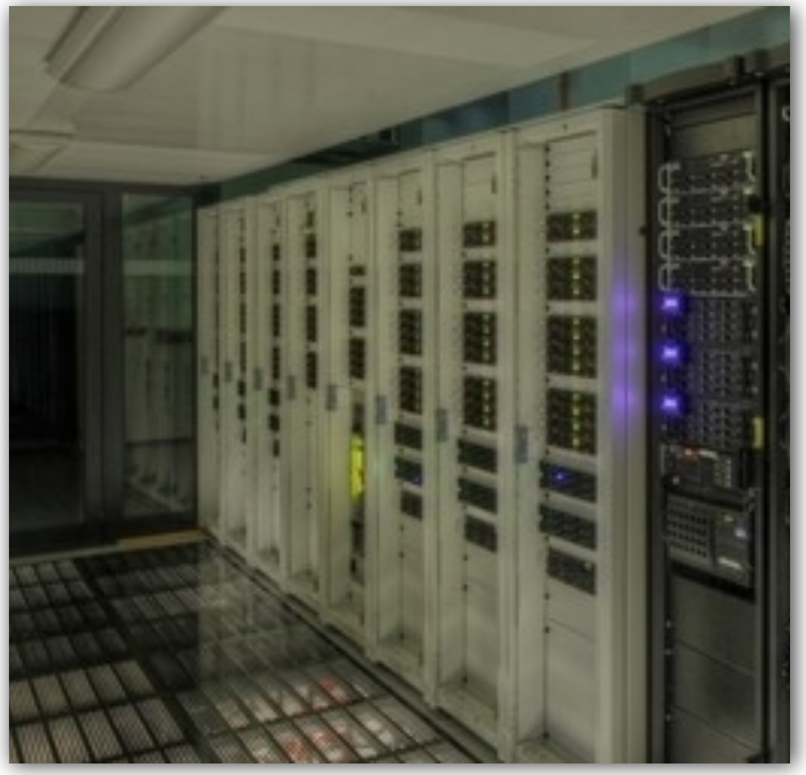

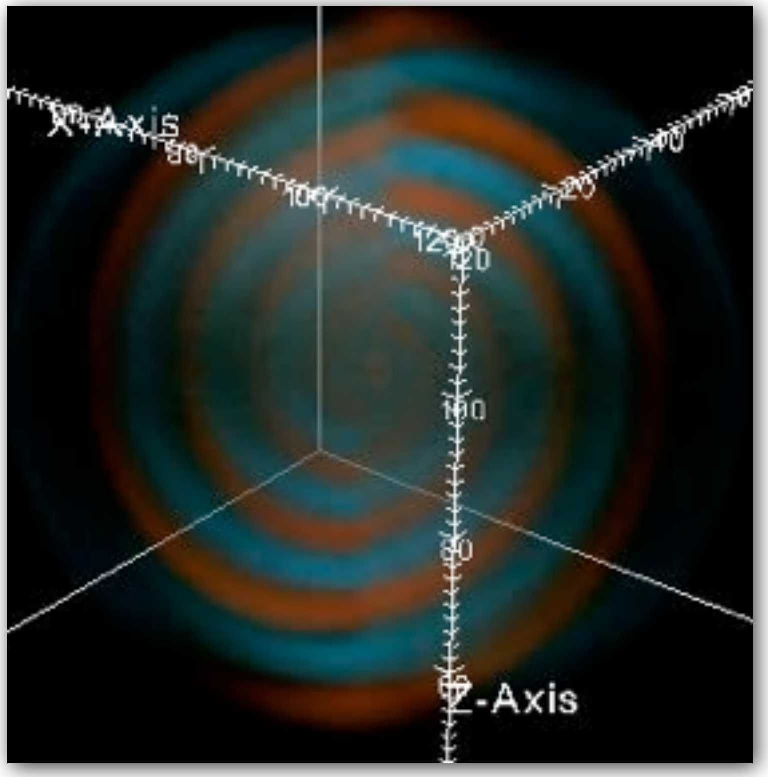

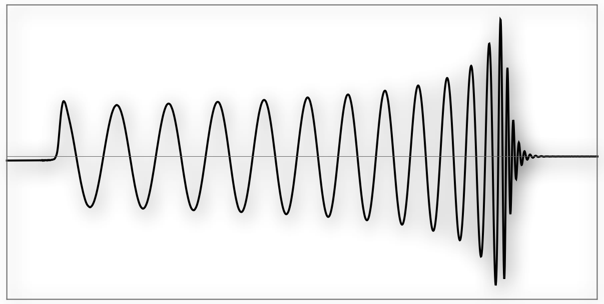

Gravitational Waveforms from Numerical Relativity

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GRavitational-wave Astronomy Meeting in PAris, September 2016

