Kinetic theory of self-gravitating systems

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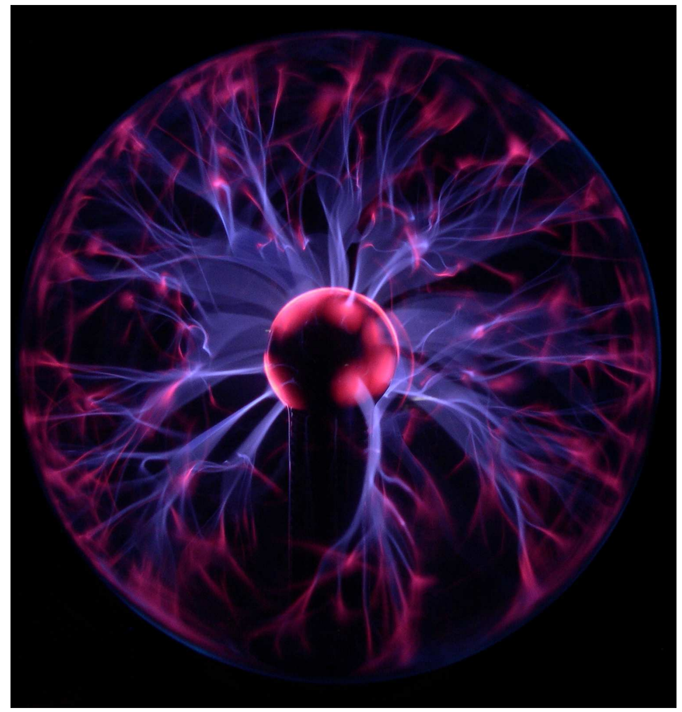
Long-term relaxation

How do systems **diffuse**?

Local
Brownian diffusion



Homogeneous
Plasma diffusion

Inhomogeneous
Galaxy diffusion

Fluctuation-Dissipation Theorem

Diffusion ⟷ **Noise**

Same process occur in galaxies, but:

Gravity is **long-range**
+ Stars follow **orbits** and **resonate**
+ Galaxies **amplify** perturbations

How do galaxies evolve on cosmic timescales?

Relaxation and Dynamics

How do stars evolve?

Ink in water

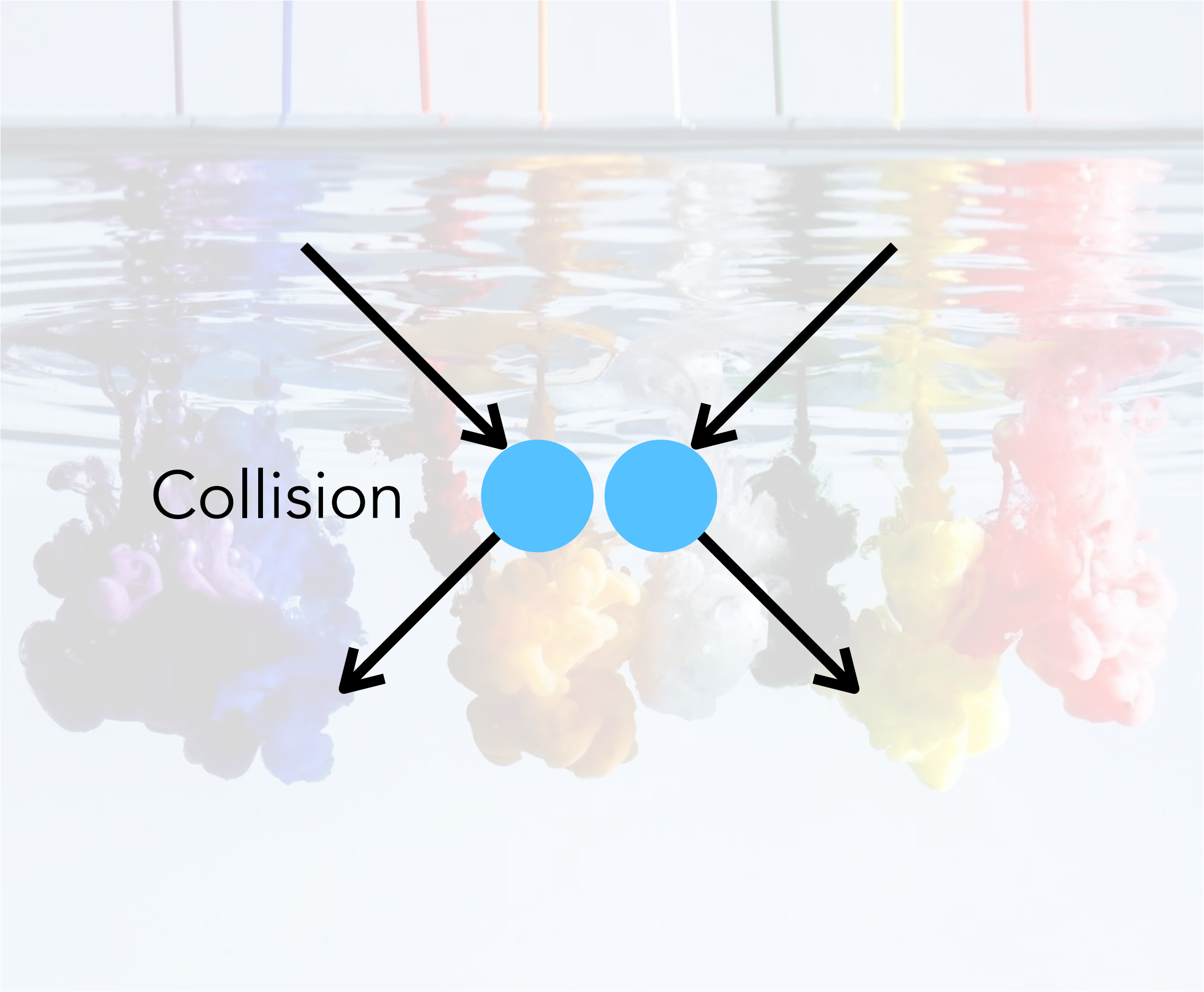
Stars in galaxies

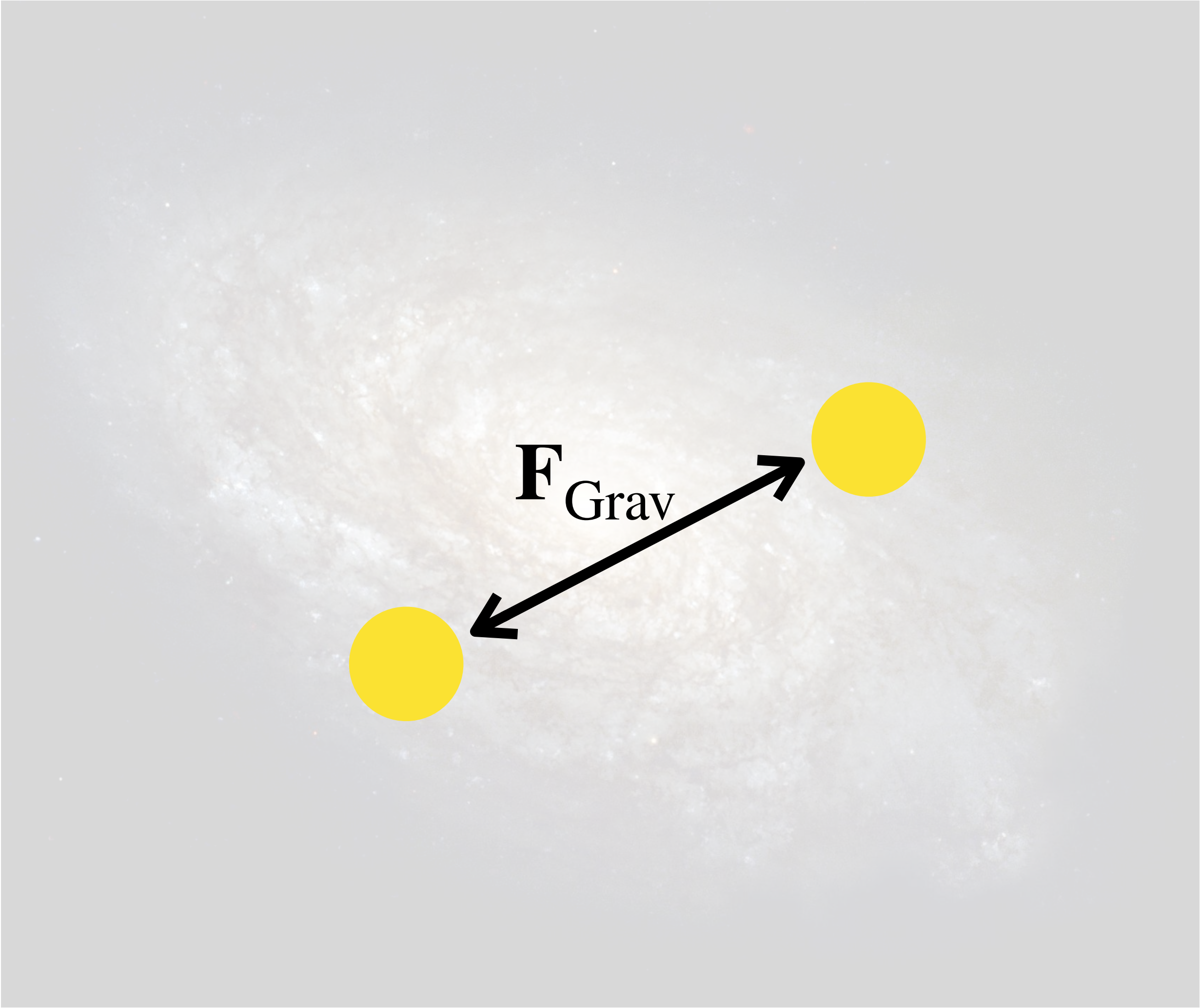
Diffusion

How do systems **diffuse**?

Ink in water

Stars in a galaxy





Local interaction

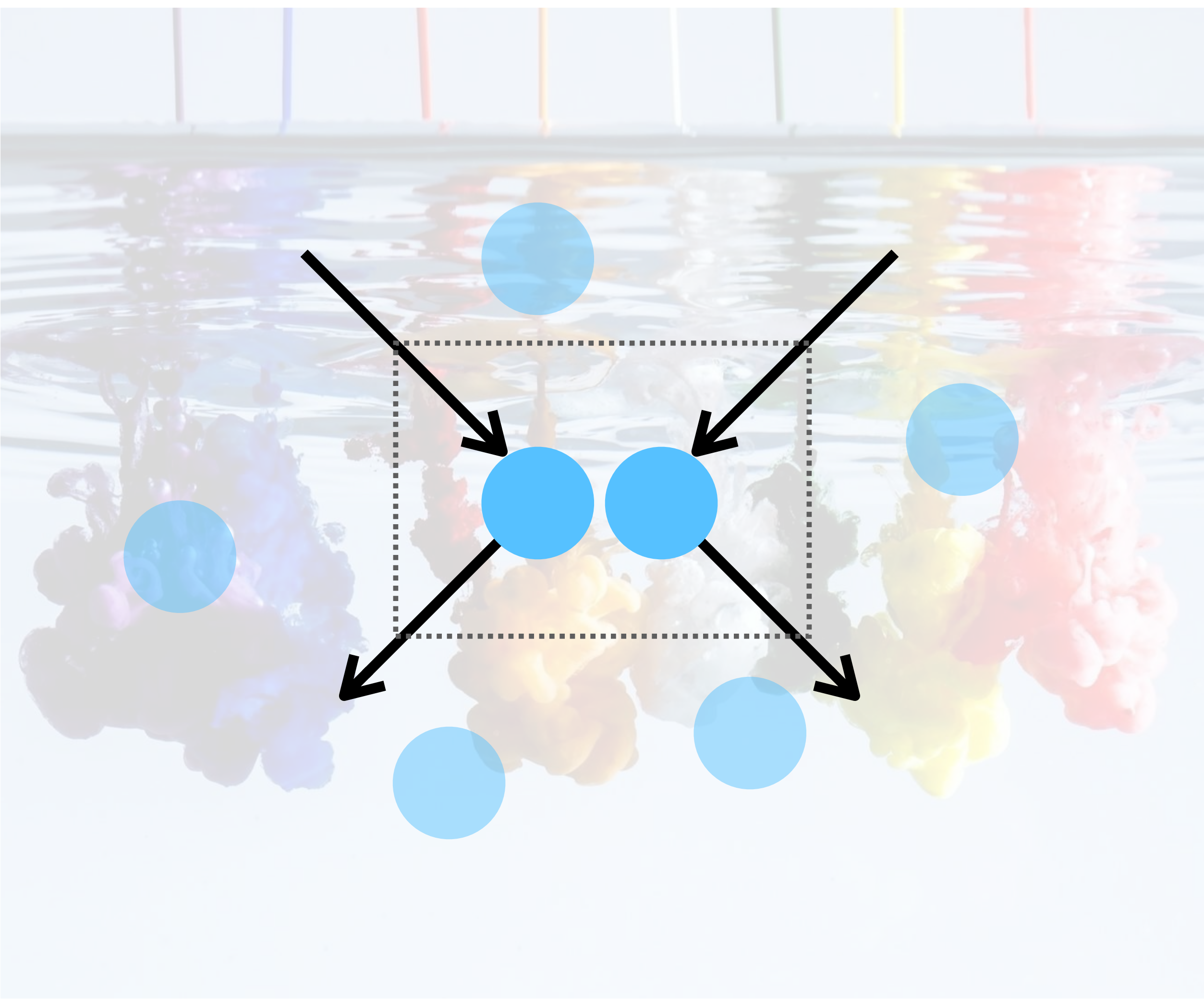
Long-range interaction

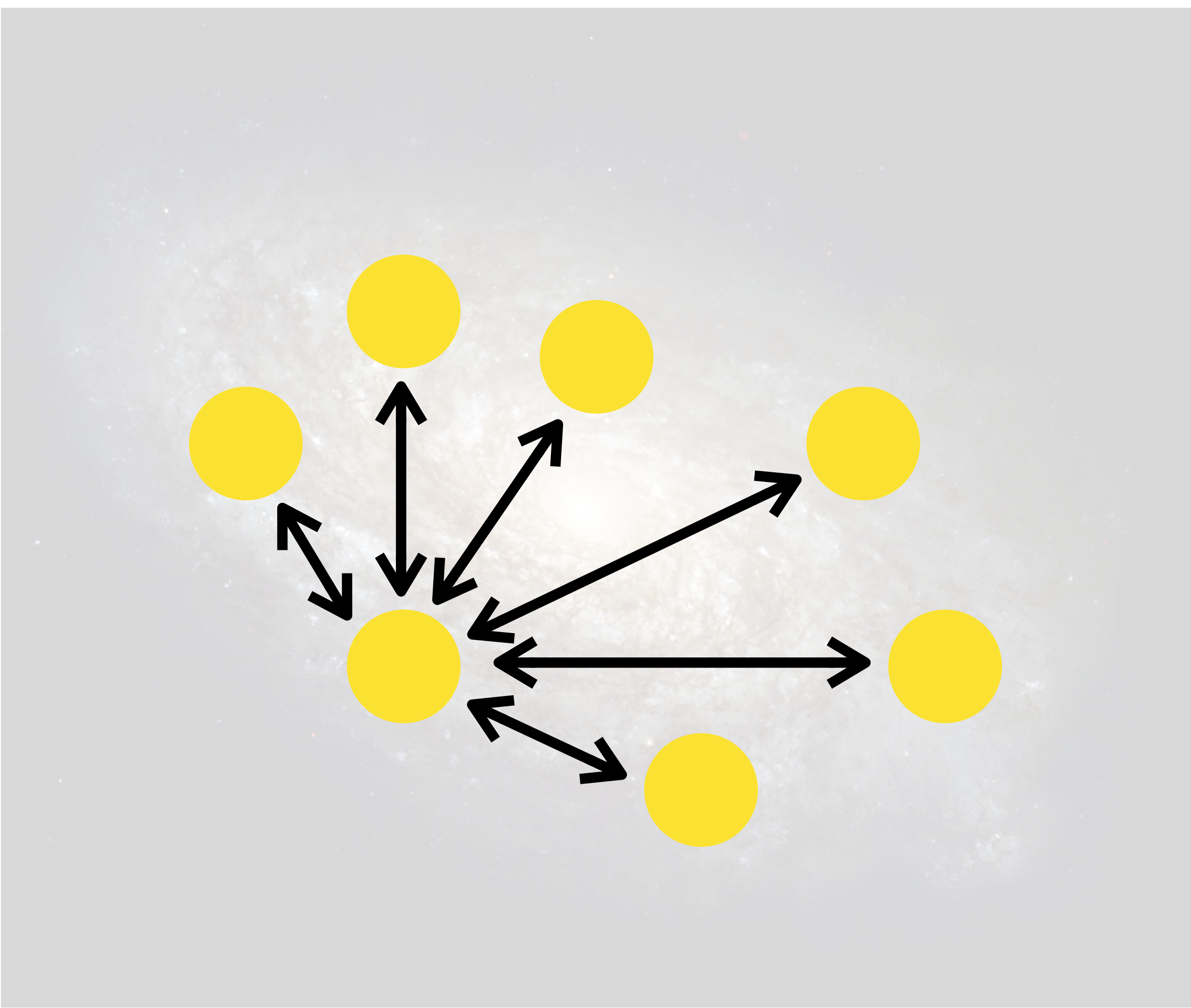
Diffusion

How do systems **diffuse**?

Ink in water

Stars in a galaxy





Pairwise interaction

Collective interaction

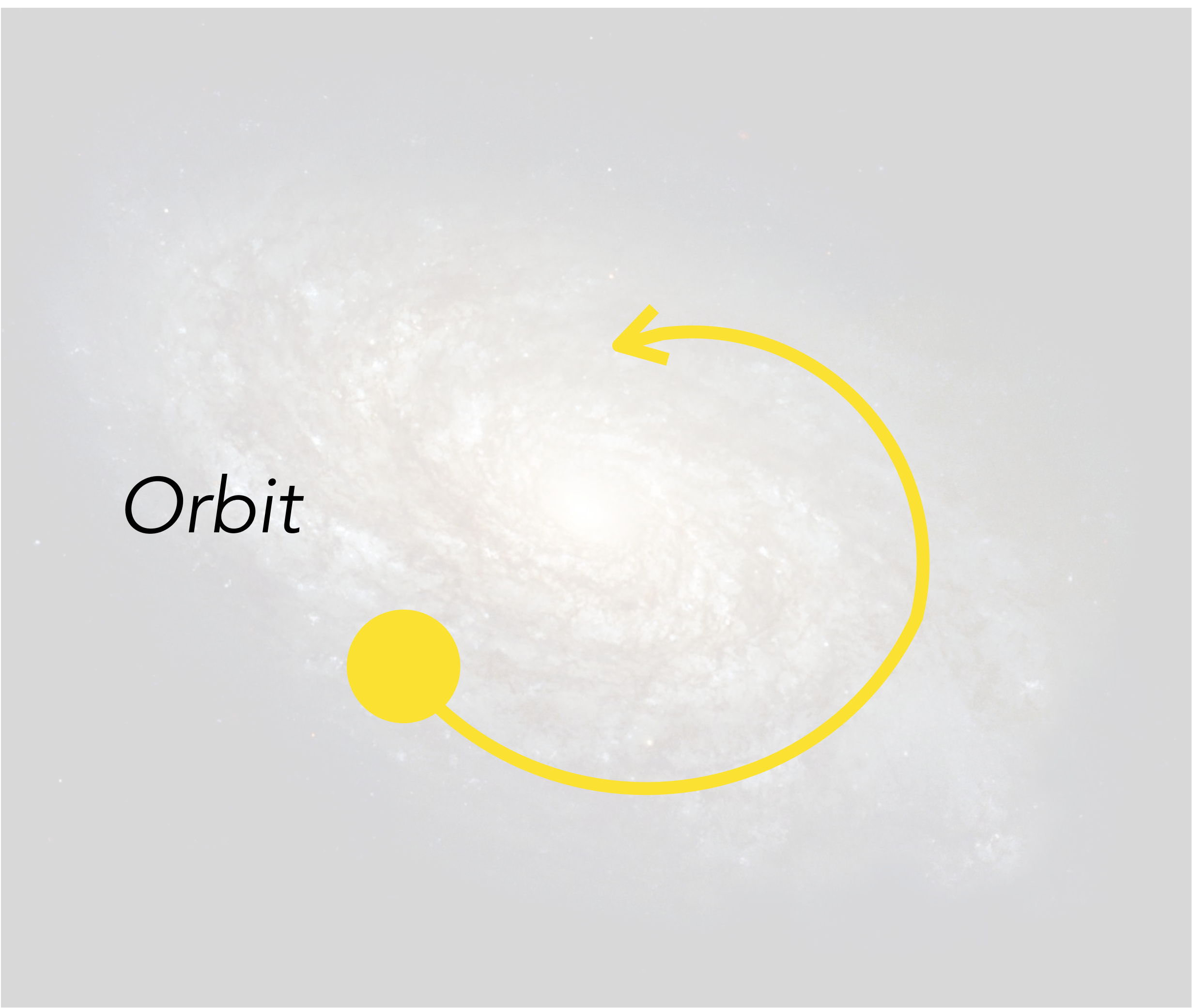
Diffusion

How do systems **diffuse**?

Ink in water

Stochastic trajectory

Stars in a galaxy



Regular trajectory

The gravitational Balescu-Lenard equation

What does it require?

Where does it come from?

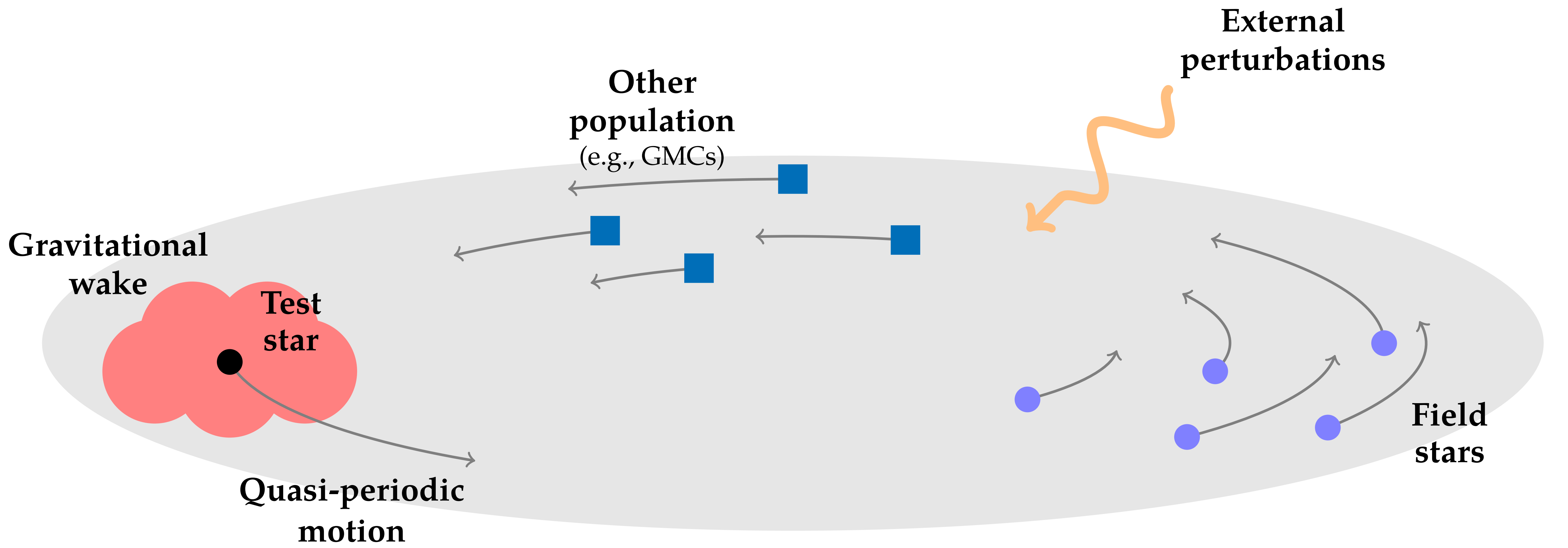
What is it?

Does it work?

What's next?

What does the Balescu-Lenard Eq. require?

Galactic evolution on cosmic timescales



Galaxies are:

- + **Inhomogeneous** (complex trajectories) — *Angle-action coordinates*
- + **Relaxed** (equilibrium states) — *Quasi-stationary states*
- + **Resonant** (orbital frequencies) — *Fast timescale vs. cosmic timescale*
- + **Degenerate** (in some regions) — *Frequency commensurability*
- + **Self-gravitating** (amplification of perturbations) — *Linear response theory*
- + **Discrete** (finite-N effects)
- + **Perturbed** (effects of the environment) — *Nature vs. Nurture*

What does it require?

Inhomogeneous

$$(\mathbf{x}, \mathbf{v}) \downarrow (\boldsymbol{\theta}, \mathbf{J})$$

Angle-Action coordinates

Relaxed

$$F = F(\mathbf{J}, t)$$

Quasi-stationary states

Resonant

$$\Omega(\mathbf{J}) = \partial H_0 / \partial \mathbf{J}$$

Fast/Slow timescale

Self-gravitating

$$\frac{1}{|E(\omega)|}$$

Linear response theory

Discrete & Perturbed

$$\frac{1}{N}$$

Finite-N effects

What does it require?

Inhomogeneous

$$(\mathbf{x}, \mathbf{v})$$
$$\downarrow$$
$$(\boldsymbol{\theta}, \mathbf{J})$$

Angle-Action coordinates

Relaxed

$$F = F(\mathbf{J}, t)$$

Quasi-stationary states

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Linear response theory

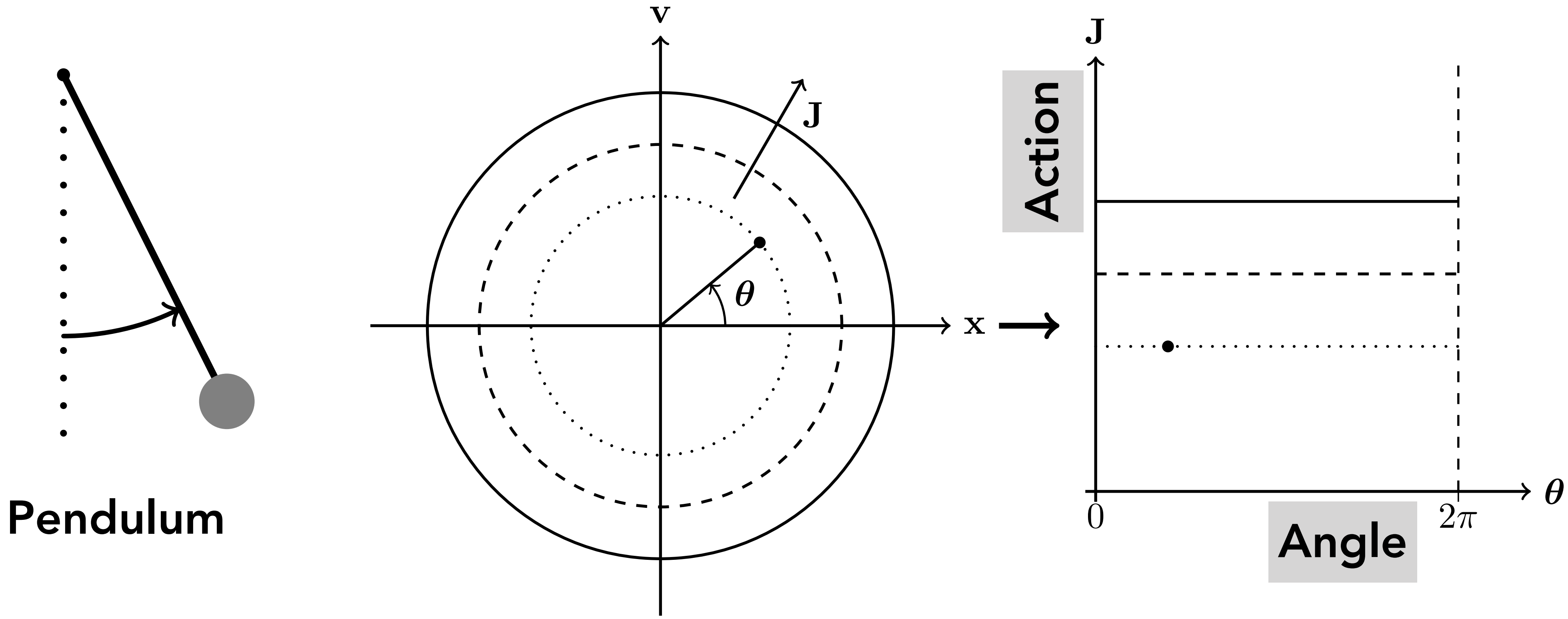
Discrete & Perturbed

$$\frac{1}{N}$$

Finite-N effects

Inhomogeneous systems

+ **Label** orbits with **integrals of motion**



+ **Angle-Action coordinates**

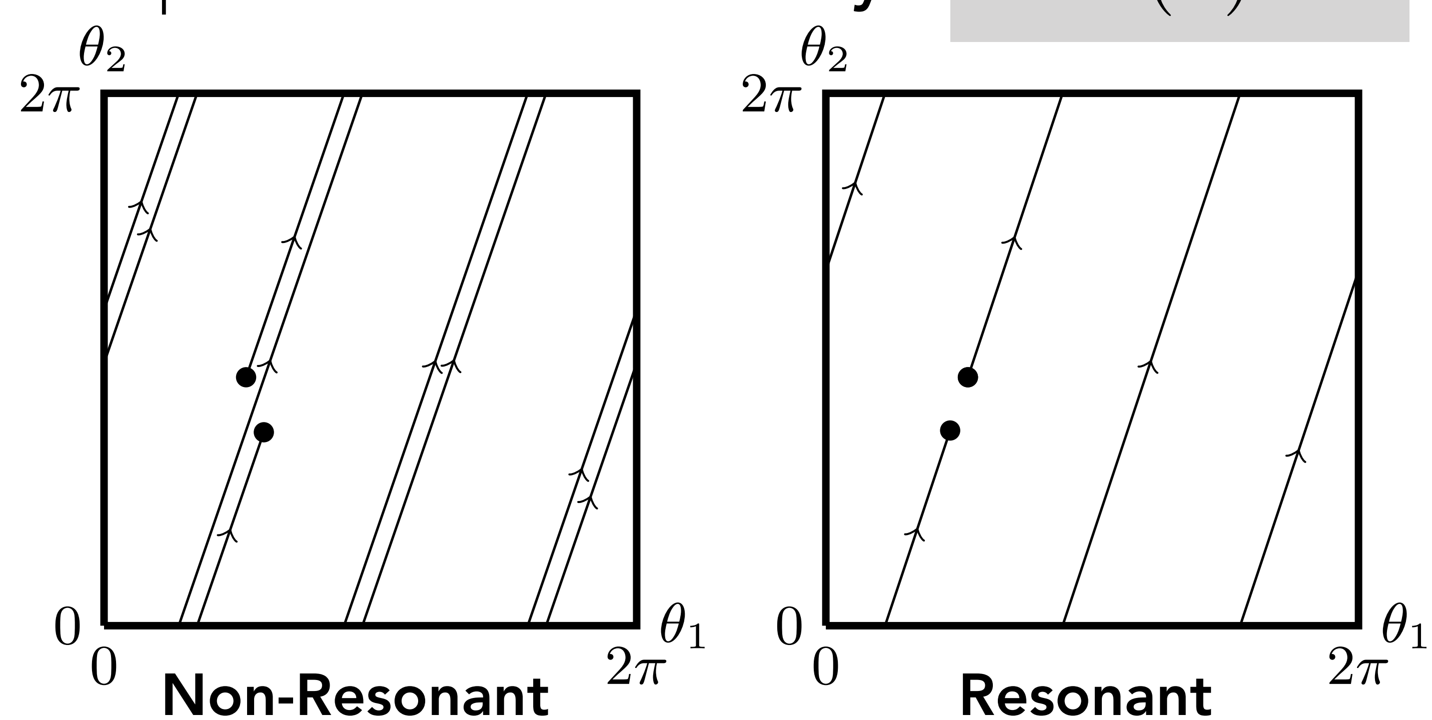
$$\begin{cases} \boldsymbol{\theta}(t) = \boldsymbol{\theta}_0 + t\,\boldsymbol{\Omega}(\mathbf{J}) \\ \mathbf{J}(t) = \text{cst.} \end{cases}$$

Trajectories become **straight lines**

+ **Relaxation**

$$\xrightarrow{(\text{few})\, t_{\text{cross}}} \quad F = F(\mathbf{J}, t)$$

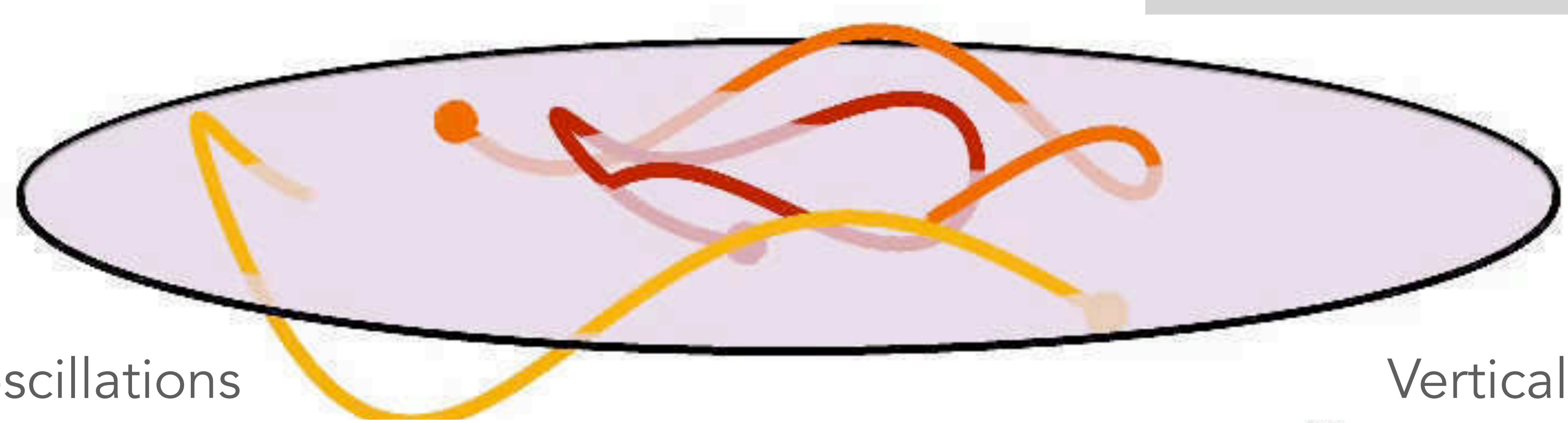
+ Frequencies' **commensurability** : $\mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J}) = 0$



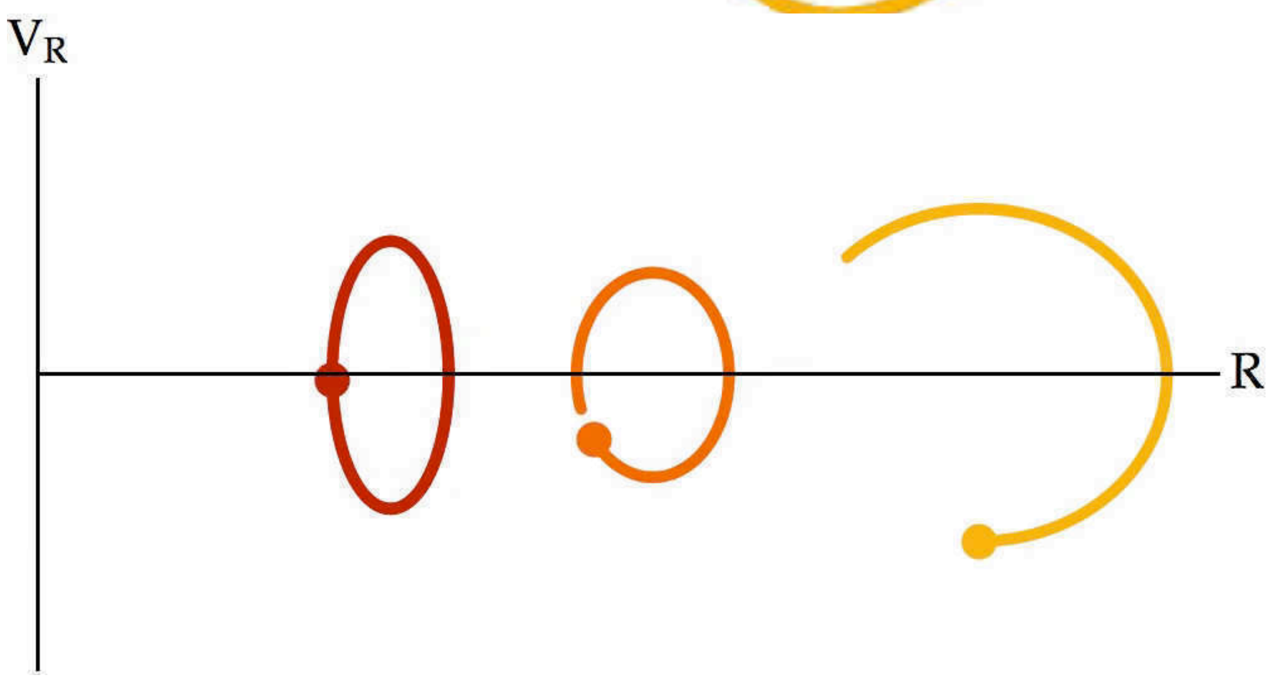
Example: Orbits in a disc

Integrable orbits $\Phi_0 = \Phi_0(R, z)$

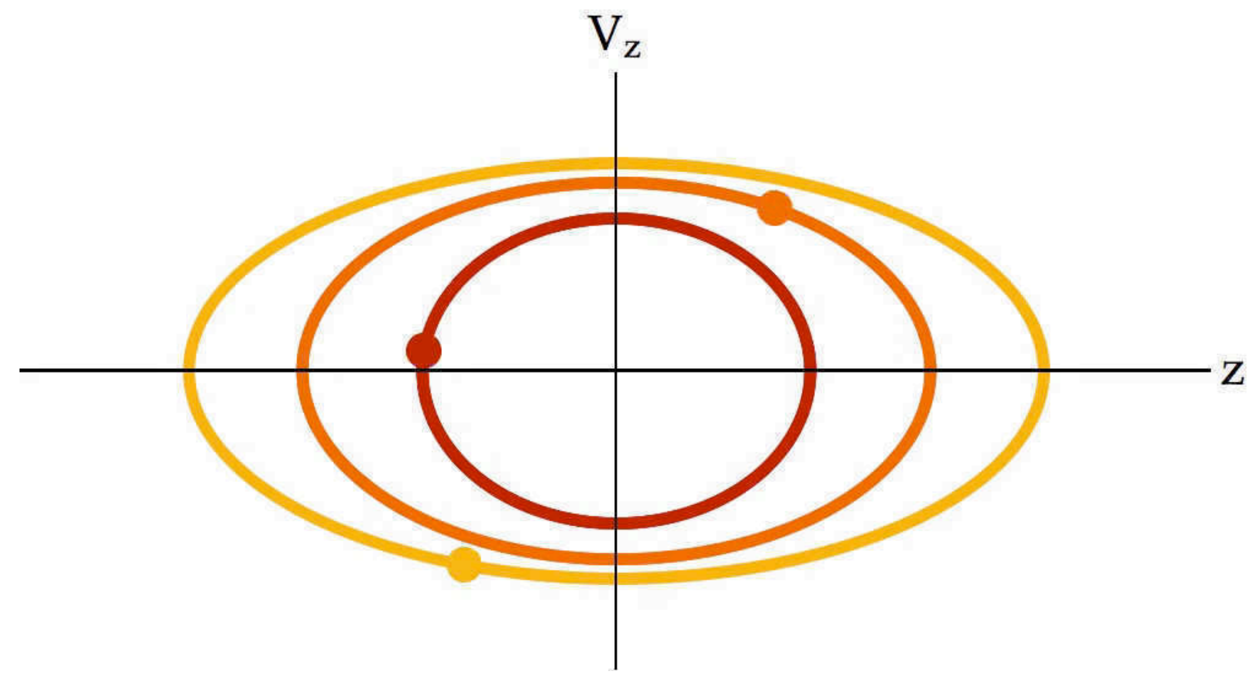
$$\begin{cases} \boldsymbol{\theta}(t) = \boldsymbol{\theta}_0 + t\,\boldsymbol{\Omega}(\mathbf{J}) \\ \mathbf{J}(t) = \mathrm{cst.} \end{cases}$$



Radial oscillations



Vertical oscillations



Actions $\mathbf{J} = \left(J_\phi\,, J_r\,, J_z\right)$

Frequencies $\boldsymbol{\Omega} = \left(\Omega_\phi, \Omega_r, \Omega_z\right)$

What does it require?

