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Higher order gravitational radiation losses in binary systems

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Summary. The rate of emission of gravitational energy from a system of two point masses is computed with an accuracy consistent with the first-order relativistic corrections in the dynamics of the system. The computations use recently-developed post-Newtonian formalisms for the dynamics of two point masses and for the generation of gravitational waves.

In the case of two point masses in (quasi) elliptic motion, standard heuristic arguments yield the expression, valid at higher relativistic order, of the rate $\dot{P}$ of decrease of the orbital period of the masses. The higher-order relativistic correction contributes to $\dot{P}$ a fractional amount of 2.15×10^{-5} in the case of the binary pulsar PSR 1913+16. This value is far smaller than the present accuracy of 1.7×10^{-2} in the measurement of $\dot{P}$ for PSR 1913+16 but this higher-order relativistic expression of $\dot{P}$ may be useful in the future for relativistic binary pulsars.

Computations valid for a system of two point masses in (quasi) hyperbolic motion are also presented. In this case, the same heuristic arguments permit the study, still valid at higher relativistic order, of the capture, by radiation emission, of two point masses moving on a quasi-hyperbolic orbit with small enough energy.

1 Introduction

The rate of emission of gravitational energy from a system of two point masses has been computed by Peters & Mathews (1963) in the case of elliptic motion, and by Hansen (1972) and Turner (1977) in the case of hyperbolic motion.

The work of Peters & Mathews has provided a remarkable explanation of the observations of the Hulse & Taylor (1974) binary pulsar PSR 1913+16 – a pulsar orbiting an unseen companion (very likely another neutron star) on a nearly elliptic orbit with orbital period

$P=7^{\mathrm{h}}45^{\mathrm{m}}$. Indeed, it has been observed with increasing precision (Taylor & Weisberg 1982; Weisberg & Taylor 1984; Taylor 1989) that the orbital period P of the pulsar is regularly decreasing as time passes by the (dimensionless) amount

$$(\dot{P})_{\mathrm{obs}}=\left(\frac{dP}{dt}\right)_{\mathrm{obs}}=(-2.40\pm 0.04)\times 10^{-12} \tag{1.1}$$

(value taken from Taylor 1989). This effect can be understood by an heuristic argument based on energy conservation. The argument is that the power of gravitational wave emission theoretically computed by Peters & Mathews (1963) should be balanced by a decrease of the Newtonian energy of the stars (the rest masses of the stars staying constant). Hence, by Kepler's third law, the gravitational wave emission should result in a steady decrease of the orbital period of the stars (Esposito & Harrison 1975; Wagoner 1975). Numerically, the argument (calculated from the data given in Taylor 1989) yields

$$(\dot{P})_{\mathrm{th}}=(-2.402\pm 0.001)\times 10^{-12}. \tag{1.2}$$

This theoretical value agrees within 1.7 per cent with the observed one (equation 1.1). Note that it can be shown independently of the heuristic energy argument by a detailed computation in general relativity (taking particularly into account the strong internal gravity of the pulsar and its companion) that the effect (equation 1.1) is a consequence of the finite velocity of the general relativistic gravitational interaction, and hence a consequence of the very existence of gravitational radiation (Damour & Deruelle 1981; Damour 1983a,b).

For the moment, the work of Hansen (1972) and Turner (1977) on the hyperbolic motion has not yielded any application to observed astrophysical systems. However, in this case the same type of heuristic argument as for the computation of $\dot{P}$ can be used. Indeed, the total energy carried off by gravitational radiation during the encounter of the two stars on the hyperbolic orbit should be balanced by a decrease of their Newtonian energy. In particular, we expect that the two stars, starting at infinity on an orbit with small enough (positive) energy, will capture each other because of the radiation emission, and form a bound system. The heuristic argument can be used to compute a 'critical' eccentricity below which an initially hyperbolic motion becomes finally elliptic (such a critical eccentricity is also known from the work by Walker & Will 1979).

In this paper we extend both the works of Peters & Mathews (1963) on the elliptic motion and of Hansen (1972) and Turner (1977) on the hyperbolic motion to include higher-order, 'post-Newtonian', corrections in the dynamics of the system. Then we use the same type of heuristic argument as in the lowest-order case to compute the post-Newtonian higher-order expression of $\dot{P}$ for a binary bound system, and the higher-order expression of a 'critical' eccentricity for a binary unbound system. We find that the post-Newtonian relative corrections in $\dot{P}$ (given by equation 4.26 below) bring in, in the case of the binary pulsar PSR 1913+16, a correction which is numerically equal to $+2.15\times 10^{-5}$. This is unfortunately far below the present accuracy in the measurement of $\dot{P}$ for PSR 1913+16 (which is 1.7×10^{-2}). However, this accuracy is steadily increasing with time and we hope that in the future the higher-order expression of $\dot{P}$ will become useful for a better fit of the observational data to the theory. This is plausible because, for instance, it has been shown that higher-order effects in PSR 1913+16 are by now significant in the measurement of another relativistic parameter, the secular periastron advance (Damour & Schäfer 1988). On the other hand, we wish to prepare the ground in anticipation of eventual discoveries of new relativistic binary pulsars. See for example the binary pulsar PSR 0021−72A, recently discovered by Ables *et al.* (1988).

Previous computations of the gravitational power emission from a binary system at the first post-Newtonian approximation have been performed by Wagoner & Will (1976) in the case of circular orbits (eccentricity $e=0$) and in the case of small angle scattering, or bremsstrahlung ($e\rightarrow\infty$). The bremsstrahlung case was then further analysed by Turner & Will (1978). These authors used the post-Newtonian wave generation formalism by Epstein & Wagoner (1975). Another computation has been done by Gal'tsov, Matiukhin & Petukhov (1980) who worked out the case of a test particle orbiting a large central mass. None of these computations apply to binary systems of stars of comparable masses in eccentric orbits. However, the case of a circular orbit is probably sufficient to study the last stages of the coalescing of a compact binary system (Clark & Eardley 1977, Krolak & Schutz 1987). Finally, we notice that the program of computing the higher-order $\dot{P}$ for arbitrary eccentricities but equal masses has been incorrectly done before by Spyrou (1981) and Spyrou & Papadopoulos (1985).

We shall do all our post-Newtonian computations in what we think is the most powerful way to do them. As a first tool, we shall use a recently implemented post-Newtonian wave generation formalism (Blanchet & Damour 1989). This formalism expresses the outgoing gravitational wave amplitude and the gravitational wave power in terms of integrals extending over the material stress-energy distribution only (see equations 3.1–6 below). This is in contrast with the previous wave generation formalism by Epstein & Wagoner (1975), and generalized by Thorne (1980b), which makes use of a pseudo stress-energy tensor for the gravitational field and, as a result, expresses the outgoing field in terms of formally divergent integrals extending over the distribution of the matter and the gravitational field. The latter Epstein–Wagoner–Thorne formalism is, however, 'formally correct' (see Blanchet & Damour 1989) and has yielded correct answers in the applications which have been made: Wagoner & Will (1976) for binary systems, and Wagoner (1977, 1979), Turner & Wagoner (1979) and Müller (1982) for collapsing or/and rotating systems. Note that the Epstein–Wagoner–Thorne formalism must be handled with care in applications because of the presence of formally infinite terms in the expressions. The use of the formalism of Blanchet & Damour (1989) will thus render our computations rigorous and also easier. It will also make clearer the passage from the 'fluid' description of the matter to the 'point-particle' limit; see the Appendix.

The second tool for our post-Newtonian computations is the use of an elegant representation of the first post-Newtonian motion of two point masses in harmonic coordinates (Damour & Deruelle 1985). This representation of the post-Newtonian motion differs from the Newtonian motion only through the appearance of three eccentricities instead of one, and of a constant measuring the secular advance of the periastron. The use of this representation will also render our computations easier. Finally, following Damour & Deruelle (1986), we shall pick up one particular post-Newtonian eccentricity, which is more directly observable because it yields the simplest timing formula for binary pulsars, to express our post-Newtonian formula for $\dot{P}$.

The paper is organized as follows. In Section 2 we recall the relevant formulas taking place at the Newtonian, lowest order. In Section 3 we derive the higher-order expression, denoted by $\mathscr{L}(t)$, of the instantaneous gravitational power emission for two point masses in terms of the relative position and velocity of the masses. In Section 4 we insert into $\mathscr{L}(t)$ the solution of the motion corresponding to the 'quasi-elliptic' (i.e. elliptic plus relativistic corrections) case and average $\mathscr{L}(t)$ over one period of the motion. This yields the higher-order post-Newtonian expression of $\dot{P}$. In Section 5 we use analytic continuation arguments to go from the quasi-elliptic to the 'quasi-hyperbolic' (i.e., hyperbolic plus relativistic corrections) motion, we compute the total energy carried off by the waves during the encounter of the stars and we determine the critical eccentricity associated with their capture. In the Appendix we consider the 'point-particle' limit of the 'fluid' formulas of the post-Newtonian wave generation.

