# Kinetic theory of self-gravitating systems

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# Problem set

**1. Isochrone potential and Bertrand's theorem.**

We consider the dynamics of a test particle within a central potential, $\Phi(r)$. Such a motion is generically integrable, and is characterised by the two actions, $J_r$ and $J_\phi = L$, respectively the radial and azimuthal actions, with associated orbital frequencies $\Omega_r$ and $\Omega_\phi$. Show that

$$\bigg(\frac{\partial J_r}{\partial E}\bigg)_{\!L} = \frac{1}{\Omega_r} \quad ; \quad \bigg(\frac{\partial J_r}{\partial L}\bigg)_{\!E} = -\frac{\Omega_\phi}{\Omega_r}, \tag{1}$$

with $E$ and $L$, respectively the energy and the angular momentum of the orbit.

In Henon (1959), Michel Hénon introduced the isochrone potential as the central potential

$$\Phi(r) = -\frac{G M_{\mathrm{tot}}}{b + \sqrt{b^2 + r^2}}, \tag{2}$$

with $b$ the system's scale length. He showed in particular that this potential is the most generic potential for which $\Omega_r(E,L) = \Omega_r(E)$, i.e. for which the radial frequency is only a function of $E$. For that particular potential, one finds

$$\Omega_r = \frac{(-2E)^{3/2}}{G M_{\mathrm{tot}}} \quad ; \quad \frac{\Omega_\phi}{\Omega_r} = \frac{1}{2}\bigg(1 + \frac{L}{\sqrt{L^2 + 4 G M_{\mathrm{tot}} b}}\bigg). \tag{3}$$

Reyling on the fact that partial derivatives commute, or otherwise, use the isochrone potential to prove Bertrand's theorem:

*The only central potentials for which all bound orbits are closed are the Keplerian and harmonic potentials.*

**2. Response matrix and dielectric function**

The response matrix of a self-gravitating system is given by

$$\widetilde{\mathbf{M}}_{pq}(\omega) = (2\pi)^d \sum_{\mathbf{k}} \!\int\! \mathrm{d}\mathbf{J}\, \frac{\mathbf{k}\cdot\partial F/\partial \mathbf{J}}{\omega - \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})}\, \psi^{(p)*}_{\mathbf{k}}(\mathbf{J})\, \psi^{(q)}_{\mathbf{k}}(\mathbf{J}). \tag{4}$$

In this exercise, we set out to show that for homogeneous systems this response matrix reduces to the dielectric function of plasma physics.

Assume that the system is placed within a periodic $3D$ box of size $L$. We also assume that the mean potential vanishes, i.e. $\Phi = 0$, so that unperturbed trajectories are straight lines. Show that the system's angle-action coordinates and the associated orbital frequencies are given by

$$\boldsymbol{\theta} = \frac{2\pi}{L}\,\mathbf{x} \quad ; \quad \mathbf{J} = \frac{L}{2\pi}\,\mathbf{v} \quad ; \quad \mathbf{\Omega} = \frac{2\pi}{L}\,\mathbf{v}. \tag{5}$$

The system's instantaneous potential and densities are linked by Poisson's equation, $\Delta\Phi = 4\pi G\rho$, as well as by the self-consistent relation, $\Phi(\mathbf{x}) = \!\int\! \mathrm{d}\mathbf{x}'\, \rho(\mathbf{x}')\, U(\mathbf{x},\mathbf{x}')$, with $U(\mathbf{x},\mathbf{x}')$ the gravitational pairwise interaction potential. Making use of the $2\pi$-periodicity of the system, and assuming that $U(\boldsymbol{\theta},\boldsymbol{\theta}')$ is translationally invariant, show that we can write

$$U(\boldsymbol{\theta},\boldsymbol{\theta}') = -\sum_{\mathbf{p}\in\mathbb{Z}^3\backslash\{0\}} \psi^{(\mathbf{p})}(\boldsymbol{\theta})\, \psi^{(\mathbf{p})*}(\boldsymbol{\theta}') \quad \text{with} \quad \psi^{(\mathbf{p})}(\boldsymbol{\theta}) = \sqrt{\frac{G}{L\pi}}\, \frac{1}{|\mathbf{p}|}\, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\boldsymbol{\theta}}. \tag{6}$$