Inhomogeneous

$$(\mathbf{x}, \mathbf{v}) \downarrow (\boldsymbol{\theta}, \mathbf{J})$$

Angle-Action coordinates

Relaxed

$$F = F(\mathbf{J}, t)$$

Quasi-stationary states

Resonant

$$\boldsymbol{\Omega}(\mathbf{J}) = \partial H_0 / \partial \mathbf{J}$$

Fast/Slow timescale

Self-gravitating

$$\frac{1}{|E(\omega)|}$$

Linear response theory

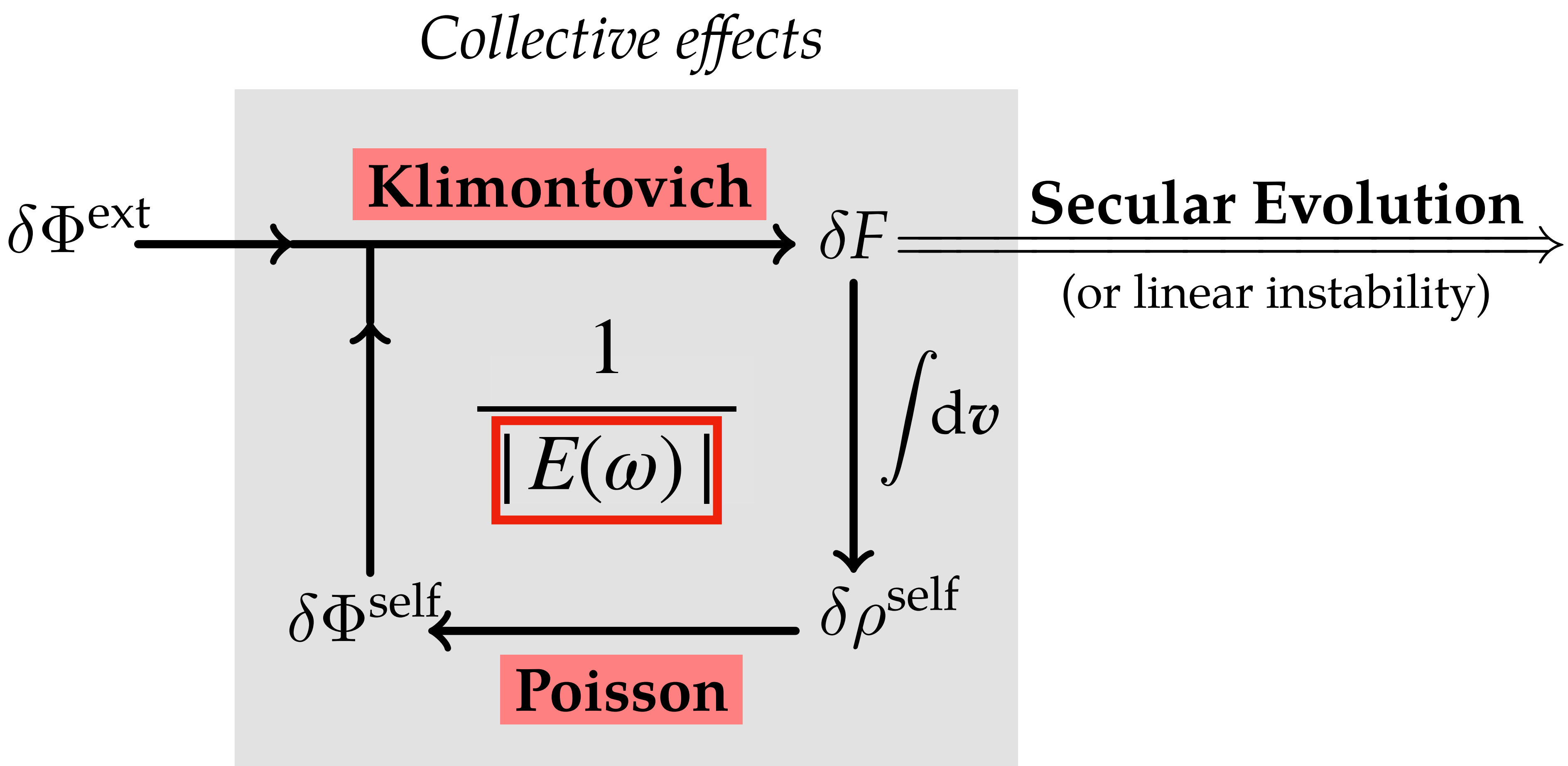
Discrete & Perturbed

$$\frac{1}{N}$$

Finite-N effects

Collective effects

Self-gravitating amplification



Gravitational polarisation essential to

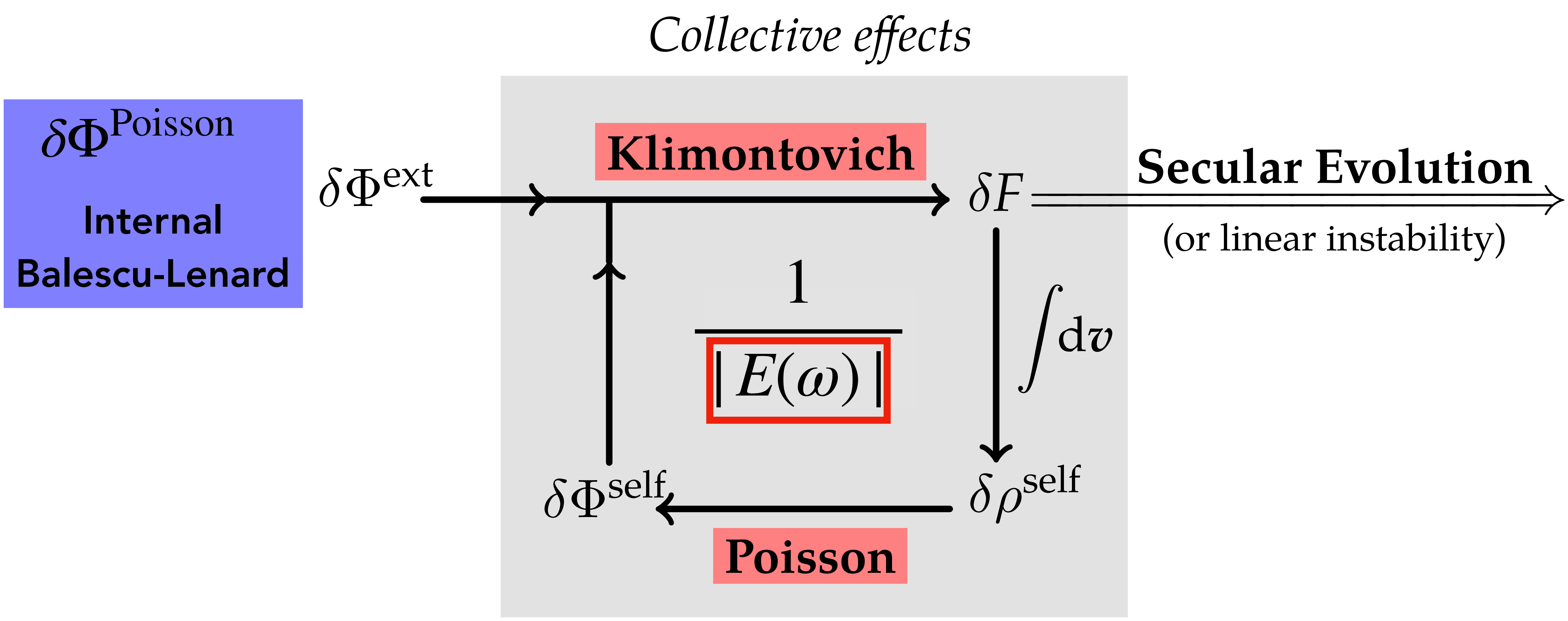
+ *Cause dynamical instabilities*

+ *Induce* ***dynamical friction*** *and* ***mass segregation***

+ ***Accelerate/Slow down*** *secular evolution*

Collective effects

Self-gravitating amplification



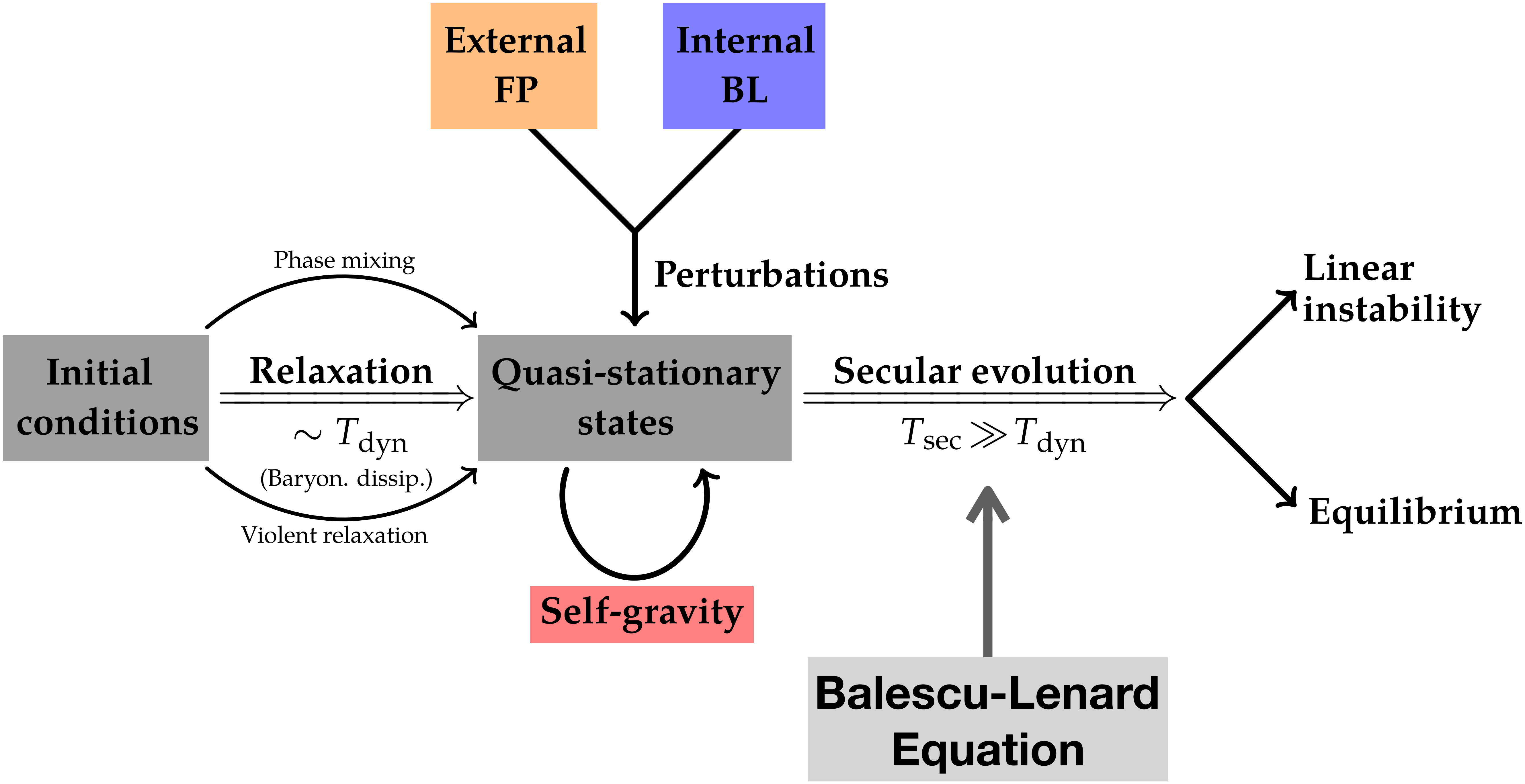
Gravitational polarisation essential to

+ *Cause dynamical instabilities*

+ *Induce* ***dynamical friction*** *and* ***mass segregation***

+ ***Accelerate/Slow down*** *secular evolution*

Typical fate of a self-gravitating system



Where does the Balescu-Lenard Eq. come from?

Balescu-Lenard via Klimontovich

Describing one **realisation** in **phase space** $\mathbf{w} = (\mathbf{x}, \mathbf{v})$

Discrete DF

$$F_{\mathrm{d}}(\mathbf{w}, t) = \sum_{i=1}^{N} m\, \delta_{\mathrm{D}}(\mathbf{w} - \mathbf{w}_i(t))$$

3D gravitational systems

$$U_{\mathrm{ext}} = \frac{|\mathbf{v}|^2}{2}$$

$$U = -\frac{G}{|\mathbf{x} - \mathbf{x}'|}$$

Discrete Hamiltonian

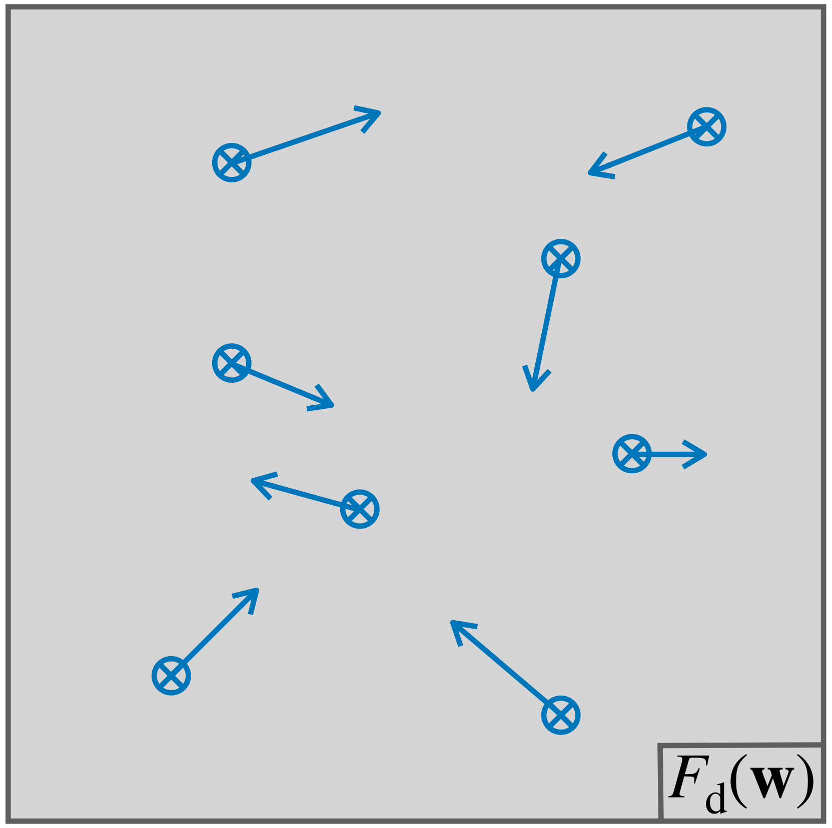
$$H_{\mathrm{d}}(\mathbf{w}, t) = U_{\mathrm{ext}}(\mathbf{w}) + \int \mathrm{d}\mathbf{w}'\, F_{\mathrm{d}}(\mathbf{w}', t)\, U(\mathbf{w}, \mathbf{w}')$$

Continuity equation in phase space

$$\frac{\partial F_{\mathrm{d}}}{\partial t} + \frac{\partial}{\partial \mathbf{w}} \cdot \left(F_{\mathrm{d}}\, \dot{\mathbf{w}} \right) = 0$$

Exact **Klimontovich** equation

$$\frac{\partial F_{\mathrm{d}}}{\partial t} + \left[F_{\mathrm{d}}, H_{\mathrm{d}} \right] = 0$$

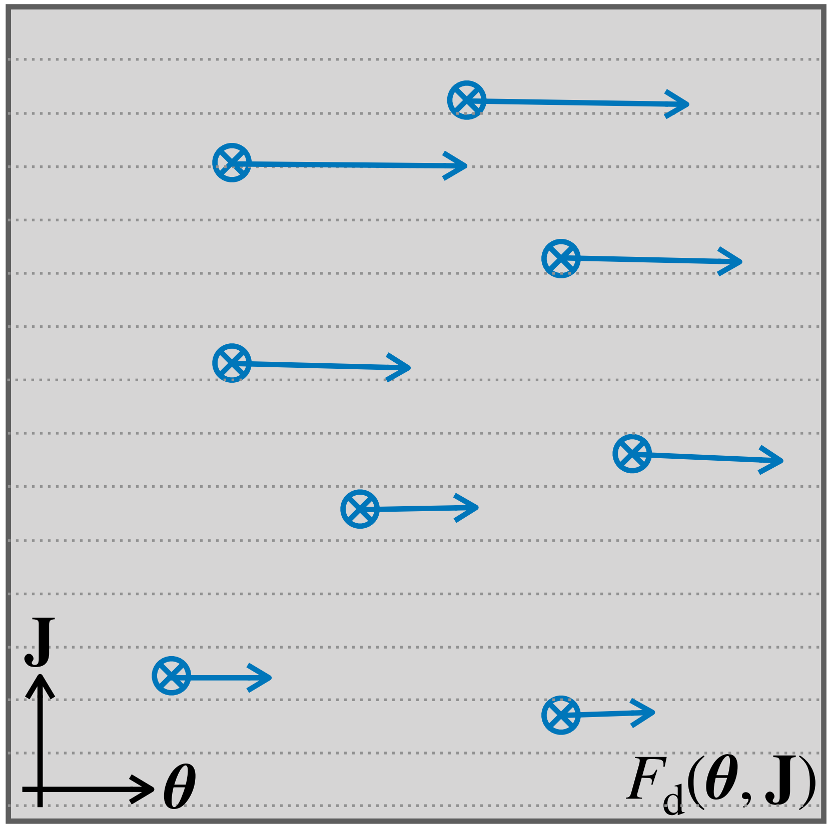


Phase space

Solving Klimontovich

Perturbative expansion

$$\begin{cases} F_{\mathrm{d}} = F_0 + \delta F \quad \text{with} \quad \langle \delta F \rangle = 0, \\ H_{\mathrm{d}} = H_0 + \delta\Phi \quad \text{with} \quad \langle \delta\Phi \rangle = 0. \end{cases}$$



Angle-Action space

Mean-field equilibrium

$$\begin{cases} F_0 = F_0(\mathbf{J}, t), \\ H_0 = H_0(\mathbf{J}, t)\,. \end{cases}$$

Quasi-linear evolution equations

$$\frac{\partial \delta F}{\partial t} + \left[\delta F, H_0\right] + \left[F_0, \delta\Phi\right] = 0$$

$$\frac{\partial F_0}{\partial t} = -\left\langle \left[\delta F, \delta\Phi\right] \right\rangle$$

Timescale separation

$$\begin{cases} T_{\delta F} \simeq T_{\mathrm{dyn}} \\ T_{F_0} \simeq \left(\sqrt{N}\right)^2 \times T_{\delta F} \end{cases}$$

Dynamics of fluctuations

Fast evolution of **perturbations** (Linearised Klimontovich Eq.)

$$\frac{\partial \delta F}{\partial t} + \left[\delta F, H_0\right] + \left[F_0, \delta\Phi\right] = 0$$

$\left[\delta F, H_0\right]$ Mean-field motion

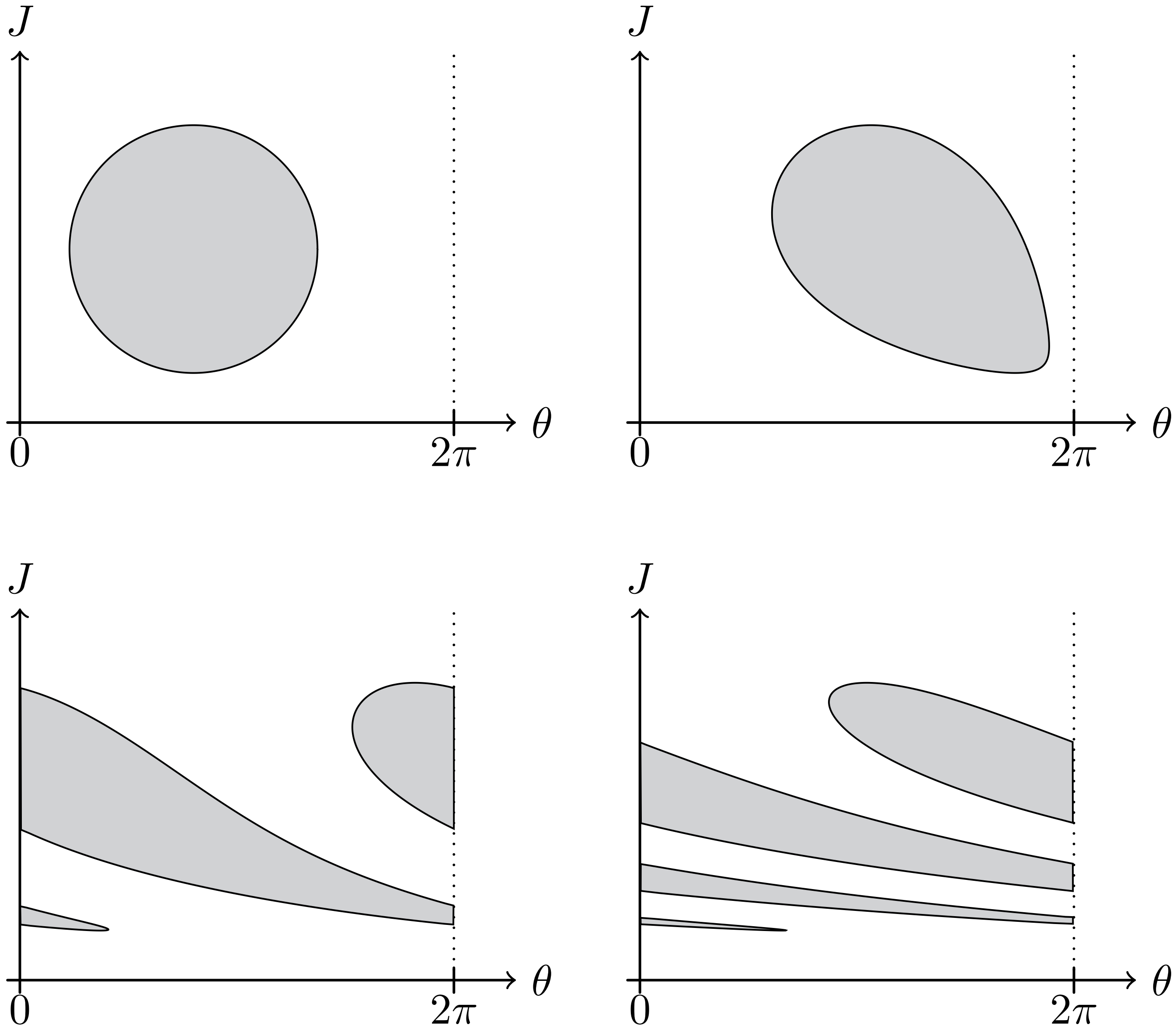
$\left[F_0, \delta\Phi\right]$ Collective effects

Self-consistent amplification

$$\delta\Phi = \delta\Phi\left[\delta F\right]$$

Timescale separation

$$\begin{cases} F_0(\mathbf{J}) = \text{cst} \\ H_0(\mathbf{J}) = \text{cst} \end{cases}$$



Phase Mixing

Solving for the fluctuations

Linear amplification

$$\delta\tilde{F}_{\mathbf{k}}(\mathbf{J},\omega) = -\frac{\delta F_{\mathbf{k}}(\mathbf{J},0)}{\mathrm{i}(\omega - \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}))} - \frac{\mathbf{k}\cdot\partial F_0/\partial\mathbf{J}}{\omega - \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})}\,\delta\tilde{\Phi}_{\mathbf{k}}(\mathbf{J},\omega)$$

Bare noise

Self-consistent amplification

with the **self-consistency**

$$\delta\Phi(\mathbf{w},t) = \int \mathrm{d}\mathbf{w}'\,\delta F(\mathbf{w}',t)\,U(\mathbf{w},\mathbf{w}')$$

Generic form of a **Fredholm equation**

$$\big[\delta\Phi(\mathbf{J})\big]_{\mathrm{dressed}} = \big[\delta\Phi(\mathbf{J})\big]_{\mathrm{bare}} + \int \mathrm{d}\mathbf{J}'\,M(\mathbf{J},\mathbf{J}')\,\big[\delta\Phi(\mathbf{J}')\big]_{\mathrm{dressed}}$$

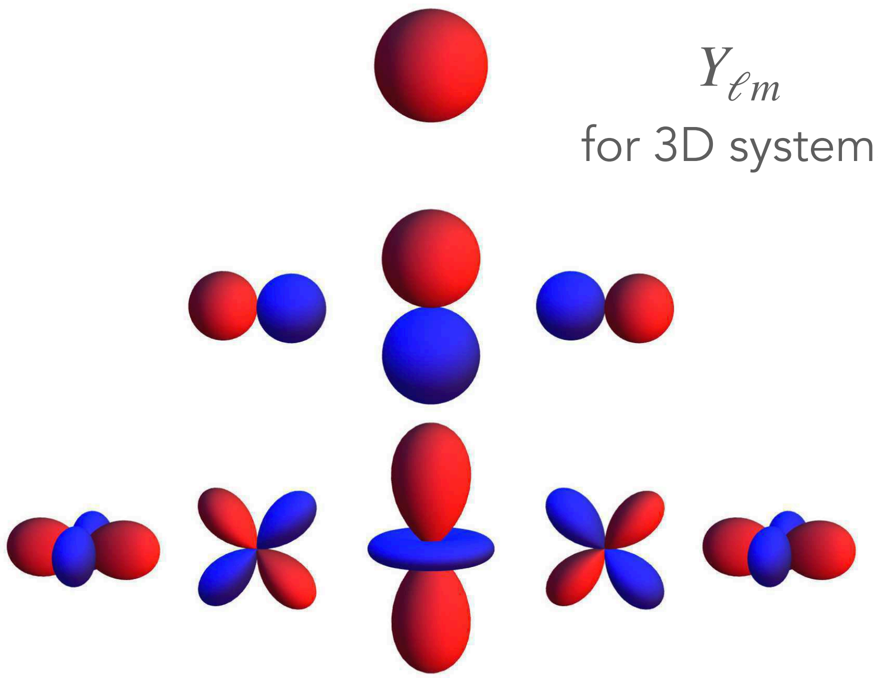
Amplification kernel

Dressing of perturbations

$$\big[\delta\Phi(\omega)\big]_{\mathrm{dressed}} \simeq \frac{[\delta\Phi(\omega)]_{\mathrm{bare}}}{1 - M(\omega)} = \frac{[\delta\Phi(\omega)]_{\mathrm{bare}}}{|\,E(\omega)\,|}$$

Basis method $\left(\psi^{(p)}(\mathbf{w}), \rho^{(p)}(\mathbf{w})\right)$

$$\begin{cases} \psi^{(p)}(\mathbf{w}) = \int \mathrm{d}\mathbf{w}' \, U(\mathbf{w}, \mathbf{w}') \, \rho^{(p)}(\mathbf{w}'), \\ \int \mathrm{d}\mathbf{w} \, \psi^{(p)}(\mathbf{w}) \, \rho^{(q)*}(\mathbf{w}) = - \, \delta_{pq} \, . \end{cases}$$



``**Separable**'' pairwise interaction

$$U(\mathbf{w}, \mathbf{w}') = - \sum_p \psi^{(p)}(\mathbf{w}) \, \psi^{(p)*}(\mathbf{w}')$$

Plasmas

$$U(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|}$$

$$\simeq \int \frac{\mathrm{d}\mathbf{k}}{|\mathbf{k}|^2} \, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}$$

Galaxies

$$\Delta\Phi = 4\pi G\rho$$

Poisson equation

Linear response theory

$$\left[\delta H(\omega)\right]_{\text{dressed}} = \frac{[\delta H(\omega)]_{\text{bare}}}{|\,E(\omega)\,|}$$

$$E_{pq}(\omega) = 1 - (2\pi)^d \sum_{\mathbf{k}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{k}\cdot\partial F_0/\partial \mathbf{J}}{\omega - \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})} \, \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \; \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Dielectric function

$$E_{pq}(\omega)$$

Two limits

$E_{pq}(\omega) \simeq 0$ ***Cold*** *regime*

$E_{pq}(\omega) \simeq 1$ ***Hot*** *regime*

Some properties

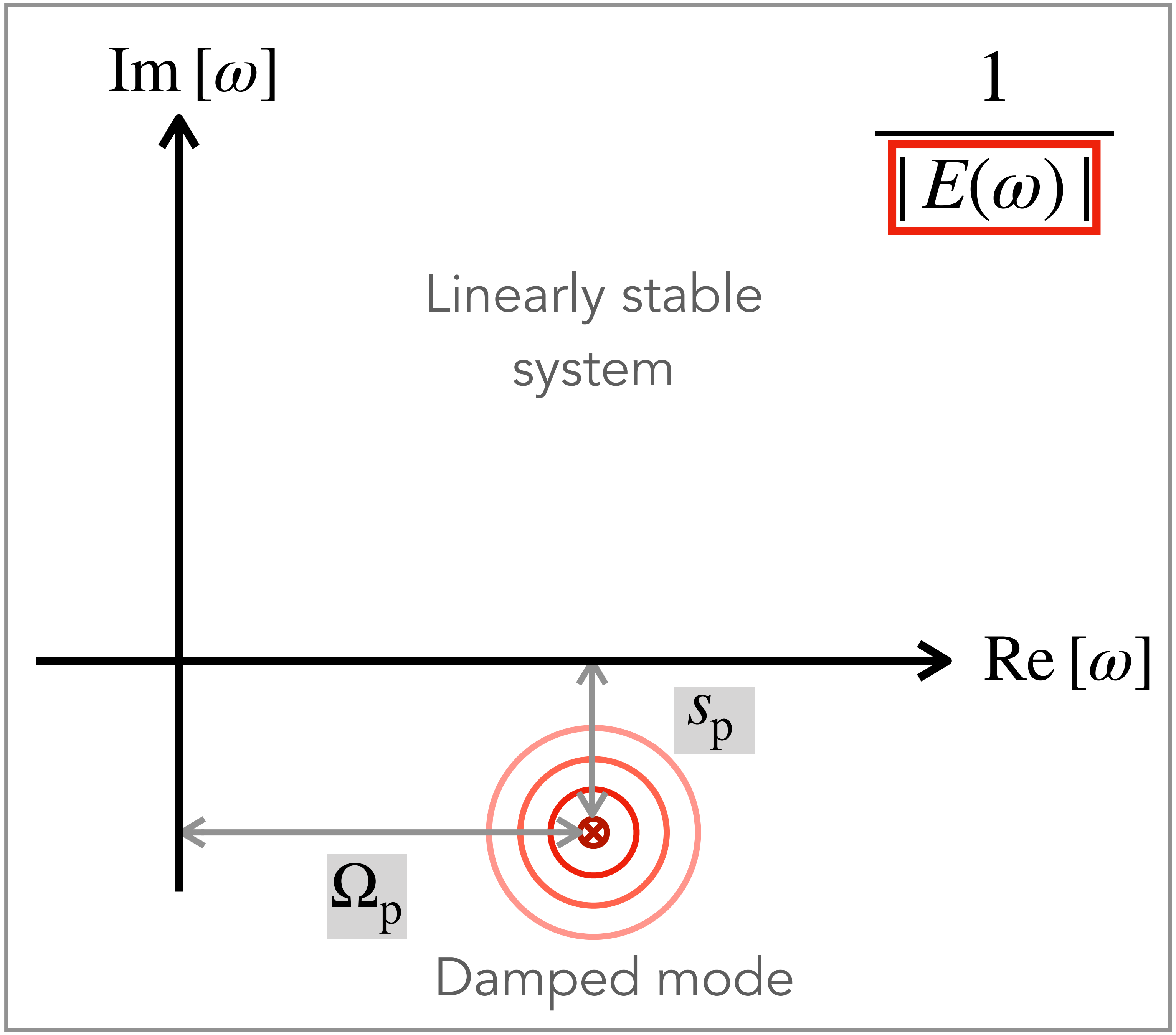
$\sum_{\mathbf{k}}$ *Sum over resonances*

$\int \mathrm{d}\mathbf{J}$ *Scan over orbital space*

$\omega - \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})$ *Resonant int.*

$\psi^{(p)} = \int \mathrm{d}\mathbf{w}' \, U \, \rho^{(p)}$ *Long-range int.*

Dielectric function



Susceptibility

$$\frac{1}{|E(\Omega_p)|} \gg 1$$

Thermalisation

$$\left[\delta\Phi(t)\right]_{\text{trans.}} \simeq e^{-s_p t}$$

Dressed long-term diffusion

Secular evolution equation

$$\frac{\partial F_0}{\partial t} = - \left\langle \left[\delta F, \delta\Phi \right] \right\rangle$$

Dressing comes twice

$$\left[\delta\Phi\right]_{\mathrm{dressed}} = \frac{\left[\delta\Phi\right]_{\mathrm{bare}}}{\boxed{|E(\omega)|}} \longrightarrow \frac{\partial F_0}{\partial t} \simeq \frac{|\delta\Phi|^2_{\mathrm{bare}}}{\boxed{|E(\omega)|^2}}$$

Bare Poisson shot noise

Relaxation time

$$|\delta\Phi|_{\mathrm{bare}} \simeq \frac{1}{\sqrt{N}} \longrightarrow T_{\mathrm{relax}} \simeq \boxed{|E|^2} N \, T_{\mathrm{dyn}}$$

Collective effects can **drastically accelerate** orbital heating,
in particular on **large scales**

The two components of diffusion

Secular evolution equation

$$\frac{\partial F_0}{\partial t} = -\left\langle \left[\delta F, \delta\Phi\right] \right\rangle = -\frac{\partial}{\partial \mathbf{J}} \cdot \mathbf{F}(\mathbf{J}, t)$$

Dynamics of **perturbations**

$$\delta\tilde{F}_{\mathbf{k}}(\mathbf{J}, \omega) = -\frac{\delta F_{\mathbf{k}}(\mathbf{J},0)}{\mathrm{i}(\omega - \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}))} - \frac{\mathbf{k}\cdot\partial F_0/\partial \mathbf{J}}{\omega - \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})}\,\delta\tilde{\Phi}_{\mathbf{k}}(\mathbf{J}, \omega)$$

Poisson noise — Potential fluctuations

Flux decomposition

$$\mathbf{F}(\mathbf{J}, t) = \mathbf{F}_1(\mathbf{J}, t) + \mathbf{F}_2(\mathbf{J}, t)$$

Dynamical friction

$$\mathbf{F}_1(\mathbf{J}, t) \propto \left\langle \delta F(0)\, \delta\Phi \right\rangle = \mathbf{D}_1(\mathbf{J})\, F_0(\mathbf{J})$$

Backreaction to DF's perturbations

Diffusion

$$\mathbf{F}_2(\mathbf{J}, t) \propto \left\langle \delta\Phi\, \delta\Phi \right\rangle = -\mathbf{D}_2(\mathbf{J}) \cdot \frac{\partial F_0}{\partial \mathbf{J}}$$

Correlations of the potential fluctuations

What is the Balescu-Lenard Eq.?

Balescu-Lenard equation

The master equation for **self-induced orbital relaxation**

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = \frac{1}{N}\frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \int \mathrm{d}\mathbf{J}' \frac{\delta_{\mathrm{D}}(\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\boldsymbol{\Omega}(\mathbf{J}'))}{|\, E_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}))\,|^2} \times \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} - \mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'} \right) F(\mathbf{J},t)\, F(\mathbf{J}',t) \right]$$

Some properties

$F(\mathbf{J},t)$ Orbital distortion in **action space**

$1/N$ Sourced by **finite-N effects**

$\partial/\partial \mathbf{J}\,\cdot$ Divergence of a **diffusion flux**

$(\mathbf{k},\mathbf{k}')$ Discrete **resonances**

$\int \mathrm{d}\mathbf{J}'$ Scan of **orbital space**

$\delta_{\mathrm{D}}(\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J})-\mathbf{k}'\cdot\boldsymbol{\Omega}(\mathbf{J}'))$ **Resonance** cond.

$1/|\, E_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega)\,|^2$ **Dressed** couplings

Fokker-Planck equation

+ **Test particle** of mass m_{t} — $P(\mathbf{J}, t)$
+ **Bath particles** of mass $m_{\mathrm{b}} = M_{\mathrm{tot}}/N$ — $F_0(\mathbf{J}, t)$

$$\frac{\partial P(\mathbf{J}, t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \int \mathrm{d}\mathbf{J}' \, \frac{\delta_{\mathrm{D}}(\mathbf{k} \cdot \mathbf{\Omega}(\mathbf{J}) - \mathbf{k}' \cdot \mathbf{\Omega}(\mathbf{J}'))}{|E_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}', \mathbf{k} \cdot \mathbf{\Omega}(\mathbf{J}))|^2} \times \left(m_{\mathrm{b}} \, \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{J}} - m_{\mathrm{t}} \, \mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{J}'} \right) P(\mathbf{J}, t) \, F_0(\mathbf{J}', t) \right]$$

Diffusion $m_{\mathrm{b}} \, \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{J}}$

Vanishes in the collisionless limit $N \to +\infty$

Sourced by **correlations** in the **potential fluctuations**

$$\mathbf{D}_{\mathrm{diff}} \propto \left\langle \delta\Phi(t) \, \delta\Phi(t') \right\rangle$$

Friction $m_{\mathrm{t}} \, \mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{J}'}$

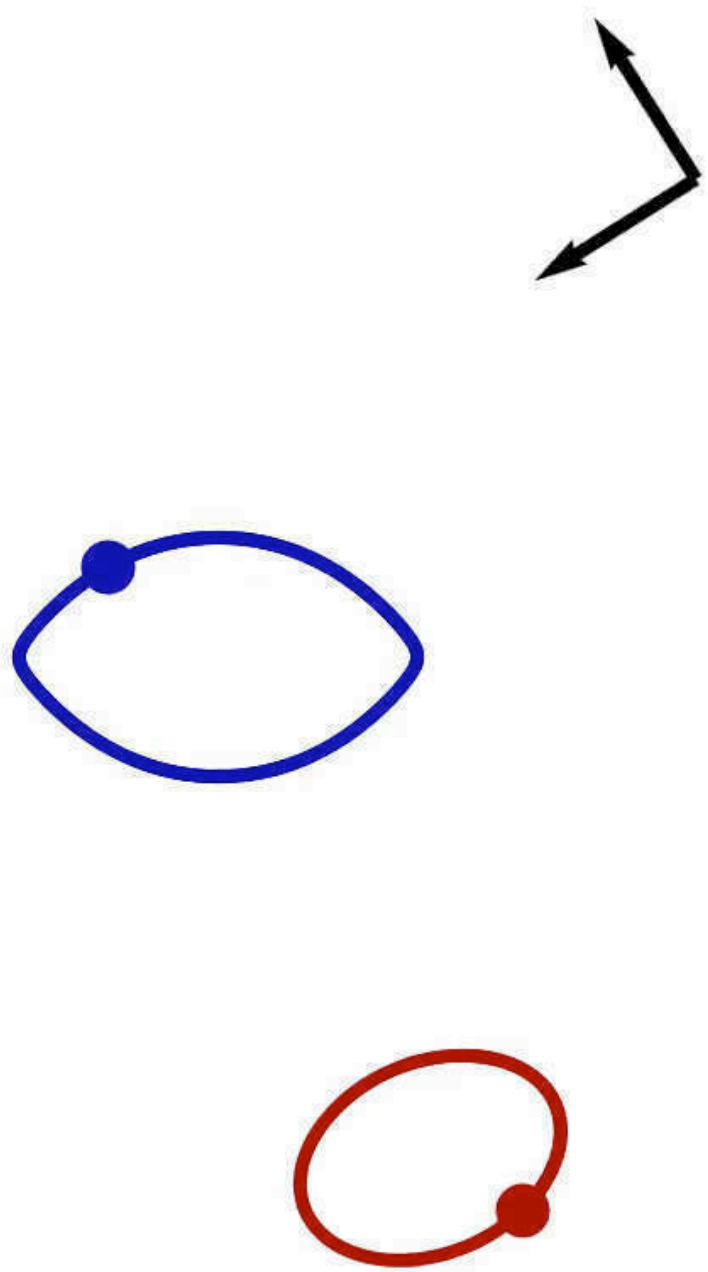
Induces **mass segregation**

Sourced by the **backreaction** of the test particle on the bath

$$\mathbf{D}_{\mathrm{fric}} \propto \left\langle \delta P(t) \, \delta\Phi(t') \right\rangle$$

Resonant encounters

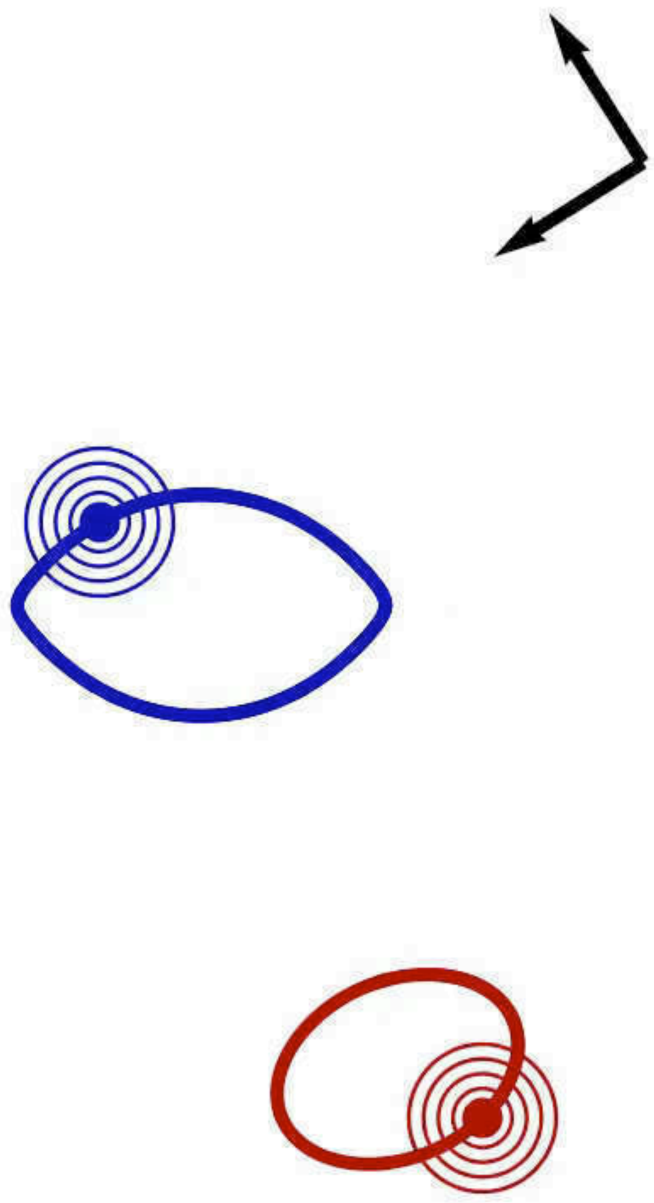
$$\delta_{\mathrm{D}}(\mathbf{k} \cdot \mathbf{\Omega}(\mathbf{J}) - \mathbf{k}' \cdot \mathbf{\Omega}(\mathbf{J}'))$$



Collisions are **resonant**, **long-range**, **correlated**

Dressed resonant encounters

$$\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})-\mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}'))$$



Fluctuations create a **wake**

$$\delta\Phi_{\mathrm{bare}} \rightarrow \frac{\delta\Phi_{\mathrm{dressed}}}{|\,E(\omega)\,|}$$

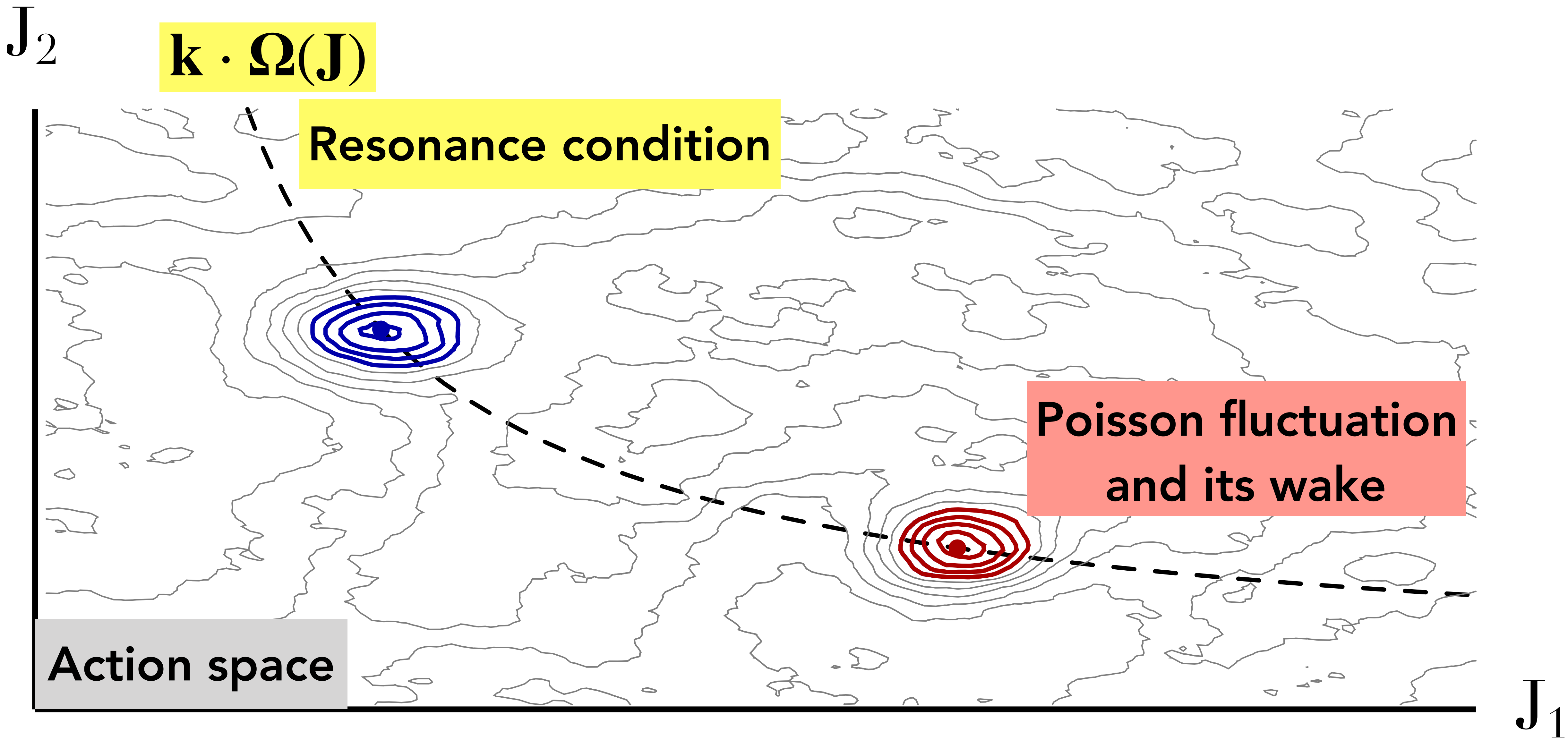
Interactions between **wakes**

$$\mathbf{D}_{\mathrm{diff}}(\mathbf{J}) \rightarrow \frac{\mathbf{D}_{\mathrm{diff}}(\mathbf{J})}{|\,E(\omega)\,|^2}$$

Collisions are **resonant**, **long-range**, **correlated,** and **dressed**

Non-local resonances

$$\delta_{\mathrm{D}}(\mathbf{k} \cdot \mathbf{\Omega}(\mathbf{J}) - \mathbf{k}' \cdot \mathbf{\Omega}(\mathbf{J}'))$$



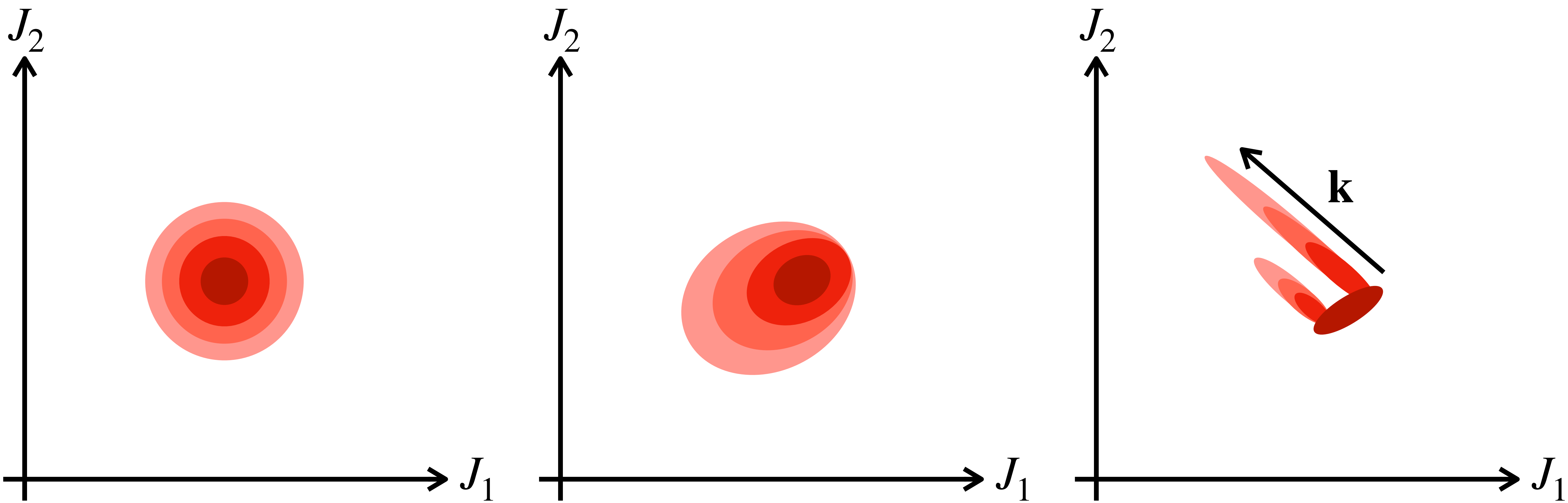
Non-local resonant couplings between **dressed** wakes

Anisotropic diffusion

Generic **diffusion equation**

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k}} \mathbf{k} \left(\mathbf{D}_{\mathbf{k}}(\mathbf{J}) \cdot \frac{\partial F}{\partial \mathbf{J}} + \ldots \right) \right]$$

Two sources of **anisotropies**



$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[D \, \frac{\partial F}{\partial \mathbf{J}} \right]$$

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[D(\mathbf{J}) \, \frac{\partial F}{\partial \mathbf{J}} \right]$$

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[\mathbf{k} \, \mathbf{D}(\mathbf{J}) \cdot \frac{\partial F}{\partial \mathbf{J}} \right]$$

Plasmas		Galaxies

Orbital coordinates

$(\mathbf{x}, \mathbf{v})$ | $(\boldsymbol{\theta}, \mathbf{J})$

Basis decomposition

$$U(\mathbf{x}, \mathbf{x}') \propto \int \frac{\mathrm{d}\mathbf{k}}{\mathbf{k}^2} \, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}$$

$$U(\mathbf{w}, \mathbf{w}') = -\sum_p \psi^{(p)}(\mathbf{w}) \, \psi^{(p)*}(\mathbf{w}')$$

Dielectric function

$$1 + \frac{1}{\mathbf{k}^2} \int \mathrm{d}\mathbf{v} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}}$$

$$\delta_{pq} - \sum_{\mathbf{k}} \int \mathrm{d}\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Resonance condition

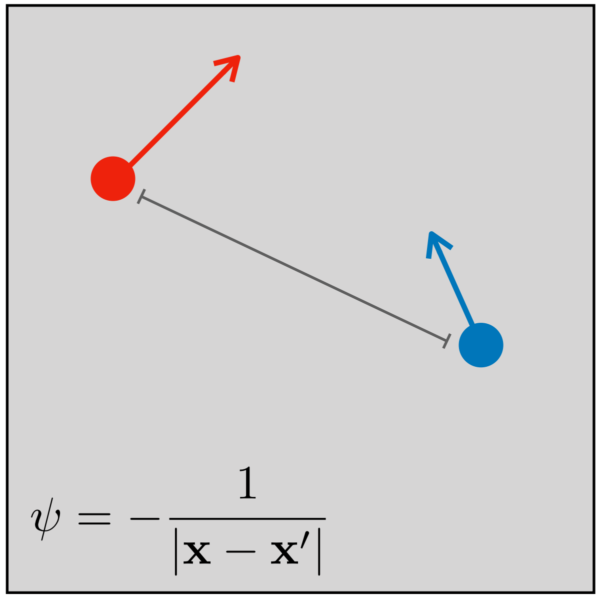
$$\delta_{\mathrm{D}}\big(\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')\big)$$

$$\delta_{\mathrm{D}}\big(\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \mathbf{k}' \cdot \boldsymbol{\Omega}(\mathbf{J}')\big)$$

Does the Balescu-Lenard Eq. work?