2 Summary of the lowest-order results

The instantaneous power (or 'luminosity') of the gravitational wave emission from a general matter system is, at lowest order, given by the usual Einstein quadrupole formula

$$\mathscr{L}(t)=\frac{G}{5c^5}\frac{d^3I_{ij}(t)}{dt^3}\frac{d^3I_{ij}(t)}{dt^3}, \tag{2.1}$$

where $I_{ij}(t)$ is the quadrupole moment of the mass distribution in the system. For a two point mass system, we have

$$I_{ij}(t)=\sum_{a=1}^{2} m_{(a)}\left[r^i_{(a)}(t)r^j_{(a)}(t)-\frac{1}{3}\delta^{ij}\mathbf{r}^2_{(a)}(t)\right], \tag{2.2}$$

where $\mathbf{r}_{(a)}(t)$ and $m_{(a)}$ are the position and mass of body (a). The quadrupole formula (2.1–2) was first derived by Einstein (1918) under the assumption of negligible self-gravity in the system. Then Landau & Lifshitz (1941) recognized that the formula also applies to a system having a weak (instead of negligible) self-gravity, thereby allowing the formula to hold for an ordinary (Newtonian) gravitationally bound star system (see e.g. the reviews by Thorne 1980a and Damour 1987a). The condition of weak self-gravity is not satisfied in the case of PSR 1913+16 because of the strong field regions around the pulsar and its companion, where the metric field is very nearly Schwarzschildian. Nevertheless, we assume that the formula (2.1–2) holds with the masses $m_{(a)}$ in equation (2.2) being the Schwarzschild masses of the stars (see Ehlers *et al.* 1976; Ehlers & Walker 1983; Damour 1983a, for critical reviews on the applicability of the quadrupole formula to the binary pulsar PSR 1913+16).

Let us first of all consider the case of a bound binary star system. We insert into equations (2.1–2) the elliptic motion of the stars and get an instantaneous expression for $\mathscr{L}(t)$ which is a periodic function of time with period P, where P is the orbital period of the system. Taking the time-average of $\mathscr{L}(t)$,

$$\langle\mathscr{L}\rangle=\frac{1}{P}\int_0^P\mathscr{L}(t)\,dt, \tag{2.3}$$

gives the averaged rate at which the system emits radiation. The net result is (Peters & Mathews 1963)

$$\langle\mathscr{L}\rangle=\frac{32G^4}{5c^5}\frac{\mu^2M^3}{a^5(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right), \tag{2.4}$$

where M and μ are the total and reduced masses of the system, with a and e the semi-major axis and the eccentricity of the ellipse, respectively. Now, as we recalled in Section 1, the argument for computing $\dot{P}$ is that $\langle\mathscr{L}\rangle$ should correspond to a decrease of the Newtonian energy $\mu E=-(G\mu M/2a)$ of the stars by the amount

$$\dot{E}\equiv\frac{dE}{dt}=-\frac{1}{\mu}\langle\mathscr{L}\rangle. \tag{2.5}$$

Thus, by Kepler's third law,

$$P=\frac{2\pi GM}{(-2E)^{3/2}}, \tag{2.6}$$

this should produce a decrease of P according to

$$\frac{\dot{P}}{P} = -\frac{3}{2}\frac{\dot{E}}{E} = +\frac{3}{2\mu E}\langle \mathscr{L} \rangle. \tag{2.7}$$

From equation (2.4) we then find

$$\frac{\dot{P}}{P} = -\frac{96G^3}{5c^5}\frac{\mu M^2}{a^4(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right). \tag{2.8}$$

This heuristic prediction (Peters & Mathews 1963; Esposito & Harrison 1975; Wagoner 1975) was confirmed by the more rigorous argumentation valid for strong field stars (Damour & Deruelle 1981; Damour 1983a,b). For PSR 1913+16, equation (2.8) gives the numerical value (equation 1.2) quoted in Section 1 which agrees within 1.7 per cent with the observational value (equation 1.1).

Let us now consider the case of an unbound binary star system. Namely we consider two stars flying past each other on a hyperbolic orbit. By insertion of the hyperbolic orbit into equations (2.1–2) we obtain an instantaneous expression for $\mathscr{L}(t)$ which in this case must be integrated over all times to give the energy,

$$\Delta\mathscr{I} = \int_{-\infty}^{+\infty} \mathscr{L}(t)\, dt, \tag{2.9}$$

of the gravitational emission for the whole process. The result is then

$$\Delta\mathscr{I} = \frac{2G}{15c^5}\frac{(GM)^6\mu^2}{J^7}\left[(37e^4+292e^2+96)\arccos\left(\frac{-1}{e}\right)+\frac{1}{3}\sqrt{e^2-1}\,(673e^2+602)\right], \tag{2.10}$$

where J is the constant (reduced by a factor μ) angular momentum of the star system (Hansen 1972, corrected by Turner 1977). Equation (2.10) can be viewed as a sort of analytic continuation for $e>1$ of equation (2.4) above (see Section 5 below). A well-studied particular case, often referred to as the bremsstrahlung case, is the limit of large eccentricity $e\to\infty$ or of small angle of deflection. In this limit, equation (2.10) reduces to

$$\Delta\mathscr{I}_{|e\to\infty} \approx \frac{37\pi G}{15c^5}\frac{(GM)^6\mu^2}{J^7}e^4 \approx \frac{37\pi G}{15c^5}\frac{(GM)^2\mu^2 V_\infty}{b^3} \tag{2.11}$$

(Ruffini & Wheeler 1971), where we have introduced the relative velocity V_∞ of the stars at infinity and their impact parameter b (such that $J=bV_\infty$ and $e\approx bV_\infty^2/GM$). Note that the bremsstrahlung limit has also been solved in the case where the field is weak but where we allow the stars to have high velocities, i.e. outside the Newtonian and post-Newtonian frameworks (Peters 1970; Kovács & Thorne 1977, 1978). Another interesting case is obtained by insertion of the value $e=1$ into equation (2.10) which yields the energy in the waves emitted by a binary system moving on a parabolic orbit

$$\Delta\mathscr{I}_{|e=1} = \frac{170\pi G}{3c^5}\frac{(GM)^6\mu^2}{J^7}. \tag{2.12}$$

However, the case of two stars in strictly parabolic motion (which means that the stars start at infinity with zero velocity) is only hypothetic, since the stars must capture each other and form a bound system because of the loss of energy due to gravitational wave emission. In spite of this, equation (2.12) can be used to compute (heuristically) the initial eccentricity below which the stars starting at infinity finally form a bound system. Indeed, by varying the relation $e^2=1+2Eh^2$ between e and the constants of motion E and $h=J/GM$, we get $\delta e=h^2\delta E$ at $E=0$ and $e=1$. Now, by the same energy balance argument as used above (equation 2.5), we expect

$$\delta E=-\frac{1}{\mu}\Delta\mathscr{I}_{|e=1}. \tag{2.13}$$

Thus we find that any orbit starting with an initial eccentricity e such that $1\leq e<1-\delta e$, with

$$\delta e=h^2\delta E=-\frac{h^2}{\mu}\Delta\mathscr{I}_{|e=1}=-\frac{170\pi G}{3c^5}\frac{\mu}{GMh^5}, \tag{2.14}$$

finally becomes a bound elliptic orbit. The latter 'critical' eccentricity $1-\delta e$ had been first computed by a different method, by Walker & Will (1979). Note that it depends only on $\Delta\mathscr{I}$, the loss of energy, and not also on the loss of angular momentum.

3 Gravitational radiation emission from a binary system

In order to compute the total power emission of waves from a binary system (either in bound or unbound motion) let us give at first the necessary formulae for the emission from a general matter system. The total power of emission, or luminosity $\mathscr{L}$, is given, with first-post-Newtonian (1PN) accuracy, by the irreducible decomposition

$$\mathscr{L}=\frac{G}{5c^5}\left\{\overset{(3)}{I}{}^{\rm rad}_{ij}\overset{(3)}{I}{}^{\rm rad}_{ij}+\frac{1}{c^2}\left[\frac{5}{189}\overset{(4)}{I}{}^{\rm rad}_{ijk}\overset{(4)}{I}{}^{\rm rad}_{ijk}+\frac{16}{9}\overset{(3)}{J}{}^{\rm rad}_{ij}\overset{(3)}{J}{}^{\rm rad}_{ij}\right]\right\}+O\left(\frac{1}{c^9}\right) \tag{3.1}$$

(see e.g. Thorne 1980b) where the moments $I^{\rm rad}_{ij}$, $I^{\rm rad}_{ijk}$ and $J^{\rm rad}_{ij}$ are some 'radiative' mass-quadrupole, mass-octupole and current-quadrupole moments. These moments by definition parametrize the asymptotic metric field in some transverse-trace-free coordinate system at Minkowskian future null infinity (see e.g. equation 4.8 of Thorne 1980b). In equation (3.1), $\mathscr{L}$ and the radiative moments are functions of some retarded time (say $t-r/c$) and we set

$$\overset{(p)}{I}=d^pI/dt^p.$$

The expression (3.1) of the luminosity $\mathscr{L}$ must be supplemented by relations linking the radiative moments to the source. These relations have been derived in Blanchet & Damour (1989) under the assumptions that the source is weakly self-gravitating, slowly moving and weakly stressed. These assumptions generally validate the various quadrupole equations. Let $(\mathbf{x}, t)$ be the harmonic coordinate system and let $T^{\mu\nu}(\mathbf{x}, t)$ be the stress-energy tensor of the system. (To be precise, the harmonic coordinate system is the one in which the inner gravitational field takes the form of equation 2.10 of Blanchet & Damour 1989). Then we define some 'active gravitational mass' σ and some 'active gravitational current' σ_i in terms of the contravariant components of $T^{\mu\nu}$ by