Show that the response matrix from Eq. (4) then becomes

$$\widetilde{\mathbf{M}}_{\mathbf{pq}}(\omega) = \delta_{\mathbf{pq}}\, \frac{G L^2}{\pi}\, \frac{1}{|\mathbf{p}|^2} \!\int\! \mathrm{d}\mathbf{v}\, \frac{\mathbf{p}\cdot\partial F/\partial\mathbf{v}}{\overline{\omega} - \mathbf{p}\cdot\mathbf{v}}, \tag{7}$$

with $\overline{\omega} = \omega L/(2\pi)$.

Assume that the system's mean distribution function (DF) follows the Maxwellian distribution

$$F(\mathbf{v}) = F(|\mathbf{v}|) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}}\, \mathrm{e}^{-|\mathbf{v}|^2/(2\sigma^2)}, \tag{8}$$

with $\rho_0$ the system's mean density, and $\sigma$ the velocity dispersion. Show that the previous expression of the response matrix reduces to

$$\widetilde{\mathbf{M}}_{\mathbf{pq}}(\omega) = \delta_{\mathbf{pq}} \left(\frac{L}{L_{\mathrm{J}}}\right)^2 \frac{1}{|\mathbf{p}|^2} \left[1 + \zeta\, Z(\zeta)\right], \tag{9}$$

with the dimensionless frequency $\zeta = \overline{\omega}/(\sqrt{2}|\mathbf{p}|\sigma)$, and $Z(\zeta) = \pi^{-1/2}\int \mathrm{d}u\, \mathrm{e}^{-u^2}/(u-\zeta)$ the plasma dispersion function. In that expression, we also introduced the Jeans length as $L_{\mathrm{J}} = \sqrt{(\pi\sigma^2)/(G\rho_0)}$.

What is the main difference between this expression for a self-gravitating system, and the analog one for an electrostatic plasma? What happens for a system with $L > L_{\mathrm{J}}$?

**3. BBGKY hierarchy and inhomogeneous Landau equation**

In this exercise, we set out to recover the inhomogeneous Landau equation through the direct resolution of the BBGKY hierarchy. We consider a Hamiltonian system in a phase space of dimension $2d$, denoting the phase space coordinates as $\mathbf{w} = (\mathbf{q}, \mathbf{p})$. We assume that the system is composed of $N$ particles, of individual mass $m = M_{\mathrm{tot}}/N$, embedded in some external potential $U_{\mathrm{ext}}(\mathbf{w})$, and coupled through each other via the long-range pairwise symmetric interaction potential $U(\mathbf{w}, \mathbf{w}')$. Under these assumptions, the system's $N$-body Hamiltonian reads

$$H_N(\mathbf{w}_1, ..., \mathbf{w}_N) = \sum_{i=1}^{N} U_{\mathrm{ext}}(\mathbf{w}_i) + \sum_{i<j}^{N} m\, U(\mathbf{w}_i, \mathbf{w}_j), \tag{10}$$

where $\mathbf{w}_i$ stands for the phase space coordinates of the $i^{\mathrm{th}}$ particle.

*a. The BBGKY hierarchy.*

We introduce the system's $N$-body probability distribution function (PDF), $P_N(\mathbf{w}_1, ..., \mathbf{w}_N, t)$, normalised so that $\int \mathrm{d}\mathbf{w}_1 ... \mathrm{d}\mathbf{w}_N P_N = 1$, and assumed to be invariant under permutations of the particles. Show that the evolution of $P_N$ is governed by the Liouville equation

$$\frac{\partial P_N}{\partial t} + \bigg[P_N, H_N\bigg]_N = 0, \tag{11}$$

where the $N$-body Poisson bracket is defined as

$$\bigg[P_N, H_N\bigg]_N = \sum_{i=1}^{N} \bigg[P_N, H_N\bigg]_{\mathbf{w}_i} \quad \text{with} \quad \bigg[P_N, H_N\bigg]_{\mathbf{w}} = \frac{\partial P_N}{\partial \mathbf{q}} \cdot \frac{\partial H_N}{\partial \mathbf{p}} - \frac{\partial P_N}{\partial \mathbf{p}} \cdot \frac{\partial H_N}{\partial \mathbf{q}}, \tag{12}$$

so that $[\,\cdot\,,\,\cdot\,]_{\mathbf{w}}$ stands for the Poisson bracket w.r.t. the coordinate $\mathbf{w}$.