Long-range interacting systems are ubiquitous

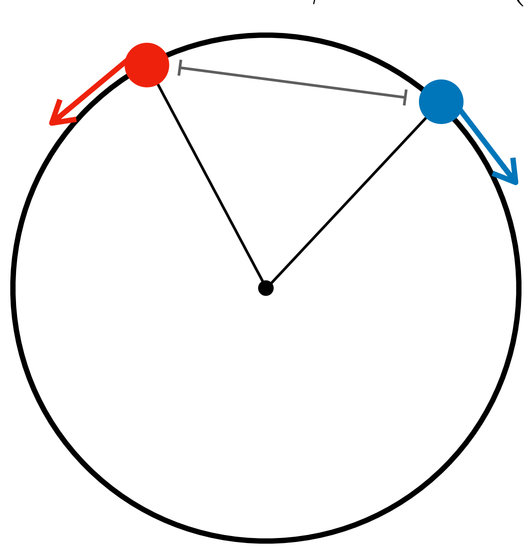
Homogeneous systems



d=3, homogeneous

Hamiltonian Mean Field Model

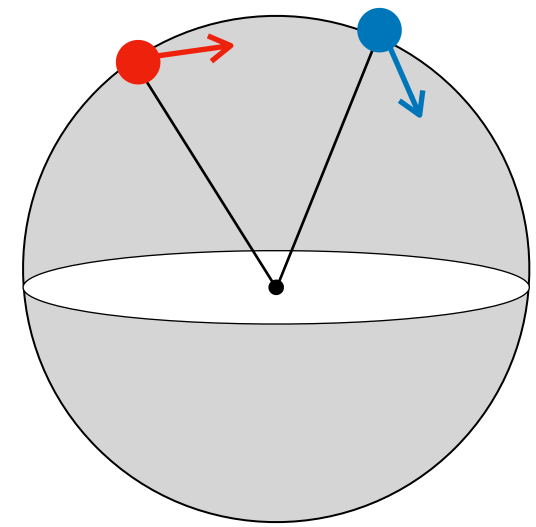
$\psi = -\cos(\theta - \theta')$



d=1, inhomogeneous

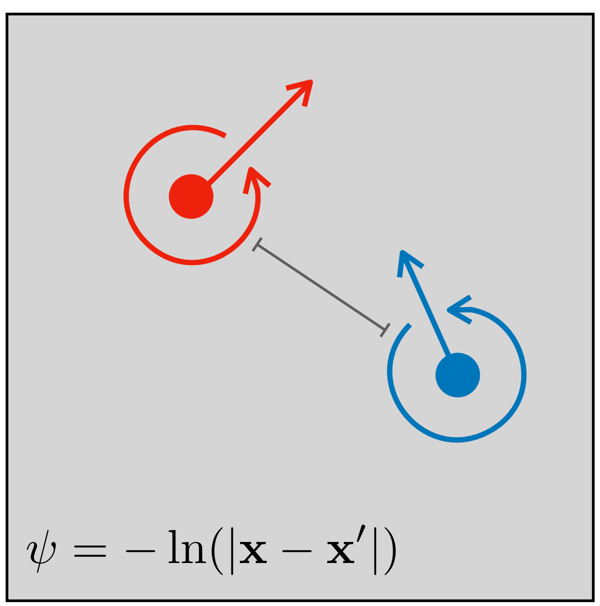
Vector Resonant Relaxation

$\psi = -V(\mathbf{s}\cdot\mathbf{s}')$



d=1, inhomogeneous, degenerate

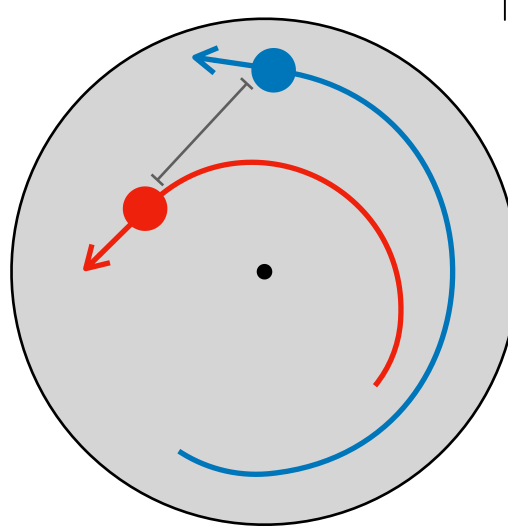
2D hydrodynamics



d=2, inhomogeneous

Self-gravitating discs

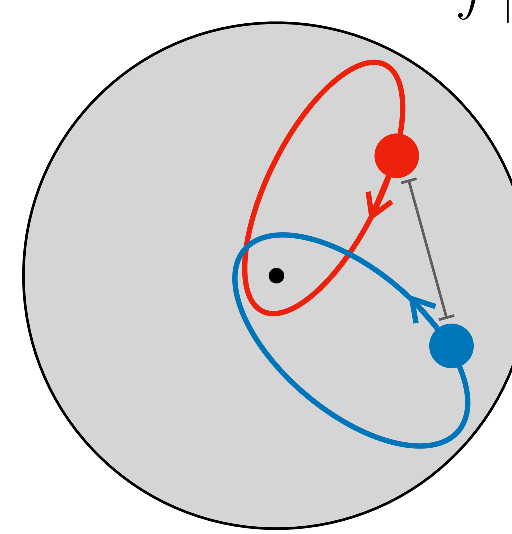
$\psi = -\frac{1}{|\mathbf{x}-\mathbf{x}'|}$



d=2, inhomogeneous

Scalar Resonant Relaxation

$\psi = -\int \frac{\mathrm{d}\theta \mathrm{d}\theta'}{|\mathbf{x}-\mathbf{x}'|}$



d=2, inhomogeneous, degenerate

The diversity of long-range interacting systems

Small dimension		Large dimension	
$d = 1$	Galactic Nuclei	$d = 2$	Globular clusters
Homogeneous		**Inhomogeneous**	
$(\mathbf{x}, \mathbf{v})$	Plasmas	$(\boldsymbol{\theta}, \mathbf{J})$	Galaxies
Hot		**Cold**	
$\frac{1}{\lvert E(\omega) \rvert} \simeq 1$	Dark matter halo	$\frac{1}{\lvert E(\omega) \rvert} \gg 1$	Galactic discs
Non-degenerate		**Degenerate**	
No global resonance	Discs	$\forall \mathbf{J},\, \mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J}) = 0$	Keplerian systems

Balescu-Lenard: A numerical nightmare

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = -\frac{\partial}{\partial \mathbf{J}} \cdot \mathbf{F}(\mathbf{J},t)$$

Balescu-Lenard equation

$$\mathbf{F}(\mathbf{J},t) = \sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \int \mathrm{d}\mathbf{J}' \, \delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}')) \, |\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}|^2 \times \left(\mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'} - \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}\right) F(\mathbf{J})\, F(\mathbf{J}')$$

Diffusion flux

$$\psi^{\mathrm{d}}(\mathbf{J},\mathbf{J}',\omega) = -\sum_{p,q} \psi^{(p)}_{\mathbf{k}}(\mathbf{J})\, \mathbf{E}^{-1}_{pq}(\omega)\, \psi^{(q)*}_{\mathbf{k}'}(\mathbf{J}')$$

Dressed susceptibility coefficients

$$\mathbf{E}_{pq}(\omega) = \delta_{pq} - \mathbf{M}_{pq}(\omega)$$

Dielectric matrix

$$\mathbf{M}_{pq}(\omega) = \sum_{\mathbf{k}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{k}\cdot\partial F/\partial \mathbf{J}}{\omega - \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})} \, \psi^{(p)*}_{\mathbf{k}}(\mathbf{J})\, \psi^{(q)}_{\mathbf{k}}(\mathbf{J})$$

Response matrix

$$\psi^{(p)}_{\mathbf{k}}(\mathbf{J}) = \int \frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^d} \, \psi^{(p)}\big(\mathbf{x}[\boldsymbol{\theta},\mathbf{J}]\big)\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}$$

Basis elements

A numerical nightmare

With also:

+ Integral over $\mathrm{d}\boldsymbol{\theta}$
+ (Double) integral over $\mathrm{d}\mathbf{J}$
+ (Triple) sum over $\mathbf{k}$
+ (Double) sum over (p, q)
+ Matrix inversion
+ Resonant denominator
+ Resonance condition

$$\mathbf{F}(\mathbf{J}, t) = \sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \int \mathrm{d}\mathbf{J}' \, \delta_{\mathrm{D}}(\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\boldsymbol{\Omega}(\mathbf{J}')) \, |\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}|^2 \times \left(\mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'} - \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}\right) F(\mathbf{J})\, F(\mathbf{J}')$$

Diffusion flux

$$\psi^{\mathrm{d}}(\mathbf{J}, \mathbf{J}', \omega) = -\sum_{p,q} \psi^{(p)}_{\mathbf{k}}(\mathbf{J}) \, \mathbf{E}^{-1}_{pq}(\omega) \, \psi^{(q)*}_{\mathbf{k}'}(\mathbf{J}')$$

Dressed susceptibility coefficients

$$\mathbf{E}_{pq}(\omega) = \delta_{pq} - \mathbf{M}_{pq}(\omega)$$

Dielectric matrix

$$\mathbf{M}_{pq}(\omega) = \sum_{\mathbf{k}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{k}\cdot\partial F/\partial \mathbf{J}}{\omega - \mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J})} \, \psi^{(p)*}_{\mathbf{k}}(\mathbf{J}) \, \psi^{(q)}_{\mathbf{k}}(\mathbf{J})$$

Response matrix

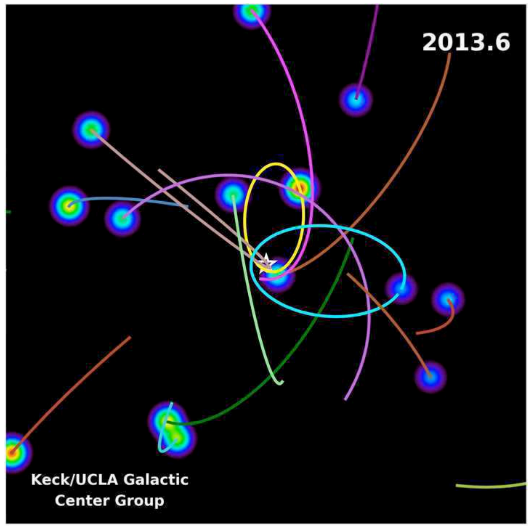
$$\psi^{(p)}_{\mathbf{k}}(\mathbf{J}) = \int \frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^d} \, \psi^{(p)}\big(\mathbf{x}[\boldsymbol{\theta}, \mathbf{J}]\big) \, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}$$

Basis elements

Does it work?

Galactic discs

Galactic nuclei



Does it work?

Galactic discs

$$\frac{1}{|E(\omega)|} \gg 1$$

Dynamically cold system

Galactic nuclei

$$U(\mathbf{w},\mathbf{w}') \mapsto \overline{U} = \int \frac{\mathrm{d}\theta}{2\pi}\frac{\mathrm{d}\theta'}{2\pi}\, U$$

Orbit-averaged interactions

Does it work?

Galactic discs

$$\frac{1}{|E(\omega)|} \gg 1$$

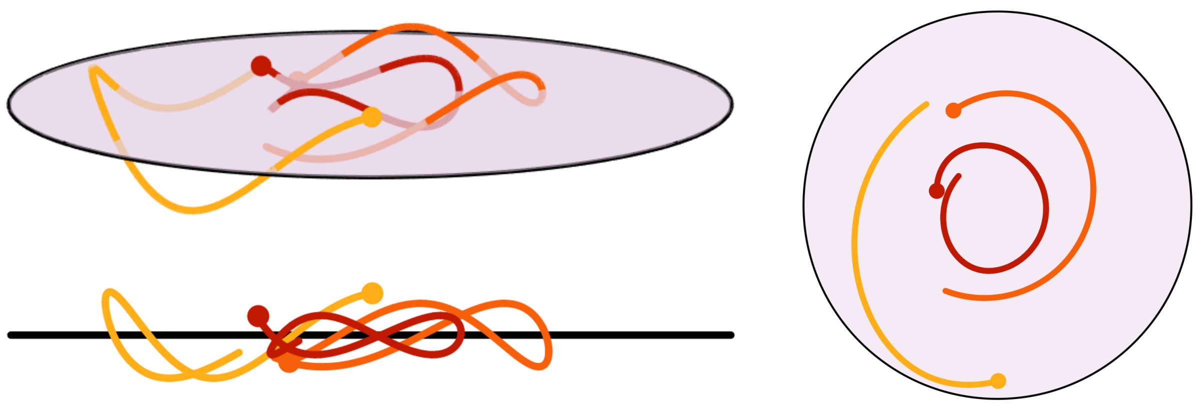
Dynamically cold system

Galactic nuclei

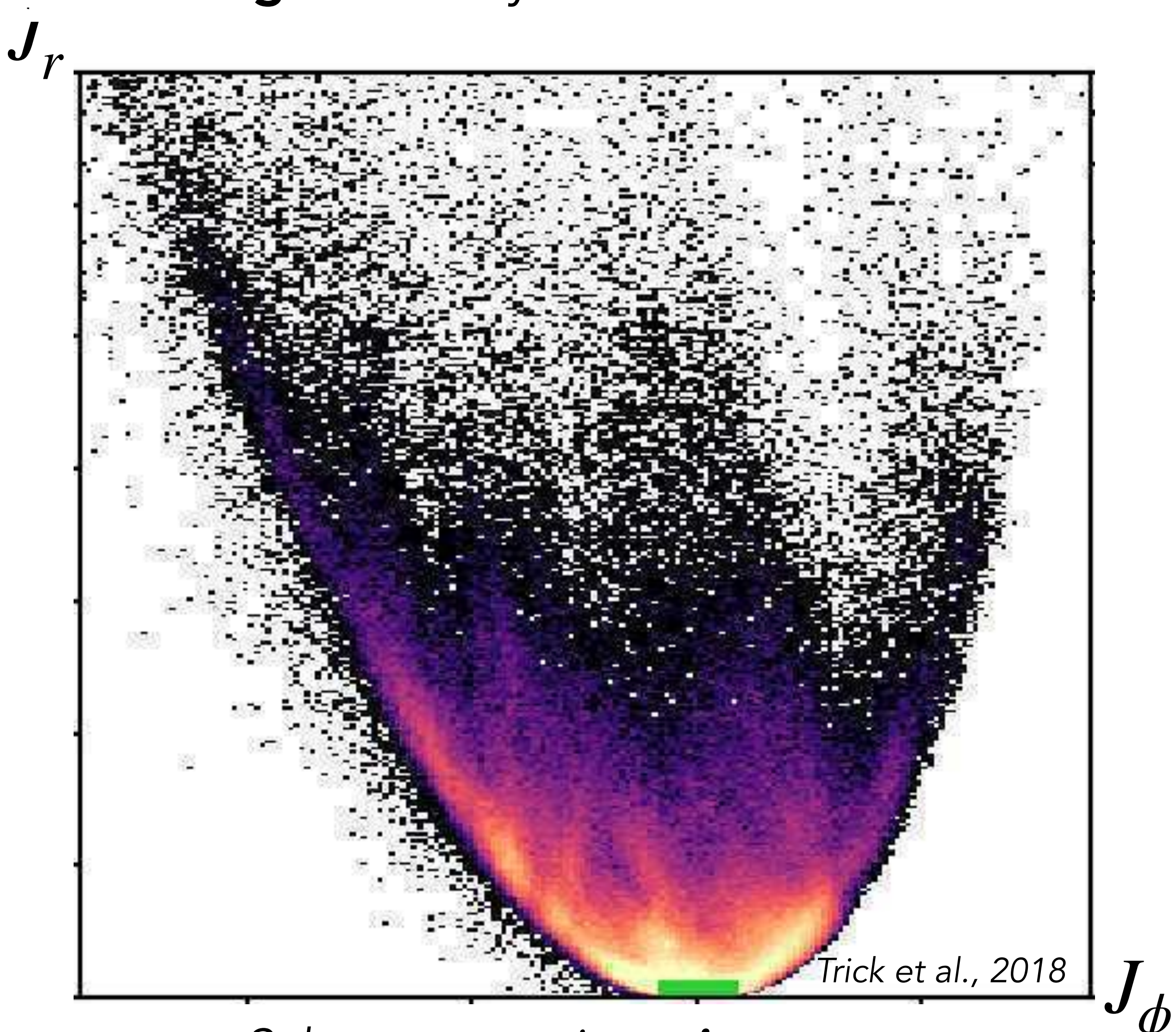
$$U(\mathbf{w},\mathbf{w}') \mapsto \overline{U} = \int \frac{\mathrm{d}\theta}{2\pi}\frac{\mathrm{d}\theta'}{2\pi}\, U$$

Orbit-averaged interactions

Galactic discs



Inhomogeneous system and intricate orbits

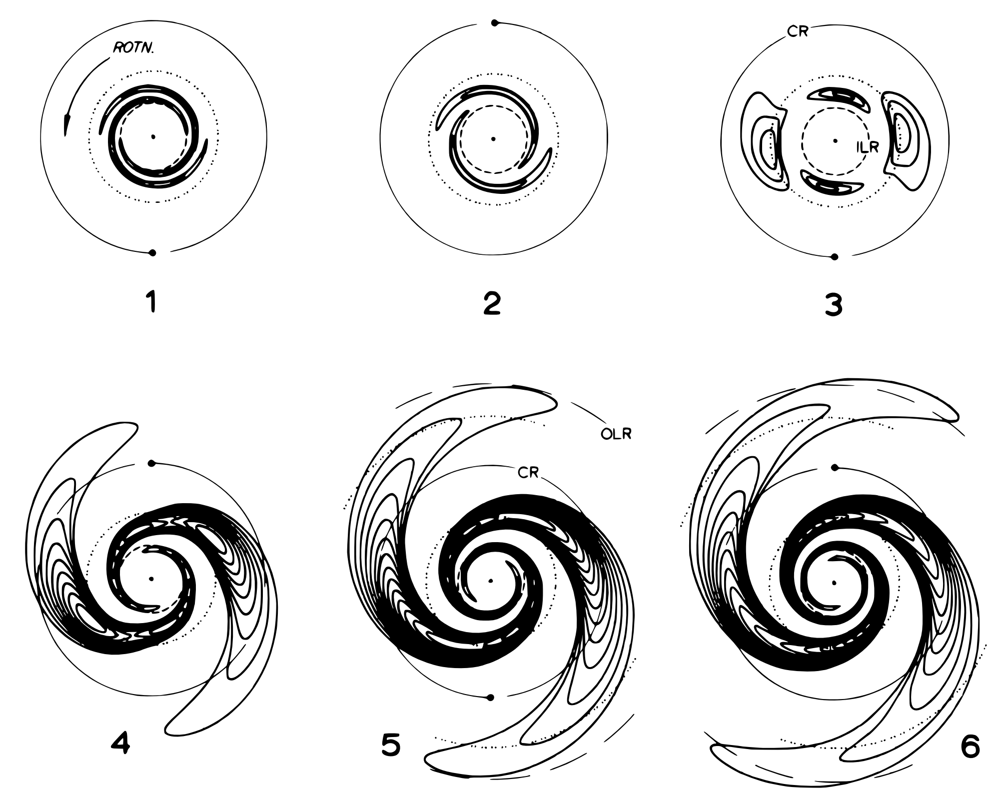


*Sub-structures in **action space**,*
as observed by GAIA

How do stars diffuse in **galactic discs**?

- + **Galactic archeology**
- + Formation of **spiral arms/bars**
- + Local **velocity anisotropies**
- + Disc **thickening**
- + Stellar **streams**

Swing amplification in **cold** discs



Toomre, 1981

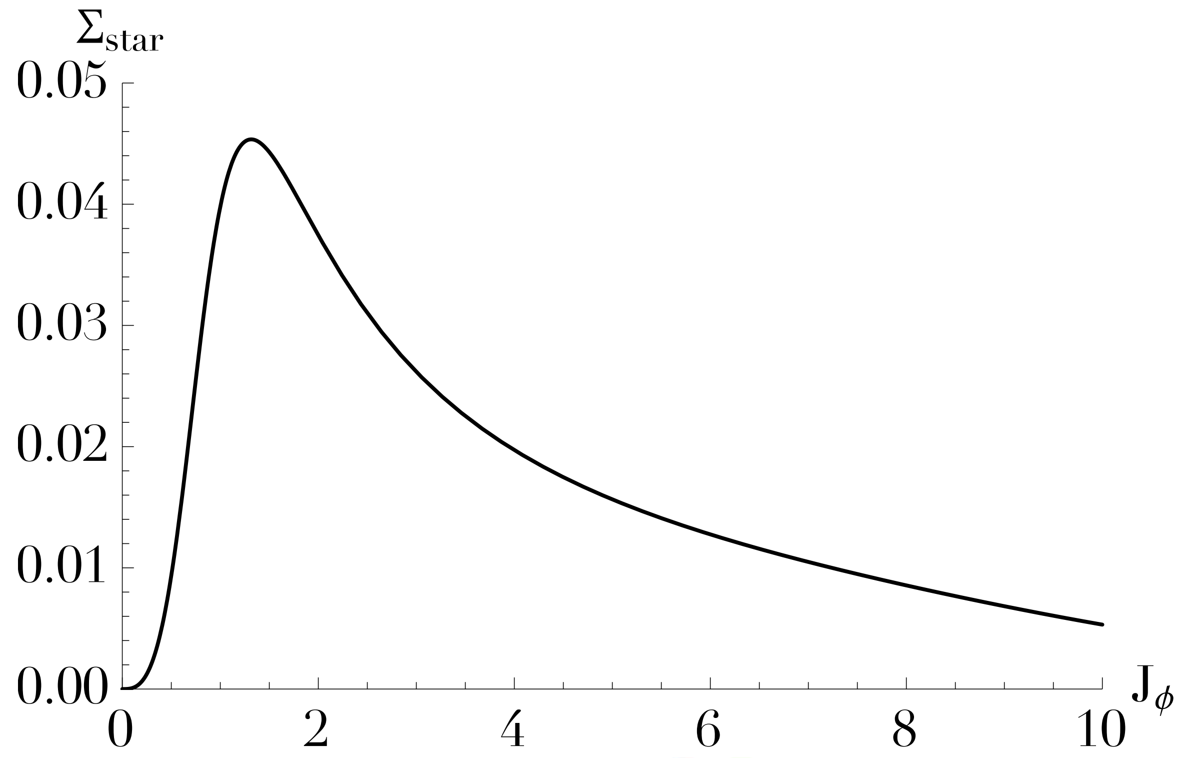
$$\frac{1}{|E(\omega)|} \simeq 30$$

Collective effects essential

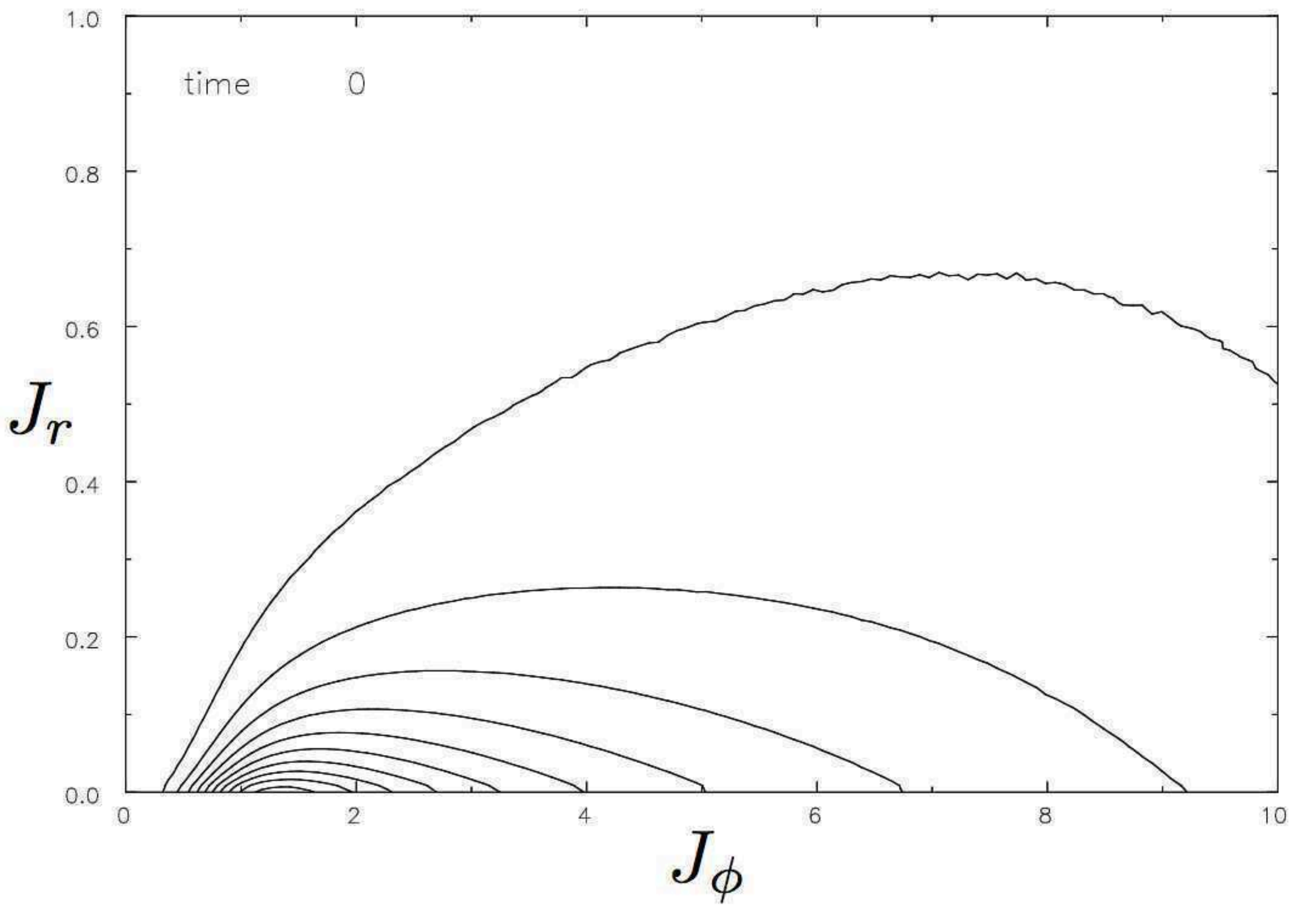
An example of secular evolution

Sellwood 2012's numerical experiment

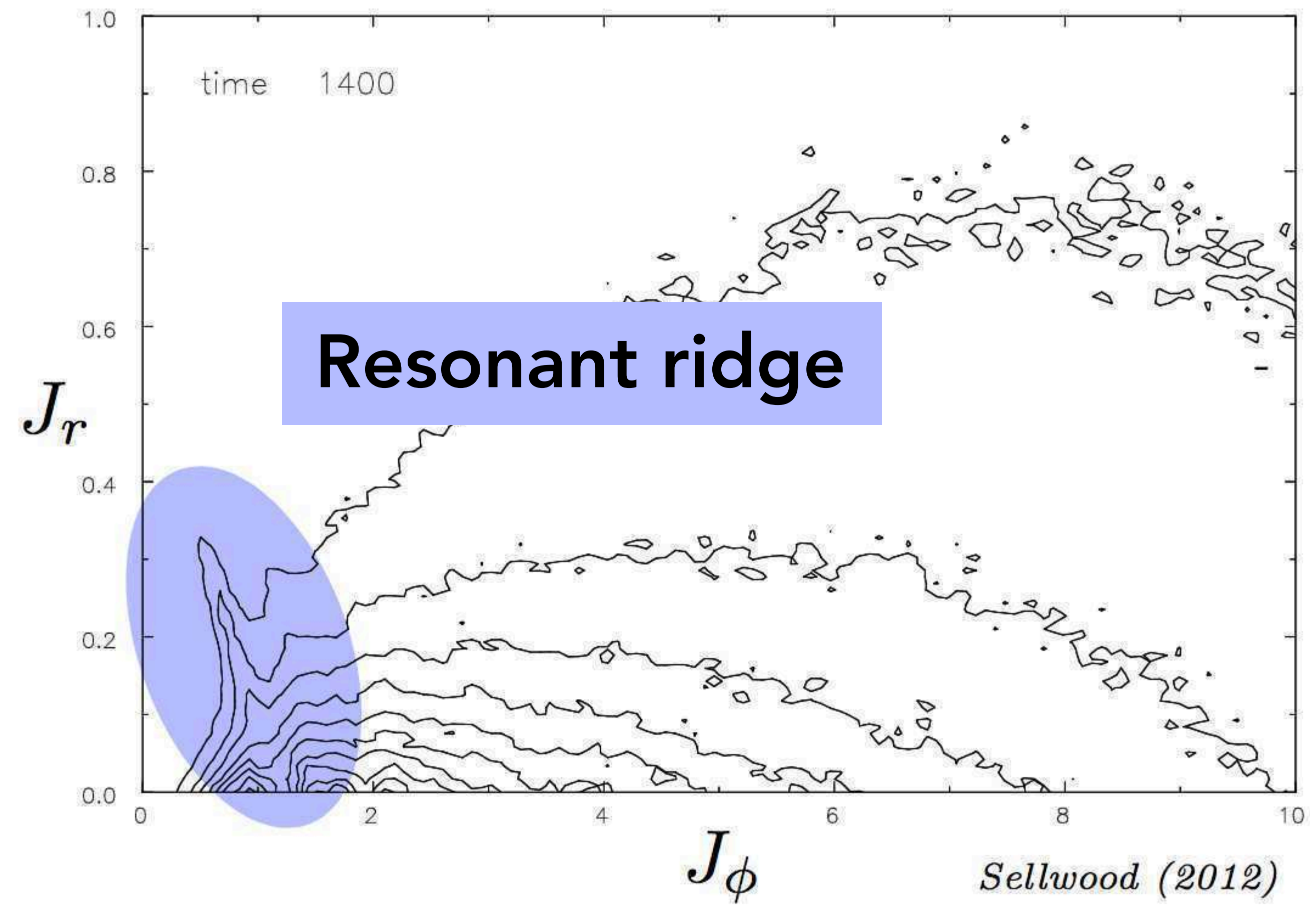
+ Stationary **stable** Mestel disc
+ Sampled with **500M particles**
+ Unavoidable **transient waves**



Initial stable/stationary DF



Evolved DF



Sellwood (2012)

In configuration space

Some remarks

+ Nothing spectacular happens

Linearly stable

+ Cannot track **disc's heating**

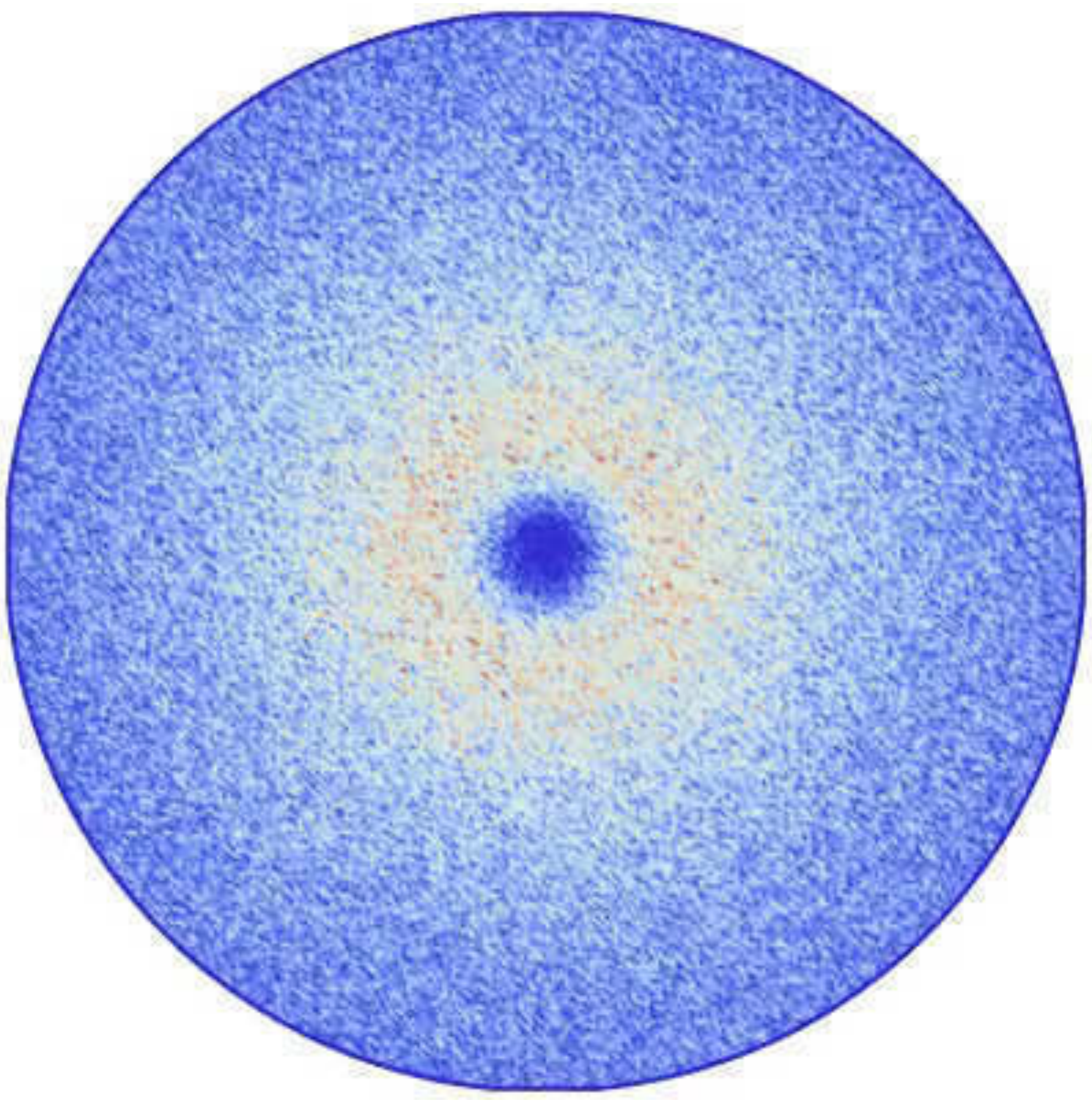
Inhomogeneous

+ **Large** transients

Self-gravitating

+ Fluctuations are **absorbed**

Resonant



In action space

Some remarks

+ **Heating** in orbital space

Inhomogeneous

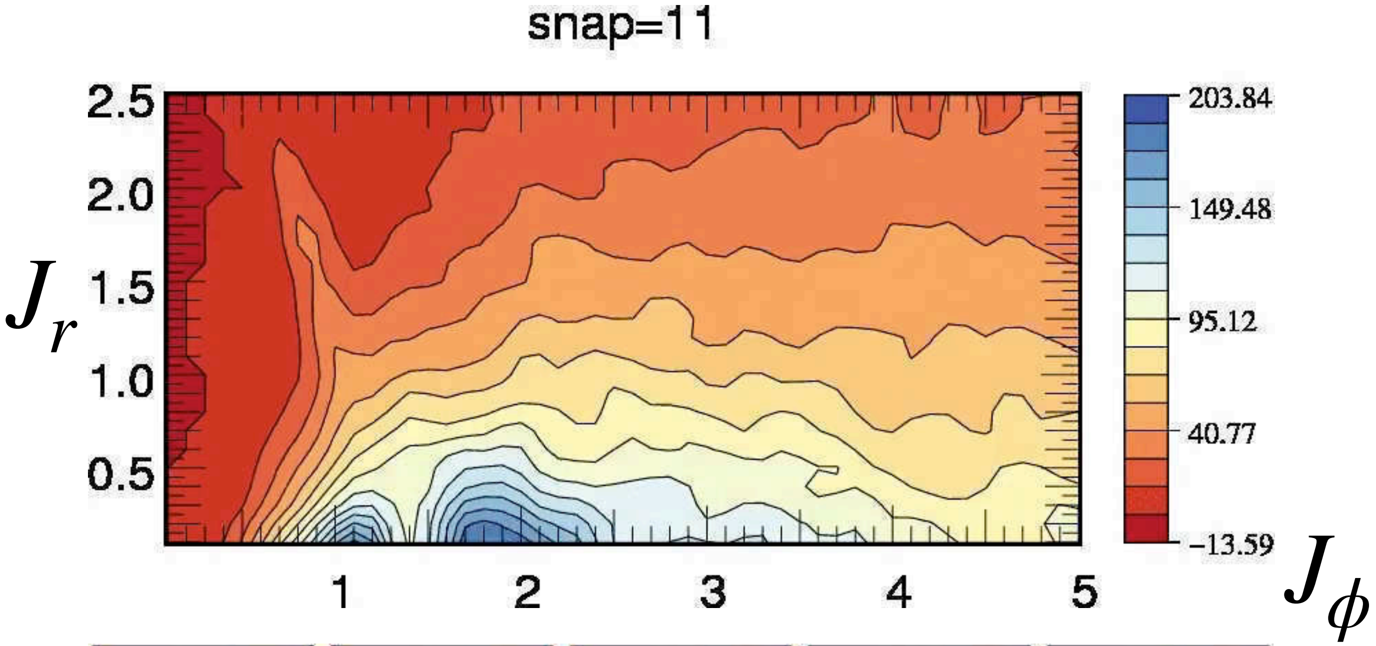
+ **Fast** heating

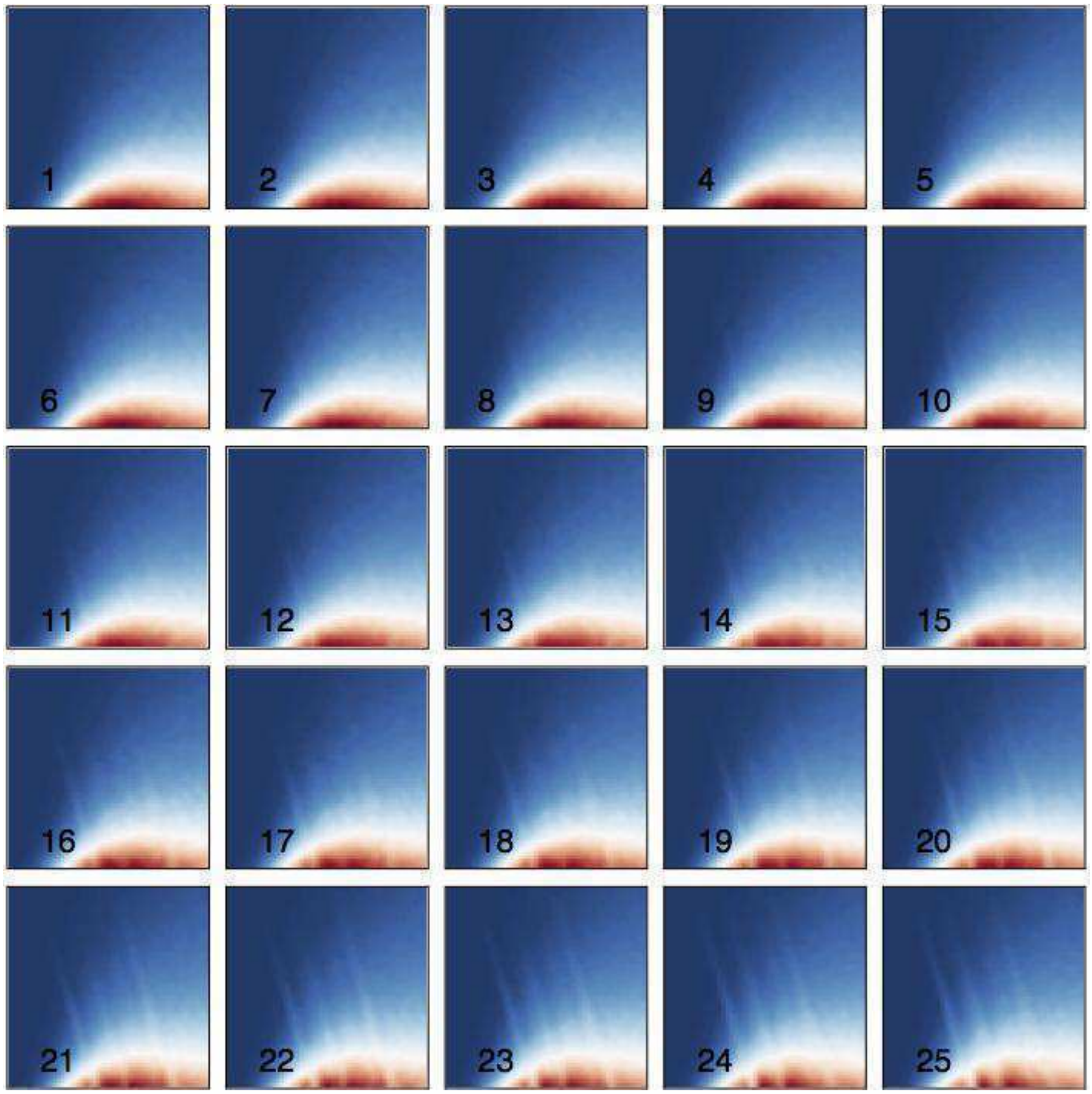
$$T_{\text{ridge}} \simeq |E|^2 N\, T_{\text{dyn}}$$