$$\sigma = \frac{1}{c^2}(T^{oo} + T^{ss}), \tag{3.2}$$

where $T^{ss} = \Sigma_i T^{ii}$ is the spatial trace of $T^{\mu\nu}$, and

$$\sigma_i = \frac{1}{c} T^{oi}. \tag{3.3}$$

Note that σ is the integrand of the Tolman mass formula for stationary systems. Then the radiative moments in equation (3.1) are given with post-Newtonian accuracy by

$$I^{\mathrm{rad}}_{ij}(t) = \int d^3\mathbf{x}\,\hat{x}_{ij}\sigma(\mathbf{x}, t) + \frac{1}{14c^2}\frac{d^2}{dt^2}\left[\int d^3\mathbf{x}\,\hat{x}_{ijk}\sigma_k(\mathbf{x}, t)\right] - \frac{20}{21c^2}\frac{d}{dt}\left[\int d^3\mathbf{x}\,\hat{x}_{ijk}\sigma_k(\mathbf{x}, t)\right] + O\left(\frac{1}{c^3}\right) \tag{3.4}$$

for the radiative mass quadrupole moment, and

$$I^{\mathrm{rad}}_{ijk}(t) = \int d^3\mathbf{x}\,\hat{x}_{ijk}\sigma(\mathbf{x}, t) + O\left(\frac{1}{c^2}\right) \tag{3.5}$$

$$J^{\mathrm{rad}}_{ij}(t) = \int d^3\mathbf{x}\,\varepsilon_{ab<i}\hat{x}_{j>a}\sigma_b(\mathbf{x}, t) + O\left(\frac{1}{c^2}\right) \tag{3.6}$$

for the other moments.★ Notice that the radiative quadrupole moment (equation 3.4) is simply the sum of three convergent integrals extending over the compactly supported distribution of the matter in the system (see Blanchet & Damour 1989 for links with the previous formalism by Epstein & Wagoner 1975).

Let us now consider the case of a system which is made of an isentropic perfect fluid whose stress-energy tensor is given by

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{3.7}$$

where

$$\varepsilon = \rho c^2\left(1 + \frac{\Pi}{c^2}\right). \tag{3.8}$$

Here, u^μ is the four-velocity of the fluid ($g_{\mu\nu}u^\mu u^\nu = -1$), and ρ, $p = p[\rho]$ and $\Pi[\rho]$ are respectively the proper rest-mass density, proper pressure and proper specific internal energy of the fluid. These quantities are linked by the thermodynamic law $d\Pi = pd\rho/\rho^2$. Let us also denote by $\rho^* = \sqrt{-g}\rho u^o$ the coordinate rest-mass density (where g is the determinant of $g_{\mu\nu}$ with signature $-+++$) satisfying the continuity equation

$$\partial_t\rho^* + \partial_i(\rho^* v_i) = 0. \tag{3.9}$$

★We use the following notation for (symmetric) trace-free (STF) products of vectors $\hat{x}_{ij} = x_{\langle ij\rangle} = x_i x_j - \frac{1}{3}\delta_{ij}\mathbf{x}^2$; $\hat{x}_{ijk} = x_{\langle ijk\rangle} = x_i x_j x_k - \frac{1}{5}\mathbf{x}^2(\delta_{ij}x_k + \delta_{jk}x_i + \delta_{ki}x_j)$; and also $R_{\langle ij\rangle} = R_i R_j - \frac{1}{3}\delta_{ij}\mathbf{R}^2$; $R_{\langle i}V_{j\rangle} = \frac{1}{2}(R_i V_j + V_i R_j) - \frac{1}{3}\delta_{ij}(RV)$ with $(RV) = R_k V_k$; etc. …

Here, $v_i = cu^i/u^o$ is the usual Eulerian coordinate velocity. Then we easily find that the mass density σ defined by equation (3.2) is

$$\sigma = \rho^{**} + \rho_v + O\left(\frac{1}{c^4}\right), \tag{3.10}$$

where ρ^{**} is the sum of ρ^* and the mass-equivalent of the usual Newtonian energy density,

$$\rho^{**} = \rho^*\left\{1 + \frac{1}{c^2}\left(\frac{\mathbf{v}^2}{2} + \Pi - \frac{U}{2}\right)\right\} = \rho^* + O\left(\frac{1}{c^2}\right), \tag{3.11}$$

with U the Newtonian potential of the system

$$U(\mathbf{x}, t) = G\int \frac{d^3\mathbf{x}'\rho^*(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|}, \tag{3.12}$$

and where ρ_v is the following mass density

$$\rho_v = \frac{1}{c^2}\rho^*\left(\mathbf{v}^2 - \frac{U}{2} + \frac{3p}{\rho^*}\right). \tag{3.13}$$

Notice that the spatial integral of ρ^{**} over the system is equal [modulo '2PN' terms of order $O(1/c^4)$] to the conserved ADM mass, $M_{\rm ADM}$, of the system, while the integral of ρ_v is equal [by the virial theorem (and still modulo 2PN terms)] to $1/2c^2$ times the second time derivative of the central moment of inertia $I = \int d^3\mathbf{x}\rho^*\mathbf{x}^2$ of the system. Hence

$$\int d^3\mathbf{x}\sigma = M_{\rm ADM} + \frac{1}{2c^2}\overset{(2)}{I} + O\left(\frac{1}{c^4}\right). \tag{3.14}$$

On the other hand, the current density σ_i defined by equation (3.3) is

$$\sigma_i = \rho^* v_i + O\left(\frac{1}{c^2}\right). \tag{3.15}$$

Hence we can rewrite the radiative moments (equations 3.4–3.6) as the following expressions, accurate up to post-Newtonian order, inclusively,

$$I^{\rm rad}_{ij}(t) = \int d^3\mathbf{x}(\rho^{**} + \rho_v)\hat{x}_{ij} + \frac{1}{14c^2}\frac{d^2}{dt^2}\left[\int d^3\mathbf{x}\rho^*\hat{x}_{ij}\mathbf{x}^2\right] - \frac{20}{21c^2}\frac{d}{dt}\left[\int d^3\mathbf{x}\rho^* v_k\hat{x}_{ijk}\right], \tag{3.16}$$

for the quadrupole mass moment, and

$$I^{\rm rad}_{ijk}(t) = \int d^3\mathbf{x}\rho^*\hat{x}_{ijk} \tag{3.17}$$

$$J^{\rm rad}_{ij}(t) = \int d^3\mathbf{x}\rho^*\varepsilon_{ab<i}\hat{x}_{j>a}v_b, \tag{3.18}$$

for the other moments. [Henceforth we no longer mention the terms $O(c^{-3})$ in equation 3.16 and $O(c^{-2})$ in equations 3.17 and 3.18 which will always be neglected].

We now wish to take in the latter formulae some 'point-particle' limit. Let us thus consider that our fluid is made of N well-separated fluid balls (labeled by $a=1\dots N$). Inspection of equations (3.11) and (3.13) leads us to expect that the point-particle form of equations (3.16–18) is given by

$$I_{ij}^{\mathrm{rad}}=\sum_{a=1}^{N} m_{(a)}\left\{r_{(a)}^{<i}r_{(a)}^{j>}\left[1+\frac{1}{c^2}\left(\frac{3}{2}\mathbf{v}_{(a)}^2-\sum_{b\neq a}\frac{Gm_{(b)}}{|\mathbf{r}_{(a)}-\mathbf{r}_{(b)}|}\right)\right]\right.$$
$$\left.+\frac{1}{14c^2}\frac{d^2}{dt^2}(\mathbf{r}_{(a)}^2 r_{(a)}^{<i}r_{(a)}^{j>})-\frac{20}{21c^2}\frac{d}{dt}(v_{(a)}^k r_{(a)}^{<i}r_{(a)}^j r_{(a)}^{k>})\right\} \tag{3.19}$$

and

$$I_{ijk}^{\mathrm{rad}}=\sum_{a=1}^{N} m_{(a)}r_{(a)}^{<i}r_{(a)}^{j}r_{(a)}^{k>}, \tag{3.20}$$

$$J_{ij}^{\mathrm{rad}}=\sum_{a=1}^{N} m_{(a)}\varepsilon^{kl<i}r_{(a)}^{j>}r_{(a)}^{k}v_{(a)}^{l}, \tag{3.21}$$

where $\mathbf{r}_{(a)}(t)$ is a 'centre of mass' position of body (a) with $\mathbf{v}_{(a)}(t)=d\mathbf{r}_{(a)}(t)/dt$, and where $m_{(a)}$ is its 'mass'. Note that the self-energy of each body in equation (3.19) has been 'renormalized' in the masses of the bodies. This fact is justified in the Appendix (under the extra assumptions of spherical symmetry and of static equilibrium of the bodies), where it is shown that the centre of mass positions $\mathbf{r}_{(a)}(t)$ and masses $m_{(a)}$ of the bodies appearing in equations (3.19–21) are given by the usual 1PN expressions (Fock 1959; Will 1974; Contopoulos & Spyrou 1976)

$$r_{(a)}^i(t)=\frac{1}{m_{(a)}}\int_{(a)} d^3\mathbf{x}\,x^i\rho^*\left[1+\frac{1}{c^2}\left(\frac{\mathbf{w}_{(a)}^2}{2}+\Pi-\frac{u_{(a)}}{2}\right)\right] \tag{3.22}$$

and

$$m_{(a)}=\int_{(a)} d^3\mathbf{x}\,\rho^*\left[1+\frac{1}{c^2}\left(\frac{\mathbf{w}_{(a)}^2}{2}+\Pi-\frac{u_{(a)}}{2}\right)\right], \tag{3.23}$$

involving the self-gravity of each object

$$u_{(a)}(\mathbf{x},t)=G\int_{(a)}\frac{d^3\mathbf{x}'\rho^*(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}. \tag{3.24}$$

In equations (3.22–23) we have posed $\mathbf{w}_{(a)}(t)=\mathbf{v}-\mathbf{v}_{(a)}(t)$.