In order to reduce the dimension of the functional space considered, we define the reduced $n$-body DFs as

$$F_n(\mathbf{w}_1, ..., \mathbf{w}_n) = m^n \frac{N!}{(N-n)!} \int \mathrm{d}\mathbf{w}_{n+1} ... \mathrm{d}\mathbf{w}_N\, P_N. \tag{13}$$

Show that for arbitrary functions $f(\mathbf{w})$ and $h(\mathbf{w})$, the Poisson bracket satisfies

$$\int \mathrm{d}\mathbf{w} \bigg[f(\mathbf{w}), h(\mathbf{w})\bigg]_{\mathbf{w}} = 0. \tag{14}$$

Using that property, show that $F_n$ evolves according to

$$\frac{\partial F_n}{\partial t} + \bigg[F_n, H_n\bigg]_n + \int \mathrm{d}\mathbf{w}_{n+1} \bigg[F_{n+1}, \delta H_{n+1}\bigg]_n = 0, \tag{15}$$

where the definitions of $H_n$ and $[\,\cdot\,,\,\cdot\,]_n$ follow from Eqs. (10) and Eq. (12), and we have introduced the interaction Hamiltonian, $\delta H_{n+1}$, as

$$\delta H_{n+1}(\mathbf{w}_1, ..., \mathbf{w}_{n+1}) = \sum_{i=1}^{n} U(\mathbf{w}_i, \mathbf{w}_{n+1}). \tag{16}$$

*b. The truncated BBGKY equations.*

Define the system's 2-body correlation function $G_2$ from $F_2$ as

$$F_2(\mathbf{w}, \mathbf{w}') = F_1(\mathbf{w})\, F_1(\mathbf{w}') + G_2(\mathbf{w}, \mathbf{w}'). \tag{17}$$

By computing $\int \mathrm{d}\mathbf{w} F_1(\mathbf{w})$ and $\int \mathrm{d}\mathbf{w}\mathrm{d}\mathbf{w}' G_2(\mathbf{w},\mathbf{w}')$, show that one has $|G_2| \ll |F_1|$ w.r.t. the small parameter $1/N$. What is an appropriate definition of $G_3$ to subsequently have $|G_3| \ll |G_2| \ll |F_1|$?

At order $1/N$, the dynamics of $F = F_1$ and $G = G_2$ are given by

$$\frac{\partial F(\mathbf{w})}{\partial t} + \Big[F(\mathbf{w}), \overline{H}(\mathbf{w})\Big]_{\mathbf{w}} + \int \mathrm{d}\mathbf{w}' \Big[G(\mathbf{w},\mathbf{w}'), U(\mathbf{w},\mathbf{w}')\Big]_{\mathbf{w}} = 0. \tag{18}$$

and

$$\begin{aligned}\frac{\partial G(\mathbf{w},\mathbf{w}')}{\partial t} + \Bigg\{ & \Big[G(\mathbf{w},\mathbf{w}'), \overline{H}(\mathbf{w})\Big]_{\mathbf{w}} \\ & + m\Big[F(\mathbf{w})\,F(\mathbf{w}'), U(\mathbf{w},\mathbf{w}')\Big]_{\mathbf{w}} \\ & + \int \mathrm{d}\mathbf{w}'' \Big[F(\mathbf{w})\,G(\mathbf{w}',\mathbf{w}''), U(\mathbf{w},\mathbf{w}'')\Big]_{\mathbf{w}} \Bigg\}_{\mathbf{w}\leftrightarrow\mathbf{w}'} = 0 \end{aligned} \tag{19}$$