Why do we need Numerical Relativity?

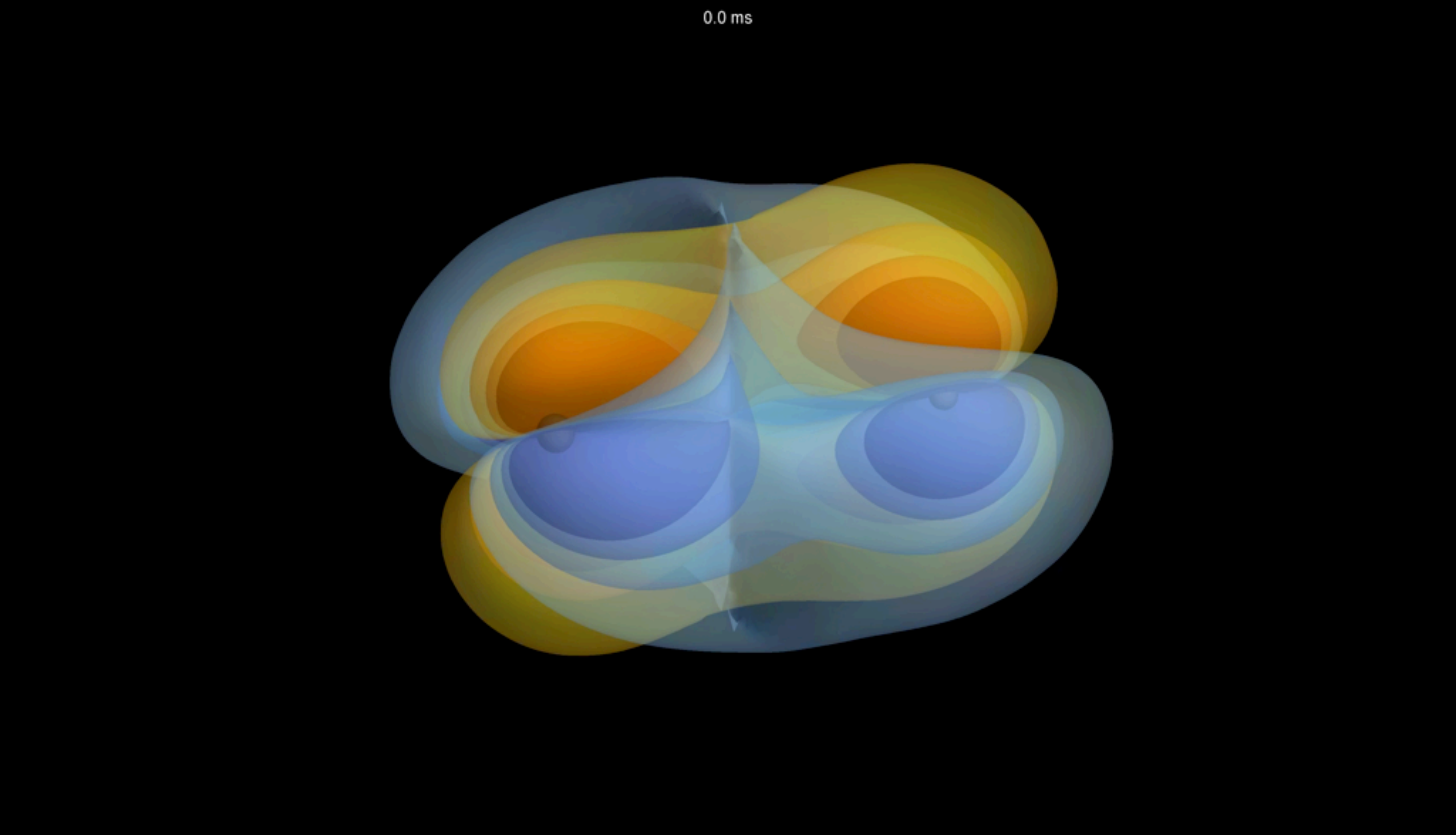
- GW150914 (loud):
    - detected from **generic transient** search: < 5 sigma
    - SEOB**NR**v2 search: > 5 sigma
- GW151226 (quiet): required matched filtering against detailed realistic **waveform models**
- Pure **post-Newtonian** waveform models: v/c << 1. Terminate before merger
- **EOB model** includes merger and ringdown; but how good is it?
- Suppose we had the **exact waveform** from GR:
    - **Test** models and **improve**
- **Numerical Relativity** closest to exact GR spacetime for compact binary coalescence
- **EOBNR/Phenom** waveform families based on NR
    - Used in LIGO searches and parameter estimation
    - Calibrated to and tested against **NR**
- **Numerical Relativity gives the final word**