Let us now further specialize the expressions (3.19–21) to the case $N=2$ of a two-body system. In this case we denote by M and μ the total and reduced masses of the system,

$$M=m_{(1)}+m_{(2)} \tag{3.25}$$

$$\mu=\frac{m_{(1)}m_{(2)}}{m_{(1)}+m_{(2)}} \tag{3.26}$$

and we introduce the ratio ν measuring the distribution of the masses among the two objects

$$\nu=\mu/M. \tag{3.27}$$

Note that $0 \le \nu \le 1/4$ where $\nu = 1/4$ holds for equal masses and $\nu = 0$ for the test-body limit. Let us also introduce the relative position of the two objects

$$\mathbf{R}(t) = \mathbf{r}_{(1)}(t) - \mathbf{r}_{(2)}(t) \tag{3.28}$$

and their relative velocity

$$\mathbf{V}(t) = d\mathbf{R}(t)/dt = \mathbf{v}_{(1)}(t) - \mathbf{v}_{(2)}(t). \tag{3.29}$$

In a mass-centred coordinate frame, one has the following relations linking the motions of the bodies to their relative motion valid through 1PN order (see e.g. equation 2.4 of Damour & Deruelle 1985):

$$\mathbf{r}_{(1)}(t) = \left[\frac{\mu}{m_{(1)}} + \frac{\mu(m_{(1)} - m_{(2)})}{2M^2c^2}\left(V^2 - \frac{GM}{R}\right)\right]\mathbf{R}(t) \tag{3.30a}$$

$$\mathbf{r}_{(2)}(t) = \left[-\frac{\mu}{m_{(2)}} + \frac{\mu(m_{(1)} - m_{(2)})}{2M^2c^2}\left(V^2 - \frac{GM}{R}\right)\right]\mathbf{R}(t) \tag{3.30b}$$

(with $R = |\mathbf{R}|$ and $V = |\mathbf{V}|$). We now replace in equations (3.19–21) the motions of the objects by their relative motion. The result is†

$$I_{ij}^{\mathrm{rad}} = \mu R_{\langle ij\rangle}\left[1 + \frac{3}{2c^2}(1 - 3\nu)V^2 - \frac{(1 - 2\nu)}{c^2}\frac{GM}{R}\right] - \frac{\mu(1 - 3\nu)}{21c^2}\left[20\frac{d}{dt}(V_k R_{\langle ijk\rangle}) - \frac{3}{2}\frac{d^2}{dt^2}(R^2 R_{\langle ij\rangle})\right] \tag{3.31}$$

and (choosing $m_{(1)} \ge m_{(2)}$ by convention)

$$I_{ijk}^{\mathrm{rad}} = -\mu\sqrt{1 - 4\nu}\,R_{\langle ijk\rangle} \tag{3.32}$$

$$J_{ij}^{\mathrm{rad}} = -\mu\sqrt{1 - 4\nu}\,\varepsilon_{kl<i}R_{j>k}V_l \tag{3.33}$$

(notice that $I_{ijk}^{\mathrm{rad}} = J_{ij}^{\mathrm{rad}} = 0$ for equal masses). It is now convenient to reduce the expression (equation 3.31) of the radiative mass quadrupole by means of the 1PN equations of motion (in harmonic coordinates). These equations read (see e.g. equation 2.5 of Damour & Deruelle 1985)

$$\frac{d\mathbf{V}}{dt} = -\frac{GM}{R^3}\mathbf{R} + \frac{GM}{c^2R^3}\mathbf{R}\left[(4 + 2\nu)\frac{GM}{R} - (1 + 3\nu)V^2 + \frac{3\nu}{2}\frac{(RV)^2}{R^2}\right] + (4 - 2\nu)\frac{GM}{c^2R^3}(RV)\mathbf{V}, \tag{3.34}$$

where we denote $(RV) = \mathbf{R}\cdot\mathbf{V}$. These equations are in fact valid in a large class of coordinate systems, including the harmonic coordinate system and also the generalized isotropic ADM coordinate system. They admit the following integral of motion of the (reduced by the factor μ) energy E = constant, with

$$E = \frac{1}{2}V^2 - \frac{GM}{R} + \frac{3}{8}(1 - 3\nu)\frac{V^4}{c^2} + \frac{GM}{2Rc^2}\left[(3 + \nu)V^2 + \nu\frac{(RV)^2}{R^2} + \frac{GM}{R}\right] \tag{3.35}$$

†We use the following identities:

$$\frac{1}{m_{(1)}^2} + \frac{1}{m_{(2)}^2} = \frac{1}{\mu^2}(1 - 2\nu); \quad \frac{1}{m_{(1)}^3} + \frac{1}{m_{(2)}^3} = \frac{1}{\mu^3}(1 - 3\nu); \quad (m_{(1)} - m_{(2)})^2 = M^2(1 - 4\nu).$$

and the integral of motion of the (reduced) angular momentum $\mathbf{J}=$ constant, with

$$\mathbf{J}=\mathbf{R}\times\mathbf{V}\left[1+\frac{1}{2}(1-3\nu)\frac{V^2}{c^2}+(3+\nu)\frac{GM}{Rc^2}\right]. \tag{3.36}$$

Then the expression of the radiative mass moment (equation 3.31) reduced by the equations of motion (equation 3.34) is

$$I_{ij}^{\text{rad}}=\mu R_{\langle ij\rangle}\left[1+\frac{29}{42c^2}(1-3\nu)V^2-\frac{1}{7c^2}(5-8\nu)\frac{GM}{R}\right]$$
$$+\frac{\mu(1-3\nu)}{21c^2}[-12(RV)R_{\langle i}V_{j\rangle}+11R^2V_{\langle ij\rangle}]. \tag{3.37}$$

Since our aim is to compute the gravitational luminosity $\mathscr{L}$ given by equation (3.1) we need to compute the third time-derivative of the mass and current quadrupoles and the fourth time-derivative of the mass octupole. The computations are straightforward and we use at each step the 1PN equations (3.34) to reduce the accelerations $d\mathbf{V}/dt$. As a result, we get for

$$\overset{(3)}{I}{}_{ij}^{\text{rad}}=d^3I_{ij}^{\text{rad}}/dt^3,\ \overset{(4)}{I}{}_{ijk}^{\text{rad}}\ \text{and}\ \overset{(3)}{J}{}_{ij}^{\text{rad}},$$

the expressions

$$\overset{(3)}{I}{}_{ij}^{\text{rad}}=6\frac{GM\mu}{R^5}(RV)R_{\langle ij\rangle}\left\{1+\frac{1}{42}(65+36\nu)\frac{V^2}{c^2}-\frac{1}{63}(440-81\nu)\frac{GM}{c^2R}-\frac{5}{21}(1-3\nu)\frac{(RV)^2}{c^2R^2}\right\}$$
$$-8\frac{GM\mu}{R^3}R_{\langle i}V_{j\rangle}\left\{1+\frac{11}{84}(5+6\nu)\frac{V^2}{c^2}-\frac{1}{84}(353-72\nu)\frac{GM}{c^2R}+\frac{1}{28}(73+12\nu)\frac{(RV)^2}{c^2R^2}\right\}$$
$$+\frac{2}{21}(241-30\nu)\frac{GM\mu}{c^2R^3}(RV)V_{\langle ij\rangle}, \tag{3.38}$$

$$\overset{(4)}{I}{}_{ijk}^{\text{rad}}=-\frac{GM\mu\sqrt{1-4\nu}}{R^5}\left\{R_{\langle ijk\rangle}\left[9V^2+12\frac{GM}{R}-45\frac{(RV)^2}{R^2}\right]+90(RV)R_{\langle ij}V_{k\rangle}-60R^2R_{\langle i}V_{jk\rangle}\right\}, \tag{3.39}$$

and

$$\overset{(3)}{J}{}_{ij}^{\text{rad}}=-\frac{GM\mu\sqrt{1-4\nu}}{R^5}\{3(RV)\varepsilon_{ab\langle i}R_{j\rangle}R_aV_b-R^2\varepsilon_{ab\langle i}V_{j\rangle}R_aV_b\}. \tag{3.40}$$

The expressions (equations 3.38–40) for the time-derivatives of the radiative moments are to be inserted in the luminosity $\mathscr{L}$ given by equation (3.1). Again the computation is straightforward and we get

$$\mathscr{L}=\frac{8}{15}\frac{G^3M^2\mu^2}{c^5R^4}\left\{12V^2-11\frac{(RV)^2}{R^2}\right\}+\frac{2}{105}\frac{G^3M^2\mu^2}{c^7R^4}\left\{(785-852\nu)V^4-2(1487-1392\nu)\right.$$
$$\times\frac{(RV)^2V^2}{R^2}+3(687-620\nu)\frac{(RV)^4}{R^4}-160(17-\nu)\frac{GMV^2}{R}$$
$$\left.+8(367-15\nu)\frac{GM(RV)^2}{R^3}+16(1-4\nu)\frac{(GM)^2}{R^2}\right\}.$$

The first term in this expression is the 'Newtonian' contribution to $\mathscr{L}$ (Peters & Mathews 1963); the other more complicated terms constitute the first post-Newtonian corrections in $\mathscr{L}$. These corrections are already known from Wagoner & Will (1976) who used the Epstein & Wagoner (1975) wave generation formalism. [Equation (3.44) agrees with equation (51) of Wagoner & Will (1976) as corrected by an Erratum (1977).] Notice that Wagoner & Will had to perform the formal manipulations inherent to the Epstein & Wagoner formalism (see the appendix of Wagoner & Will). Our computation thus sets on a solid footing the latter expression for $\mathscr{L}(t)$ found by Wagoner & Will.