where we have introduced the symmetrising notation $\{f(\mathbf{w},\mathbf{w}')\}_{(\mathbf{w}\leftrightarrow\mathbf{w}')} = f(\mathbf{w},\mathbf{w}') + f(\mathbf{w}',\mathbf{w})$, and did not write explicitly the time dependence to shorten the notations. In these equations, we have also introduced the system's mean Hamiltonian as

$$\overline{H}(\mathbf{w}) = U_{\mathrm{ext}}(\mathbf{w}) + \int \mathrm{d}\mathbf{w}'\, U(\mathbf{w},\mathbf{w}')\, F(\mathbf{w}'). \tag{20}$$

Assume that the mean system is integrable, so that there exists some angle-action coordinates $\mathbf{w} = (\boldsymbol{\theta}, \mathbf{J})$ so that $F(\mathbf{w}) = F(\mathbf{J})$, and $\overline{H}(\mathbf{w}) = \overline{H}(\mathbf{J})$. Assume also that the system is dynamically hot, so that collective effects can be neglected, i.e. so that correlations are not submitted to the potential perturbations that they generate themselves. In that limit, show that the two previous evolution equations can be rewritten as

$$\frac{\partial F(\mathbf{J})}{\partial t} + \frac{\partial}{\partial \mathbf{J}} \cdot \bigg[\int \frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^d} \int \mathrm{d}\mathbf{w}'\, \frac{\partial G(\mathbf{w},\mathbf{w}')}{\partial \boldsymbol{\theta}}\, U(\mathbf{w},\mathbf{w}')\bigg] = 0, \tag{21}$$

and

$$\frac{\partial G(\mathbf{w},\mathbf{w}')}{\partial t} + \bigg\{ \frac{\partial G(\mathbf{w},\mathbf{w}')}{\partial \boldsymbol{\theta}} \cdot \boldsymbol{\Omega}(\mathbf{J}) - m\, F(\mathbf{J}')\, \frac{\partial F(\mathbf{J})}{\partial \mathbf{J}} \cdot \frac{\partial U(\mathbf{w},\mathbf{w}')}{\partial \boldsymbol{\theta}} \bigg\}_{\mathbf{w}\leftrightarrow\mathbf{w}'} = 0, \tag{22}$$

where $\boldsymbol{\Omega}(\mathbf{J})$ are the mean-field orbital frequencies.

*c. The dynamics of correlations.*

Relying on the $2\pi$-periodicity of the angles $\boldsymbol{\theta}$, we expand the correlation function $G(\mathbf{w},\mathbf{w}')$ and the pairwise interaction, $U(\mathbf{w},\mathbf{w}')$, as (pay attention to the conventions)

$$\begin{aligned} G(\mathbf{w},\mathbf{w}') &= \sum_{\mathbf{k},\mathbf{k}'} G_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\theta} + \mathbf{k}'\cdot\boldsymbol{\theta}')}, \\ U(\mathbf{w},\mathbf{w}') &= \sum_{\mathbf{k},\mathbf{k}'} \psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\theta} - \mathbf{k}'\cdot\boldsymbol{\theta}')}. \end{aligned} \tag{23}$$

Justify that $F(\mathbf{J})$ and $\boldsymbol{\Omega}(\mathbf{J})$ can be taken to be constant on the timescale over which $G(t)$ evolves. In that limit, show that the time-evolution of the two-body correlation can be explicitly solved as

$$G_{-\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',t) = m\psi^*_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\, \frac{\mathrm{e}^{\mathrm{i}\Delta\Omega t} - 1}{\Delta\Omega} \bigg(\mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'} - \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}\bigg)\, F(\mathbf{J})\, F(\mathbf{J}'), \tag{24}$$

where we have assumed that the particles are initially decorrelated, i.e. $G(t=0) = 0$, and have introduced the frequency resonance condition $\Delta\Omega = \mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\boldsymbol{\Omega}(\mathbf{J}')$.