Self-gravitating

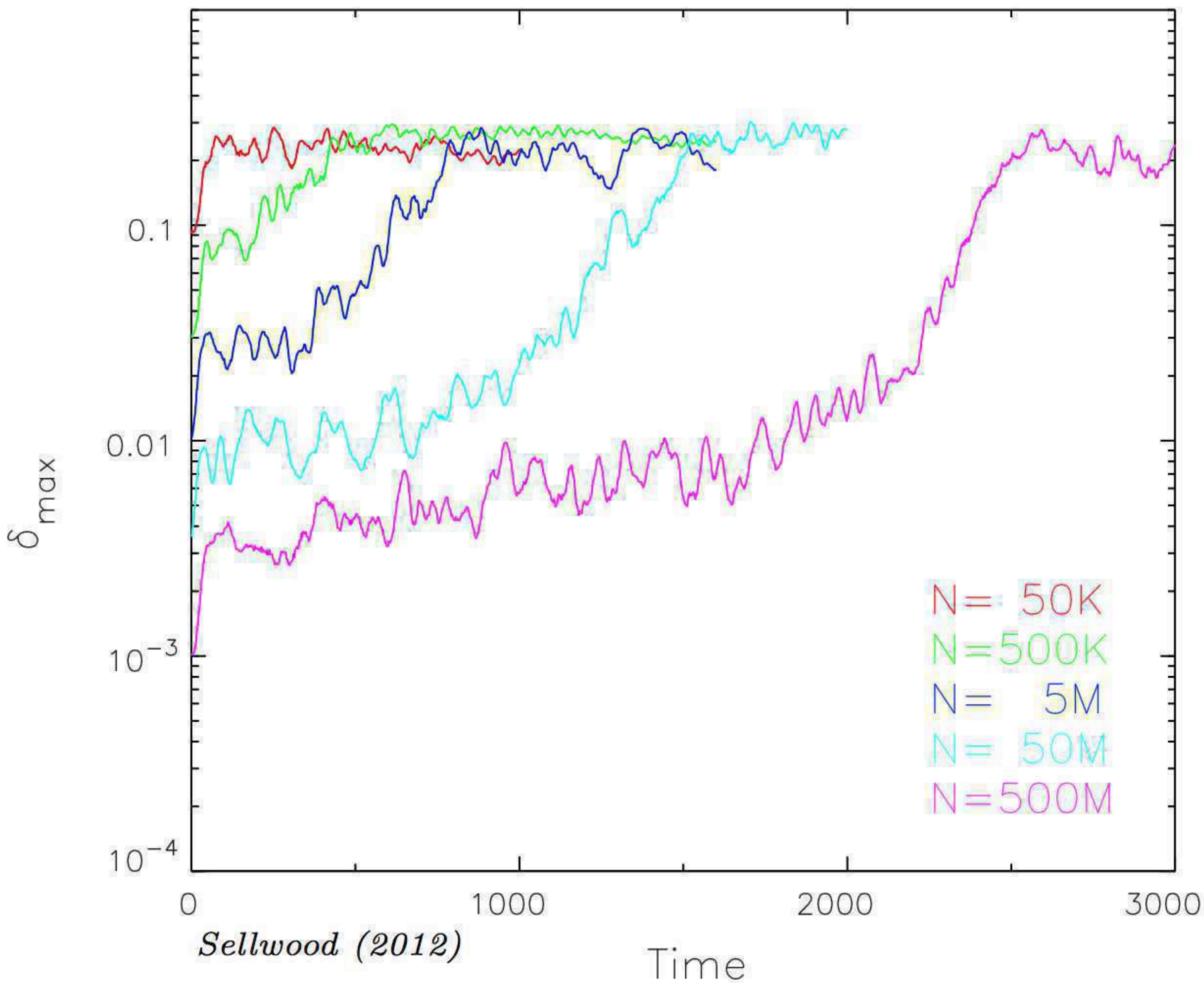
+ **Localised heating**

Resonant





A dynamics sourced by finite-N effects



The larger the number of particles, the slower the effect

Needed ingredients

Needed ingredients for that dynamics

+ Disc is frozen on **dynamical times**

  Linearly stable

+ Disc is **isolated**

  No external perturbations

+ Disc has **internal** fluctuations

  Internal Poisson noise

+ Fluctuations are **large**

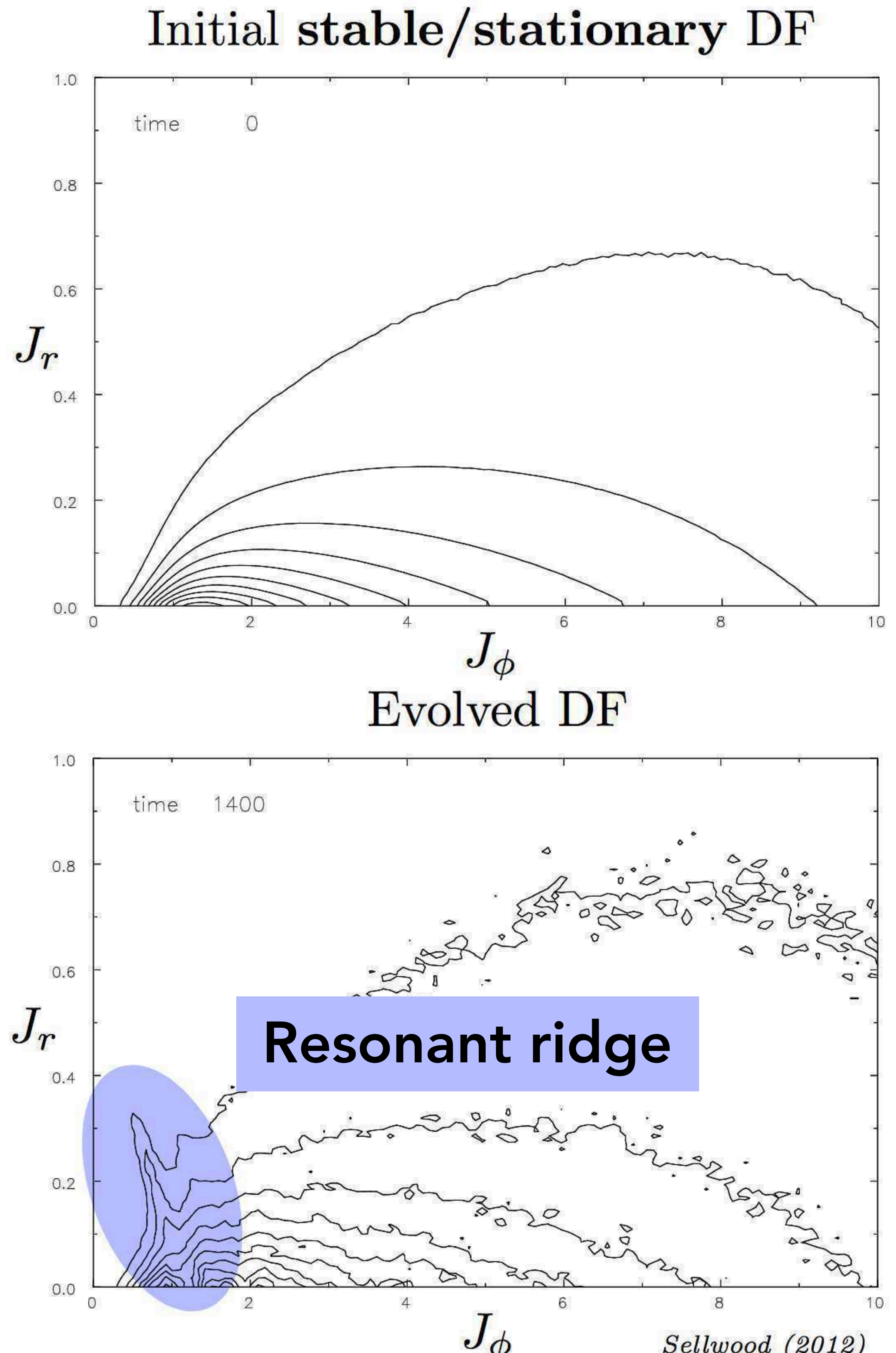
  Self-gravitating

+ Heating happens in **orbital space**

  Inhomogeneous

+ Heating is **localised**

  Resonant



Balescu-Lenard equation

The master equation for **self-induced orbital relaxation**

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = \frac{1}{N}\frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \int \mathrm{d}\mathbf{J}' \frac{\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}'))}{|E_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}))|^2} \times \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} - \mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'}\right) F(\mathbf{J},t)\,F(\mathbf{J}',t)\right]$$

Some properties

$F(\mathbf{J},t)$ Orbital distorsion in **action space**

$1/N$ Sourced by **finite-N effects**

$\partial/\partial\mathbf{J}\,\cdot$ Divergence of a **diffusion flux**

$(\mathbf{k},\mathbf{k}')$ Discrete **resonances**

$\int \mathrm{d}\mathbf{J}'$ Scan of **orbital space**

$\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})-\mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}'))$ **Resonance** cond.

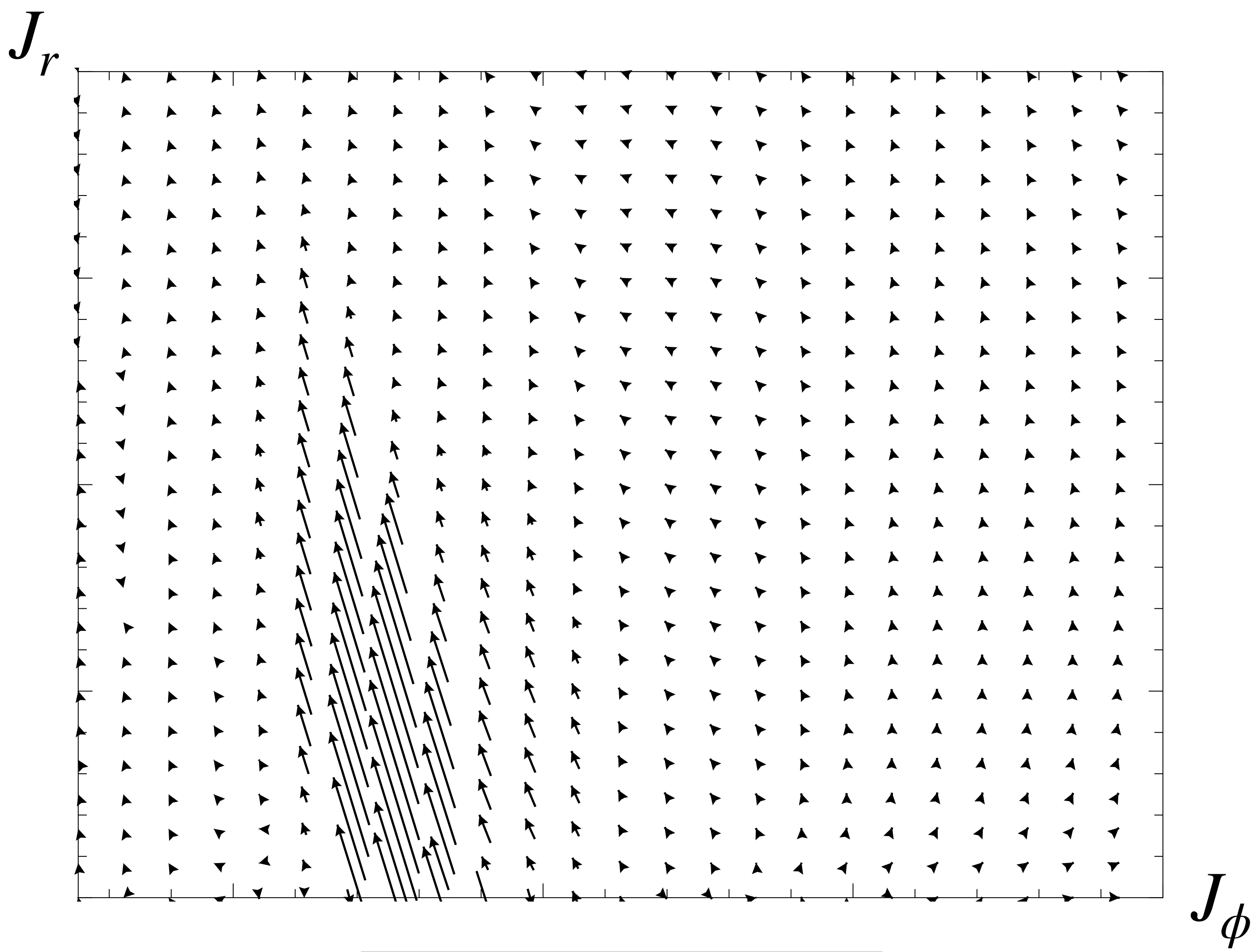
$1/|E_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega)|^2$ **Dressed** couplings

Prediction for the diffusion

Diffusion flux in action space

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = -\frac{\partial}{\partial \mathbf{J}} \cdot \mathbf{F}(\mathbf{J},t)$$

Spontaneous formation of **anisotropic** sub-structures in action space



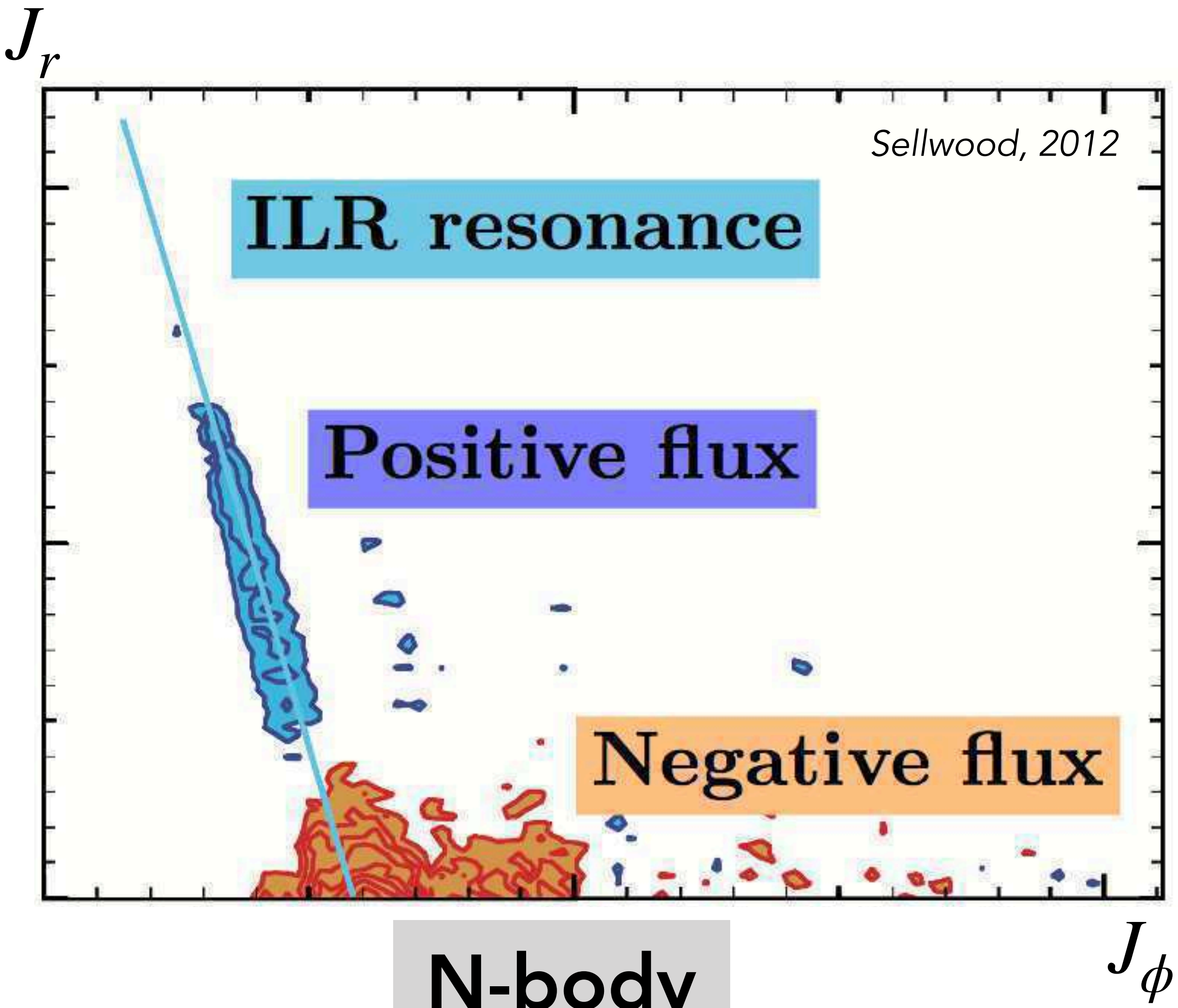
Balescu-Lenard

Prediction for the diffusion

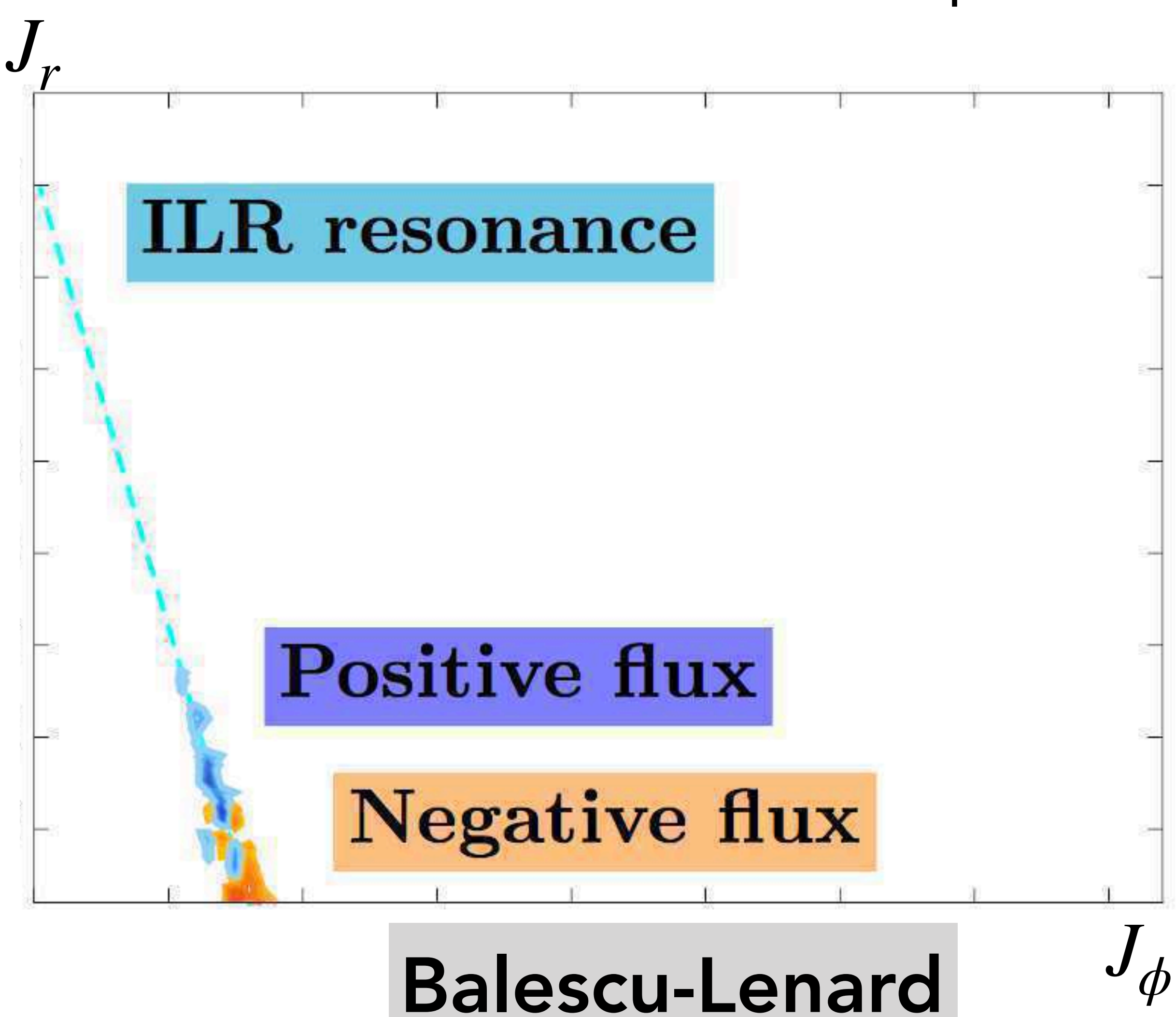
Diffusion flux in action space

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = -\frac{\partial}{\partial \mathbf{J}} \cdot \mathbf{F}(\mathbf{J},t)$$

Spontaneous formation of **anisotropic** sub-structures in action space



N-body



Balescu-Lenard

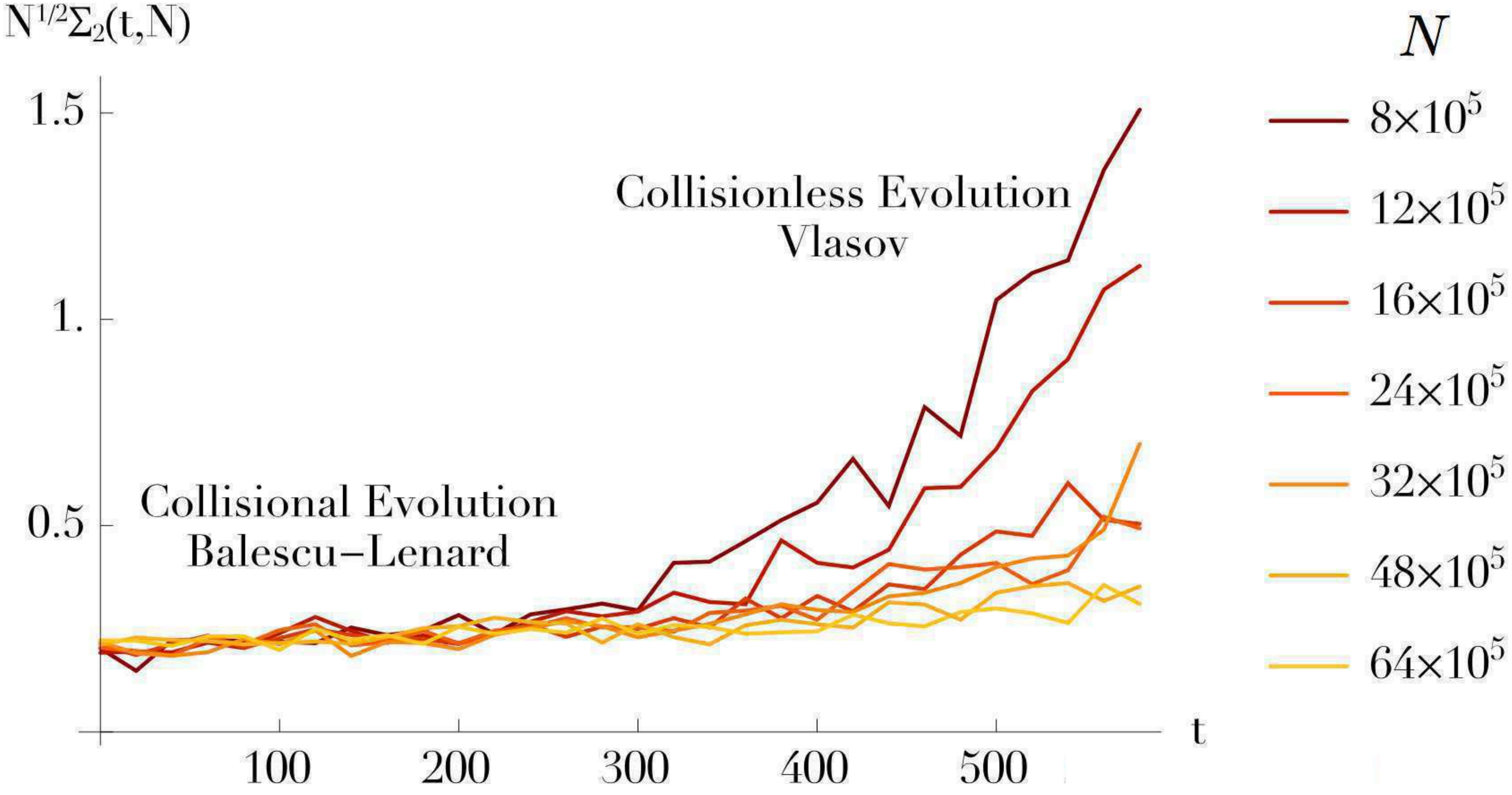
Kinetic theory can predict **localised anisotropic heatings**

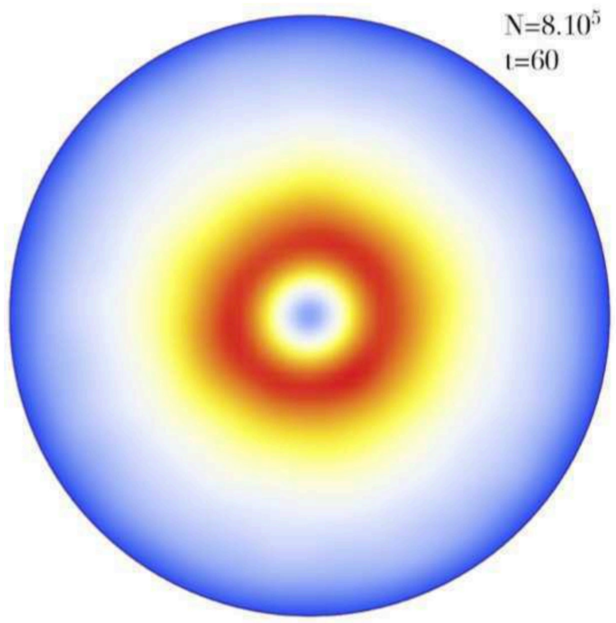
What happens after the ridge formation?

When the ridge gets large enough

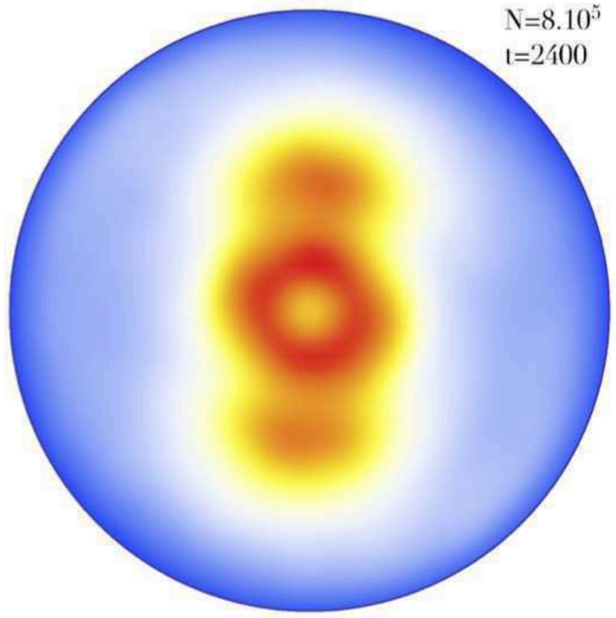
$$\frac{1}{|E(\omega)|} = +\infty$$

Linear instability **Balescu-Lenard ⟶ Vlasov**





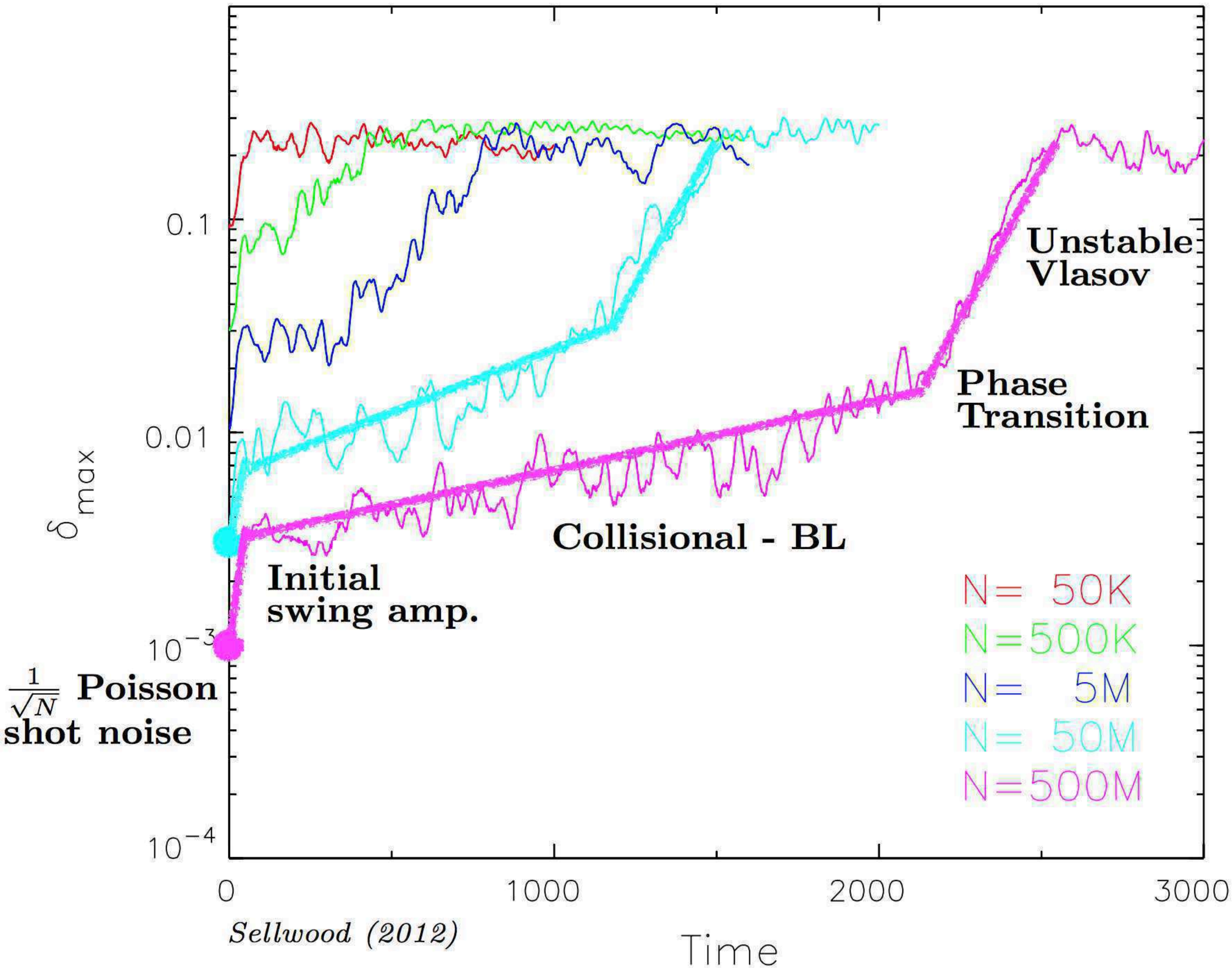
Initial times



Late times

2-body **resonant** relaxation creates **small-scale structures** in the DF

The fate of secular evolution



Sellwood (2012)

Does it work?

Galactic discs

$$\frac{1}{|\varepsilon(\omega)|} \gg 1$$

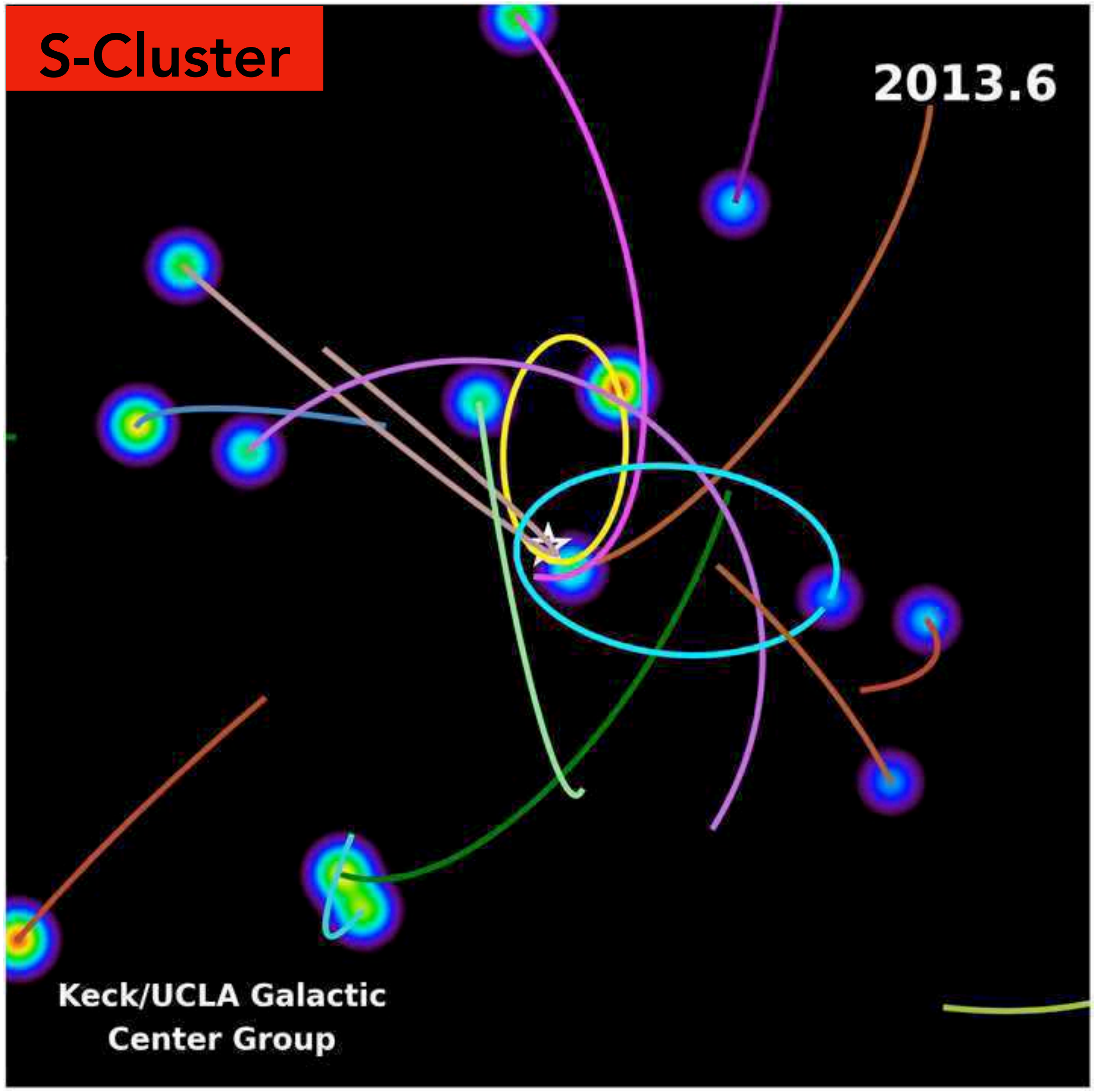
Dynamically cold system

Galactic nuclei

$$U(\mathbf{w},\mathbf{w}') \mapsto \overline{U} = \int \frac{\mathrm{d}\theta}{2\pi}\frac{\mathrm{d}\theta'}{2\pi}\, U$$

Orbit-averaged interactions

Galactic centres



*S-Cluster of **SgrA****
***Densest** stellar system of the galaxy*
*Dynamics dominated by the **central black hole***

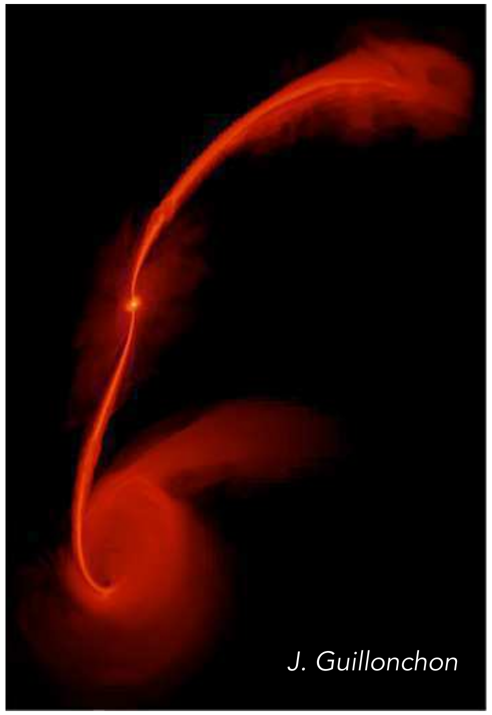
What is the diet of a **supermassive black hole**?