Now that we have the expression of $\mathscr{L}(t)$ in terms of the relative motion of the two bodies, we must use explicit post-Newtonian solutions of this motion. We deal in the next section with the quasi-elliptic motion of the two bodies, and in the following section with the quasi-hyperbolic one.

4 Application to quasi-elliptic motion

In this section we replace in $\mathscr{L}(t)$ (equation 3.44) the relative motion **R**, **V** of the two stars by explicit expressions corresponding to the 1PN quasi-elliptic motion. It is very convenient to use a representation of the 1PN motion in harmonic coordinates which has a nearly Newtonian form (Damour & Deruelle 1985). Let $R(t)$, $\theta(t)$ be the planar relative motion of the two stars in usual polar coordinates R, θ associated with the harmonic coordinates. Following Damour & Deruelle (1985), we write the radial motion $R(t)$ in parametrized form

$$R = a_R(1 - e_R \cos u) \tag{4.1}$$

$$n(t - t_0) = u - e_t \sin u, \tag{4.2}$$

where u is some 'eccentric anomaly', parametrizing the motion. The constants a_R, e_R, e_t, n and t_0 are some 1PN 'semi-major axis', some 'radial eccentricity', some 'time eccentricity', some 'mean motion' and some initial instant, respectively. As to the angular motion $\theta(t)$, it is given by

$$\theta(t) = \theta_0 + 2K \arctan\left[\left(\frac{1+e_\theta}{1-e_\theta}\right)^{1/2} \tan\frac{u}{2}\right], \tag{4.3}$$

in which θ_0 = constant and where K and e_θ are some 'periastron precession' constant and some 'angular eccentricity' constant, respectively. By differentiating equation (4.3) we get

$$d\theta = K(1 - e_\theta^2)^{1/2} \frac{du}{1 - e_\theta \cos u}. \tag{4.3$'$}$$

Notice that this representation of the solution of the 1PN quasi-elliptic motion differs from the Newtonian elliptic motion only through the occurrence of three types of eccentricity e_R, e_t, e_θ (differing from each other by small post-Newtonian corrections) and of the constant K (differing from one by a small post-Newtonian correction).

The semi-major axis a_R and the mean motion n are related to the 1PN constant of energy E (equation 3.35) by

$$a_R = -\frac{GM}{2E}\left[1 - \frac{1}{2}(\nu - 7)\frac{E}{c^2}\right] \tag{4.4}$$

$$n=\frac{(-2E)^{3/2}}{GM}\left[1-\frac{1}{4}(\nu-15)\frac{E}{c^2}\right] \tag{4.5}$$

(Damour & Deruelle 1985). An important point for us is the fact that the mean motion n (and also the semi-major axis a_R) depends, at 1PN order, only on the constant of energy E and not on the constant of angular momentum J. (This is one of the sources of errors in the work of Spyrou (1981) and Spyrou & Papadopoulos (1985) who find n to depend both on E and J). Hence we shall be able below to compute the change in the orbital period $P=2\pi/n$, namely $\dot{P}=dP/dt$, as a function of $\dot{E}=dE/dt$ only and hence as a function of the average $\langle \mathscr{L} \rangle$ only. According to Damour & Deruelle (1985), the constant K is related to J by

$$K=1+\frac{3}{c^2h^2}, \tag{4.6}$$

where we have introduced the reduced angular momentum

$$h=J/GM. \tag{4.7}$$

Note that K measures the angle of precession of the periastron per rotation $\Delta\theta=2\pi(K-1)=6\pi/c^2h^2$. Finally, the various eccentricities e_R, e_t, e_θ are related to E and h by

$$e_R^2=1+2Eh^2+[2(\nu-6)+5(\nu-3)Eh^2]\frac{E}{c^2} \tag{4.8}$$

$$e_t^2=1+2Eh^2+[4(-\nu+1)+(-7\nu+17)Eh^2]\frac{E}{c^2} \tag{4.9}$$

$$e_\theta^2=1+2Eh^2+[-12+(\nu-15)Eh^2]\frac{E}{c^2} \tag{4.10}$$

(see Damour & Deruelle 1985).

The replacement of the 1PN quasi-elliptic motion (equations 4.1–3) along with the relations (equations 4.4–10) in the 'instantaneous' luminosity $\mathscr{L}(t)$ is now straightforward. We use equation (4.1) and also the following easily checked consequences of equations (4.1–3),

$$V^2=\frac{(RV)^2}{R^2}+\frac{4E^2h^2}{(1-e_R\cos u)^2}\left[1+(7\nu-9)\frac{E}{c^2}-8(\nu-2)\frac{E}{c^2}\frac{1}{1-e_R\cos u}\right] \tag{4.11}$$

$$\frac{(RV)^2}{R^2}=-2E\left[\frac{-(1-e_R^2)}{(1-e_R\cos u)^2}+\frac{2}{1-e_R\cos u}-1\right]$$
$$\times\left[1+\frac{3}{2}(3\nu-1)\frac{E}{c^2}-2(3\nu-8)\frac{E}{c^2}\frac{1}{1-e_R\cos u}\right] \tag{4.12}$$

to express $\mathscr{L}$ as a polynomial in $(1-e_R\cos u)^{-1}$. We find that this polynomial is of the type

$$\mathscr{L}=\frac{du}{ndt}\sum_{N=2}^{6}\frac{\alpha_N(E,h)}{(1-e_R\cos u)^{N+1}}, \tag{4.13}$$

where for convenience we have factorized out $du/ndt = (1 - e_t \cos u)^{-1}$. A direct computation shows that the coefficients $\alpha_N(E, h)$ in equation (4.13) take the form

$$\alpha_N(E, h) = \frac{(\mu/M)^2}{Gc^5} (-E)^5 \beta_N(E, h), \tag{4.14}$$

where

$$\beta_2 = -\frac{256}{15} + \left(\frac{29\,824}{105} - \frac{15\,488}{105}\nu\right)\frac{E}{c^2} \tag{4.15a}$$

$$\beta_3 = \frac{512}{15} + \left(-\frac{26\,368}{35} + \frac{19\,968}{35}\nu\right)\frac{E}{c^2} \tag{4.15b}$$

$$\beta_4 = -\frac{5632}{15}Eh^2 + \left[\frac{1024}{7} + \frac{126\,464}{15}Eh^2 - \left(\frac{3072}{7} + \frac{31\,744}{7}Eh^2\right)\nu\right]\frac{E}{c^2} \tag{4.15c}$$

$$\beta_5 = \left(-\frac{1\,654\,784}{105} + \frac{47\,616}{7}\nu\right)\frac{E^2h^2}{c^2} \tag{4.15d}$$

$$\beta_6 = \left(-\frac{351\,744}{35} + \frac{63\,488}{7}\nu\right)\frac{E^3h^4}{c^2}. \tag{4.15e}$$

Notice that only the coefficients β_2, β_3 and β_4 are non-zero at the Newtonian order. The luminosity $\mathscr{L}(t)$ is a periodic function of time with period $P = 2\pi/n$. Hence we average in time over one period P,

$$\langle \mathscr{L} \rangle = \frac{1}{P}\int_0^P \mathscr{L}(t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{ndt}{du}\right) \mathscr{L}(u)\, du. \tag{4.16}$$

Using equation (3.13), this gives

$$\langle \mathscr{L} \rangle = \sum_{N=2}^{6} \alpha_N(E, h) \frac{1}{2\pi}\int_0^{2\pi} \frac{du}{(1 - e_R \cos u)^{N+1}}. \tag{4.17}$$

The integrals appearing in equation (4.17) are easily computed; the more convenient form for our purpose is the following expression (valid for $0 \le e < 1$)

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{du}{(1 - e\cos u)^{N+1}} = \frac{(-)^N}{N!}\left[\left(\frac{d}{dx}\right)^N \left(\frac{1}{\sqrt{x^2 - e^2}}\right)\right]_{|x=1} \tag{4.18}$$

in which the right-hand side means that one must differentiate N times with respect to x (with $e < x \le 1$) the function $(x^2 - e^2)^{-1/2}$ and take afterwards the value $x = 1$.‡ It can also be recognized in these integrals the Laplace second integrals for the Legendre polynomials (see

‡Equation (4.18) follows from N differentiations with respect to x of the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{du}{x - e\cos u} = \frac{1}{\sqrt{x^2 - e^2}}.$$

e.g. Whittaker & Watson 1927, page 314). Hence we have

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{du}{(1-e\cos u)^{N+1}}=\frac{1}{(1-e^2)^{(N+1)/2}}P_N\left(\frac{1}{\sqrt{1-e^2}}\right), \tag{4.19}$$

where P_N is the usual Legendre polynomial. We can now insert the expressions (equations 4.18 or 4.19) of the integrals into equation (4.17) and use the explicit values (equations 4.14–15) for the coefficients. We have found

$$\langle\mathscr{L}\rangle=\frac{1024}{5Gc^5}\frac{\nu^2(-E)^5}{(1-e_R^2)^{7/2}}\left\{1+\frac{73}{24}e_R^2+\frac{37}{96}e_R^4+\frac{(-E)}{168c^2(1-e_R^2)}\right.$$
$$\left.\times\left[13-6414e_R^2-\frac{27405}{4}e_R^4-\frac{5377}{16}e_R^6+\left(-840-\frac{6419}{2}e_R^2-\frac{5103}{8}e_R^4+\frac{259}{8}e_R^6\right)\nu\right]\right\}. \tag{4.20}$$