What exactly *is* Numerical Relativity?

- **Direct solution** of the full nonlinear Einstein equations using **numerical methods**
- Ideal case:
  - Solution plus error estimate. Error can be made **arbitrarily small**. Price is computational **cost**.
  - Compare post-Newtonian (not in strong field), or perturbation theory (close to exact solutions)
- Non-ideal case:
  - **Continuum problem** incomplete? e.g. boundary conditions, initial data.
  - Compare experiment and simulation:
    - Experiment: **random error** and **systematic error**
    - Simulation: **numerical error** and **continuum approximation error**

Overview

- Introduction
- Numerical Relativity
- Waveforms from Numerical Relativity
- Recent results
- Summary



1. Numerical Relativity

Image: Simulation of merger of GW150914, Weyl scalar ψ_4
- Barry Wardell, Einstein Toolkit

Mathematical formulation

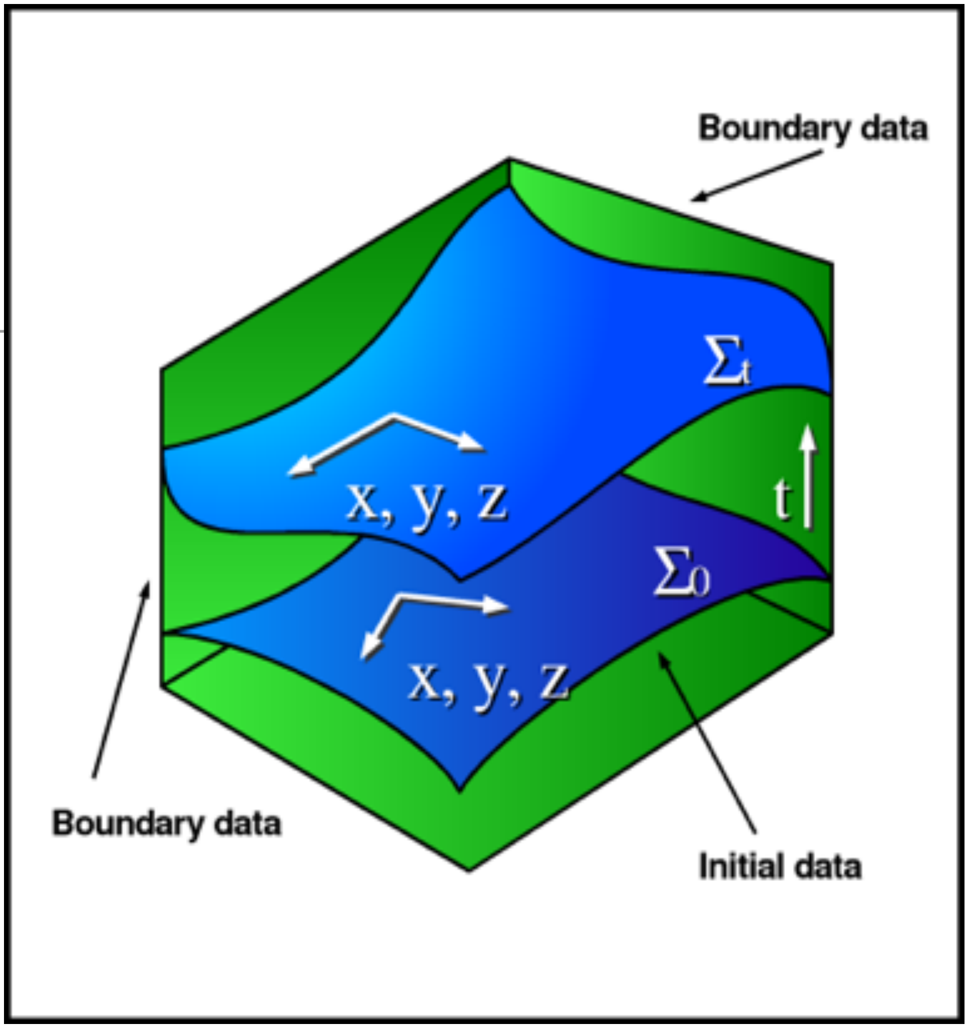
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$$

