blanchet & Schäfer (1989) using BD moments

1PN

Poisson (1993) perturbative limit $\nu \to 0$
Wiseman (1993)
Blanchet & Schäfer (1993)

1.5PN (tail)

Blanchet, Damour, Iyer, Will & Wiseman (1995)
Blanchet, Iyer, Will & Wiseman (1996) waveform
Blanchet (1996) 2.5PN tail
Arun, Blanchet, Iyer & Qusailah (2004) 2.5PN waveform

2PN + 2.5PN

Blanchet (1998) 3PN tail-of-tail
Blanchet, Iyer & Joguet (2001) Hadamard reg. 3 ambiguity parameters ξ, κ, ζ
Blanchet & Iyer (2004) Had. reg., general orbits
Blanchet, Damour, Esposito-Farèse & Iyer (2005) Dim. reg. computation of ξ, κ, ζ

3PN + 3.5PN