This is the extension to post-Newtonian accuracy of the 'Newtonian' expression (equation 2.4). Particular cases of this expression are already known from previous authors: Wagoner & Will (1976) in the case of a circular ($e_R=0$) orbit (see their equation 81), and Gal'tsov *et al.* (1980) in the case of $\nu=0$ in the higher-order corrections [see their equation 18 with their Schwarzschild-coordinate eccentricity ε related to our eccentricity e_R by $\varepsilon=e_R(1+2E/c^2)$]. Let us now use equation (4.8) relating e_R to the 1PN constants of motion to express $\langle\mathscr{L}\rangle$ in terms of E and h. We find

$$\langle\mathscr{L}\rangle=\frac{1}{15Gc^5}\frac{\nu^2(-2E)^{3/2}}{h^7}\left\{425+732Eh^2+148E^2h^4\right.$$
$$+\frac{1}{c^2h^2}\left[\frac{40341}{8}+10065Eh^2+\frac{85047}{14}E^2h^4+\frac{6278}{7}E^3h^6\right.$$
$$\left.\left.-\left(\frac{5635}{2}+\frac{32225}{4}Eh^2+5415E^2h^4+481E^3h^6\right)\nu\right]\right\}. \tag{4.21}$$

The straightforward application of the latter computation of $\langle\mathscr{L}\rangle$ is the (heuristic) determination of the rate at which the orbital period P of the binary system decays by emission of gravitational radiation. Indeed we have already noticed that the orbital period $P=2\pi/n$ is a function of the total constant of energy E only via the equation

$$P=\frac{2\pi GM}{(-2E)^{3/2}}\left[1+\frac{1}{4}(\nu-15)\frac{E}{c^2}\right] \tag{4.22}$$

(generalizing equation 2.6). From equation (4.22) and from the expectation $\dot{E}=-\langle\mathscr{L}\rangle/\mu$ (equation 2.5) we then find, similarly to equation (2.8),

$$\frac{\dot{P}}{P}=\frac{3}{2\mu E}\left[1-\frac{1}{6}(\nu-15)\frac{E}{c^2}\right]\langle\mathscr{L}\rangle. \tag{4.23}$$

We now use the expression (equation 4.21) computed above to get $\dot{P}$ as a function of the masses and of the 1PN constants of energy and of angular momentum:

$$\dot{P} = -\frac{\pi}{5c^5}\frac{\nu}{(-E)h^7}\left\{425 + 732Eh^2 + 148E^2h^4 + \frac{1}{c^2h^2}\left[\frac{40341}{8} + \frac{38135}{4}Eh^2 + \frac{72237}{14}E^2h^4 + \frac{4983}{7}E^3h^6 - \left(\frac{5635}{2} + \frac{48125}{6}Eh^2 + 5354E^2h^4 + \frac{1406}{3}E^3h^6\right)\nu\right]\right\}. \tag{4.24}$$

Notice that this equation relates an observable (relativistic) parameter $\dot{P}$ to the conserved energy E and angular momentum h of the relative motion and thus is independent of the coordinate system we used to derive it (the harmonic coordinate system). But since $\dot{P}$ is actually measured in PSR 1913+16, it is better to express it not in terms of E and h but in terms of other directly observable parameters, namely the orbital period P (or orbital frequency $n = 2\pi/P$) and some eccentricity. Following Damour & Deruelle (1986) we shall use a 'proper-time' eccentricity e_T, which is associated with the object which is timed and which yields the simplest relativistic 'timing formula'. The relation between e_T and E, h is

$$e_T^2 = 1 + 2Eh^2 + \left[-1 + 3\frac{\delta m}{M} + 2\nu + \left(7 + 6\frac{\delta m}{M} + 5\nu\right)Eh^2\right]\frac{E}{c^2}, \tag{4.25}$$

where $\delta m = m_p - m_c$, $[(\delta m/M)^2 = 1 - 4\nu]$, with p denoting the object which is timed, and c its companion (see Damour & Deruelle 1985, 1986). The final expression of $\dot{P}$ as a function of n and e_T (and of the masses) is then obtained from equations (4.5), (4.24) and (4.25). We have found:

$$\dot{P} = -\frac{192\pi}{5c^5}\frac{\nu(GMn)^{5/3}}{(1-e_T^2)^{7/2}}\left\{1 + \frac{73}{24}e_T^2 + \frac{37}{96}e_T^4 + \frac{(GMn)^{2/3}}{336c^2(1-e_T^2)} \times\left[1273 + \frac{16495}{2}e_T^2 + \frac{42231}{8}e_T^4 + \frac{3947}{16}e_T^6 - \left(924 + 3381e_T^2 + \frac{1659}{4}e_T^4 - \frac{259}{4}e_T^6\right)\nu + \left(3297e_T^2 + 4221e_T^4 + \frac{2331}{8}e_T^6\right)\frac{\delta m}{M}\right]\right\}. \tag{4.26}$$

Let us compute the numerical value of the relative post-Newtonian correction in equation (4.26) in the case of the binary pulsar PSR 1913+16, for which we have $n = 2.2515 \times 10^{-4}$ s^{-1}, $e_T = 0.617$, pulsar mass: $m_p = 1.44M_\odot$, companion mass: $m_c = 1.39M_\odot$ (masses computed from the data of Taylor 1989). Then we find that

$$\dot{P} = -\frac{192\pi}{5c^5}\frac{\nu(GMn)^{5/3}}{(1-e_T^2)^{7/2}}\left(1 + \frac{73}{24}e_T^2 + \frac{37}{96}e_T^4\right)(1 + X_{PN}), \tag{4.27}$$

where the relative post-Newtonian correction is numerically equal to $X_{PN} = +2.15 \times 10^{-5}$ (this is -60 times the result of Spyrou & Papadopoulos 1985). This is unfortunately far below the present accuracy in the measurement of $\dot{P}$ (which is 1.7×10^{-2}). However, as the precision on the measurement of $\dot{P}$ steadily increases, we hope that the expression (equation 4.26) will be useful in the future for a better interpretation of the data from PSR 1913+16, or from other binary pulsars. Notice that for PSR 1913+16, X_{PN} is only 19 times smaller than the precision in equation (1.2).

5 Application to quasi-hyperbolic motion

We consider in this section the emission of gravitational radiation during the encounter of two stars, with ratio ν arbitrary, moving on a quasi-hyperbolic orbit with arbitrary eccentricity $e>1$. The two stars are still supposed to be slowly moving. Let us use the results of the previous section to compute the total gravitational energy emitted in the waves during the encounter.

The quasi-hyperbolic 1PN motion can easily be deduced from the quasi-elliptic one (equations 4.1–3) by setting the eccentric anomaly to be $u=iv$ and consider v as a new (real) parameter along the orbit, and by putting $\bar{n}=in$ and consider $\bar{n}$ as a new (real) mean motion (see Damour & Deruelle 1985). Therefore, the instantaneous luminosity $\mathscr{L}(t)$ given by equation (4.13) with equations (4.14–15) in the quasi-elliptic case becomes, in the quasi-hyperbolic case,

$$\mathscr{L}=\frac{dv}{\bar{n}dt}\sum_{N=2}^{6}(-)^N\frac{\alpha_N(E,h)}{(e_R\mathrm{ch}v-1)^{N+1}}, \tag{5.1}$$

where ch$v$ is the hyperbolic cosine of v, and where the coefficients $\alpha_N(E,h)$ are the same functions of E (now positive) and h, as in the elliptic case (equations 4.14–15). [Note the factor $(-)^N$ in equation 5.1]. The total energy in the emission is then given by

$$\Delta\mathscr{I}=\int_{-\infty}^{+\infty}\mathscr{L}(t)\,dt=\int_{-\infty}^{+\infty}\left(\frac{dt}{dv}\right)\mathscr{L}(v)\,dv \tag{5.2}$$

and hence by equation (5.1), we have

$$\Delta\mathscr{I}=\frac{1}{\bar{n}}\sum_{N=2}^{6}(-)^N\alpha_N(E,h)\int_{-\infty}^{+\infty}\frac{dv}{(e_R\mathrm{ch}v-1)^{N+1}}. \tag{5.3}$$

Exactly like in the elliptic case, our problem is reduced to the computation of each one of the integrals in the right-hand side of equation (5.3). We have a formula for these integrals paralleling equation (4.18), namely

$$\int_{-\infty}^{+\infty}\frac{dv}{(e\mathrm{ch}v-1)^{N+1}}=\frac{2}{N!}\left[\left(\frac{d}{dx}\right)^N\left(\frac{1}{\sqrt{e^2-x^2}}\arccos\left(-\frac{x}{e}\right)\right)\right]_{|x=1}, \tag{5.4}$$

where $x=1$ is set in the right-hand side after the N differentiations.§

In a way similar to equation (4.19), one also has the following expression involving the

§Equation (5.4) follows from N differentiations with respect to x of the integral

$$\int_{-\infty}^{+\infty}\frac{dv}{e\mathrm{ch}v-x}=\frac{2}{\sqrt{e^2-x^2}}\mathrm{arc\,cos}\left(-\frac{x}{e}\right)\left(=\frac{4}{\sqrt{e^2-x^2}}\mathrm{arc\,tan}\sqrt{\frac{e+x}{e-x}}\right).$$