- 10 coupled nonlinear 2nd order partial differential equations:

$$\begin{aligned} {}^{(4)}R_{\mu\nu} &\equiv \frac{1}{2} g^{\sigma\rho}(g_{\sigma\nu,\mu\rho} + g_{\mu\rho,\sigma\nu} - g_{\sigma\rho,\mu\nu} - g_{\mu\nu,\sigma\rho}) \\ &\quad + g^{\sigma\rho}({\Gamma^m}_{\mu\rho}\Gamma_{m\sigma\nu} - {\Gamma^m}_{\mu\nu}\Gamma_{m\sigma\rho}) \\ {\Gamma^\mu}_{\nu\sigma} &\equiv \frac{1}{2} g^{\mu\rho}(g_{\rho\nu,\sigma} + g_{\rho\sigma,\nu} - g_{j\sigma,\rho}) \end{aligned}$$

- Formulate as **initial value problem** by projecting onto a **foliation** of 3D t=const slices:

$$\frac{\partial}{\partial t} u(t, x^i) = F\left(u(t, x^i), \partial u(t, x^i), \partial^2 u(t, x^i)\right)$$



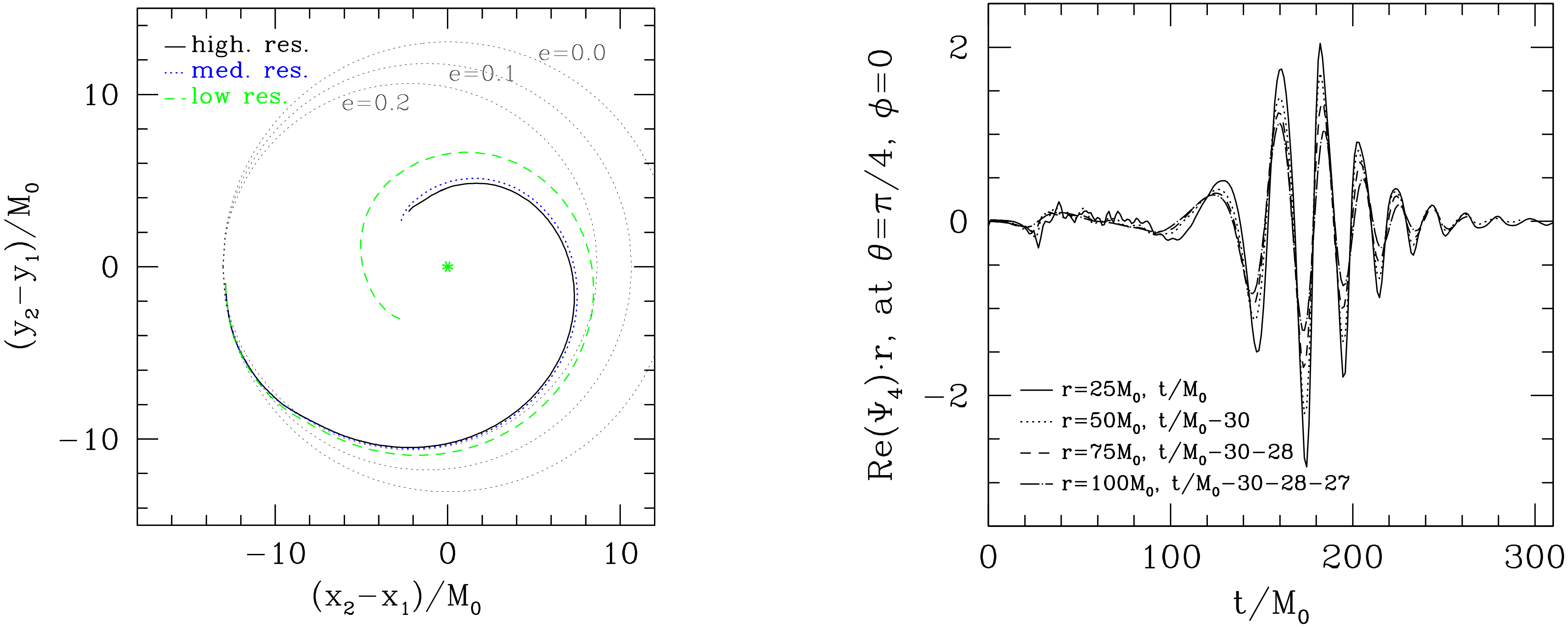
- Initial data (t=0) evolved forward in time with **evolution equations**
- Also get **constraint equations** on each t=const slice