*d. The inhomogeneous Landau equation.*

Similarly, show that Eq. (21) can be rewritten as

$$\frac{\partial F(\mathbf{J})}{\partial t} = (2\pi)^d \frac{\partial}{\partial \mathbf{J}} \cdot \bigg[\sum_{\mathbf{k},\mathbf{k}'} \mathrm{i}\,\mathbf{k} \int \mathrm{d}\mathbf{J}'\, G_{-\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\, \psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\bigg]. \tag{25}$$

Relying on the asymptotic formula

$$\lim_{t\to+\infty} \frac{\mathrm{e}^{\mathrm{i}\Delta\Omega t} - 1}{\Delta\Omega} = -\mathcal{P}\bigg(\frac{1}{\Delta\Omega}\bigg) + \mathrm{i}\pi\delta_{\mathrm{D}}(\Delta\Omega), \tag{26}$$

with $\mathcal{P}$ Cauchy's principal value, recover the inhomogeneous Landau equation

$$\frac{\partial F(\mathbf{J})}{\partial t} = -\pi(2\pi)^d\, m\, \frac{\partial}{\partial \mathbf{J}} \cdot \Bigg[ \sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \!\int\! \mathrm{d}\mathbf{J}' \left|\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\right|^2 \delta_{\mathrm{D}}(\mathbf{k}\!\cdot\!\mathbf{\Omega}(\mathbf{J}) - \mathbf{k}'\!\cdot\!\mathbf{\Omega}(\mathbf{J}')) \times \left( \mathbf{k}'\!\cdot\!\frac{\partial}{\partial \mathbf{J}'} - \mathbf{k}\!\cdot\!\frac{\partial}{\partial \mathbf{J}} \right) F(\mathbf{J})\, F(\mathbf{J}') \Bigg]. \tag{27}$$

**4. Conservations, H-Theorem, and Balescu-Lenard equation**

Up to prefactors, the total mass, energy, and entropy of a stellar system are given by

$$\begin{aligned} M(t) &= \int \mathrm{d}\mathbf{J}\, F(\mathbf{J},t), \\ E(t) &= \int \mathrm{d}\mathbf{J}\, H(\mathbf{J})\, F(\mathbf{J},t), \\ S(t) &= \int \mathrm{d}\mathbf{J}\, s(F(\mathbf{J},t)), \end{aligned} \tag{28}$$

where $H(\mathbf{J})$ is the system's mean Hamiltonian, and $s(F) = -F\ln(F)$ is Boltzmann's entropy function. Show that the Balescu-Lenard equation ensures the conservation of total mass and total energy, and satisfies a H-Theorem, i.e. $\mathrm{d}S/\mathrm{d}t \geq 0$. Should it exist, we introduce the inhomogeneous Boltzmann's distribution as $F_{\mathrm{B}}(\mathbf{J}) \propto \mathrm{e}^{-H(\mathbf{J})}$. Compute $\partial F_{\mathrm{B}}/\partial t$ as driven by the Balescu-Lenard equation, and comment.

**5. Kinetic blockings.**

We consider a generic long-range interacting Hamiltonian system. Assume that the system is (i) one-dimensional, i.e. $d = 1$, and (ii) with a symmetric pairwise interaction, i.e. the Fourier-transformed basis elements satisfy $\psi_k^{(p)} \propto \delta_k^p$. Show that the Balescu-Lenard equation can be rewritten as

$$\frac{\partial F(J,t)}{\partial t} = -2\pi^2\, m\, \frac{\partial}{\partial J} \left[ \int \mathrm{d}J' \left|\psi^{\mathrm{tot}}(J,J')\right|^2 \delta_{\mathrm{D}}(\Omega(J) - \Omega(J')) \left( \frac{\partial}{\partial J'} - \frac{\partial}{\partial J} \right) F(J,t)\, F(J',t) \right], \tag{29}$$

and give the expression of the coupling coefficient $\left|\psi^{\mathrm{tot}}(J,J')\right|^2$.

Comment on what happens for a system with a monotonic frequency profile, $J \mapsto \Omega(J)$.