Legendre function of the second kind Q_N (see e.g. Whittaker & Watson 1927, page 319):

$$\int_{-\infty}^{+\infty} \frac{dv}{(e\operatorname{ch}v-1)^{N+1}} = \frac{2}{i^{N+1}(e^2-1)^{(N+1)/2}}\, Q_N\left(\frac{i}{\sqrt{e^2-1}}\right). \tag{5.5}$$

By inserting the expressions (equations 5.4 or 5.5) into equation (5.3), and by using the values (equations 4.14 and 4.15) of the coefficients, we can get straightforwardly the looked-for expression of $\Delta\mathscr{I}$. Notice that from the structure of equation (5.4), one easily deduces that $\Delta\mathscr{I}$ will involve a term having as a factor $\arccos(-1/e_R)$ plus other terms involving only radicals and powers of e_R. Furthermore, since the coefficient of $\arccos(-1/e)$ in the right-hand side of equation (5.4) is given by

$$\frac{2}{N!}\left[\left(\frac{d}{dx}\right)^N\left(\frac{1}{\sqrt{e^2-x^2}}\right)\right]_{|x=1},$$

which is $2(-)^{N+1}i$ (where $i=\sqrt{-1}$) the right-hand side of equation (4.18), we find that the total coefficient of $\arccos(-1/e_R)$ in $\Delta\mathscr{I}$ will simply be given by $2i\langle\mathscr{L}\rangle/\bar{n}=2\langle\mathscr{L}\rangle/n$. Namely, we shall have

$$\Delta\mathscr{I} = \left(\frac{2}{n}\langle\mathscr{L}\rangle\right)\arccos\left(-\frac{1}{e_R}\right) + C(E, e_R), \tag{5.6}$$

where $(2/n)\langle\mathscr{L}\rangle$ can be directly read from our expression (equation 4.20) of $\langle\mathscr{L}\rangle$, and where $C(E, e_R)$ involves only square roots of e_R^2-1 and powers of e_R. We find

$$\begin{aligned}\Delta\mathscr{I} = \frac{2G}{15c^5}\frac{\mu^2}{GMh^7}\Bigg\{&(37e_R^4+292e_R^2+96)\arccos\left(-\frac{1}{e_R}\right)+\frac{1}{3}\sqrt{e_R^2-1}\,(673e_R^2+602)\\ &+\frac{1}{c^2h^2}\Bigg[\frac{1}{56}\Big(e_R^6(17\,933-8288\nu)+e_R^4(94\,542-78\,148\nu)+e_R^2(117\,288-61\,936\nu)\\ &+52\,624-9408\nu\Big)\arccos\left(\frac{-1}{e_R}\right)+\frac{1}{840}\Big(e_R^4(1\,271\,421-803\,040\nu)\\ &+e_R^2(1\,447\,788-1\,251\,460\nu)+1\,516\,596-312\,200\nu\Big)\sqrt{e_R^2-1}\Bigg]\Bigg\}.\end{aligned} \tag{5.7}$$

This expression generalizes equation (2.10) to post-Newtonian accuracy. Notice that the resulting expression can be checked to be C^∞ in a neighbourhood of the 'parabolic' case $e_R=1$ (or $E=0$). This can be proven from a well-known property of the Legendre function of the second kind Q_N appearing in equation (5.5), namely that $Q_N(z)$ admits when $z\to+\infty$ an asymptotic expansion in powers of z^{-1} starting with z^{-N-1}. The fact that $\Delta\mathscr{I}$ is C^∞ near $e_R=1$ provides us with a check of the coefficients in equation (5.7).

The energy emission $\Delta\mathscr{I}$ in the (hypothetic) parabolic case $e_R=1$ is then explicitly given by

$$\Delta\mathscr{I}_{|e_R=1} = \frac{2\pi G}{15c^5}\frac{\mu^2}{GMh^7}\left\{425+\frac{7}{8c^2h^2}[5763-3220\nu]\right\}. \tag{5.8}$$

This expression can be used, as in Section 1, to compute the 'critical' value at which any one of the post-Newtonian eccentricities below an initially unbound orbit becomes bound because of the emission of radiation. For instance, it is found from equation (5.8) that the 'critical' radial eccentricity is $1-\delta e_R$, with

$$\delta e_R = -\frac{170\pi G}{3c^5}\frac{\mu}{GMh^5}\left\{1+\frac{3}{850c^2h^2}\left[\frac{6647}{4}-1595\nu\right]\right\}. \tag{5.9}$$

Finally, let us end this section by considering the emission of radiation in the bremsstrahlung limit ($e_R \to \infty$). In this case, equation (5.7) becomes

$$\Delta\mathcal{J}_{|e_R\to\infty} = \frac{148\pi G}{15c^5}\frac{\mu^2E^2}{GMh^3}\left\{1+\frac{E}{1036c^2}(2393-3108\nu)\right\}. \tag{5.10}$$

We then recover the result computed by Wagoner & Will (1976) (see their equations 13–14 corrected by an erratum, 1977).

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Appendix: The point-particle limit

The aim of this Appendix is to work out the 'point-particle' limit for the fluid presentation of the radiative quadrupole moment (equation 3.16), namely

$$I_{ij}^{\rm rad}=\int d^3\mathbf{x}(\rho^{**}+\rho_{\rm v})\hat{x}_{ij}+\frac{1}{14c^2}\frac{d^2}{dt^2}\left[\int d^3\mathbf{x}\rho^*\hat{x}_{ij}\mathbf{x}^2\right]-\frac{20}{21c^2}\frac{d}{dt}\left[\int d^3\mathbf{x}\rho^*v_k\hat{x}_{ijk}\right], \tag{A1}$$

where we have put

$$\rho^{**}=\rho^*\left[1+\frac{1}{c^2}\left(\frac{\mathbf{v}^2}{2}+\Pi-\frac{U}{2}\right)\right] \tag{A2}$$

$$\rho_{\rm v}=\frac{1}{c^2}\rho^*\left(\mathbf{v}^2-\frac{U}{2}+\frac{3p}{\rho^*}\right), \tag{A3}$$

ρ^* being the conserved coordinate rest-mass density satisfying $\partial_t\rho^*+\partial_i(\rho^*v_i)=0$, and U being the Newtonian potential of the system

$$U(\mathbf{x},t)=G\int\frac{d^3\mathbf{x}'\rho^*(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}. \tag{A4}$$

Let us consider the case where our fluid is made of N well-separated objects, the objects having a typical diameter d (much larger than their Schwarzschild radii) and a typical separation D. We assume that the separation ratio between the objects is small, namely

$$\alpha = d/D \ll 1. \tag{A5}$$

Let us define (following many previous workers, including Fock 1959; Will 1974; Contopoulos & Spyrou 1976) a 'centre of mass' world-line within each body, according to

$$r^i_{(a)}(t) = \frac{1}{m_{(a)}} \int_{(a)} d^3\mathbf{x} x^i \rho^* \left[1 + \frac{1}{c^2}\left(\frac{\mathbf{w}^2_{(a)}}{2} + \Pi - \frac{u_{(a)}}{2}\right)\right], \tag{A6}$$

where

$$m_{(a)} = \int_{(a)} d^3\mathbf{x} \rho^* \left[1 + \frac{1}{c^2}\left(\frac{\mathbf{w}^2_{(a)}}{2} + \Pi - \frac{u_{(a)}}{2}\right)\right]. \tag{A7}$$

Equations (A6–A7) can equivalently be written as

$$0 = \int_{(a)} d^3\mathbf{y}_{(a)} y^i_{(a)} \rho^*_{(a)} \left[1 + \frac{1}{c^2}\left(\frac{\mathbf{w}^2_{(a)}}{2} + \Pi_{(a)} - \frac{u_{(a)}}{2}\right)\right]. \tag{A8}$$

Our notations are: $y^i_{(a)}$ for the position within body (a) relative to the centre of mass line

$$y^i_{(a)} = x^i - r^i_{(a)}(t); \tag{A9}$$

$w^i_{(a)}$ for the internal velocity within body (a)

$$w^i_{(a)} = dy^i_{(a)}/dt = v^i - v^i_{(a)}(t) \tag{A10}$$

with $v^i_{(a)}(t) = dr^i_{(a)}(t)/dt$; $u_{(a)}$ for the internal self-gravity in (a),

$$u_{(a)}(\mathbf{x}, t) = G\int_{(a)} \frac{d^3\mathbf{x}' \rho^*(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} \tag{A11}$$

[the integration being limited to body (a)]; and $\rho^*_{(a)}(\mathbf{y}_{(a)}) = \rho^*[\mathbf{y}_{(a)} + \mathbf{r}_{(a)}(t)]$; $\Pi_{(a)}(\mathbf{y}_{(a)}) = \Pi[\mathbf{y}_{(a)} + \mathbf{r}_{(a)}(t)]$.