Stellar diffusion in galactic centres

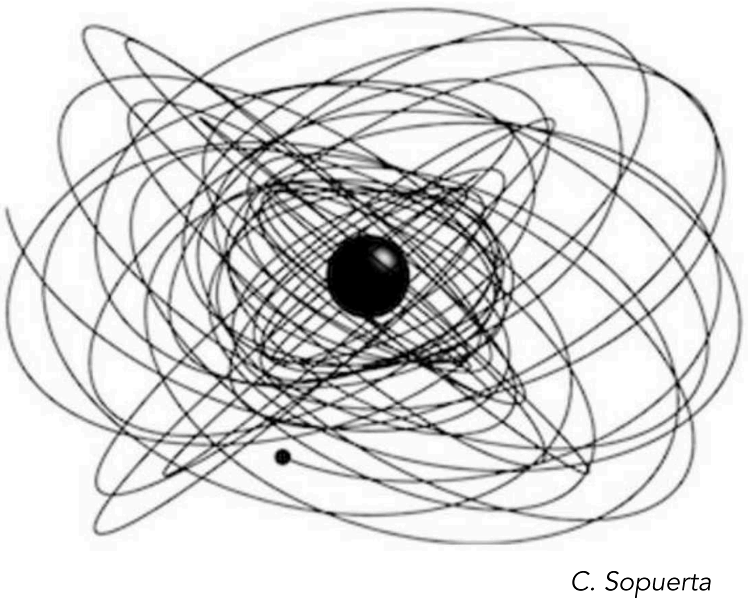
- *+ **Origin and structure** of SgrA**
- *+ Relaxation in **eccentricity**, **orientation***

Sources of **gravitational waves**

- *+ BHs-binary mergers*
- *+ TDE, EMRIs*



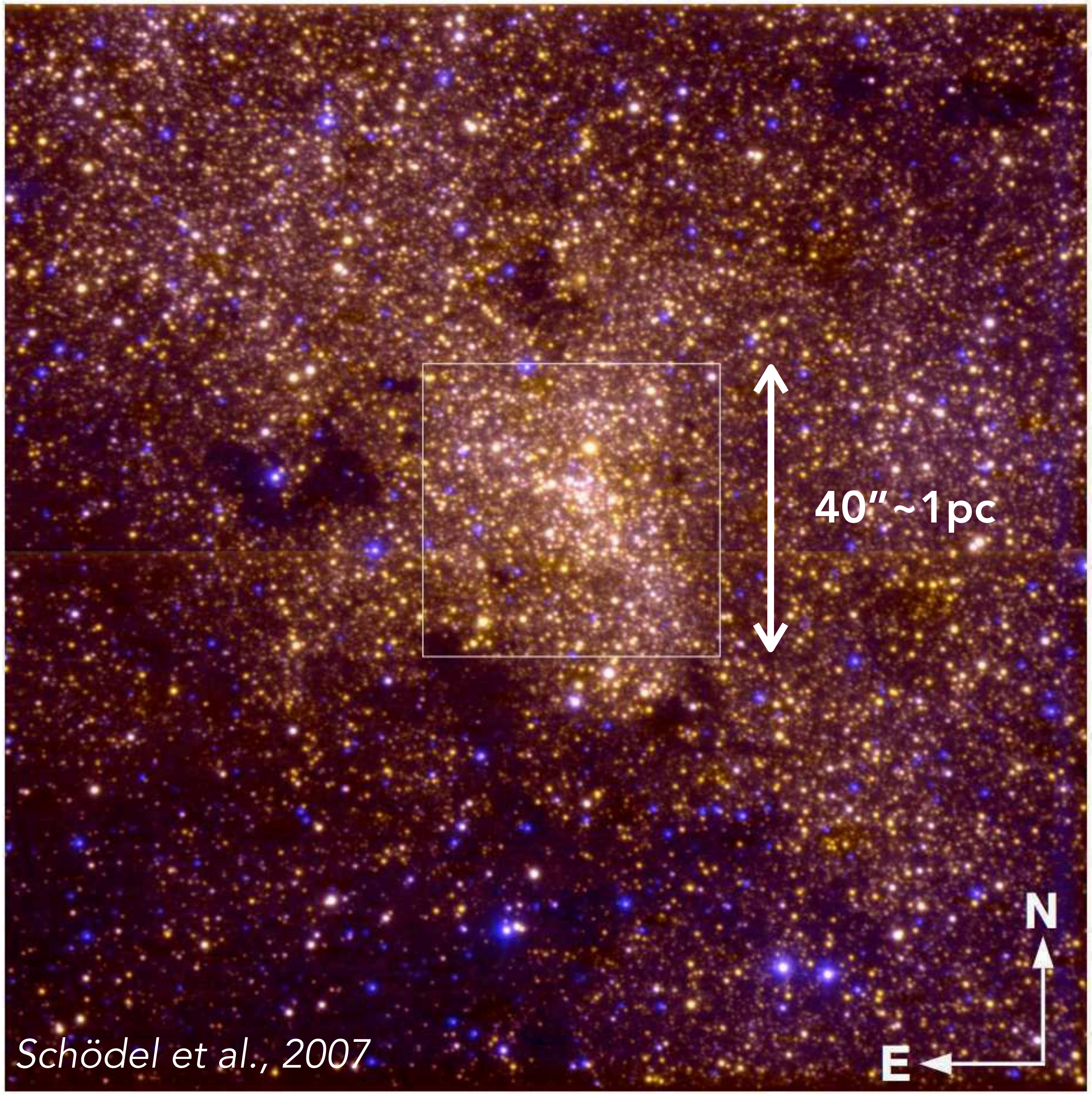
Tidal Disruption Event

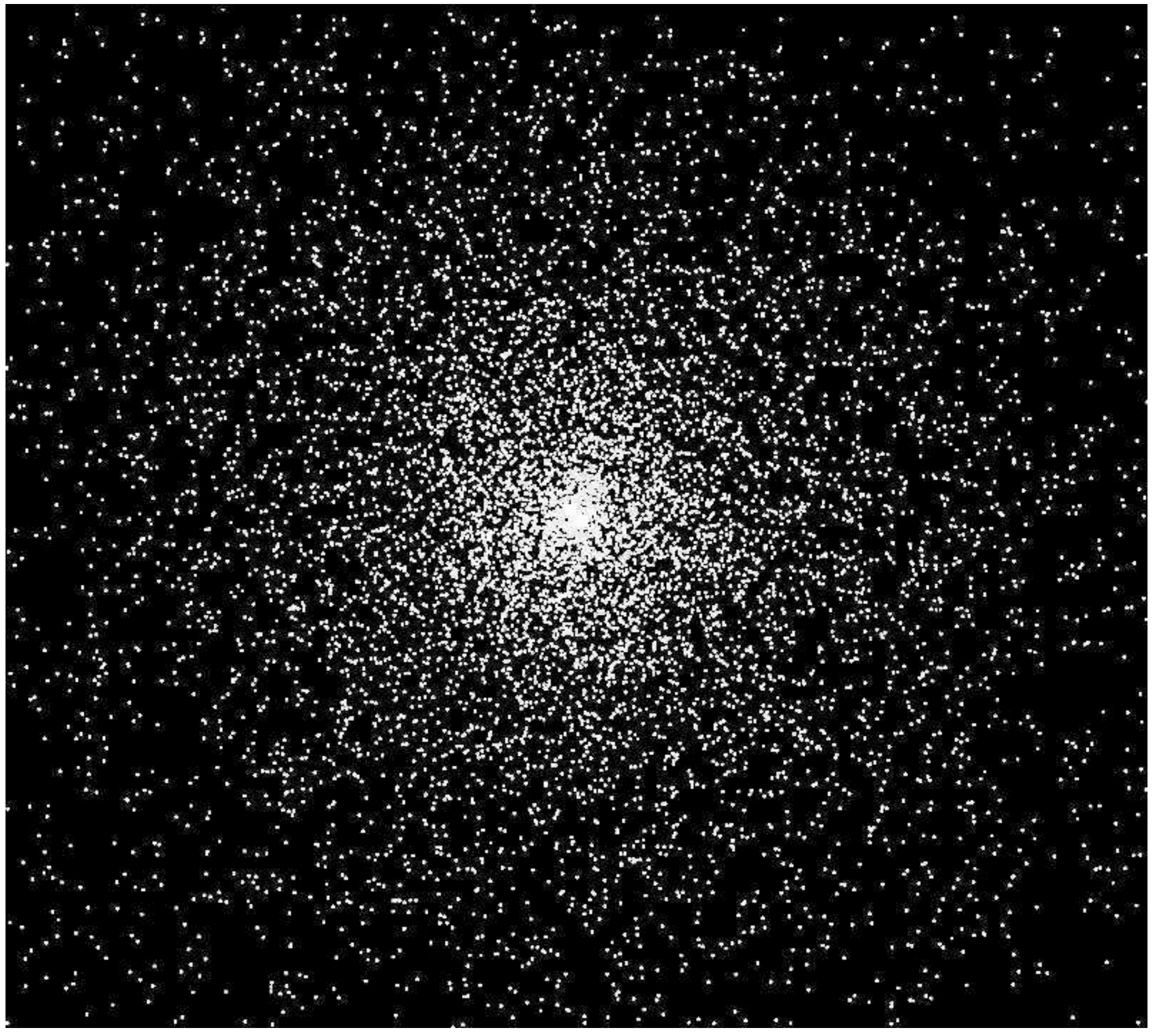


Extreme Mass Ratio Inspiral

What is the long-term dynamics of stars in these very dense systems?

Galactic centres are extremely dense





VLT observations

N-body simulations *(B. Bar-Or)*

Perfect ``lab" to investigate the **statistical physics** of a stellar system

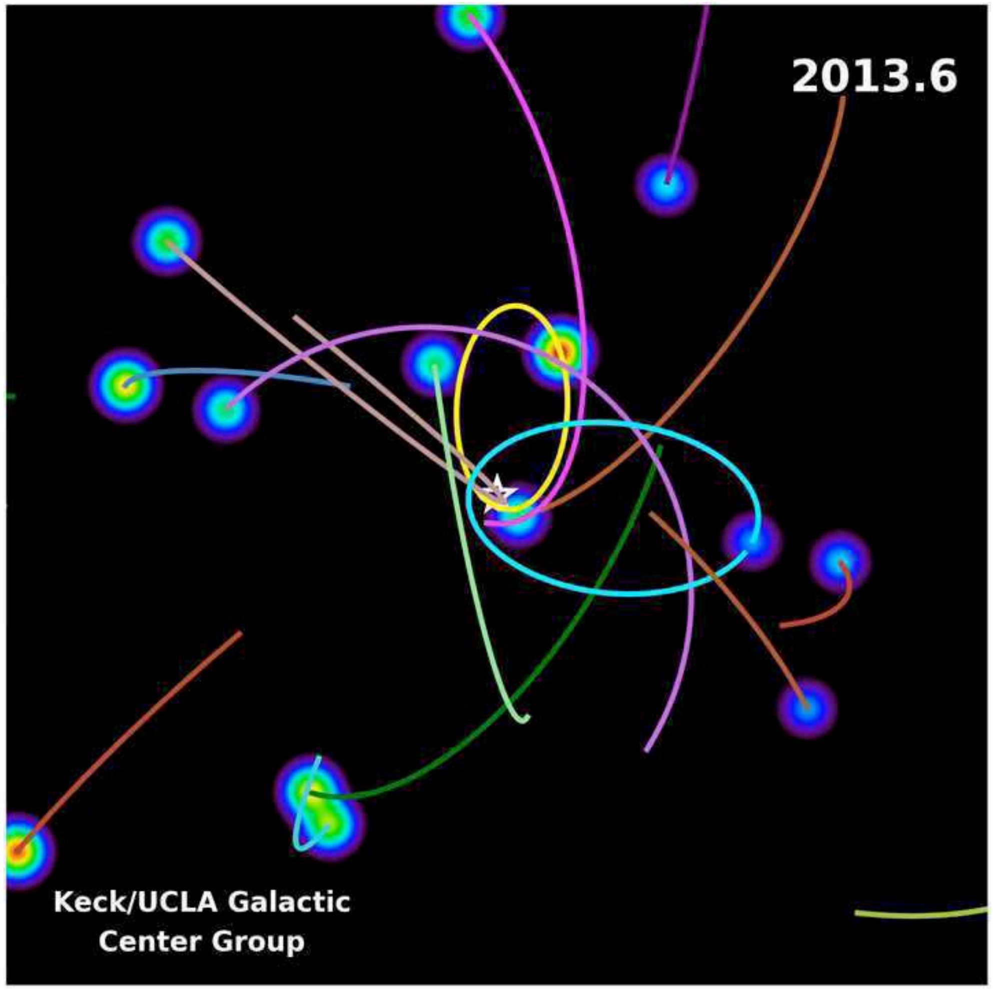
Galactic centers are degenerate

Potential dominated by the SMBH:

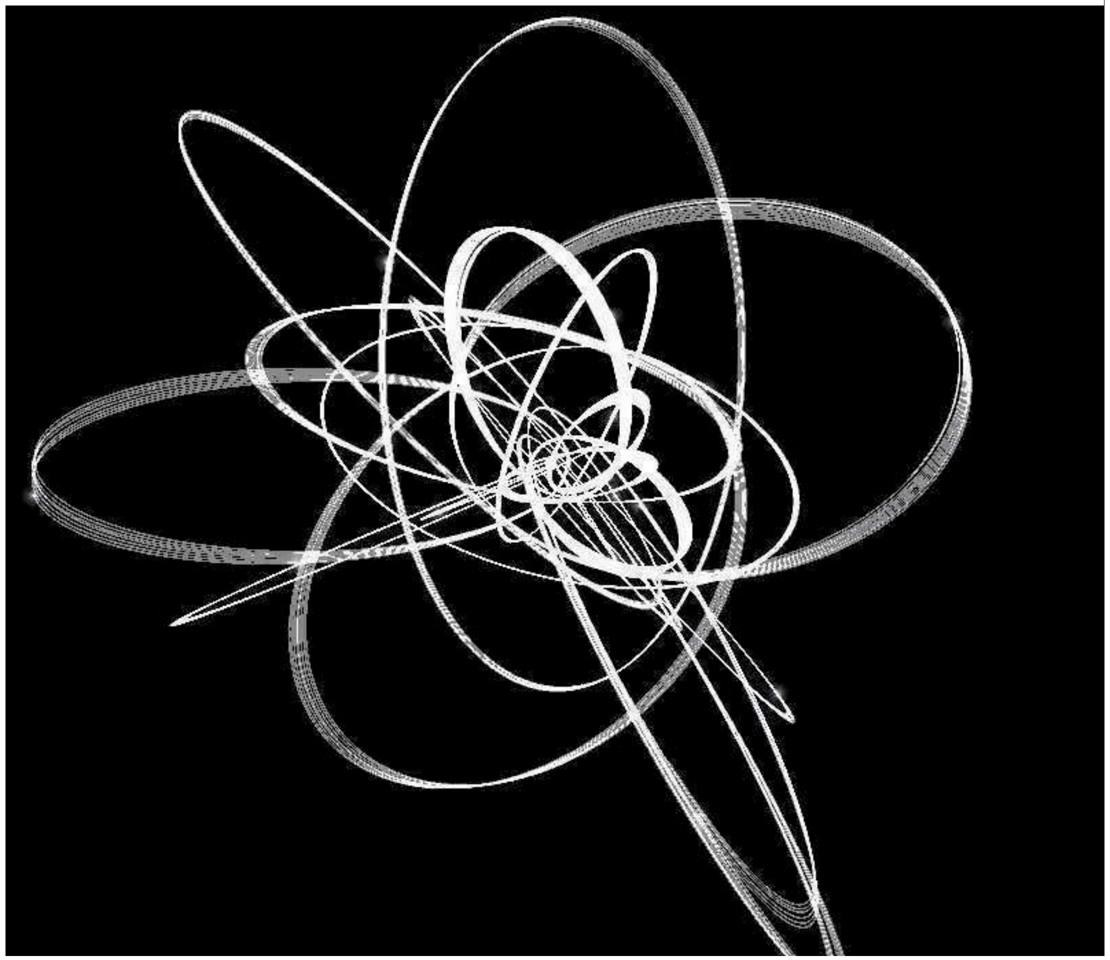
+ Keplerian orbits are **closed**

$$\varepsilon = M_\star / M_\bullet \ll 1$$

Dynamical degeneracy: $\forall \mathbf{J}\,,\ \mathbf{n}\cdot\boldsymbol{\Omega}_{\mathrm{Kep}}(\mathbf{J}) = 0$



KECK observations

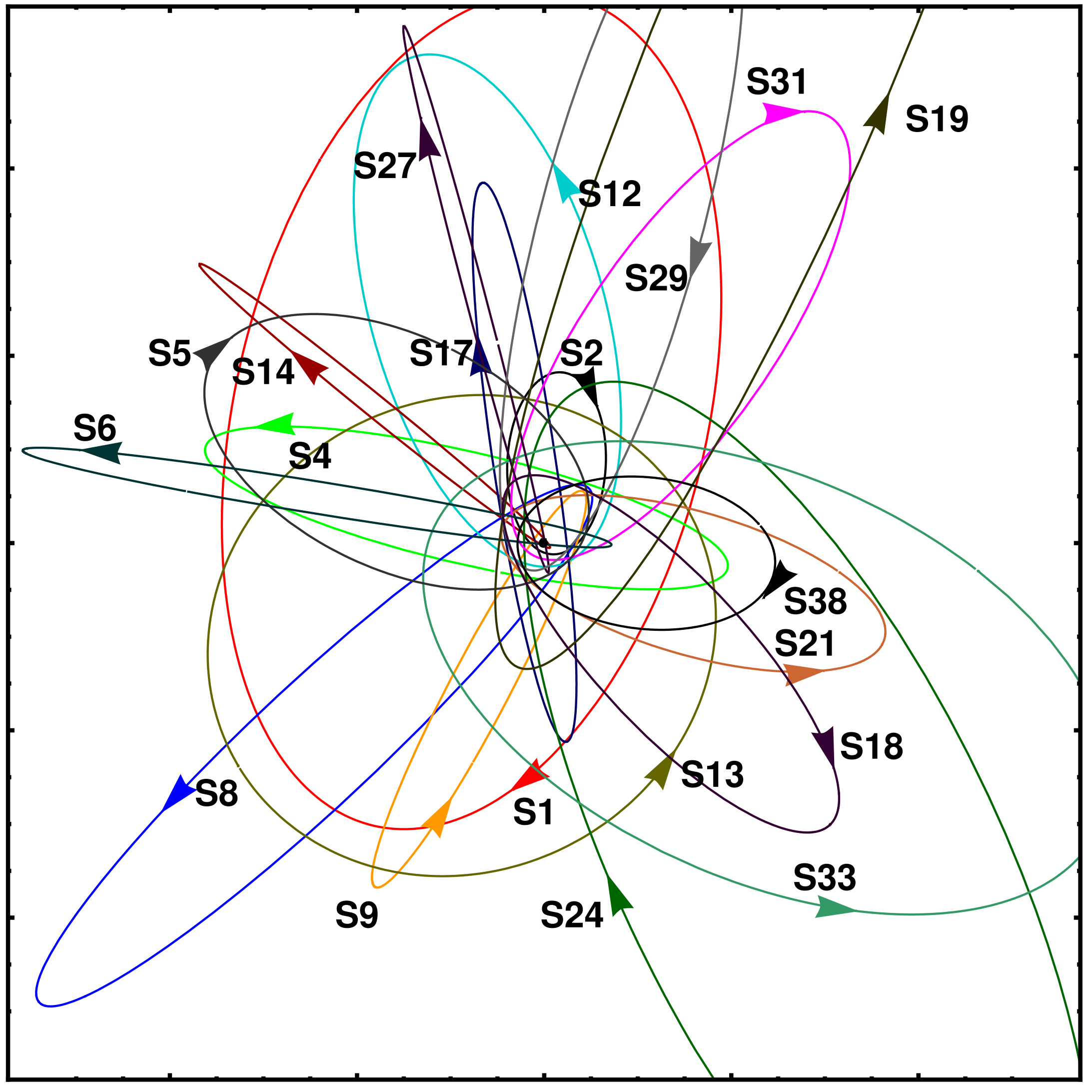


N-body simulations *(B. Bar-Or)*

Orbit-average: stars are replaced by Keplerian wires

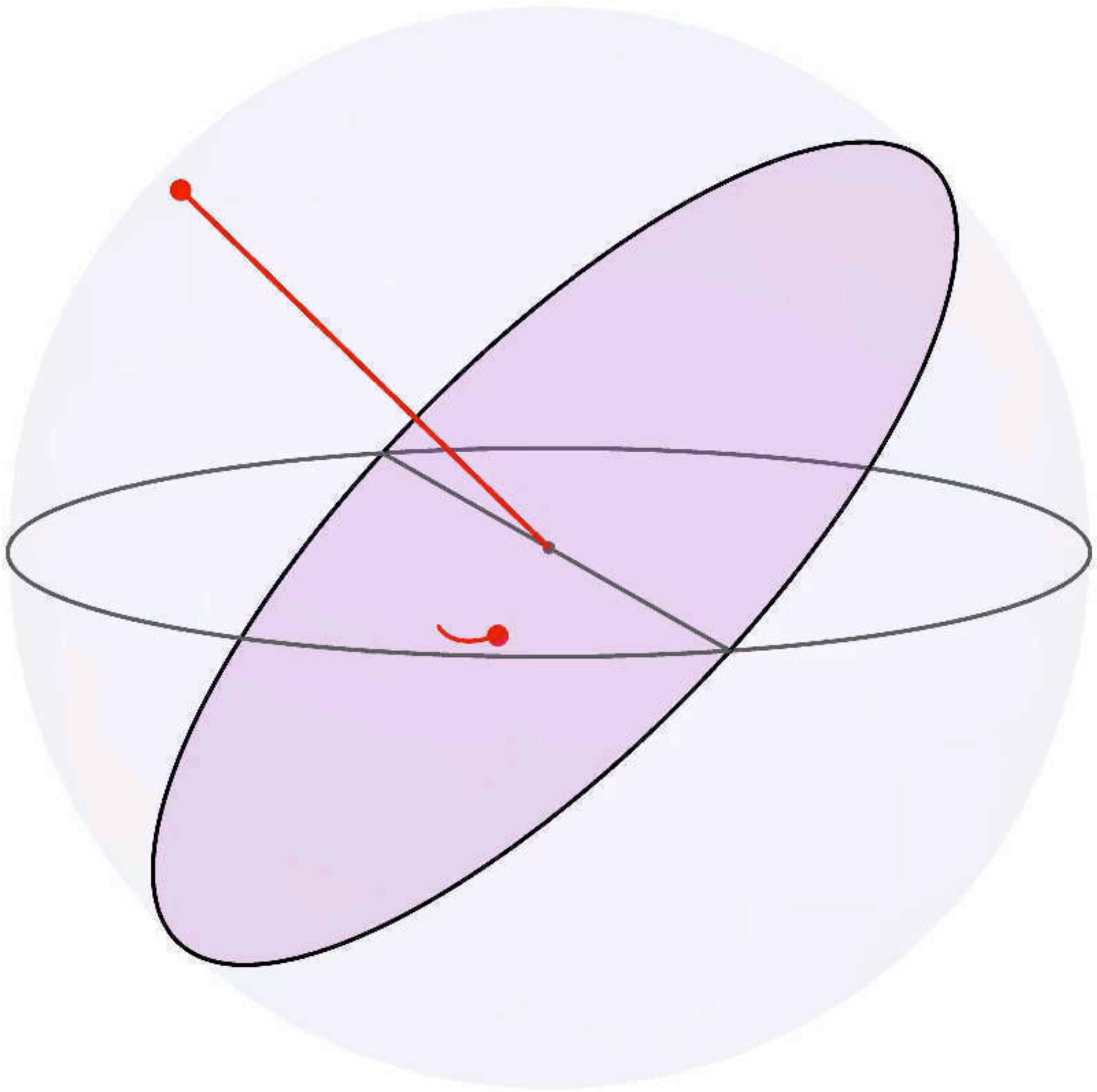
Keplerian orbits

The BH dominates the dynamics



Gillessen et al., 2009

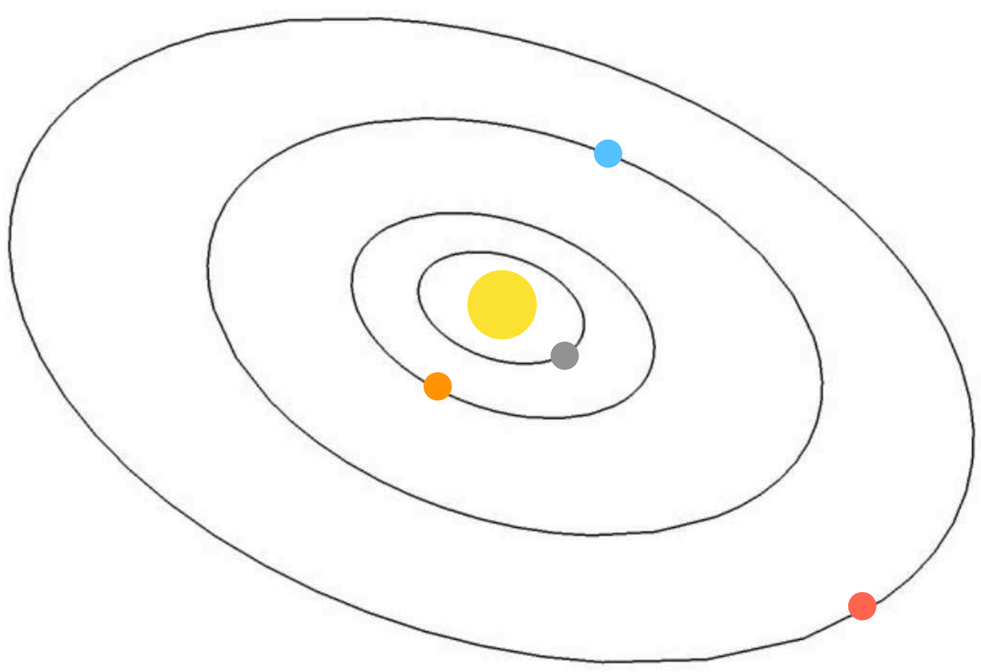
VLT observations

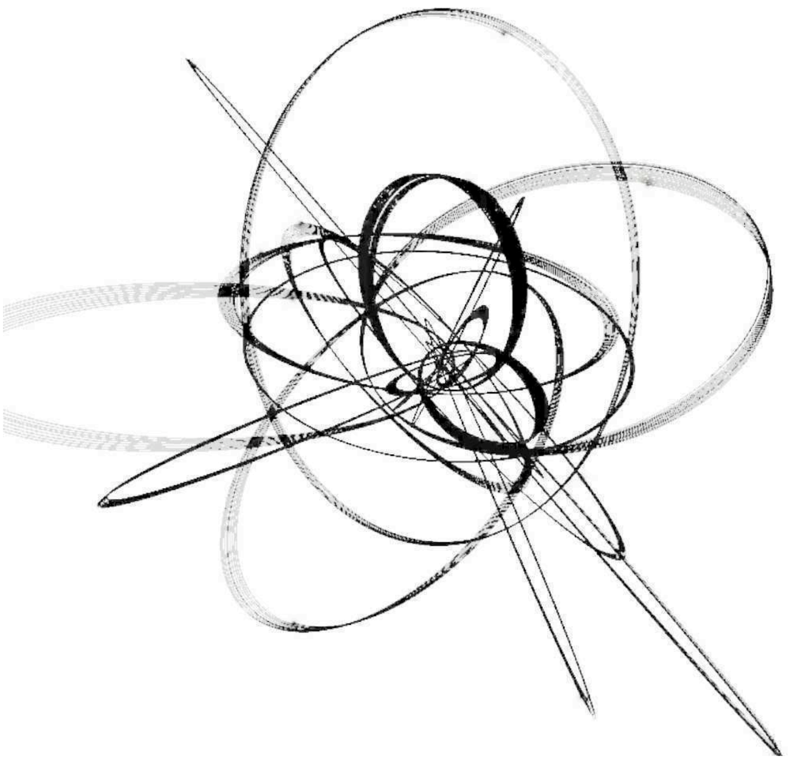


Typical orbit

Keplerian systems

Solar System		Galactic Centre



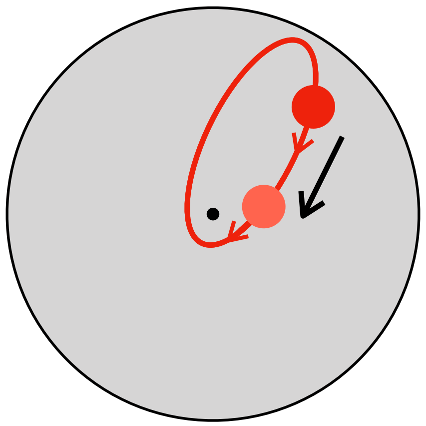


Solar System		Galactic Centre
Planets	⟷ Light object	**Stars**
Sun	⟷ Heavy objects	**Black hole**
$N \simeq 10$	⟷ Number of ``particles''	$N \simeq 10^6$
Planar symmetry	⟷ Shape of the system	**Spherical** symmetry
Quasi-circular orbits	⟷ Shape of the orbits	**Eccentric** orbits

Keplerian orbits

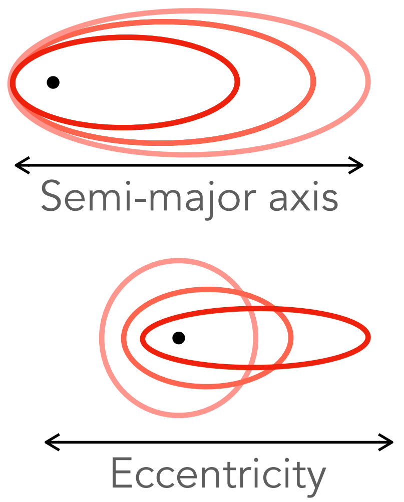
How to describe an **orbit**

Position of the star

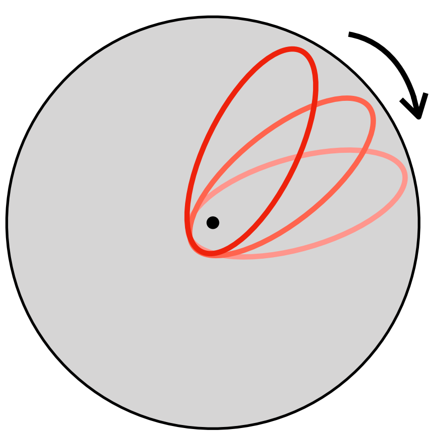


Dynamical motion

Shape of the orbit

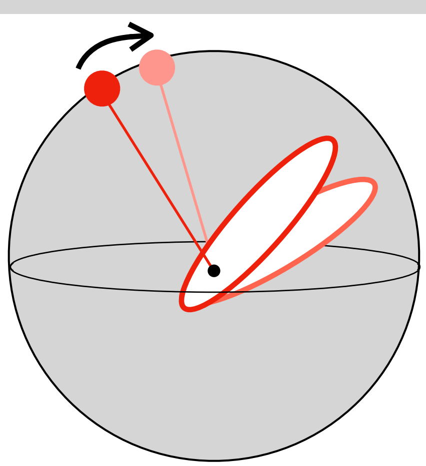


Phase of the orbit

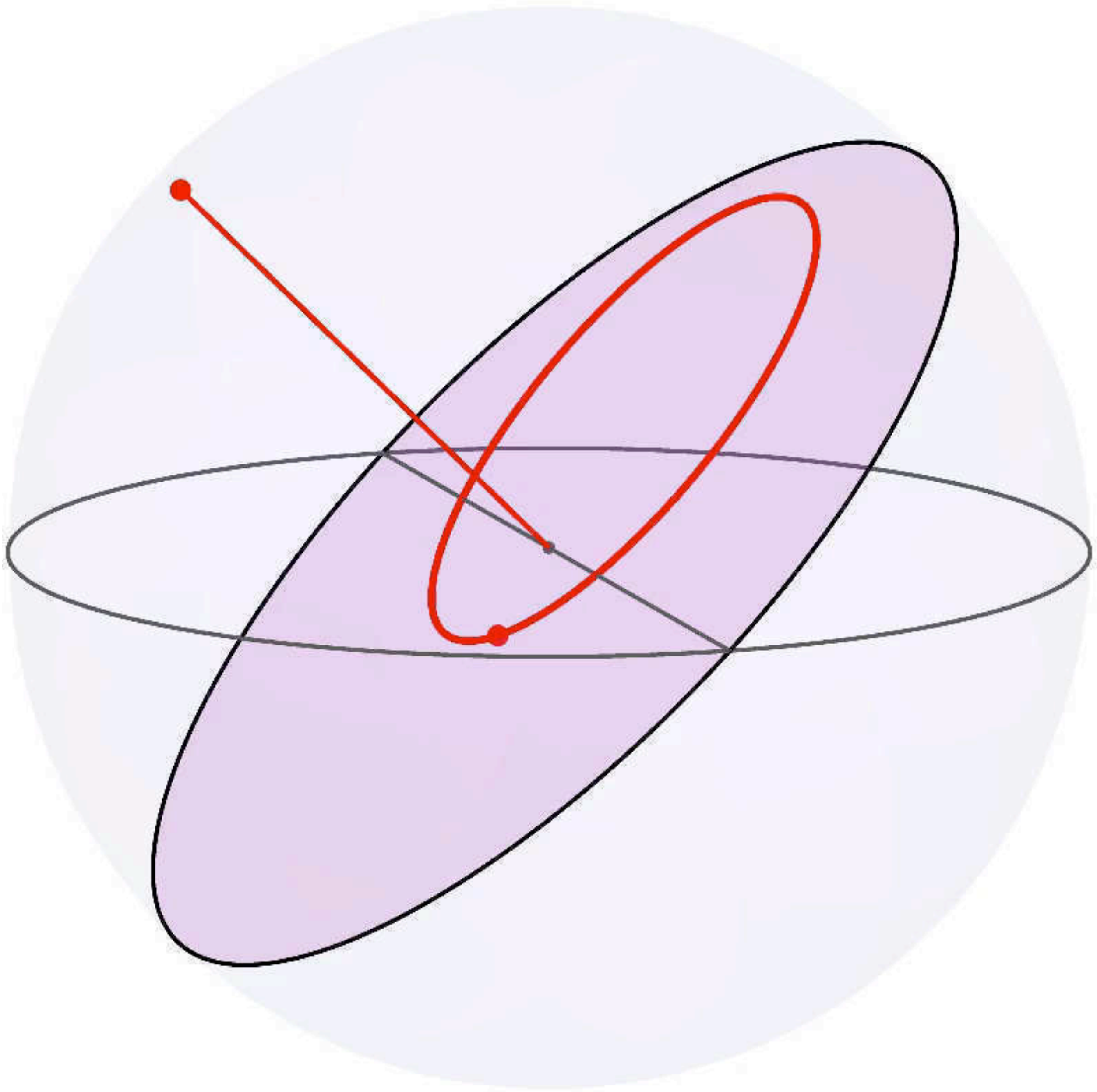


Pericentre phase

Orientation of the orbit



Spatial orientation

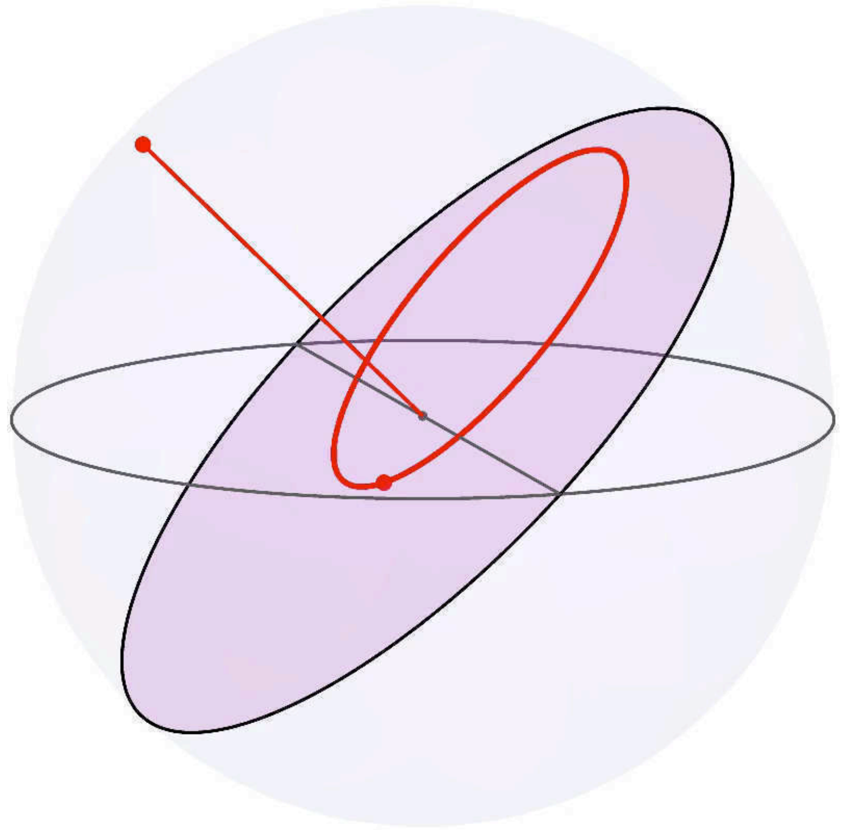


Keplerian orbit

Wires dynamics

Orbit Average

$$J_{\rm fast} = I(a)$$ **adiabatically conserved**



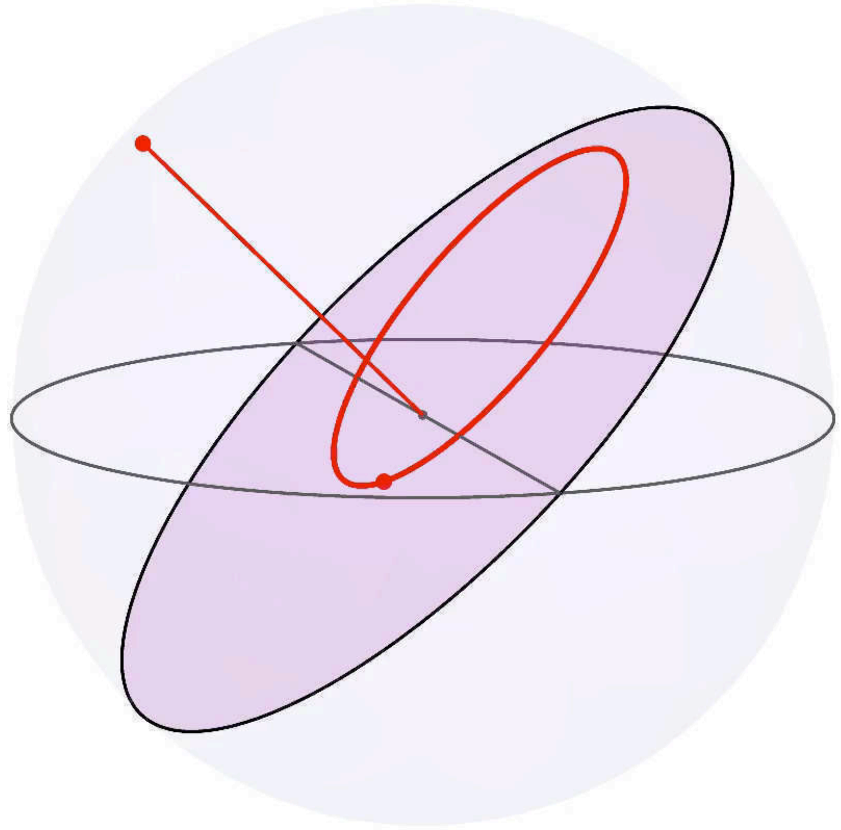
Wires may **precess constructively**:

+ In-plane precessions

- Spherical cluster mass
- 1PN relativistic **Schwarzschild precession**

$$\dot{\omega} = \Omega_{\rm prec} \;\; ; \quad \hat{\mathbf{L}} = \mathrm{cst}\,.$$

+ Out-of-plane precessions



- Triaxial cluster mass
- 1.5PN relativistic **Lense-Thirring precession**

$$\dot{\hat{\mathbf{L}}} = \mathbf{\Omega}_{\rm prec} \;\; ; \quad L = \mathrm{cst}\,.$$

Wires may also **jitter stochastically**

- Finite-N effects $\dot{\hat{\mathbf{L}}} = \eta(t)$

Long-term dynamics of wires

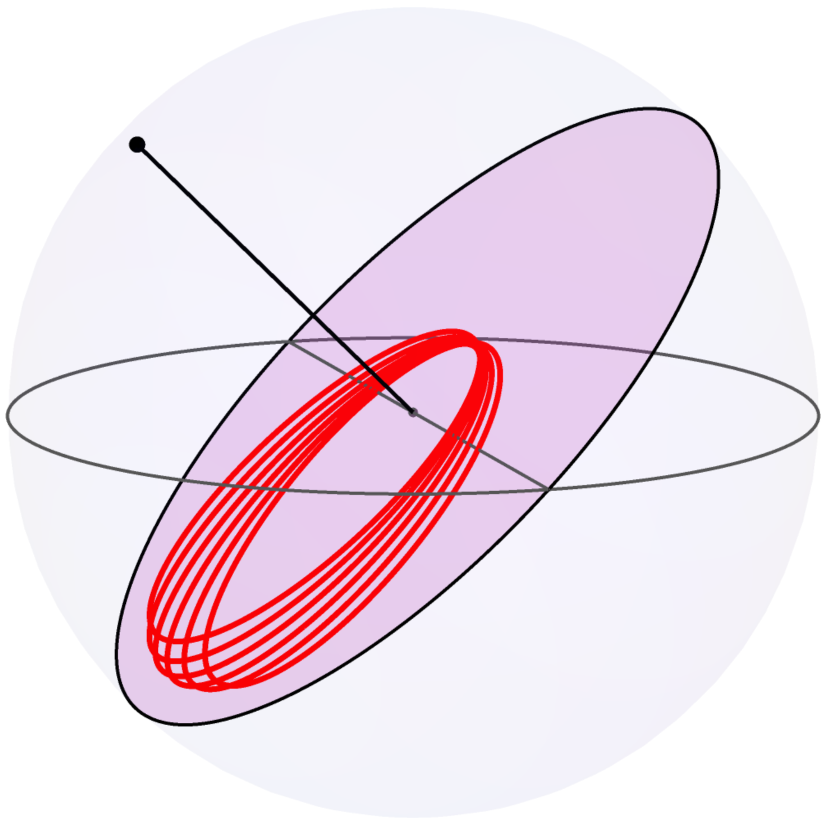
In-plane precessions (L, ω)

Constructive mean field motion

$$\Omega^{\mathrm{prec}} = \Omega^{\mathrm{prec}}_{\mathrm{self}} + \Omega^{\mathrm{prec}}_{\mathrm{rel}} + \Omega^{\mathrm{prec}}_{\mathrm{ext}}$$

Long-term diffusion of $L = L(e)$

Scalar Resonant Relaxation



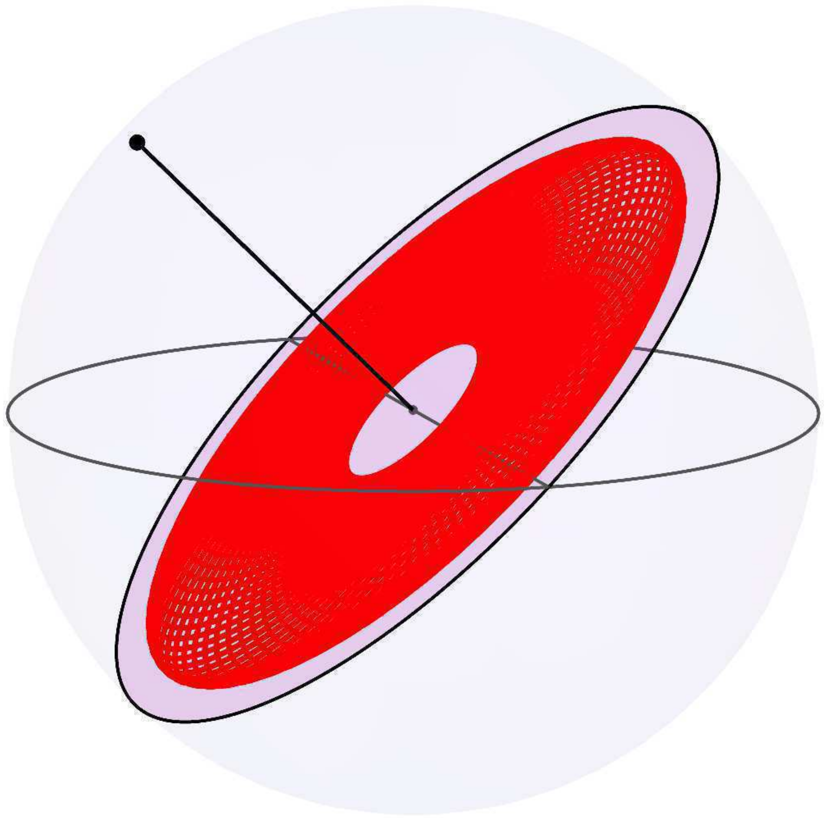
Out-of-plane precessions $\hat{\mathbf{L}}$

No mean field motion

$$\langle \Omega^{\mathrm{prec}} \rangle = 0$$

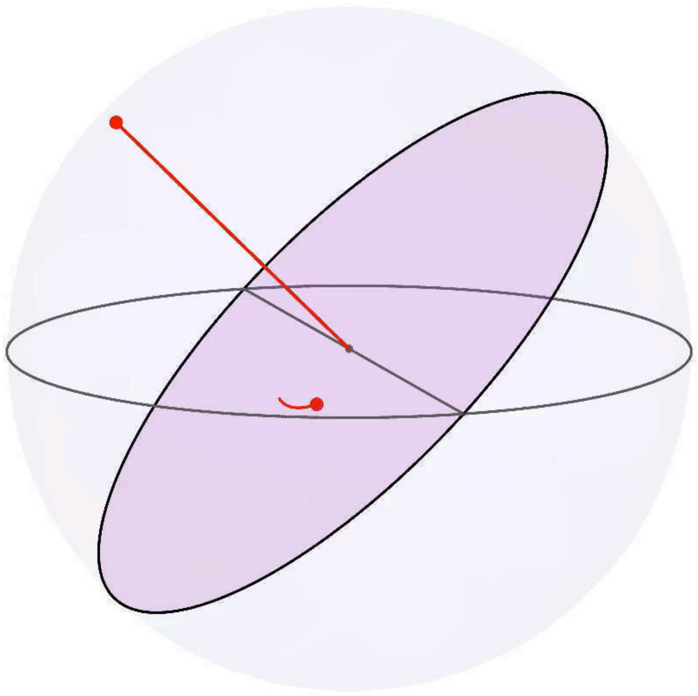
Random walk on the sphere of $\hat{\mathbf{L}}$