Milestones of Numerical Relativity - Pre-revolutionary

1959	Arnowitt, Deser and Misner - ADM formalism: **initial value problem** for GR
1964	Hahn and Lindquist, also Smarr and Eppley: **first numerical solution** to the Einstein equations: attraction between two wormholes in axisymmetry;
1980s	Piran, Stark - **gravitational waves** in axisymmetry from formation of axisymmetric BH
1980s	Choptuik - **Critical collapse** with adaptive mesh refinement
1990s	Binary Black Hole Grand Challenge - **Head-on BBH collision**
to 2005	Development of **coordinate conditions** and **excision** techniques, wave extraction formalisms.
	Finite simulation lifetime, solutions unstable, much frustration

Milestones of Numerical Relativity - The revolution begins

2005 **Pretorius** is the first to successfully evolve **more than one orbit of a BBH** through **merger and ringdown** and compute the **gravitational waveform**



Milestones of Numerical Relativity - The Golden Age

2005	Pretorius, long-term stable method for orbit using **excision**, **finite difference** methods and **adaptive mesh refinement**, **generalised harmonic formulation**
2005	Goddard and Brownsville groups: **Moving puncture method** (no excision): **finite differences**, **BSSN formulation**
2006	Buonanno, Cook and Pretorius: Detailed **comparison with PN**
2007	Campanelli, Lousto, Zlochower, Merritt, and Gonzalez, Hannam, Sperhake, Bruegmann, Husa - Unexpectedly high "**super-kick**" of merging BHs for certain spin orientations
2008	**Inspiral** waveform from the **SpEC** code (**pseudo-spectral methods,** dual coordinate frames, excision, generalised harmonic formulation)
2009	**Inspiral-merger-ringdown** simulation from the **SpEC** code
2011	Lovelace, Scheel, Szilagyi - Breaking the **high spin limit** (~0.93) of Bowen-York conformally flat initial data
2015	Waveform models built on NR results used in LIGO searches and parameter estimation for first GW detection

Approaches to the BBH problem 1

- Initial data:
  - **Elliptic** constraint equations
  - **Junk** radiation
- **Formulations** (BSSN, CCZ4, generalised harmonic):
  - 3+1 decomposition of the Einstein equations is **not unique**
  - Well-posedness