We now split the expression (equation A1) of the radiative moment into the sum over the N volumes of the bodies and change into each volume the position variable according to equation (A9). Then $I^{\rm rad}_{ij}(t)$ is the sum of four terms:

$$I^{\rm rad}_{ij}(t) = A_{ij}(t) + B_{ij}(t) + C_{ij}(t) + D_{ij}(t), \tag{A12}$$

given by the following expressions. A_{ij} is

$$A_{ij} = \sum_{a=1}^{N} \int_{(a)} d^3\mathbf{y}_{(a)} \rho^*_{(a)} (r^{\langle i}_{(a)} + y^i_{(a)})(r^j_{(a)} + y^{j\rangle}_{(a)}) \times \left\{1 + \frac{1}{c^2}\left[\frac{1}{2}(\mathbf{w}_{(a)} + \mathbf{v}_{(a)})^2 + \Pi_{(a)} - \frac{1}{2}(u_{(a)} + U_{(a)})\right]\right\}, \tag{A13}$$

where we have introduced the Newtonian potential $U_{(a)}$ acting on (a) by all bodies except (a):

$$U_{(a)}(\mathbf{y}_{(a)}) = U - u_{(a)} = \sum_{b \neq a} G \int_{(b)} \frac{d^3\mathbf{x}' \rho^*(\mathbf{x}', t)}{|\mathbf{x}' - \mathbf{y}_{(a)} - \mathbf{r}_{(a)}(t)|}. \tag{A14}$$

The second term is

$$B_{ij}=\frac{1}{c^2}\sum_{a=1}^{N}\int_{(a)} d^3\mathbf{y}_{(a)}\rho^*_{(a)}(r^{\langle i}_{(a)}+y^i_{(a)})(r^j_{(a)}+y^{j\rangle}_{(a)})\times\left[(\mathbf{w}_{(a)}+\mathbf{v}_{(a)})^2-\frac{1}{2}(u_{(a)}+U_{(a)})+\frac{3p_{(a)}}{\rho^*_{(a)}}\right] \tag{A15}$$

[where $p_{(a)}(\mathbf{y}_{(a)})=p(\mathbf{r}_{(a)}+\mathbf{y}_{(a)})$]. The third term is

$$C_{ij}=\frac{1}{14c^2}\frac{d^2}{dt^2}\sum_{a=1}^{N}\int_{(a)} d^3\mathbf{y}_{(a)}\rho^*_{(a)}(\mathbf{r}_{(a)}+\mathbf{y}_{(a)})^2(r^{\langle i}_{(a)}+y^i_{(a)})(r^j_{(a)}+y^{j\rangle}_{(a)}), \tag{A16}$$

and the fourth one is

$$D_{ij}=-\frac{20}{21c^2}\frac{d}{dt}\sum_{a=1}^{N}\int_{(a)} d^3\mathbf{y}_{(a)}\rho^*_{(a)}(v^k_{(a)}+w^k_{(a)})(r^{\langle i}_{(a)}+y^i_{(a)})(r^j_{(a)}+y^j_{(a)})(r^k_{(a)}+y^{k\rangle}_{(a)}). \tag{A17}$$

Let us work out the latter expressions of $A_{ij}\ldots D_{ij}$ to second order in α, i.e. let us neglect terms in these expressions which are of relative order $O(\alpha^2)$ smaller than the dominant ones. First of all, as is well known, the external potential $U_{(a)}$ takes, since the lines $\mathbf{r}_{(a)}(t)$ are mass-centred, a 'point-particle' form modulo $O(\alpha^2)$ terms:

$$U_{(a)}=\sum_{b\neq a}\frac{Gm_{(b)}}{|\mathbf{r}_{(a)}(t)-\mathbf{r}_{(b)}(t)|}+O(\alpha^2) \tag{A18}$$

(see e.g. Damour 1987b, page 139). Secondly, we can neglect in equations (A13, A15–A17) all terms involving at least the product of two vectors $y^i_{(a)}$ since these terms will be of relative order $O(\alpha^2)$. Let us also use the definition (equation A7) of the mass $m_{(a)}$ and the following consequences of equation (A8):

$$\int_{(a)} d^3\mathbf{y}_{(a)}y^i_{(a)}\rho^*_{(a)}=O(1/c^2) \tag{A19}$$

and

$$\frac{d}{dt}\int_{(a)} d^3\mathbf{y}_{(a)}\rho^*_{(a)}y^i_{(a)}=\int_{(a)} d^3\mathbf{y}_{(a)}\rho^*_{(a)}w^i_{(a)}=O(1/c^2). \tag{A20}$$

We then find [discarding '2PN' terms of order $O(c^{-4})$] that A_{ij} takes the following form modulo $O(\alpha^2)$:

$$A_{ij}=\sum_{a=1}^{N}\left\{m_{(a)}r^{\langle i}_{(a)}r^{j\rangle}_{(a)}\left[1+\frac{1}{c^2}\left(\frac{1}{2}\mathbf{v}^2_{(a)}-\frac{1}{2}\sum_{b\neq a}\frac{Gm_{(b)}}{|\mathbf{r}_{(a)}-\mathbf{r}_{(b)}|}\right)\right]+\frac{2}{c^2}v^k_{(a)}r^{\langle i}_{(a)}\int d^3\mathbf{y}_{(a)}\rho^*_{(a)}y^{j\rangle}_{(a)}w^k_{(a)}+O(\alpha^2)\right\}. \tag{A21}$$

Similarly, using the virial theorem,

$$\frac{1}{2}\frac{d^2}{dt^2}\int_{(a)} d^3\mathbf{y}_{(a)}\rho^*_{(a)}\mathbf{y}^2_{(a)}=\int d^3\mathbf{y}_{(a)}\rho^*_{(a)}\left[\mathbf{w}^2_{(a)}-\frac{1}{2}u_{(a)}+\frac{3p_{(a)}}{\rho^*_{(a)}}\right], \tag{A22}$$

we find that B_{ij} is given by

$$B_{ij}=\frac{1}{c^2}\sum_{a=1}^{N}\left\{m_{(a)}r^{\langle i}_{(a)}r^{j\rangle}_{(a)}\left(\mathbf{v}^2_{(a)}-\frac{1}{2}\sum_{b\neq a}\frac{Gm_{(b)}}{|\mathbf{r}_{(a)}-\mathbf{r}_{(b)}|}\right)+\frac{1}{2}r^{\langle i}_{(a)}r^{j\rangle}_{(a)}\frac{d^2}{dt^2}\int_{(a)}d^3\mathbf{y}_{(a)}\rho^*_{(a)}\mathbf{y}^2_{(a)}\right.$$

$$\left.+2r^{\langle i}_{(a)}\int_{(a)}d^3\mathbf{y}_{(a)}\rho^*_{(a)}y^{j\rangle}_{(a)}\times\left[\mathbf{w}^2_{(a)}+2\mathbf{w}_{(a)}\cdot\mathbf{v}_{(a)}-\frac{1}{2}u_{(a)}+\frac{3p_{(a)}}{\rho^*_{(a)}}\right]+O(\alpha^2)\right\}. \tag{A23}$$

Notice that the term involving the self-gravity $u_{(a)}$ of body (a) in the last term of equation (A23) is *a priori* of the same order of magnitude $\approx m^2L$ as the term involving the external Newtonian potential in the first term. This is the reason why we have considered in equations (A21 and A23) terms which are *a priori* of relative order α smaller than the dominant ones. We shall now assume that our N bodies are spherically symmetric (in the coordinate system we are using) and in static equilibrium.

The first assumption of spherical symmetry kills such terms as the one in equation (A23), which involves the self-gravity $u_{(a)}$; and the second assumption of static equilibrium discards all terms involving $w^i_{(a)}$ and also the second term in equation (A23), for which we have made use of the virial theorem (equation A22). With these assumptions, equations (A21 and A23) become

$$A_{ij}=\sum_{a=1}^{N}m_{(a)}r^{\langle i}_{(a)}r^{j\rangle}_{(a)}\left[1+\frac{1}{c^2}\left(\frac{1}{2}\mathbf{v}^2_{(a)}-\frac{1}{2}\sum_{b\neq a}\frac{Gm_{(b)}}{|\mathbf{r}_{(a)}-\mathbf{r}_{(b)}|}\right)\right] \tag{A24}$$

and

$$B_{ij}=\frac{1}{c^2}\sum_{a=1}^{N}m_{(a)}r^{\langle i}_{(a)}r^{j\rangle}_{(a)}\left(\mathbf{v}^2_{(a)}-\frac{1}{2}\sum_{b\neq a}\frac{Gm_{(b)}}{|\mathbf{r}_{(a)}-\mathbf{r}_{(b)}|}\right). \tag{A25}$$

As for C_{ij} and D_{ij}, they are readily found to be

$$C_{ij}=\frac{1}{14c^2}\frac{d^2}{dt^2}\sum_{a=1}^{N}m_{(a)}\mathbf{r}^2_{(a)}r^{\langle i}_{(a)}r^{j\rangle}_{(a)} \tag{A26}$$

and

$$D_{ij}=-\frac{20}{21c^2}\frac{d}{dt}\sum_{a=1}^{N}m_{(a)}v^k_{(a)}r^{\langle i}_{(a)}r^j_{(a)}r^{k\rangle}_{(a)}. \tag{A27}$$

The sum of $A_{ij}\ldots D_{ij}$ then gives us the expression (equation 3.19) for the radiative quadrupole moment which is used in the text. Evidently, we would have got the same result by using the formal procedure $\rho^*=\Sigma m_{(a)}\delta[\mathbf{x}-\mathbf{r}_{(a)}(t)]$ and dropping all self-energy contributions.

Within their approach, Spyrou & Papadopoulos (1985) have performed the point-particle limit incorrectly in two aspects. The corresponding correct procedure can be found, e.g., in the paper by Wagoner & Will (1976); in particular, see there the equations (36a) and (36c).