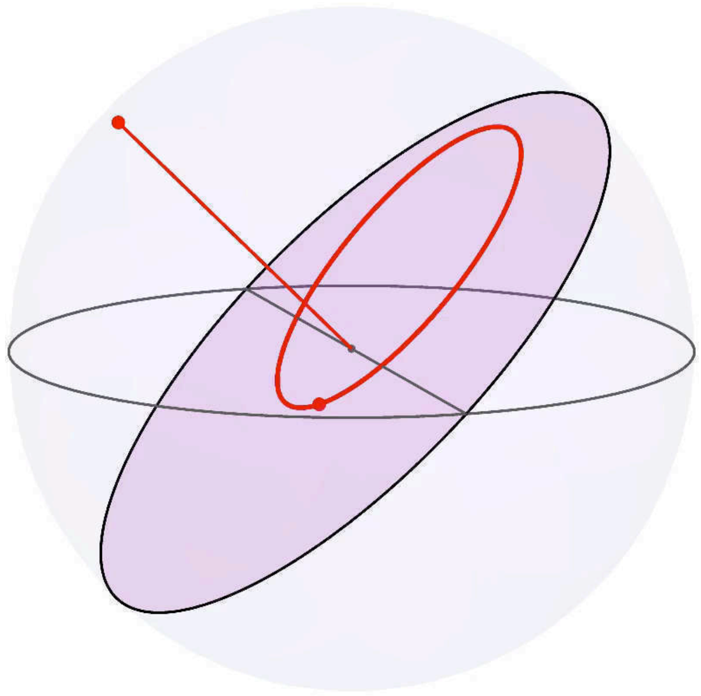
Vector Resonant Relaxation

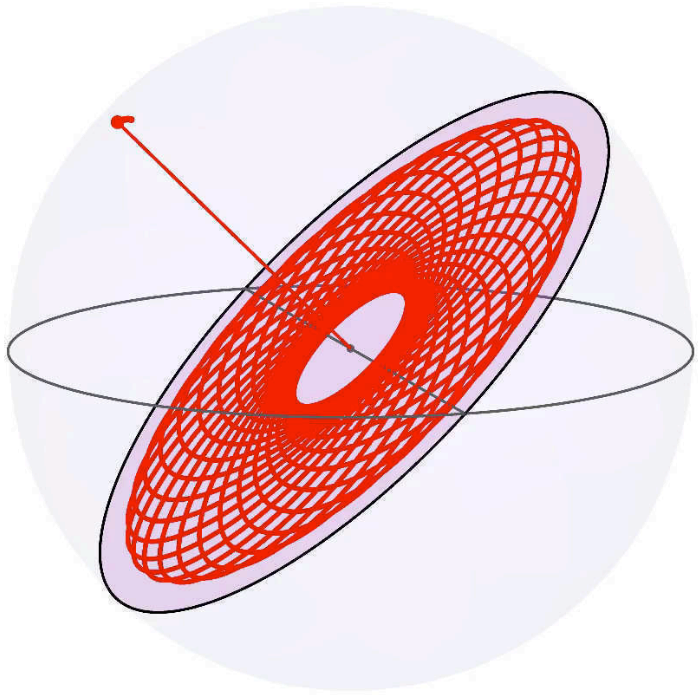


Stellar dynamics

Stars	Ellipses	Annuli







~10 years	~30,000 years	~1,000,000 years
Orbital motion	**Pericentre** precession	**Orientation** precession

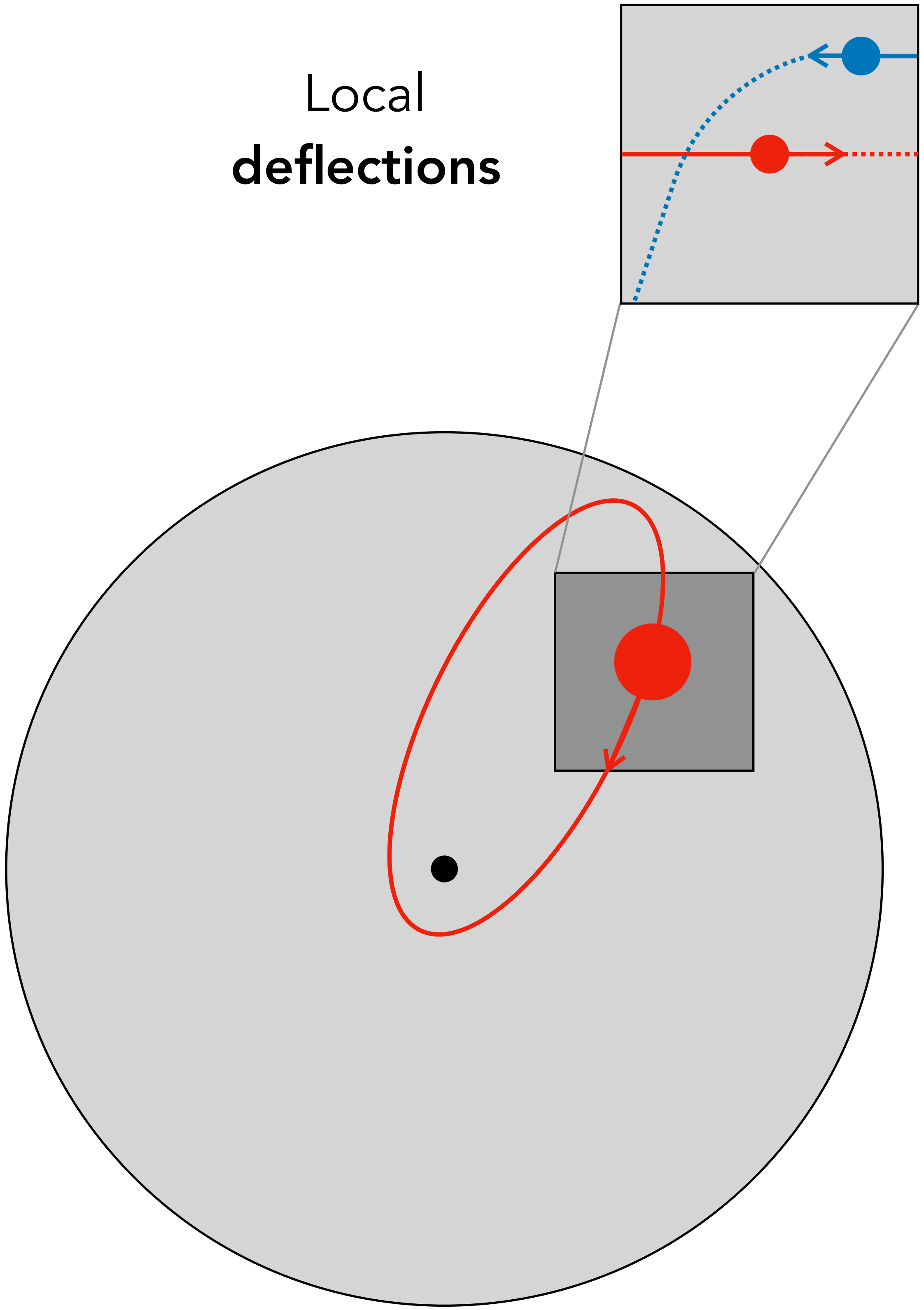
SgrA* is 10,000,000,000 years old, we can wait longer…

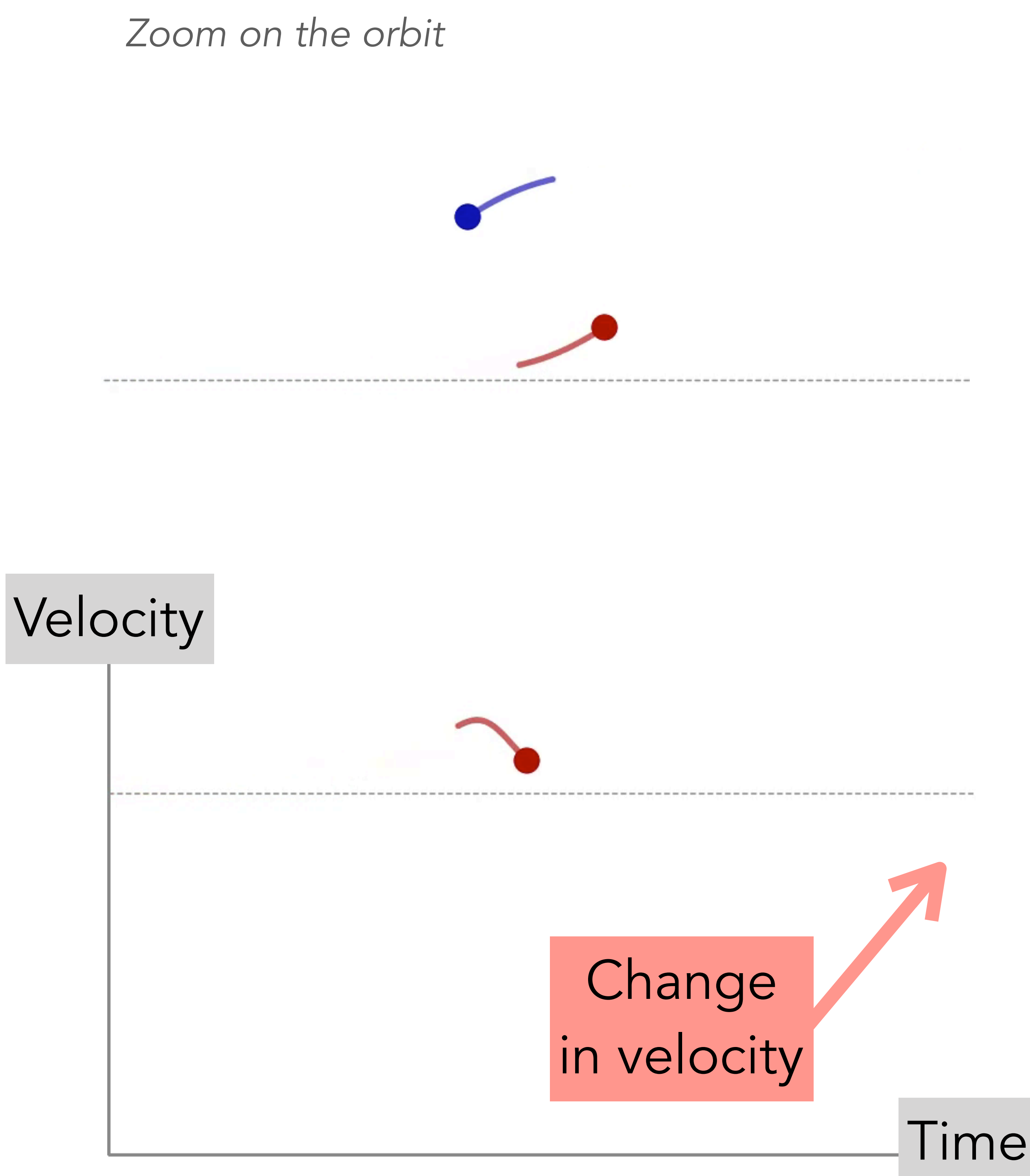
Deflections

What happens along the **stellar orbit**?

Typical timescale
~1,000,000,000 years

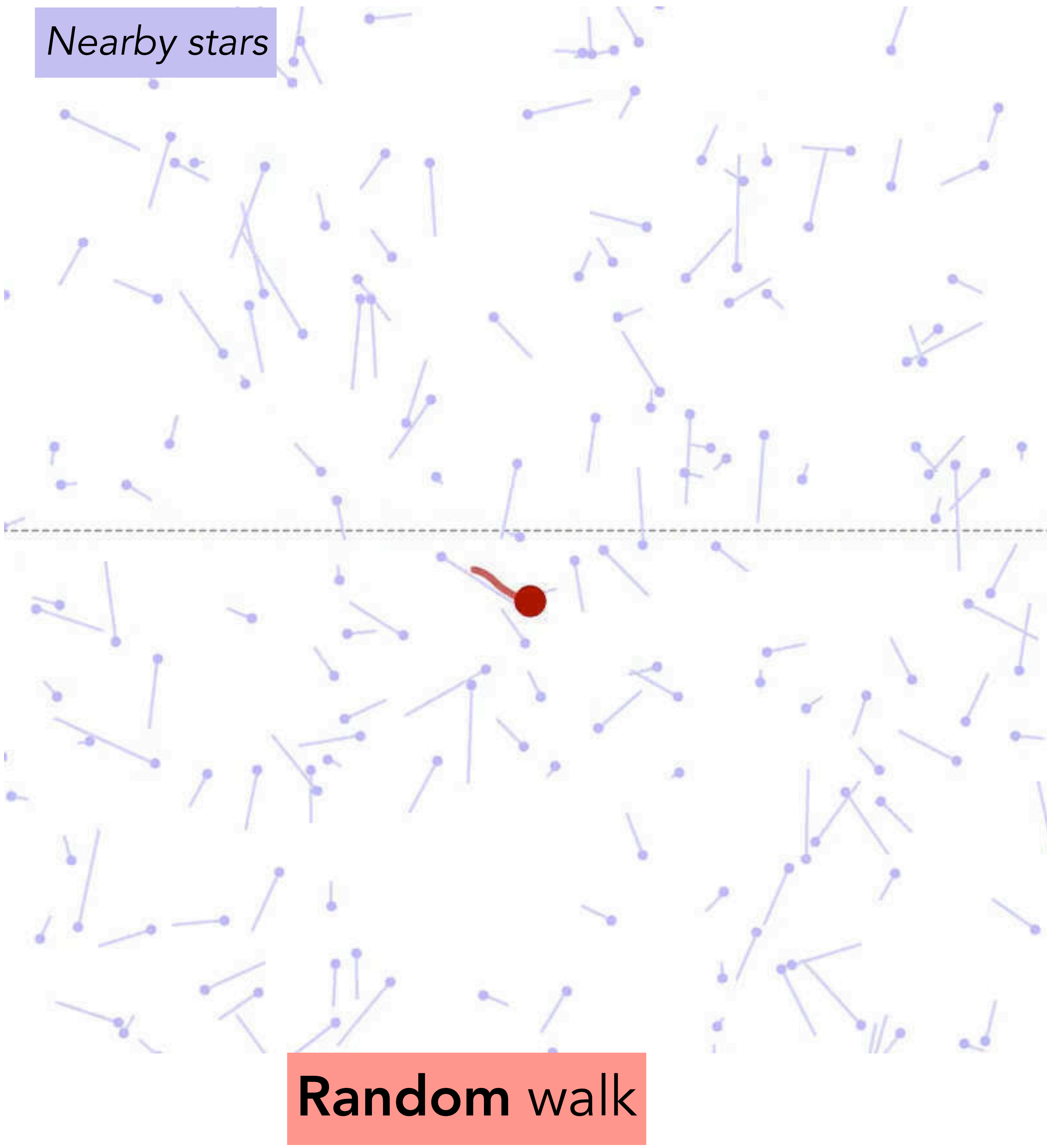
Local
deflections



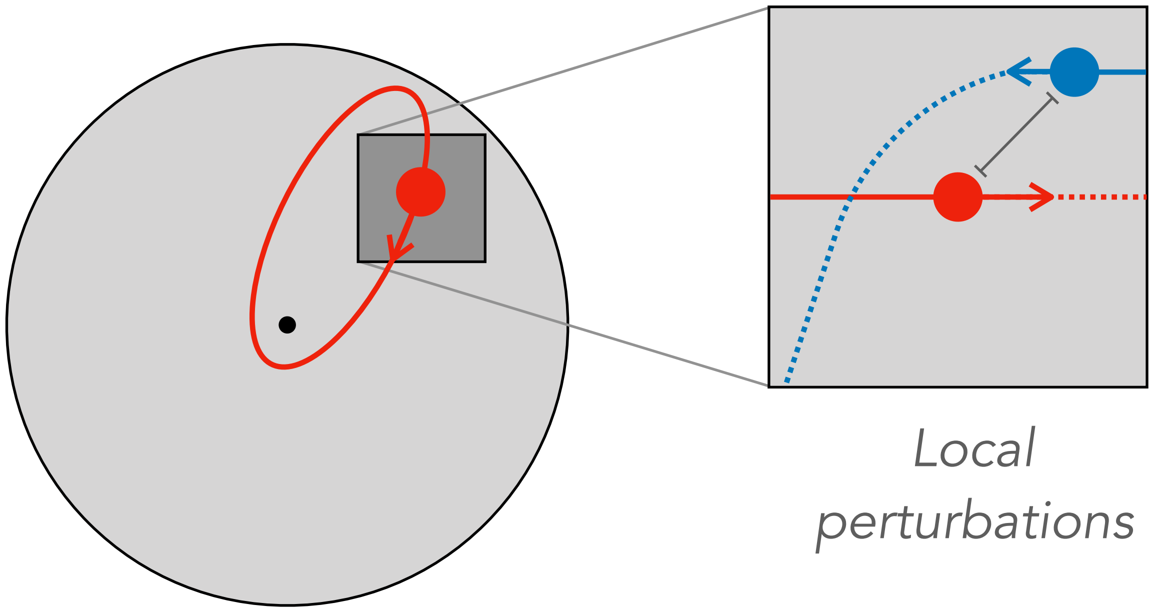


Deflections

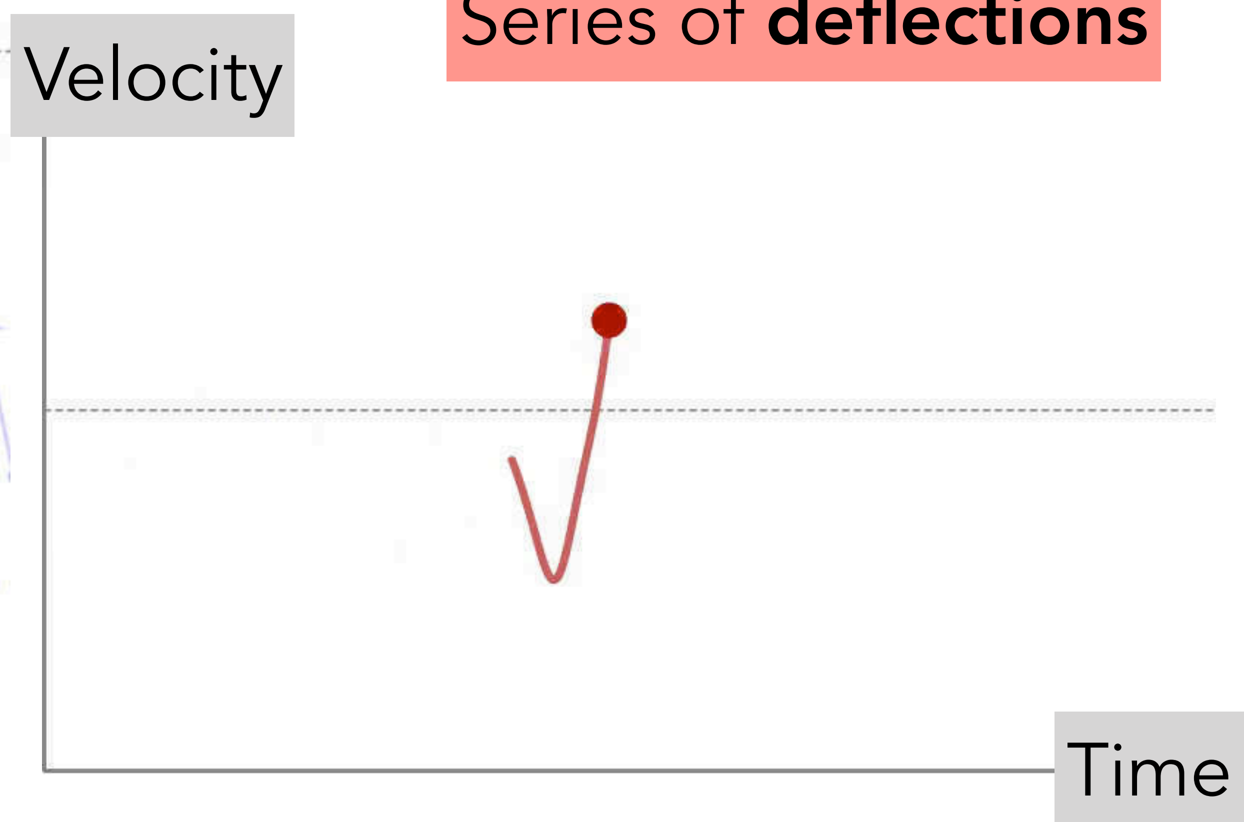
The star has **a lot of neighbours**



Typical timescale ~1,000,000,000 years



Series of **deflections**



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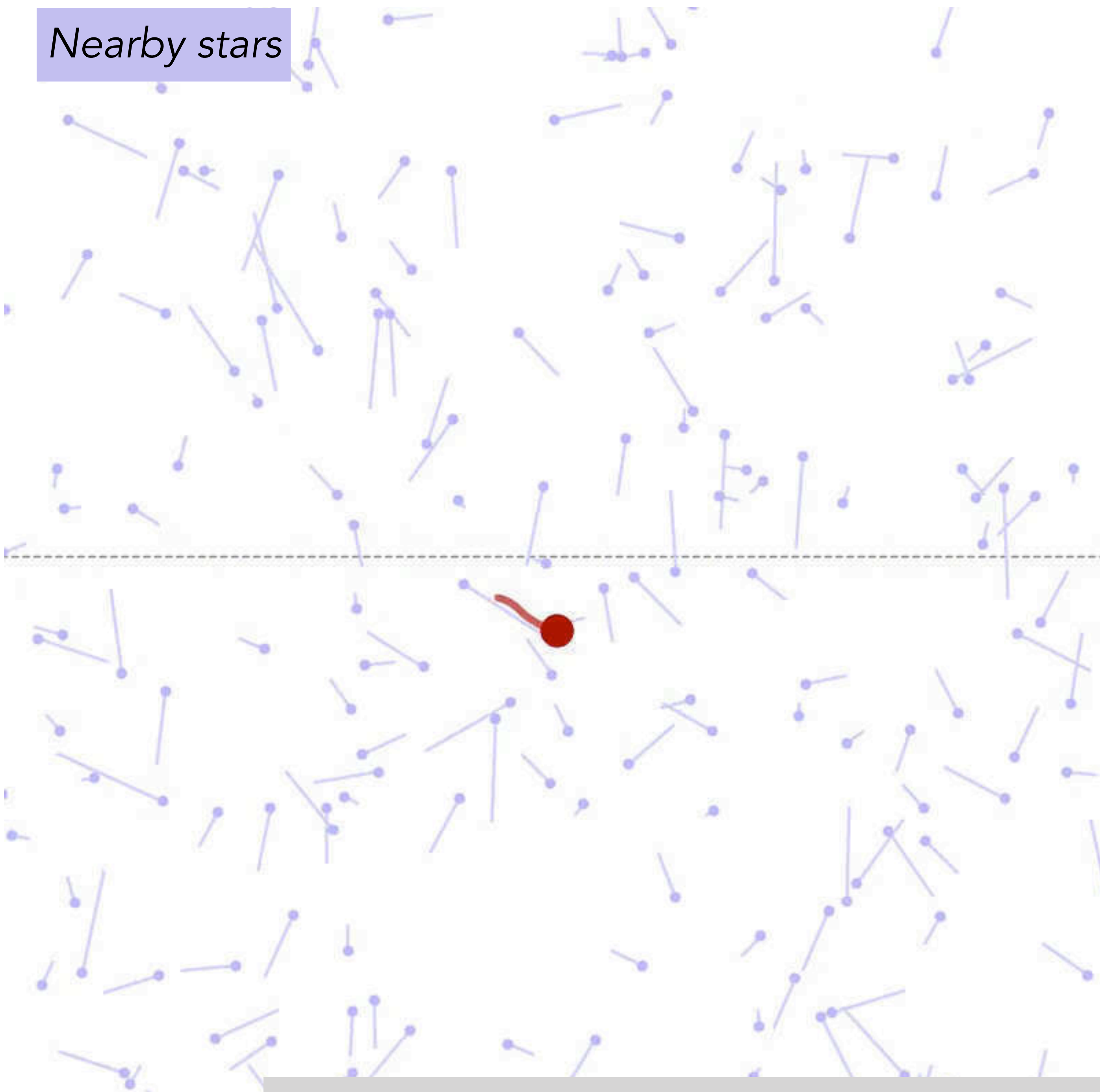
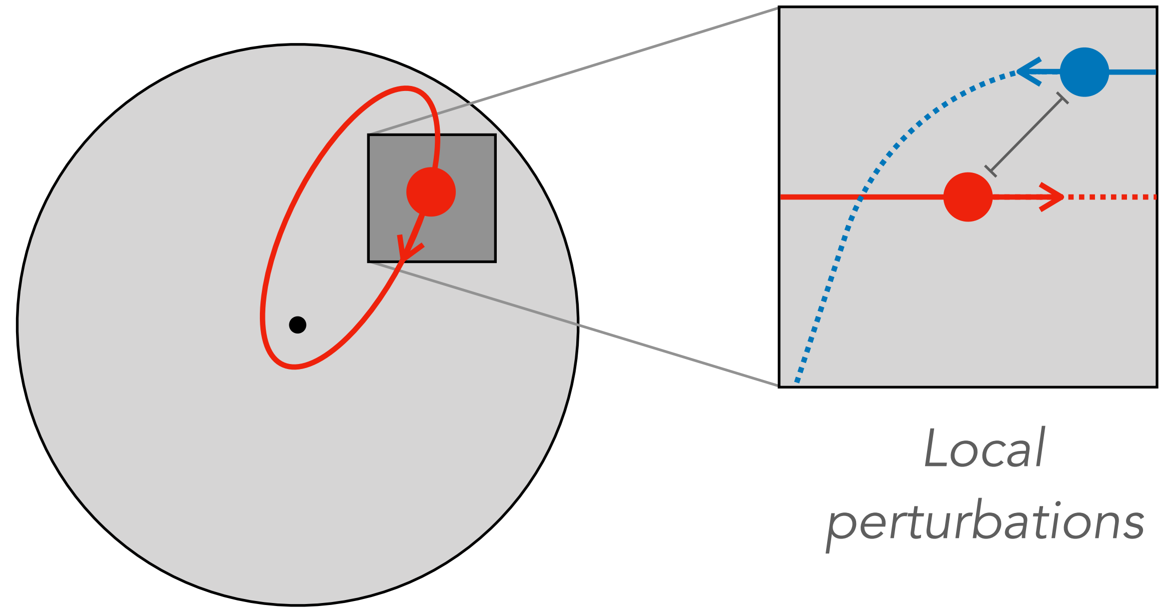
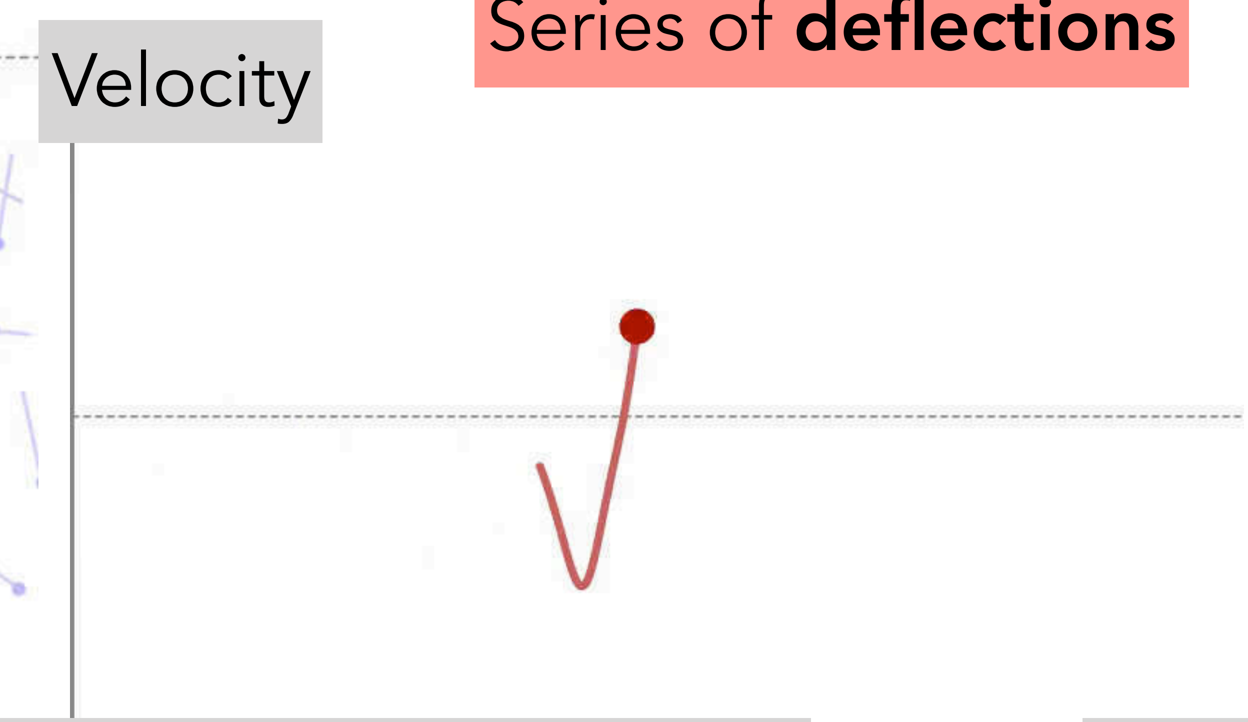




Series of **deflections**



Characterised by the **homogeneous Landau equation**

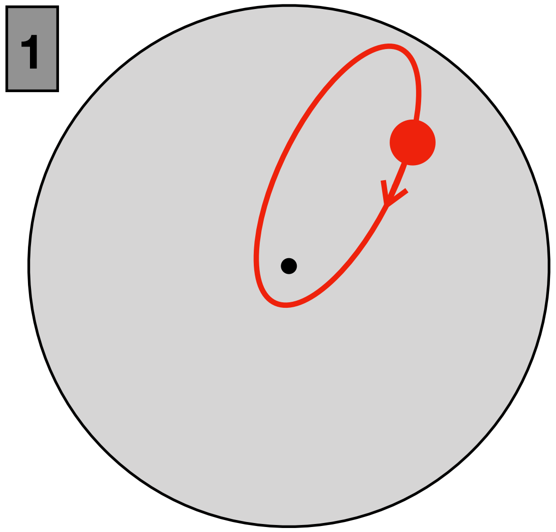
Timescales are highly hierarchical

1. Dynamical time

Fast orbital motion induced by the BH

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \Omega_{\mathrm{Kep}}$$

1



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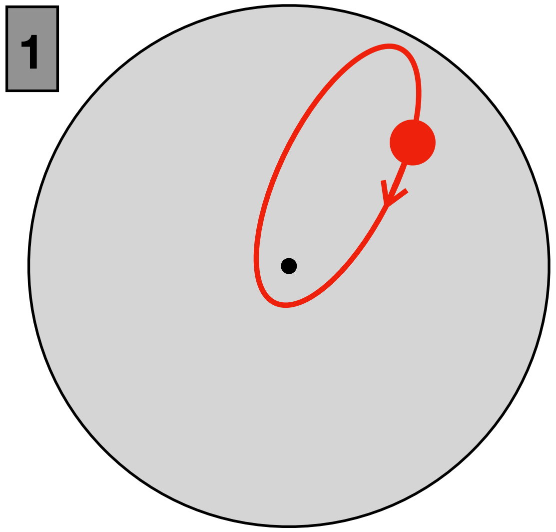
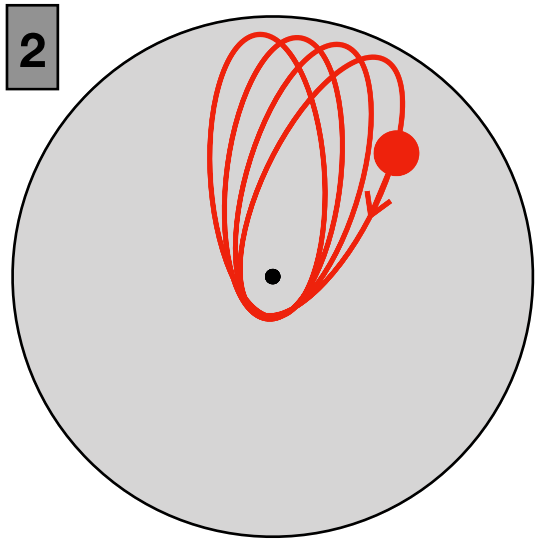
2. Precession time

In-plane precession (mass + relativity)

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \Omega_{\mathrm{prec}}$$





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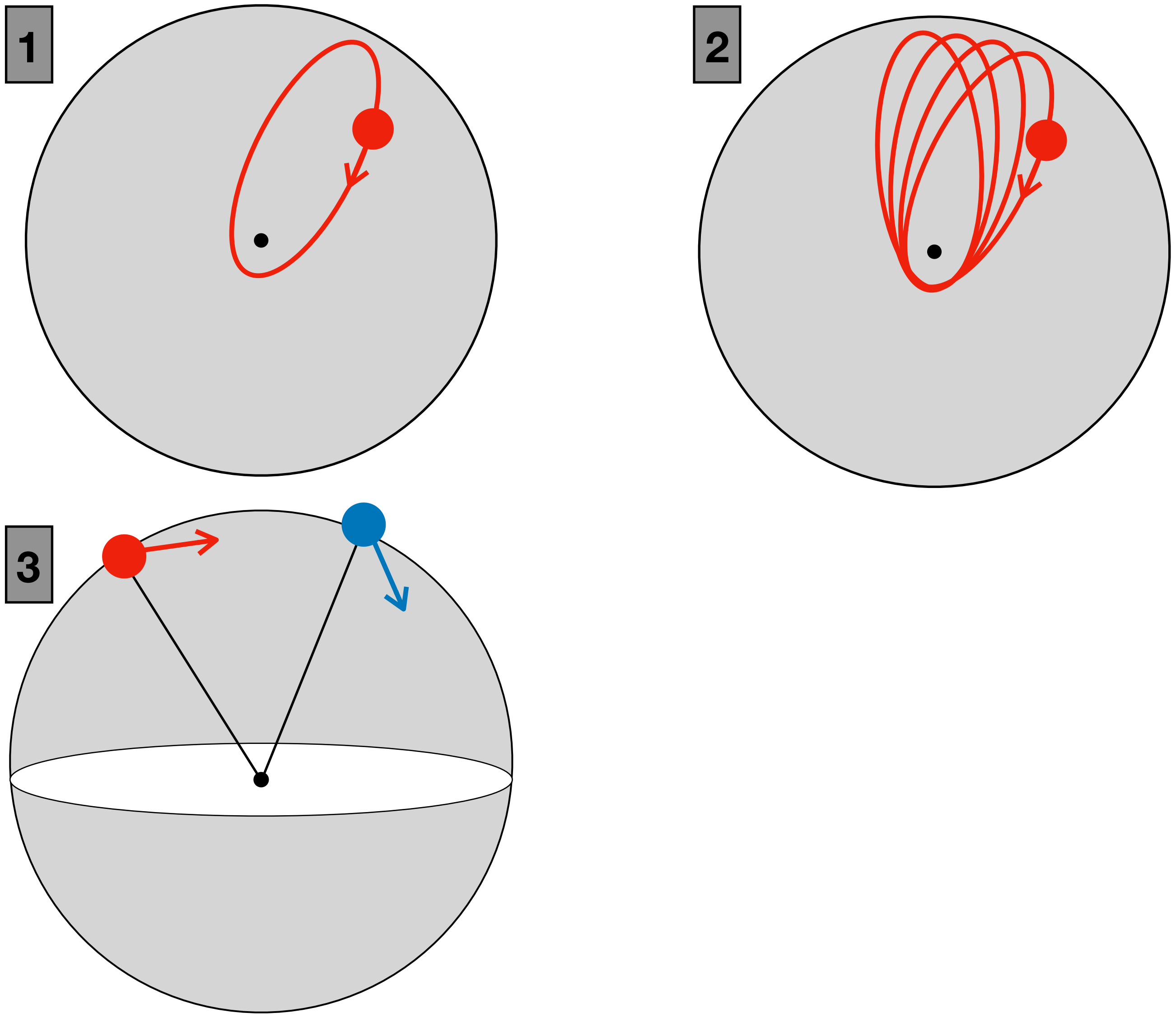
3. Vector Resonant Relaxation
Non-spherical torque coupling

$$\frac{\mathrm{d}\hat{\mathbf{L}}}{\mathrm{d}t} = \eta(\hat{\mathbf{L}}, t)$$



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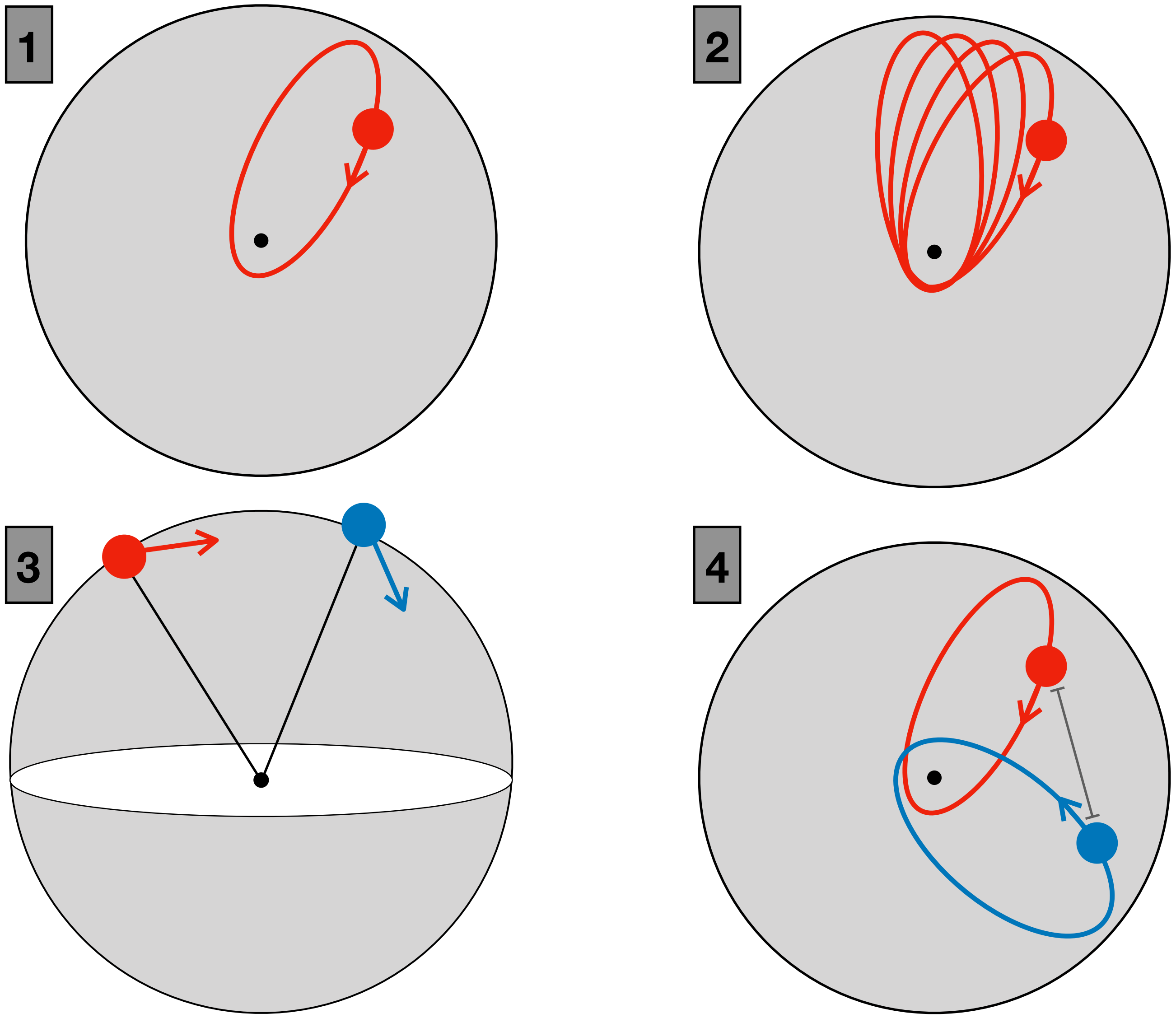
Resonant coupling on precessions

$$\frac{\mathrm{d}\,|\mathbf{L}|}{\mathrm{d}t} = \eta(|\mathbf{L}|, t)$$



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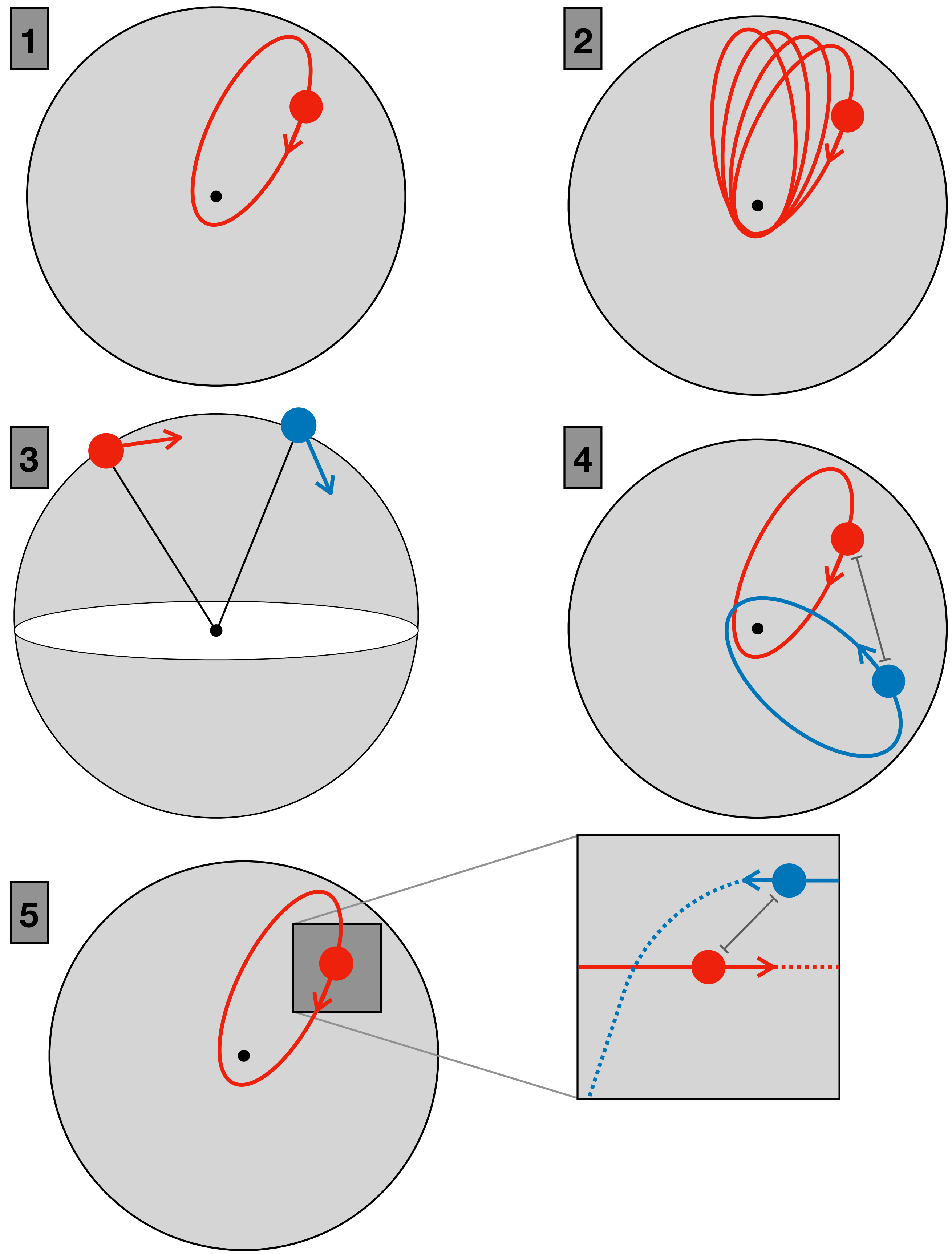
Resonant coupling on precessions

$$\frac{\mathrm{d}\,|\,\mathbf{L}\,|}{\mathrm{d}t} = \eta(\,|\,\mathbf{L}\,|\,, t)$$