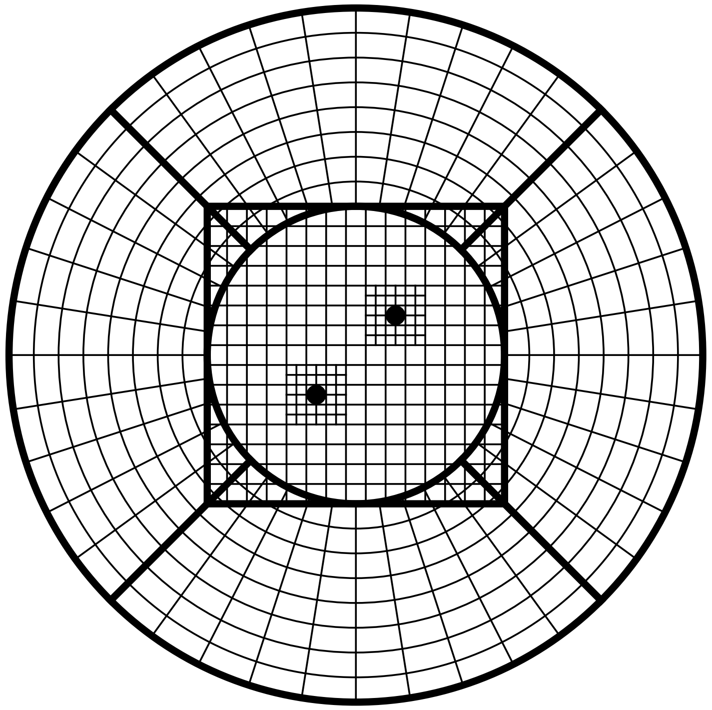
- **Coordinate freedom** of GR:
  - Well-behaved coordinates
  - Choose dynamically by evolving along with the spacetime
- Physical black hole **singularities**?
  - Excision
  - Punctures

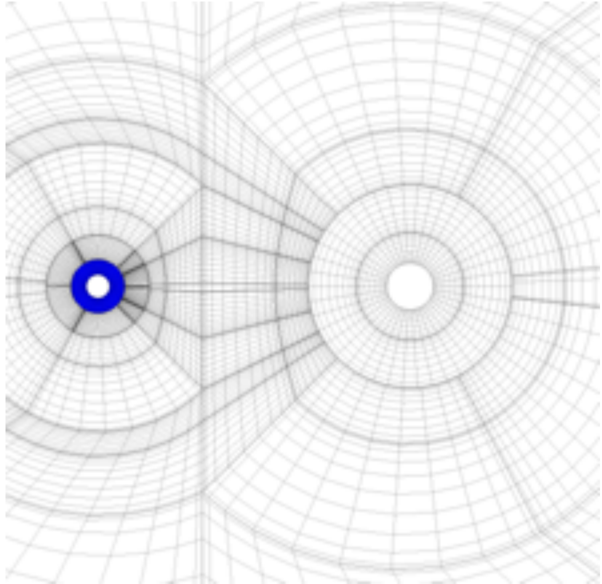
Approaches to the BBH problem 2

- Numerical methods? Two main approaches:
  - **Finite differences**: spatial derivatives from subtracting neighbouring points.
    e.g. error = $O(\Delta x^8)$.
  - **Spectral**: expand **solution** in basis functions. Spatial derivatives of basis functions analytic.
    e.g. error = $O(e^{-cN})$

Approaches to the BBH problem 3

- What type of **numerical grid**?
  - Regular Cartesian grid patches
  - Boxes of high resolution around the BHs - **mesh refinement**
  - Angular grids (r, th, ph) for the **wave zone**
  - Complex grid geometries **adapted** to the shape of the binary
  - **Rotate the grid** (dual frame method) with the binary to reduce errors





NR codes

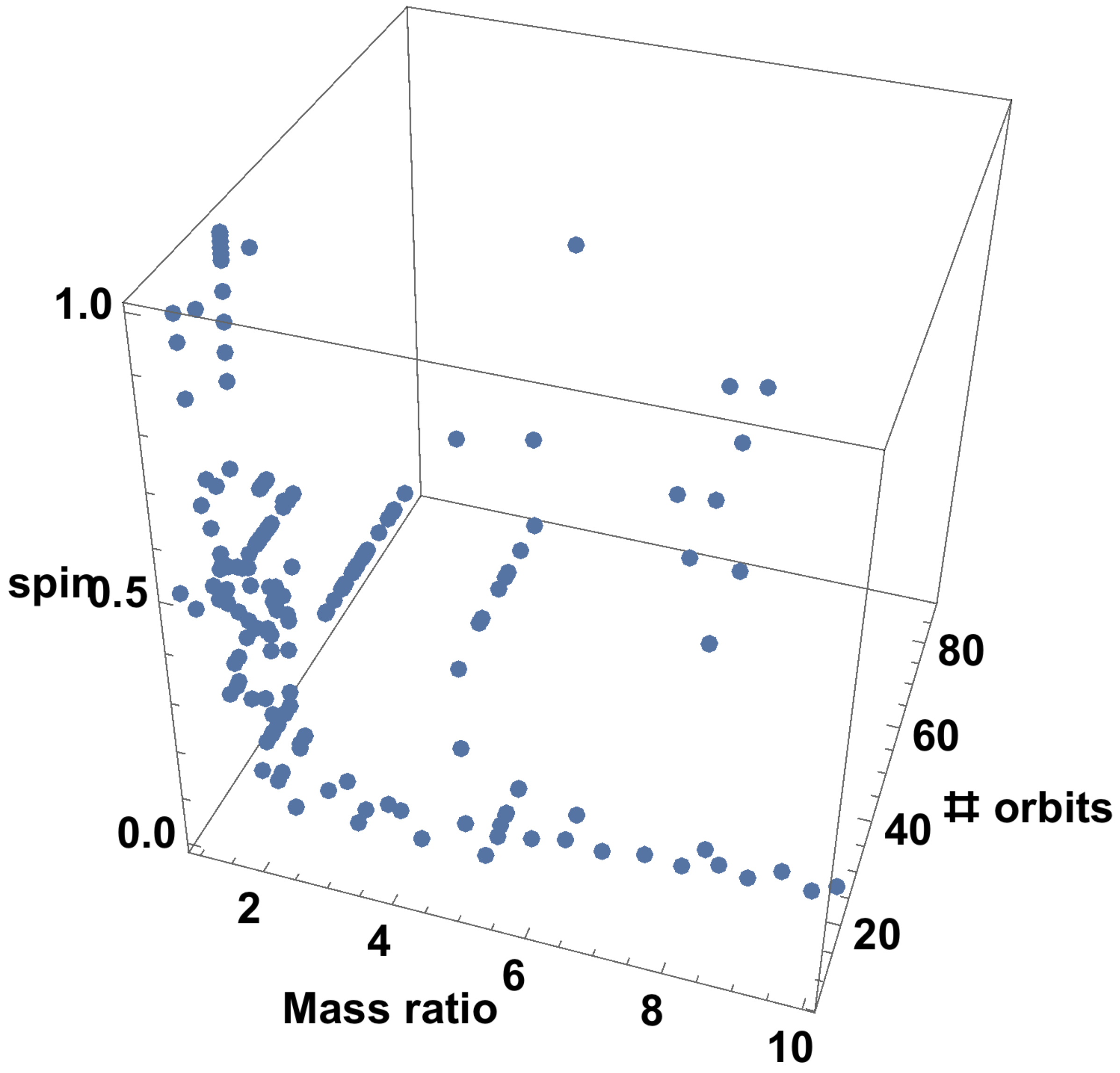
- Finite difference codes:
    - **Cactus**-based: Einstein Toolkit, Maya, LazEv, Illinois
    - SACRA
    - BAM
    - GR**Chombo:**
- Pseudospectral:
    - SpEC
- + others (apologies)

What can we do today?

- **Stable** evolutions of **moderate** BBH configurations:
  - Mass ratio $q = m_1/m_2 < \sim 10$
  - Spins chi $= S/m^2 < \sim 0.6$
  - Number of orbits $N < \sim 15$
- Main problem: different **length scales**
- Different **codes** have different strengths:
  - SpEC: large numbers of orbits with **high phase accuracy**
  - Moving puncture finite difference: extremely robust

How high can we go....