5. Non-Resonant Relaxation
Local two-body encounters

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \eta(a, t)$$



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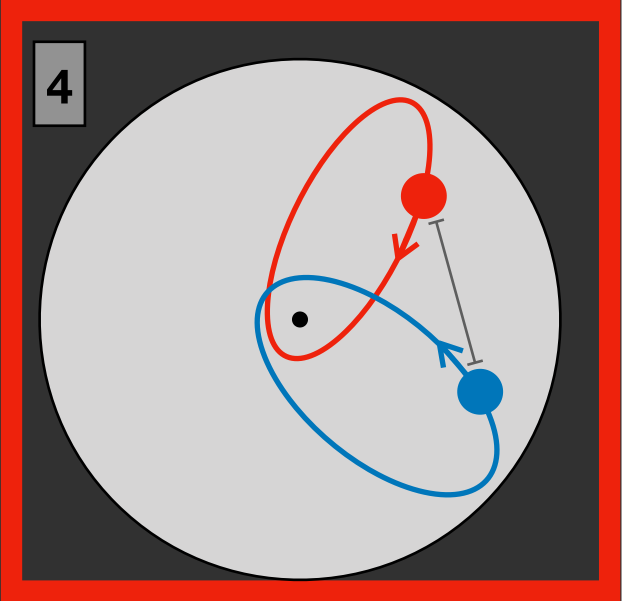
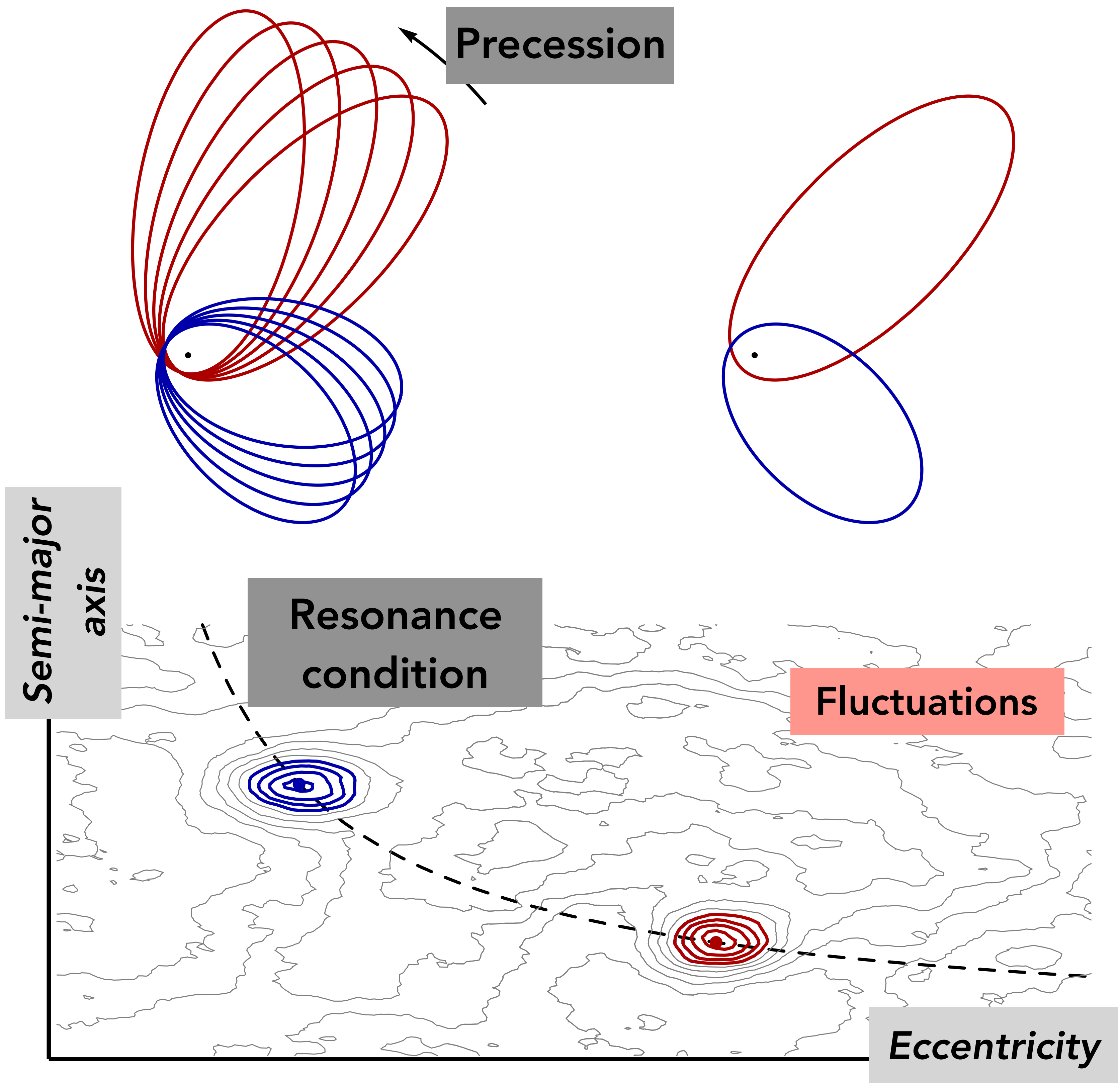


Non-local resonances



Non-local resonances

between wires

The (degenerate) Balescu-Lenard equation

The master equation of **scalar resonant relaxation**

$$\frac{\partial F(L,a,t)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial L}\left[L\, D_{LL}(L,a)\,\frac{\partial}{\partial L}\,\frac{F(L,a,t)}{L}\right]$$

Anisotropic diffusion coefficients

$$D_{LL}(L,a) \propto \frac{1}{N_\star}\sum_{n,n'} n^2 \int \mathrm{d}L'\mathrm{d}a'\, \delta_{\mathrm{D}}(n\Omega^{\mathrm{prec}}(L,a) - n'\Omega^{\mathrm{prec}}(L',a')) \times \left|A_{nn'}(L,a,L',a')\right|^2 F^{\mathrm{Cluster}}(L',a',t)$$

Some properties

- $F(L,a,t)$ *Orbital distortion*
- $\partial/\partial L$ *Adiabatic invariance*
- $D_{LL}(L,a)$ *Anisotropic diffusion*
- $1/N_\star$ *Finite-N effects*
- n *Resonance numbers*
- $\int \mathrm{d}L'\mathrm{d}a'$ *Scan of orbital space*
- $\delta_{\mathrm{D}}(n\Omega^{\mathrm{prec}} - n'\Omega^{\mathrm{prec}\,\prime})$ *Resonance condition*
- $\left|A_{nn'}(L,a,L',a')\right|^2$ *Coupling coefficients*

Timescales are highly hierarchical

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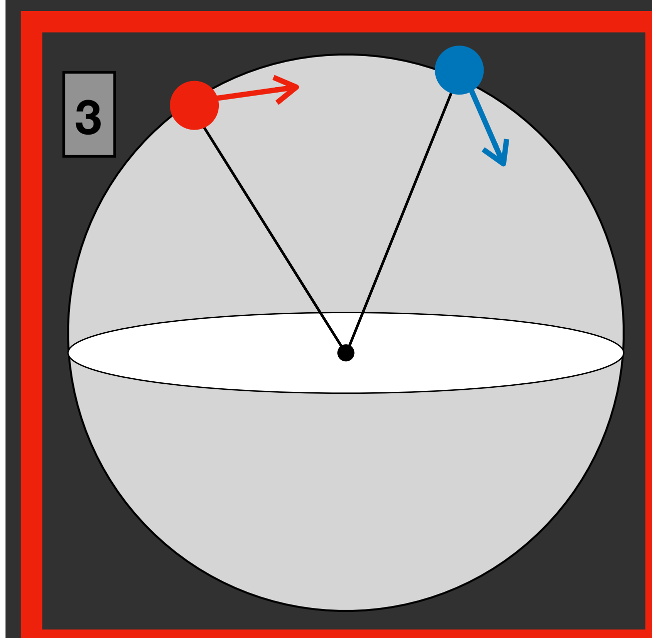
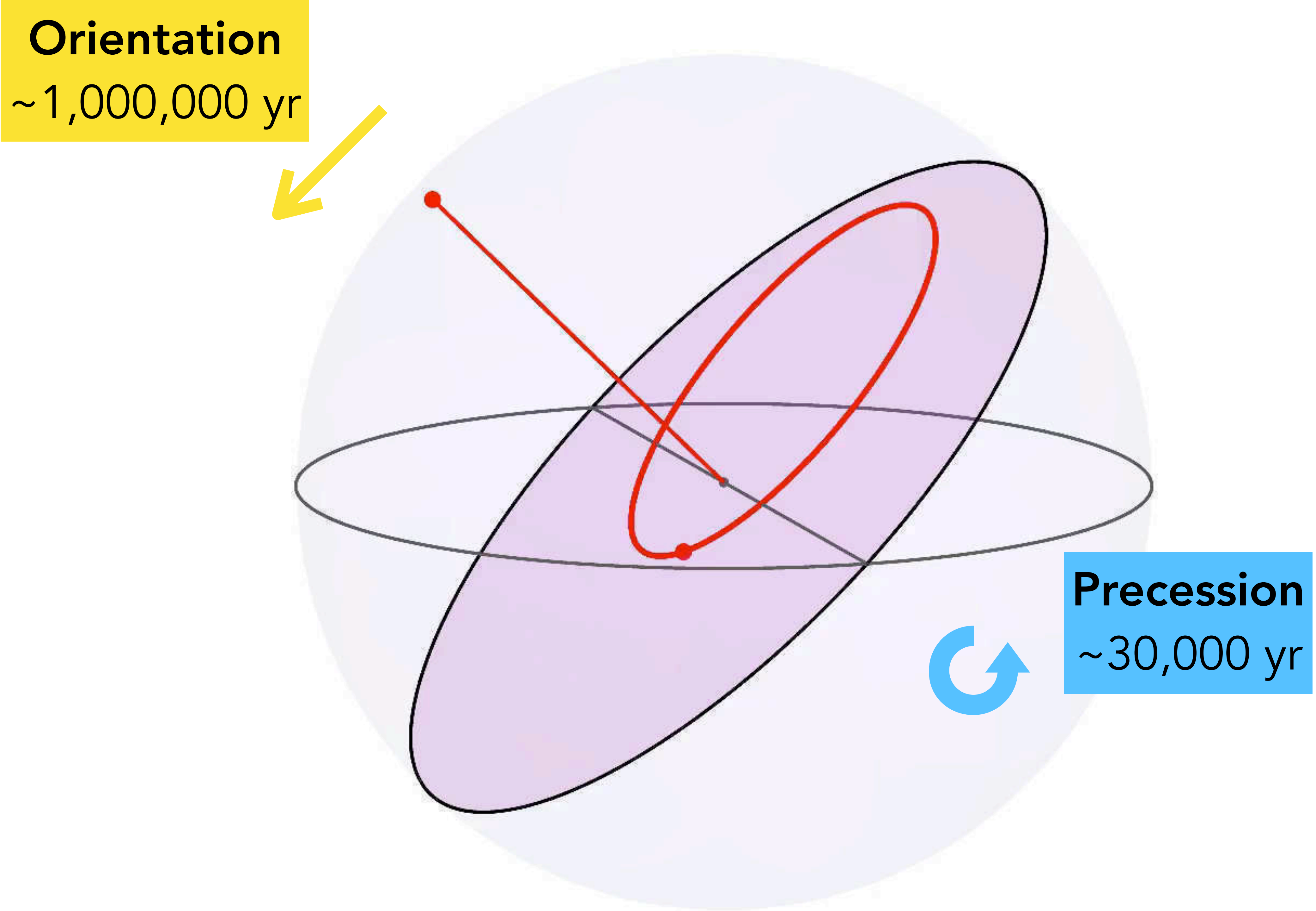
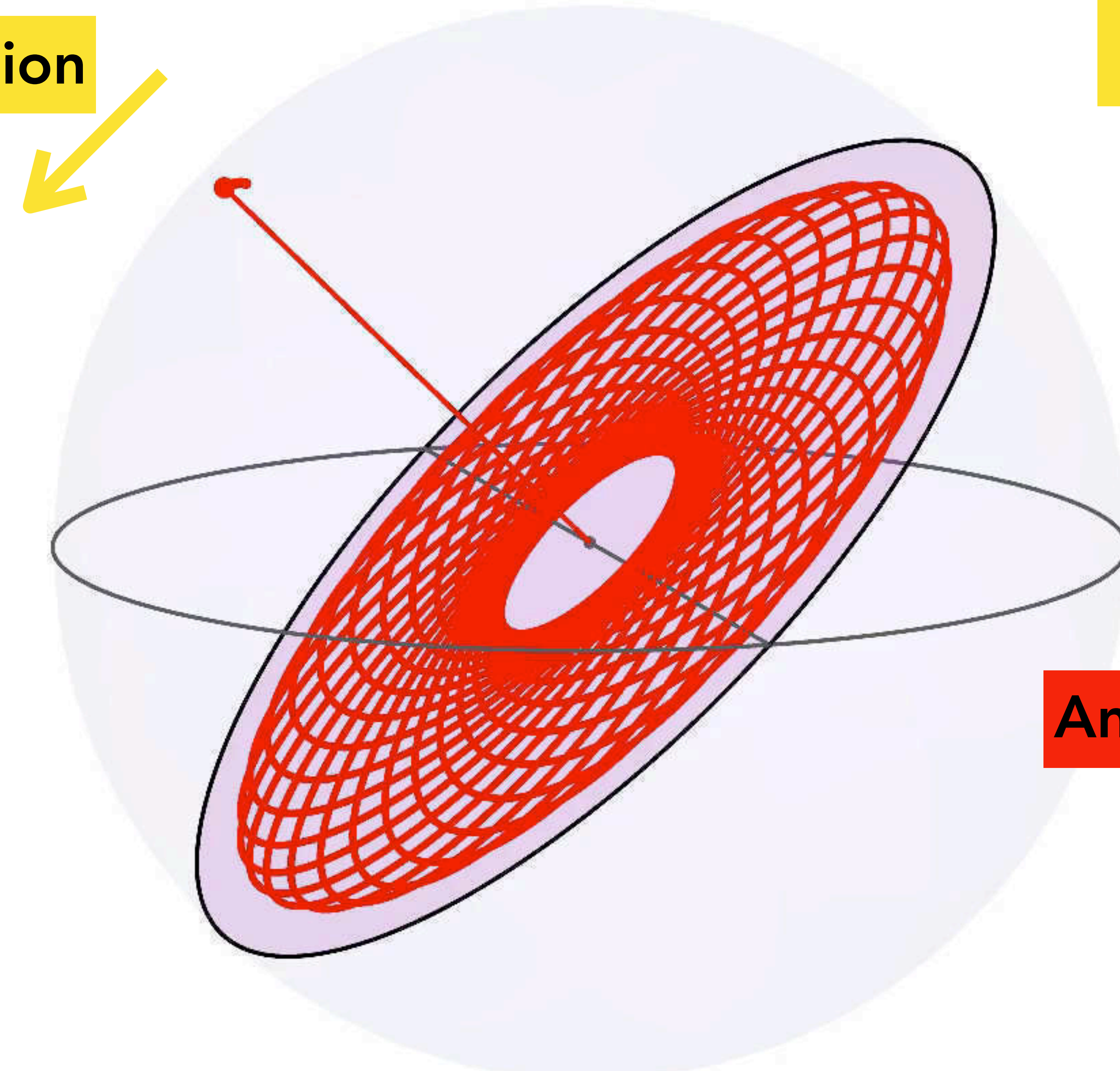


Orbital orientations



Two timescales : **Precession** $\ll$ **Orientation**

Orbital orientations

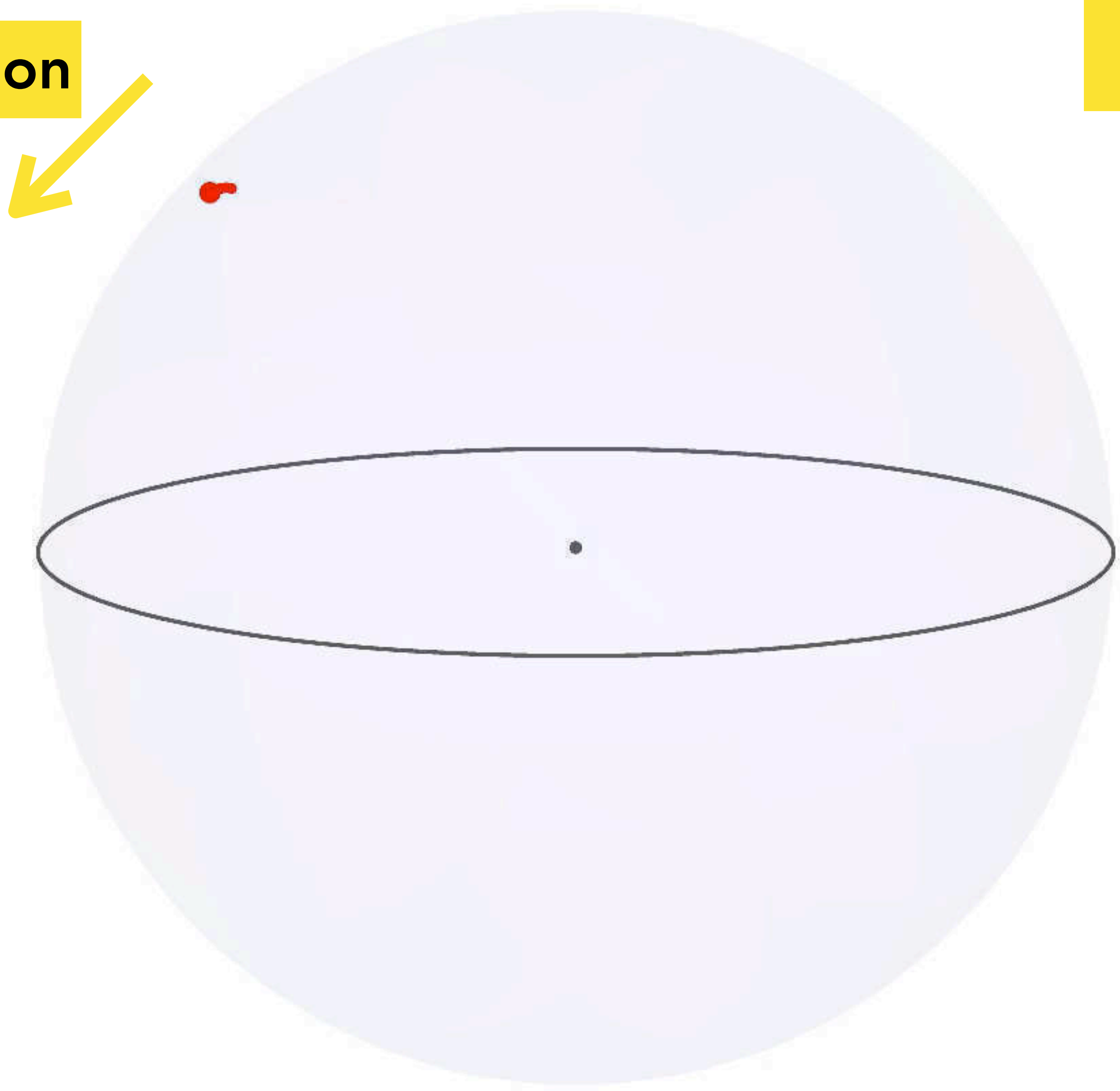


After one full precession, **ellipses** become **annuli**.

Orbital orientations

Typical timescale
~1,000,000 yr

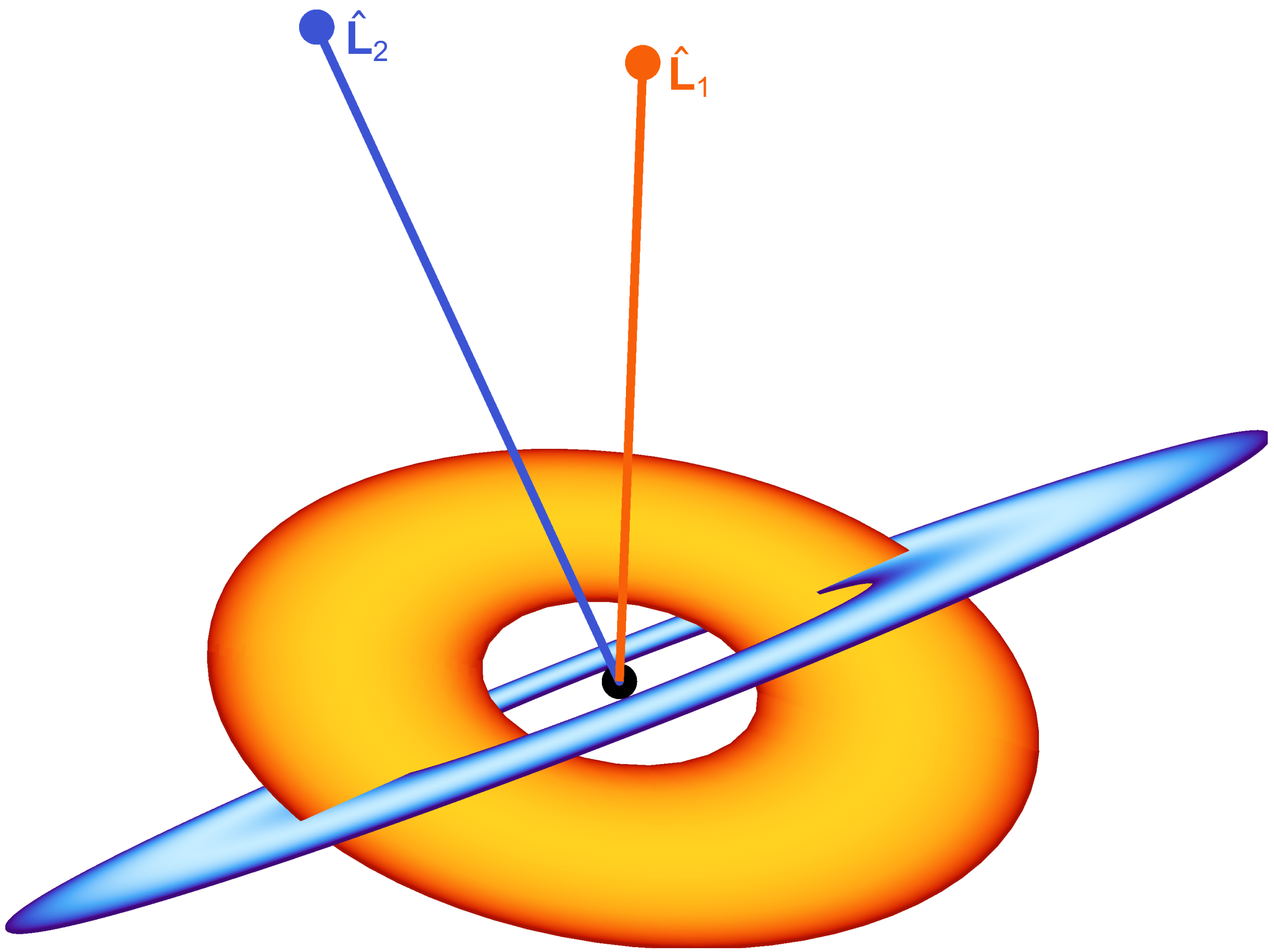
Orientation



The orbital orientation simply becomes **a point on the unit sphere**.

Vector Resonant Relaxation

Random walk of the stars' orientations



Pairwise coupling between two annuli

+ **Long-range** Hamiltonian system

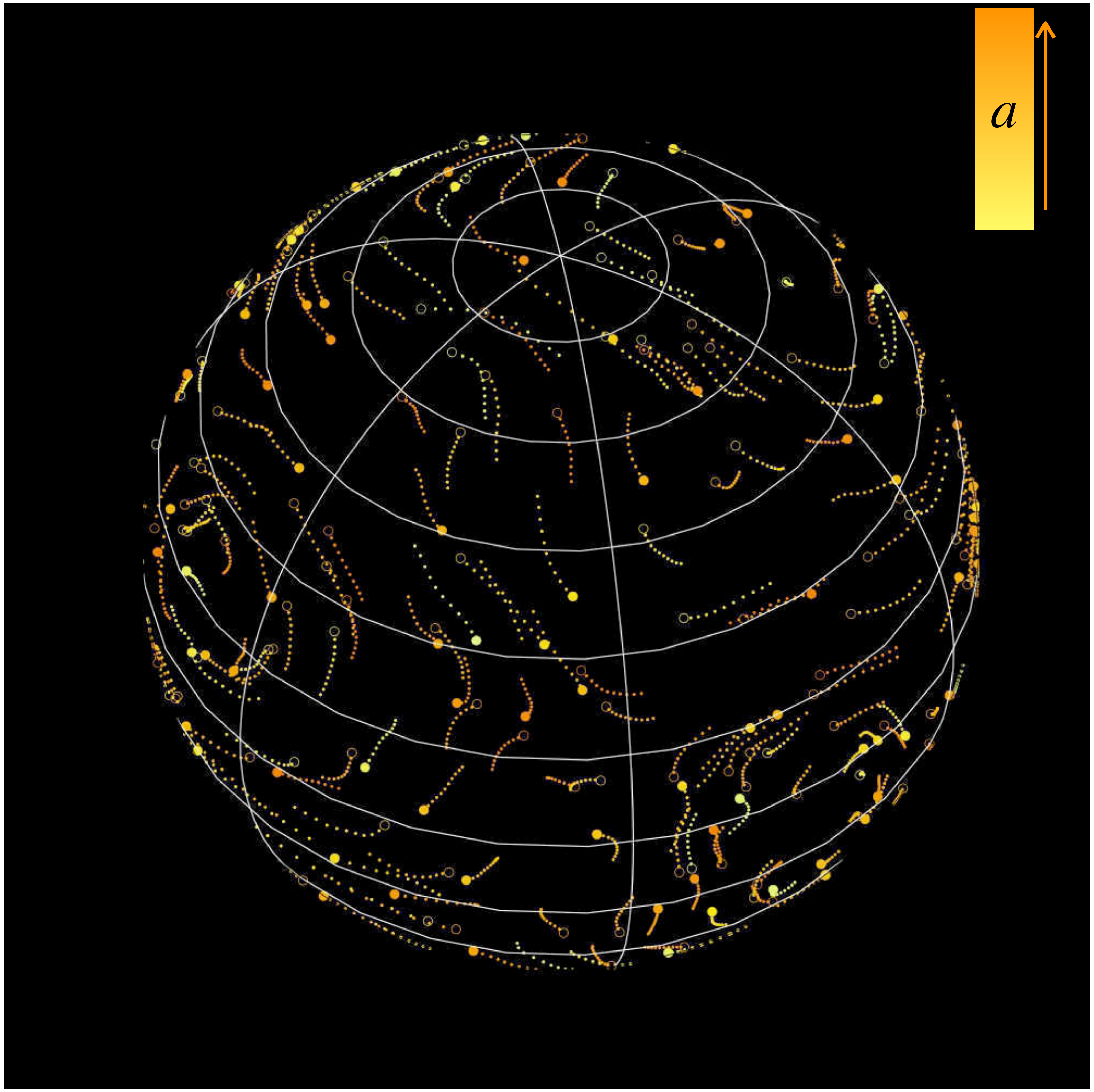
$$H = \sum_{i<j} A(a_i, e_i, a_j, e_j)\, U(\hat{\mathbf{L}}_i \cdot \hat{\mathbf{L}}_j)$$

+ Dynamical variables - **orientations**: $\hat{\mathbf{L}}$

+ Some properties

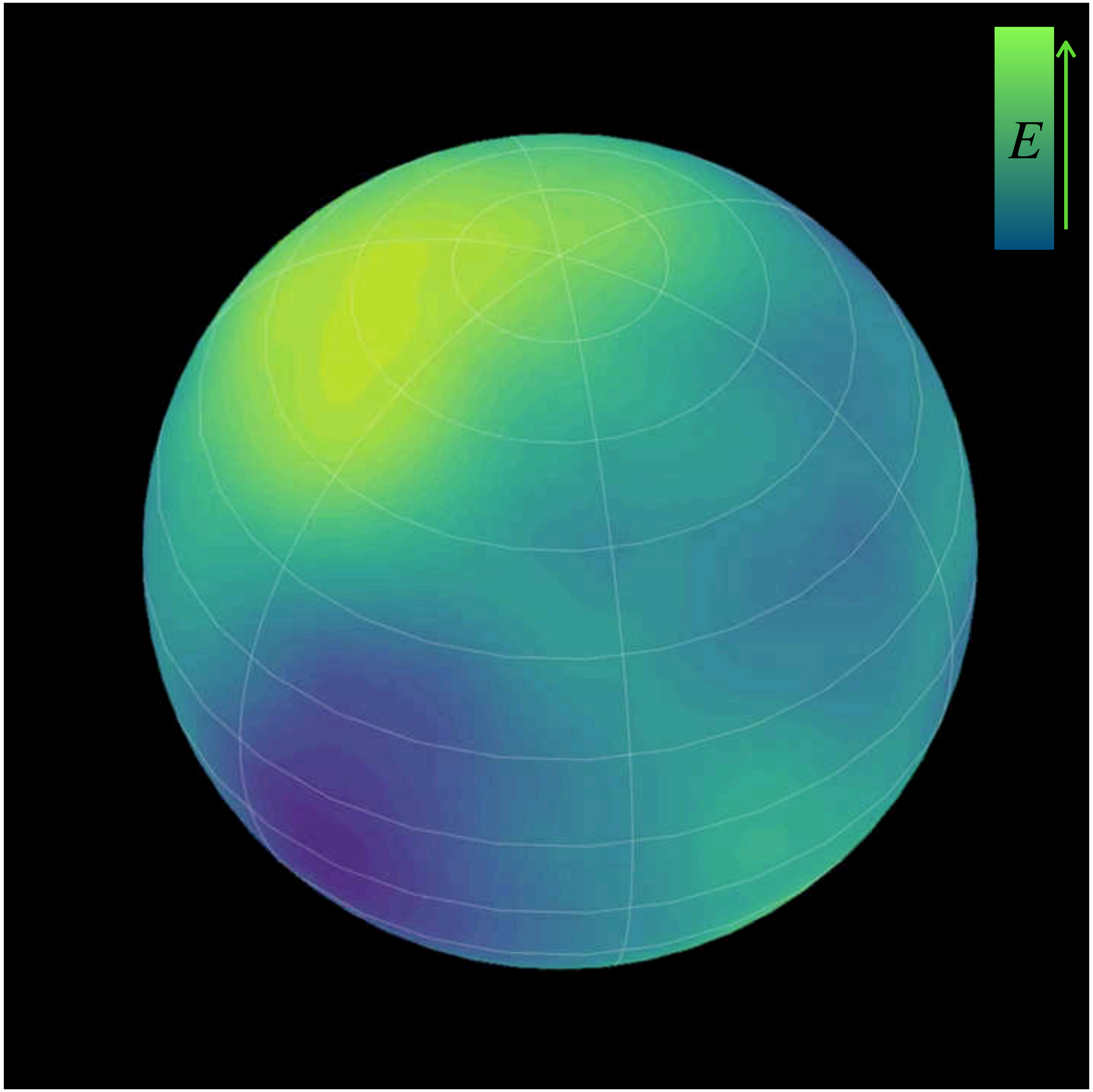
  - No **kinetic energy**
  - Vanishing **mean field** $\langle H \rangle = 0$
  - Additional "labels" (a, e)
  - **Rotational invariance** $\hat{\mathbf{L}}_i \cdot \hat{\mathbf{L}}_j$

Vector Resonant Relaxation



+ Motion coherent on large scales
  - **Long-range interacting system**
+ Motion smooth on short times
  - **Time-correlated noise**
+ Particles have "preferred friends"
  - **Parametric coupling** (a, e)
+ System in statistical equilibrium
  - **Time stationarity** $(t - t')$
  - **Rotation invariance** $(\hat{\mathbf{L}} \cdot \hat{\mathbf{L}}')$

Vector Resonant Relaxation



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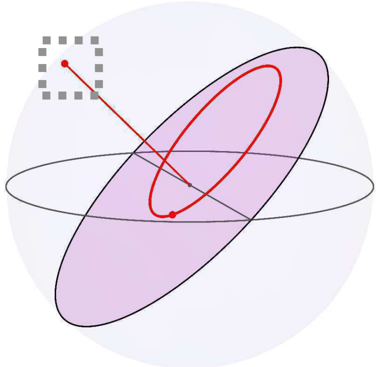
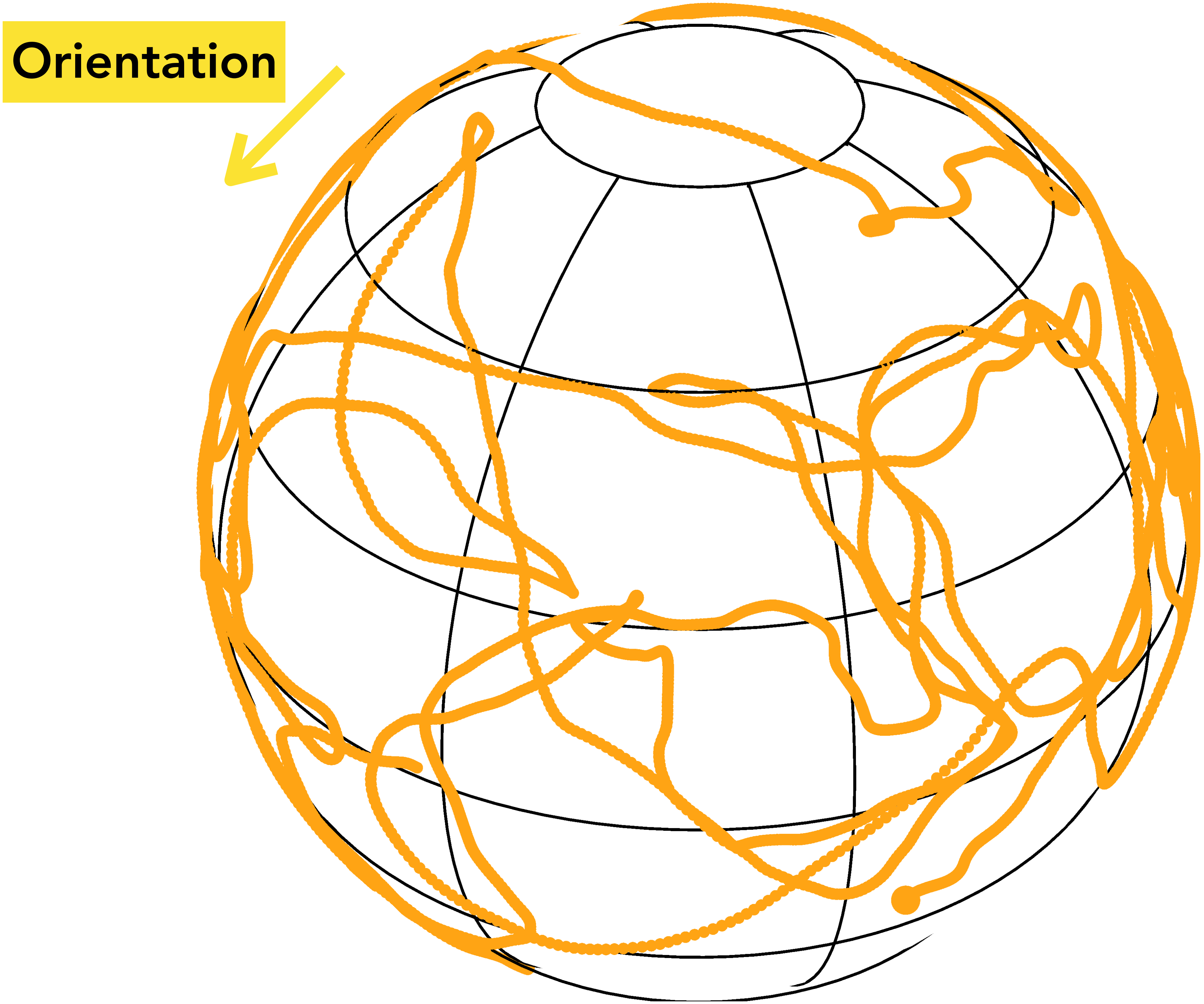
Typical evolution of an orientation

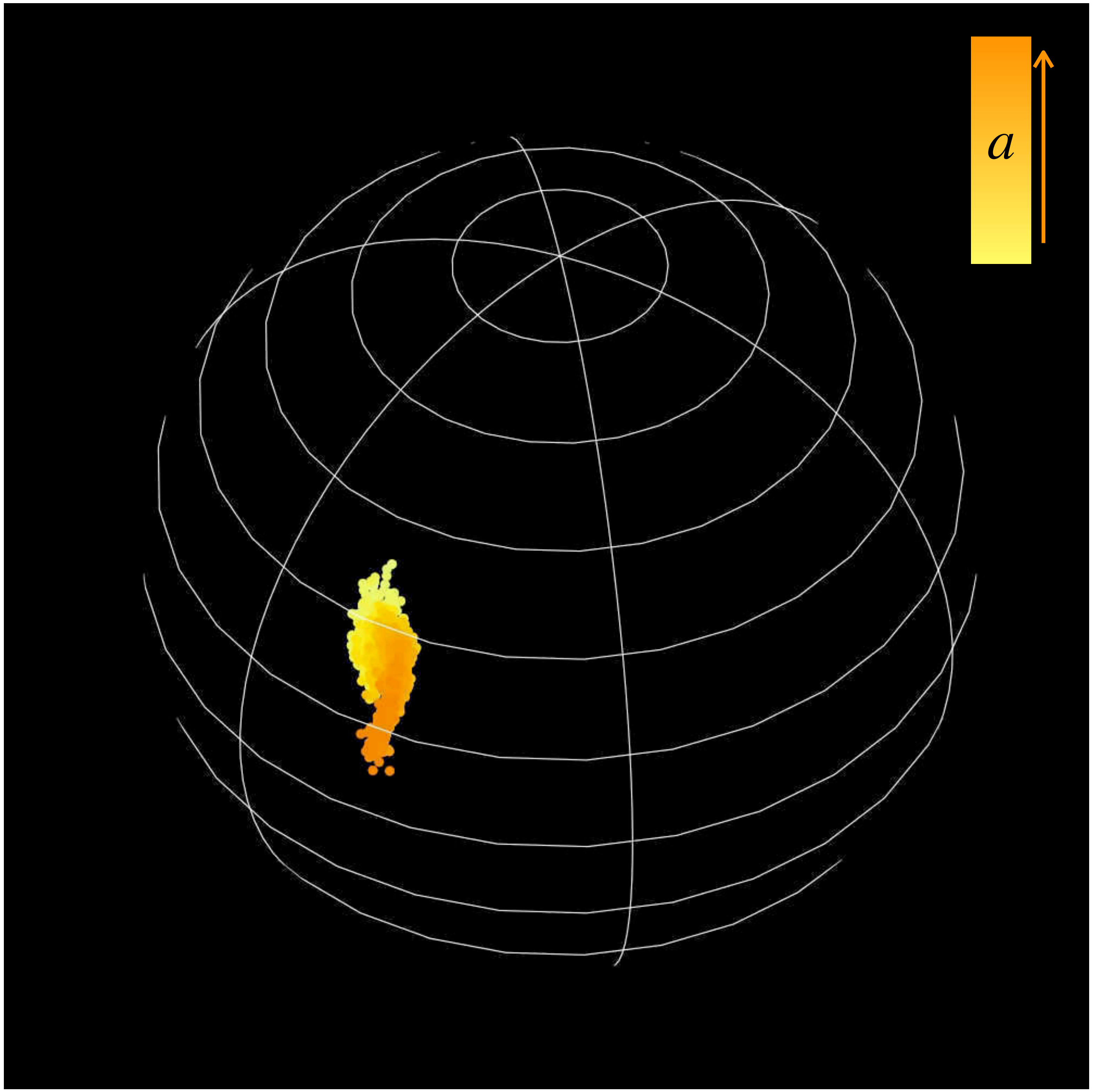




Typical timescale
~1,000,000 yr

The orbital orientation follows a **correlated random walk**.

Vector Resonant Relaxation



+ How "**neighbors**" get separated

$$\frac{\mathrm{d}\hat{\mathbf{L}}_i}{\mathrm{d}t} = \eta(\hat{\mathbf{L}}_i, t)$$

+ Evolution sourced by a **shared**, **spatially-extended** and **time-correlated** noise

$$\big\langle \eta(a_i, \hat{\mathbf{L}}_i, t)\, \eta(a_j, \hat{\mathbf{L}}_j, t') \big\rangle = C(a_i, a_j, \hat{\mathbf{L}}_i \cdot \hat{\mathbf{L}}_j, t - t')$$

+ Two joint sources of **separation**

- **Parametric** separation

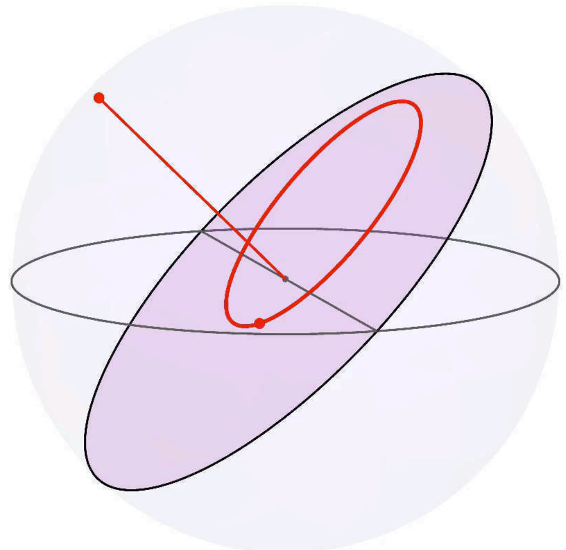
$$a_i \neq a_j$$

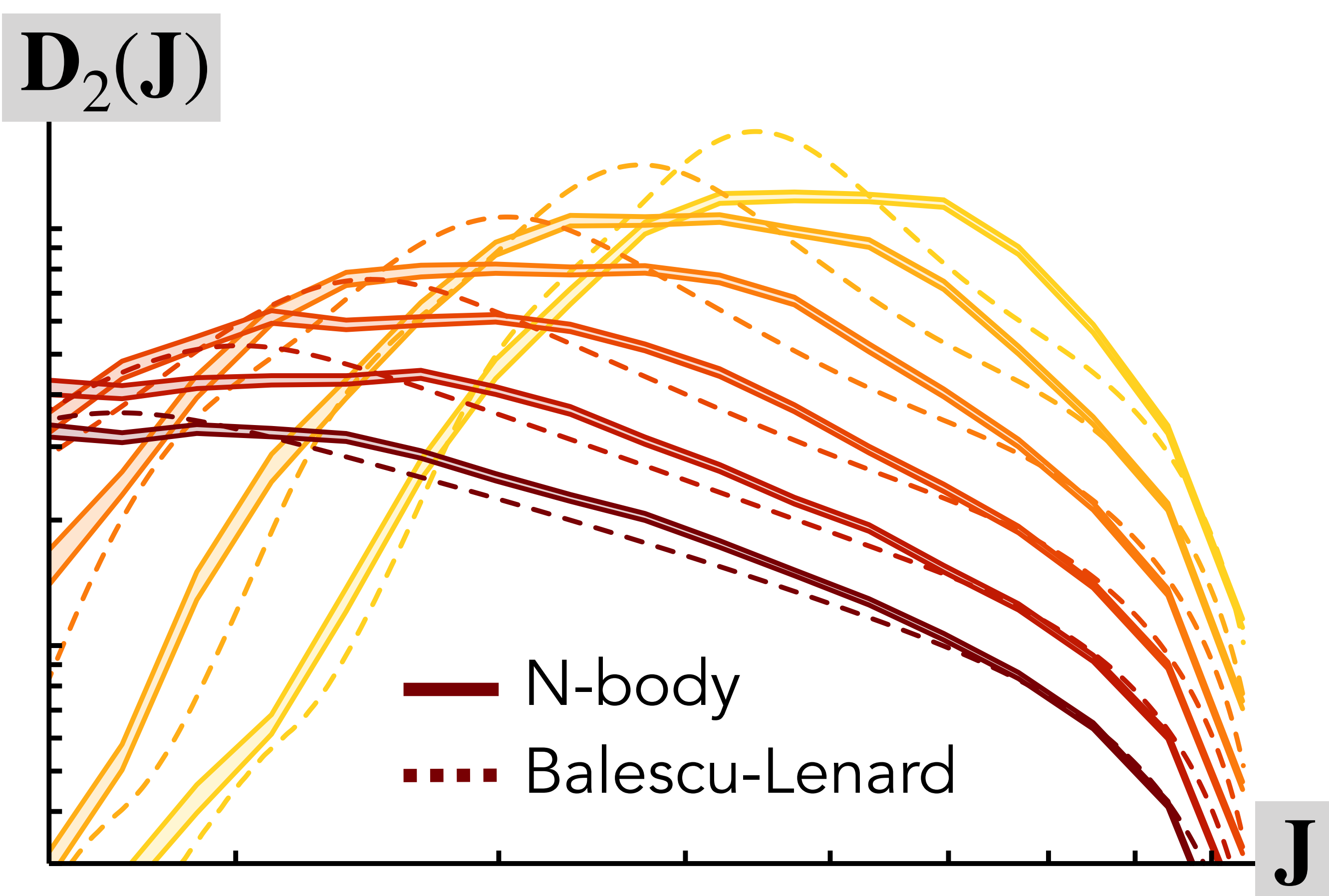
- **Angular** separation

$$\hat{\mathbf{L}}_i \neq \hat{\mathbf{L}}_j$$

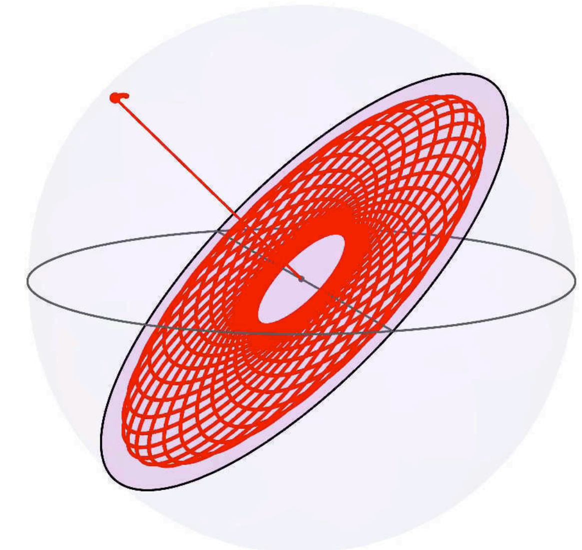
Resonant Relaxation in Galactic nuclei

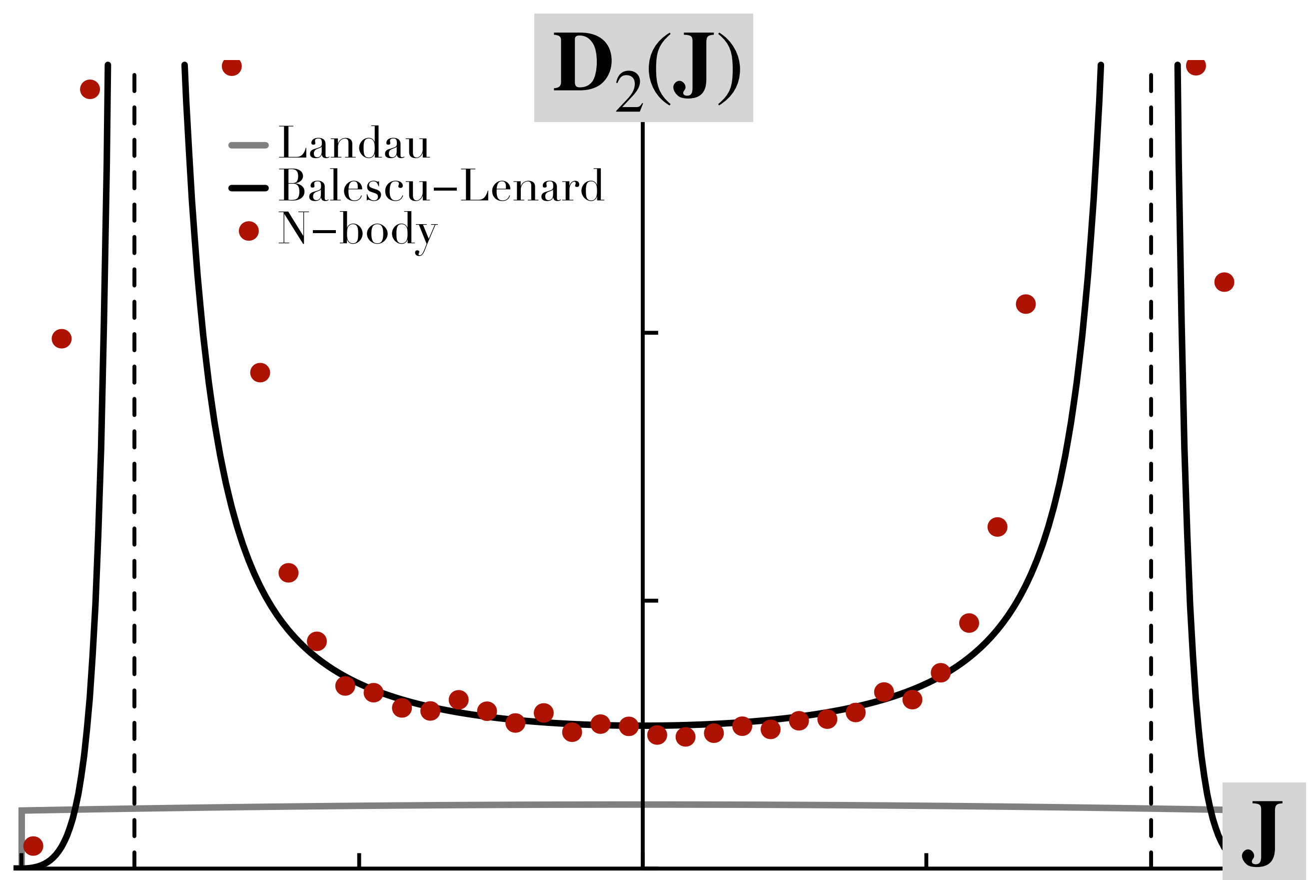
Relaxation
of **eccentricities**





Relaxation
of **orientations**





Kinetic theory can predict **long-term relaxations**.

What's next?

What's next?

Resonances

$$\mathbf{k}, \mathbf{k}' \to +\infty$$

$$\mathbf{\Omega}(\mathbf{J}) = \mathrm{cst}$$

Kinetic blockings

$$d = 1 \quad \text{and} \quad \frac{1}{N^2}$$

Deviations

$$\frac{\partial F_{\mathrm{d}}}{\partial t} \quad \text{vs} \quad \frac{\partial \langle F_{\mathrm{d}} \rangle}{\partial t}$$

Integrability

$$\Phi(\mathbf{x}, t) \neq \Phi(\mathbf{J}, t)$$

What's next?

Resonances

$$\mathbf{k}, \mathbf{k}' \to +\infty$$

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Integrability

$$\Phi(\mathbf{x}, t) \neq \Phi(\mathbf{J}, t)$$

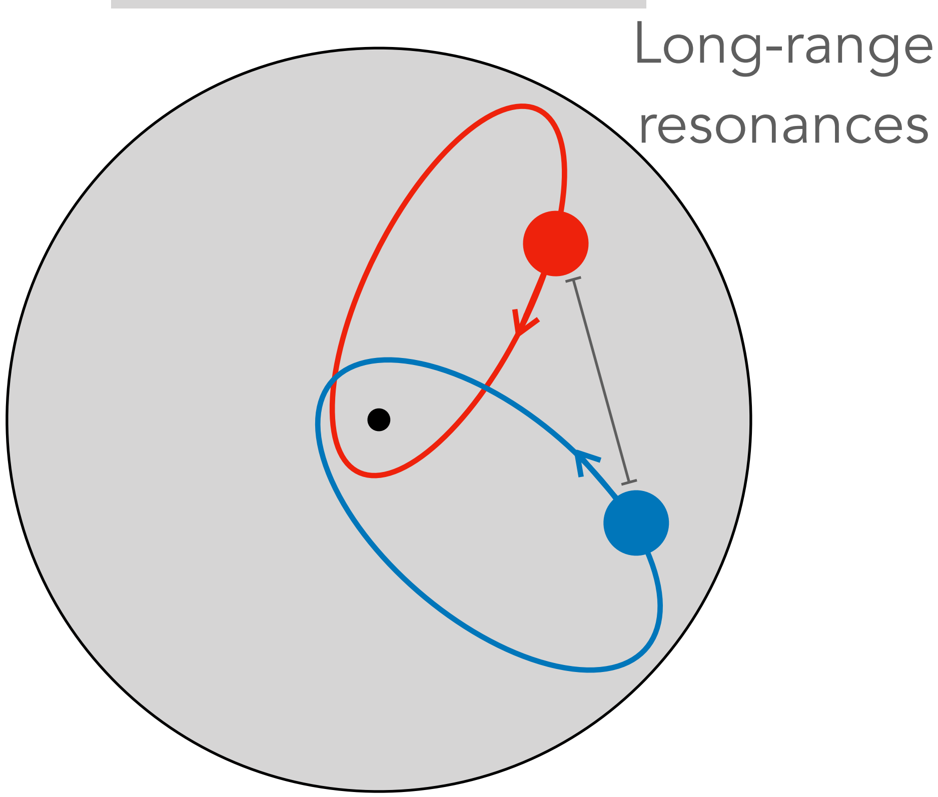
(Non)-resonant relaxation

What about **high-order resonances**?

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = -\frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k},\mathbf{k}'} \left(\cdots \right) \right]$$

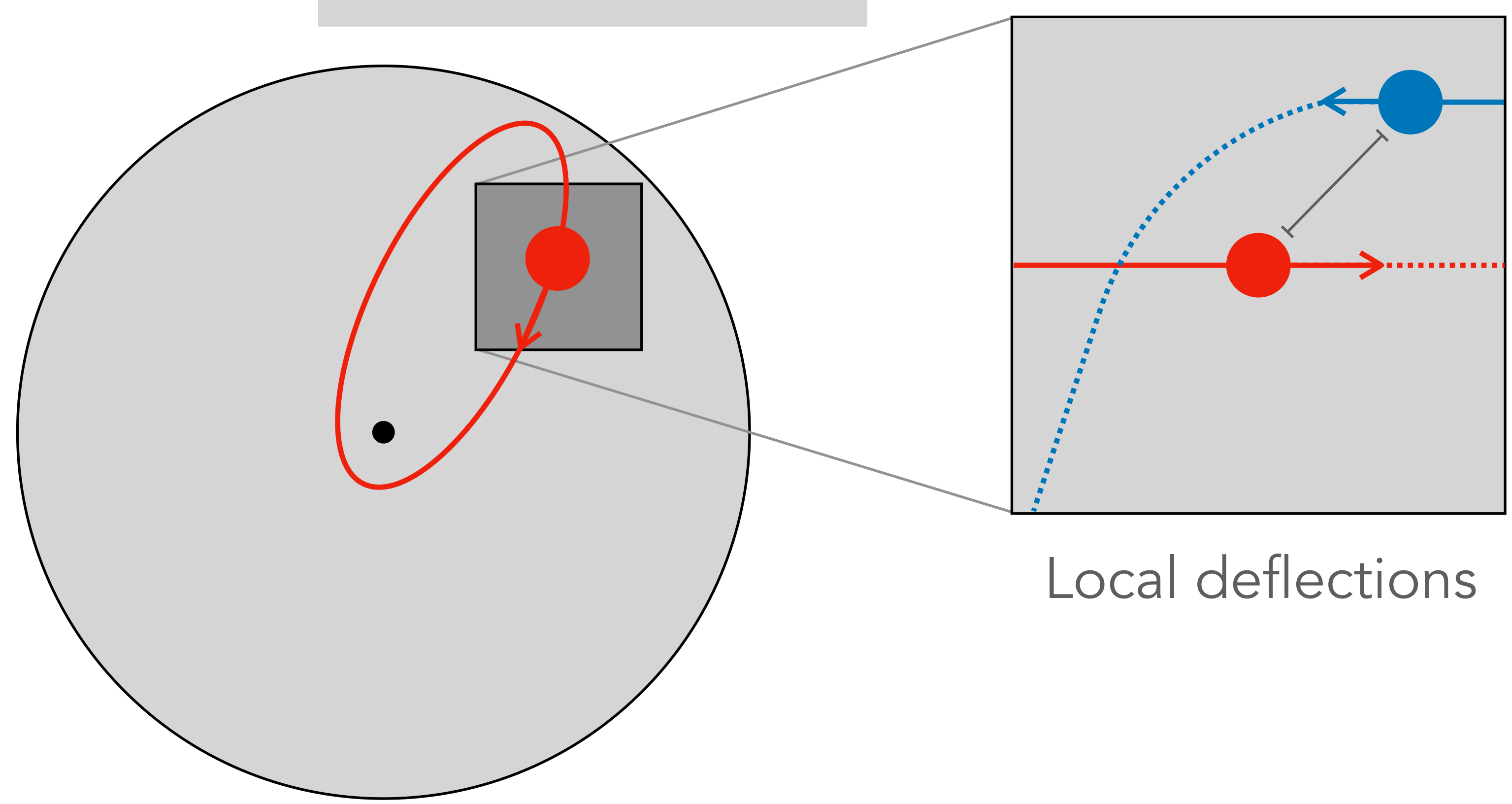
Resonant Relaxation

$$|\,\mathbf{k}\,|\,,\,|\,\mathbf{k}'|\,\simeq\,1$$



Non-Resonant Relaxation

$$|\,\mathbf{k}\,|\,,\,|\,\mathbf{k}'|\,\gg\,1$$



Where is the **Coulomb logarithm**?

$$\ln \Lambda = \ln(k_{\mathrm{min}}/k_{\mathrm{max}})$$

Fundamental degeneracies

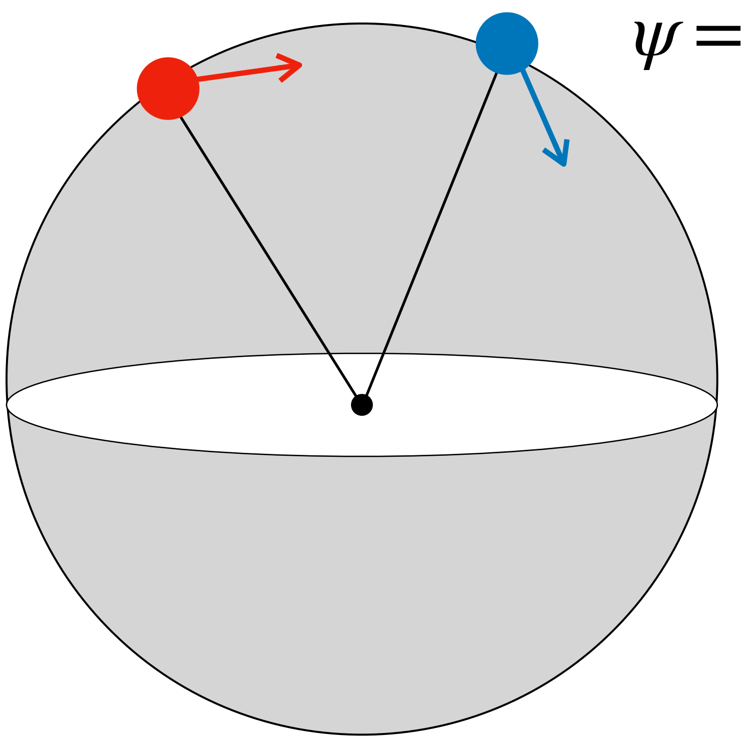
Dynamics in **degenerate** frequency profiles

$$\delta_{\mathrm{D}}(\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J})-\mathbf{k}'\cdot\boldsymbol{\Omega}(\mathbf{J}'))$$

Resonance condition

$$\forall \mathbf{J},\;\; \boldsymbol{\Omega}(\mathbf{J})=0$$

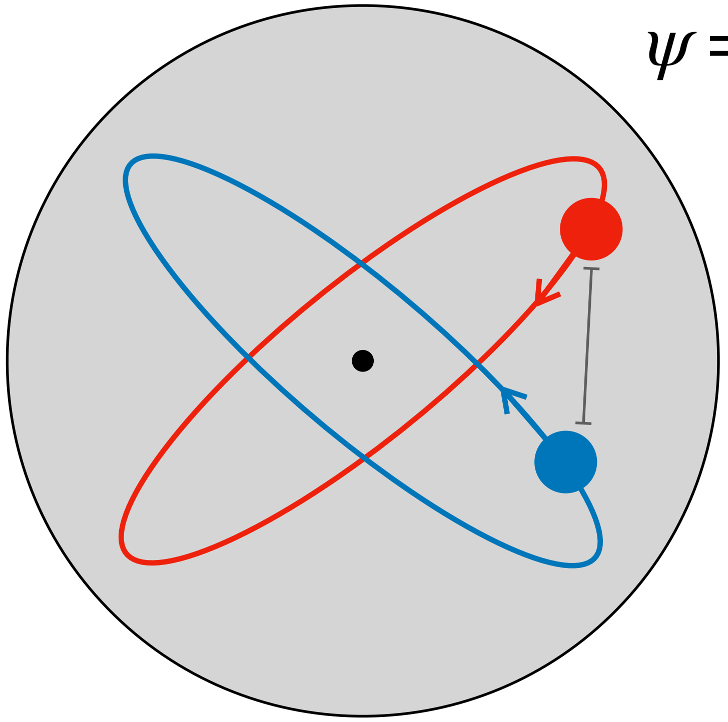
$$\psi = U(\hat{\mathbf{L}}\cdot\hat{\mathbf{L}}')$$



Vector Resonant Relaxation

$$\forall \mathbf{J},\;\; \boldsymbol{\Omega}(\mathbf{J})=\boldsymbol{\Omega}_0$$

$$\psi = -\int \frac{\mathrm{d}\theta\,\mathrm{d}\theta'}{|\mathbf{x}-\mathbf{x}'|}$$



Harmonic potential

How does relaxation occur in **degenerate systems**?

What's next?

Resonances

$$\mathbf{k}, \mathbf{k}' \to +\infty$$

$$\Omega(\mathbf{J}) = \mathrm{cst}$$

Kinetic blockings

$$d = 1 \quad \text{and} \quad \frac{1}{N^2}$$

Deviations

$$\frac{\partial F_{\mathrm{d}}}{\partial t} \quad \text{vs} \quad \frac{\partial \langle F_{\mathrm{d}} \rangle}{\partial t}$$

Integrability

$$\Phi(\mathbf{x}, t) \neq \Phi(\mathbf{J}, t)$$

Kinetic blockings

Generic **Balescu-Lenard** equation

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = \frac{1}{N}\frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \int \mathrm{d}\mathbf{J}' \, \frac{\delta_{\mathrm{D}}(\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\boldsymbol{\Omega}(\mathbf{J}'))}{\boxed{|\,E_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}))\,|^2}} \times \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} - \mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'}\right) F(\mathbf{J},t)\, F(\mathbf{J}',t)\right]$$

What happens in **1D systems**?

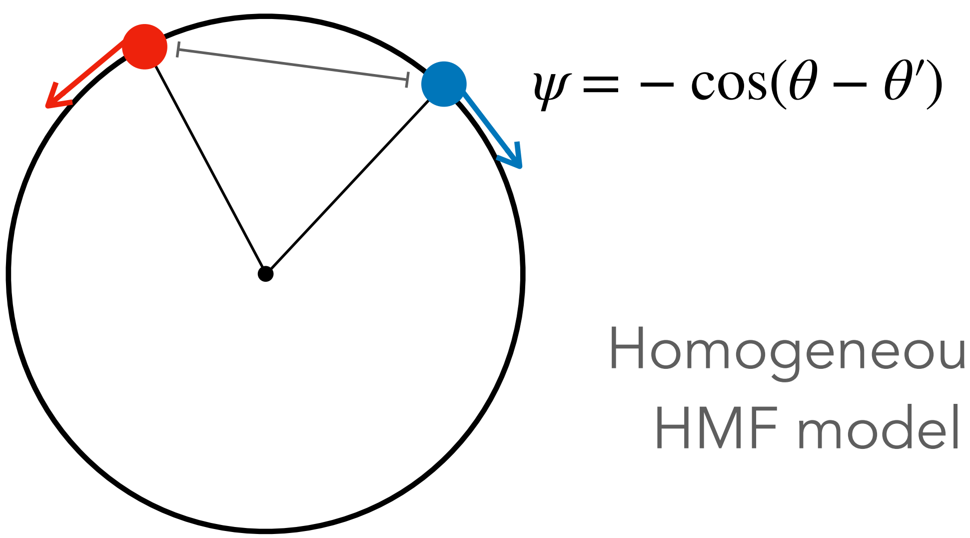
$$\begin{cases} \mathbf{k} = \mathbf{k}' = k \\ \mathbf{J} = \mathbf{J}' = J \end{cases} \longrightarrow$$

No relaxation!

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = \frac{1}{N} \times 0$$

Conspiracy for 2-body effects in 1D

$$\begin{cases} v_1 + v_2 = \mathrm{cst} \\ v_1^2 + v_2^2 = \mathrm{cst} \end{cases}$$



Homogeneous
HMF model

Kinetic theory at order $1/N^2$

$1/N^2$ kinetic equation

Without collective effects | *Without inhomogeneity*

$$\frac{\partial F(v)}{\partial t} = \frac{1}{N^2} \frac{\partial}{\partial v} \Bigg[\sum_{k_1,k_2} U(k_1,k_2)\, \mathcal{P}\!\int \frac{\mathrm{d}v_1}{(v-v_1)^4} \times \int \mathrm{d}v_2\, \delta_{\mathrm{D}}\big[(k_1+k_2)\,v - k_1\,v_1 - k_2\,v_2\big] \times \bigg((k_1+k_2)\frac{\partial}{\partial v} - k_1 \frac{\partial}{\partial v_1} - k_2 \frac{\partial}{\partial v_2} \bigg) F(v)\, F(v_1)\, F(v_2) \Bigg]$$

+ How do **collective effects** contribute?
+ How do **frequency profiles** contribute?
+ What is the structure of kinetic theories at **higher order** $1/N^s$?

What’s next?

Resonances

$$\mathbf{k}, \mathbf{k}' \to +\infty$$

$$\Omega(\mathbf{J}) = \mathrm{cst}$$

Kinetic blockings

$$d = 1 \quad \text{and} \quad \frac{1}{N^2}$$

Deviations

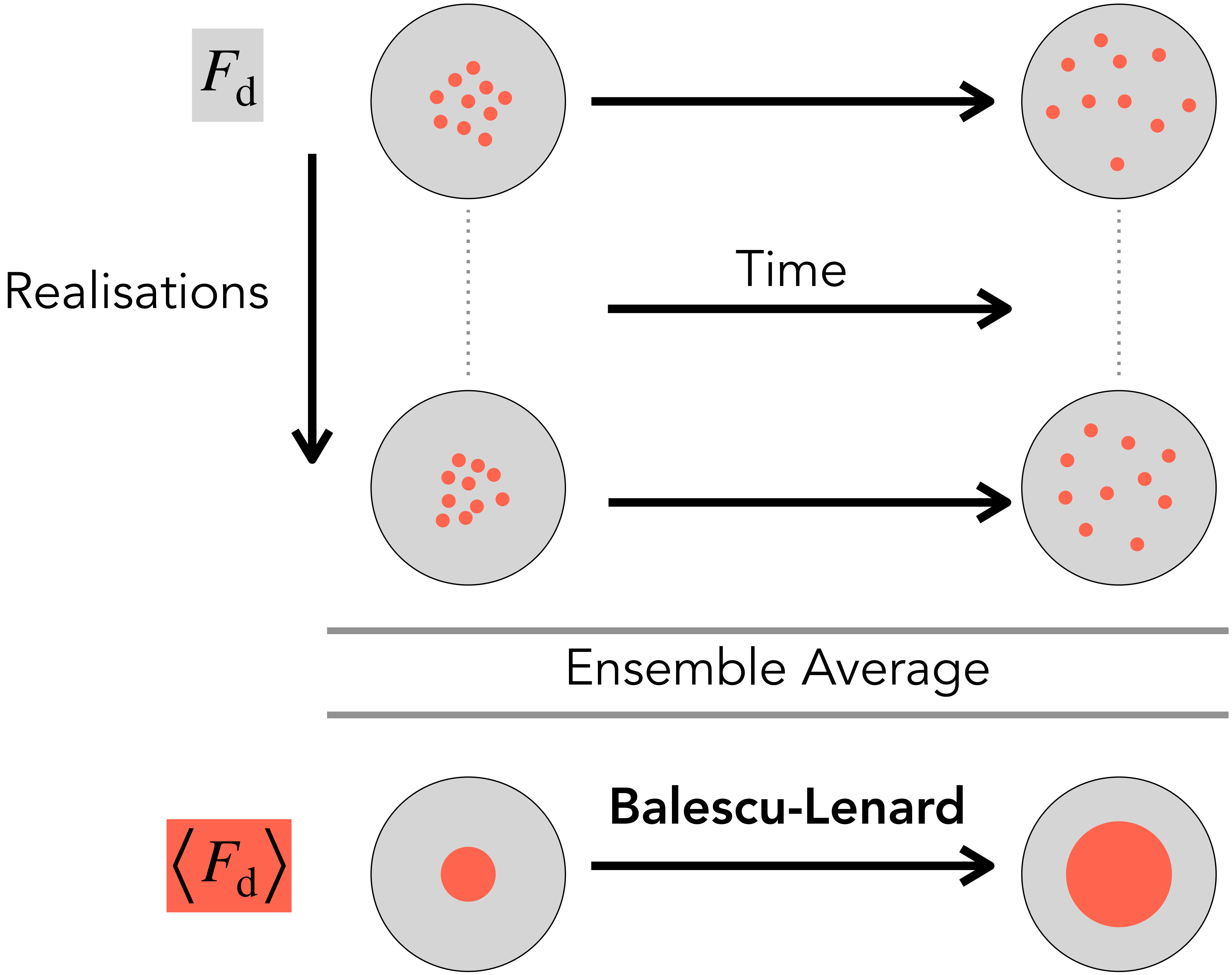
$$\frac{\partial F_{\mathrm{d}}}{\partial t} \quad \text{vs} \quad \frac{\partial \langle F_{\mathrm{d}} \rangle}{\partial t}$$

Integrability

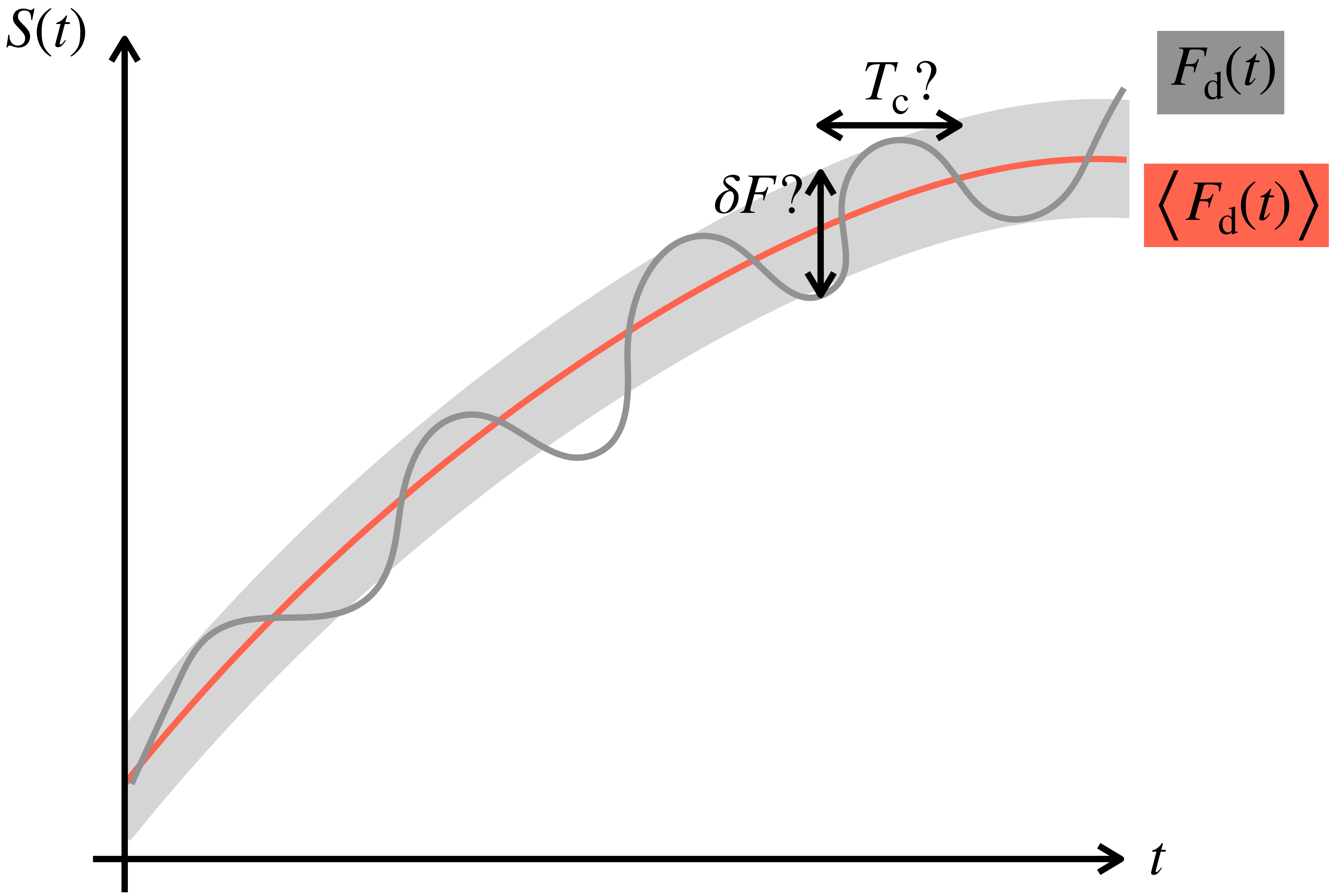
$$\Phi(\mathbf{x}, t) \neq \Phi(\mathbf{J}, t)$$

Faking the dynamics

Kinetic theory predicts the **ensemble average** dynamics

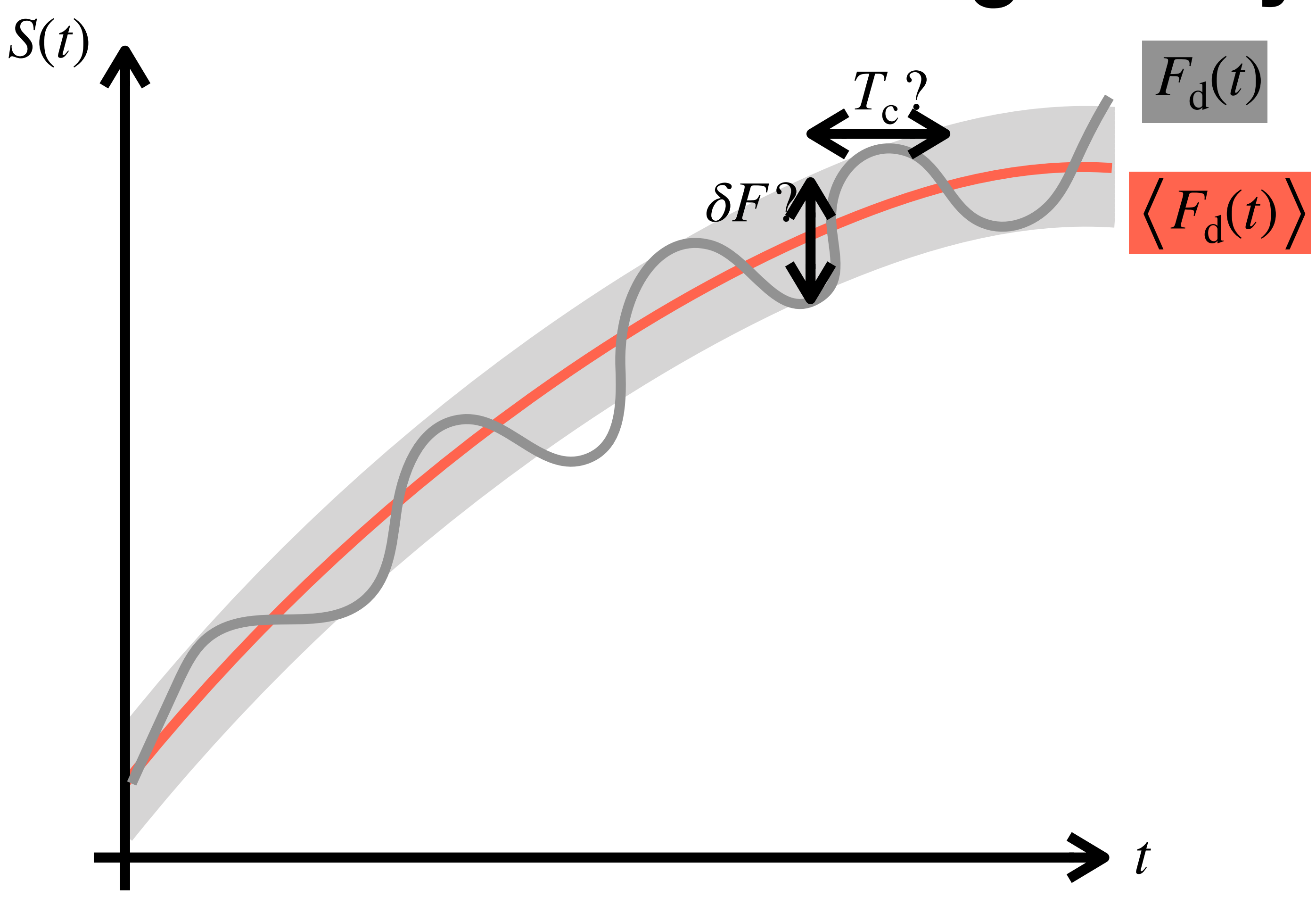


Faking the dynamics



One realisation vs. **mean kinetic prediction**

Faking the dynamics



What is the statistics of **(large) deviations**?

Probability of a given realisation?

$$\mathbb{P}\big(F_d(t) = F_0(t)\big)$$

Most likely realisation?

$$\mathbb{P}\big(F_d(t) = \langle F_d(t) \rangle\big)$$

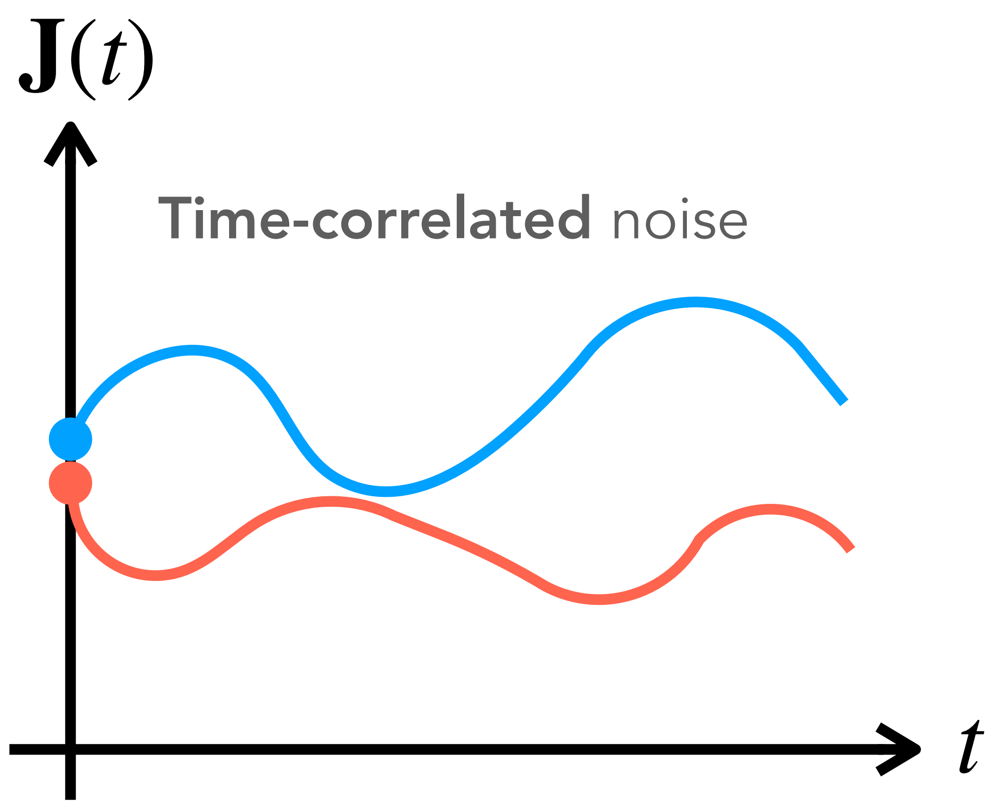
Can one **fake** realisations?

$$\frac{\partial F_d}{\partial t} = \mathrm{BL}[F_d(t)] + \eta[F_d(t)]$$

with the noise

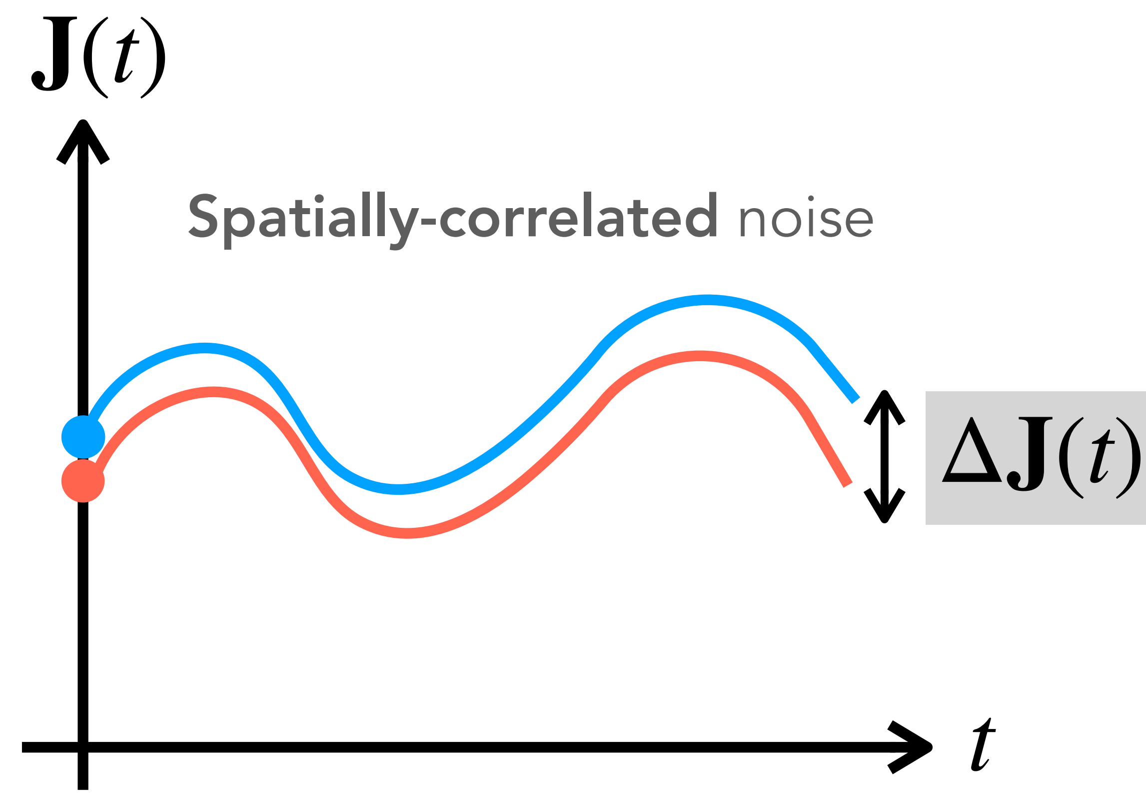
$$\big\langle \eta[F_d]\, \eta[F_d] \big\rangle = ??$$

Faking the dynamics



Independent random walks

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = \eta(\mathbf{J}(t), t)$$



Simultaneous random walks

$$\frac{\mathrm{d}\Delta\mathbf{J}}{\mathrm{d}t} = \eta(\mathbf{J}_1(t), \mathbf{J}_2(t), t)$$

How do **nearby particles** separate?

What's next?

Resonances

$$\mathbf{k}, \mathbf{k}' \to +\infty$$

$$\mathbf{\Omega}(\mathbf{J}) = \mathrm{cst}$$

Kinetic blockings

$$d = 1 \quad \text{and} \quad \frac{1}{N^2}$$

Deviations

$$\frac{\partial F_{\mathrm{d}}}{\partial t} \quad \text{vs} \quad \frac{\partial \langle F_{\mathrm{d}} \rangle}{\partial t}$$

Integrability

$$\Phi(\mathbf{x}, t) \neq \Phi(\mathbf{J}, t)$$

Going beyond isolated, integrable, resonant

Systems are not always **isolated**

$$\begin{cases} N = N(t) \\ \big[\delta H(t)\big]_{\text{tot}} = \big[\delta H(t)\big]_{\text{Poisson}} + \big[\delta H(t)\big]_{\text{ext}} \end{cases}$$

Structure formation
Open clusters
Collisionless relaxation

Systems are not always **integrable**

$$\left[\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t}\right]_{\text{tot}} = \left[\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t}\right]_{\text{resonant}} + \left[\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t}\right]_{\text{chaotic}}$$

Thickened discs
Barred galaxies
Flattened halos

Systems are not always ``nicely'' **resonant**

$$\mathbf{\Omega}(\mathbf{J}) = \big(\Omega_1(\mathbf{J})\,,\epsilon\,\Omega_2(\mathbf{J})\big)$$

Mean-motion resonances
Eviction resonances
Precession resonances

Conclusion

Kinetic theory of self-gravitating systems

Long-range interacting systems are ubiquitous

Inhomogeneous	Self-gravitating	Resonant
$(\mathbf{x}, \mathbf{v}) \downarrow (\boldsymbol{\theta}, \mathbf{J})$	$\dfrac{1}{\lvert E(\omega) \rvert}$	$\mathbf{k} \cdot \mathbf{\Omega}(\mathbf{J})$

Master equation for **dressed resonant relaxation**

$$\frac{\partial F(\mathbf{J}, t)}{\partial t} = \frac{1}{N} \frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \int \mathrm{d}\mathbf{J}' \, \frac{\delta_{\mathrm{D}}(\mathbf{k} \cdot \mathbf{\Omega}(\mathbf{J}) - \mathbf{k}' \cdot \mathbf{\Omega}(\mathbf{J}'))}{\lvert E_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}', \mathbf{k} \cdot \mathbf{\Omega}(\mathbf{J})) \rvert^2} \times \left(\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{J}} - \mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{J}'} \right) F(\mathbf{J}, t) \, F(\mathbf{J}', t) \right]$$