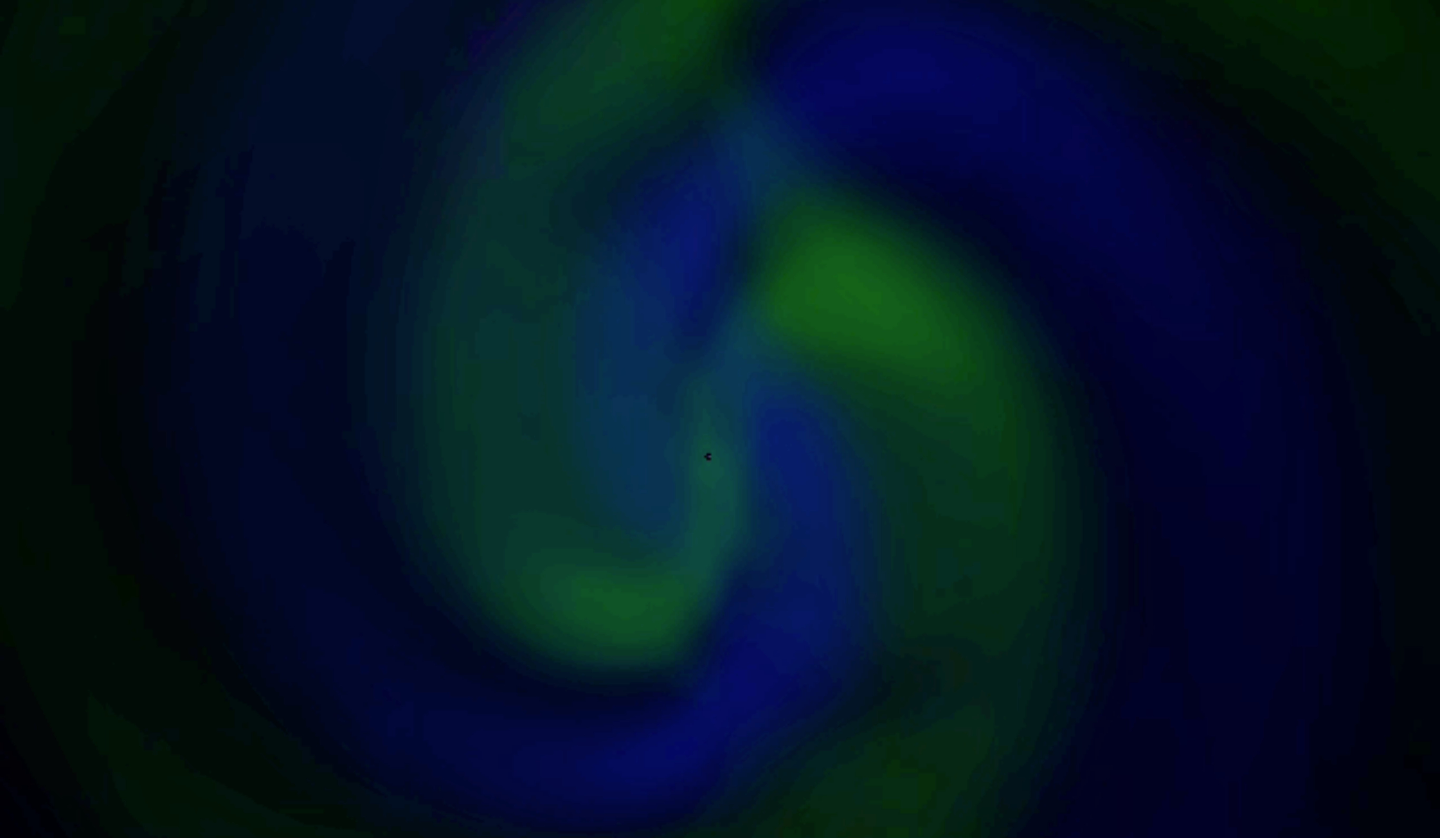
- ...in mass ratio?
  - **q=100** for ~ 1 orbit
  - **q=18** for 10 orbits
- ...in number of orbits?
  - **175 orbits** (q=7)
- ...in spins?
  - **S/m2 = 0.994** for 25 orbits, q=1



SXS public simulations catalogue

Some conceptual issues

- **Extrapolation** of waves to Scri
- Asymptotic **frame** / centre of mass
- **Spin** direction



2. Waveforms from Numerical Relativity

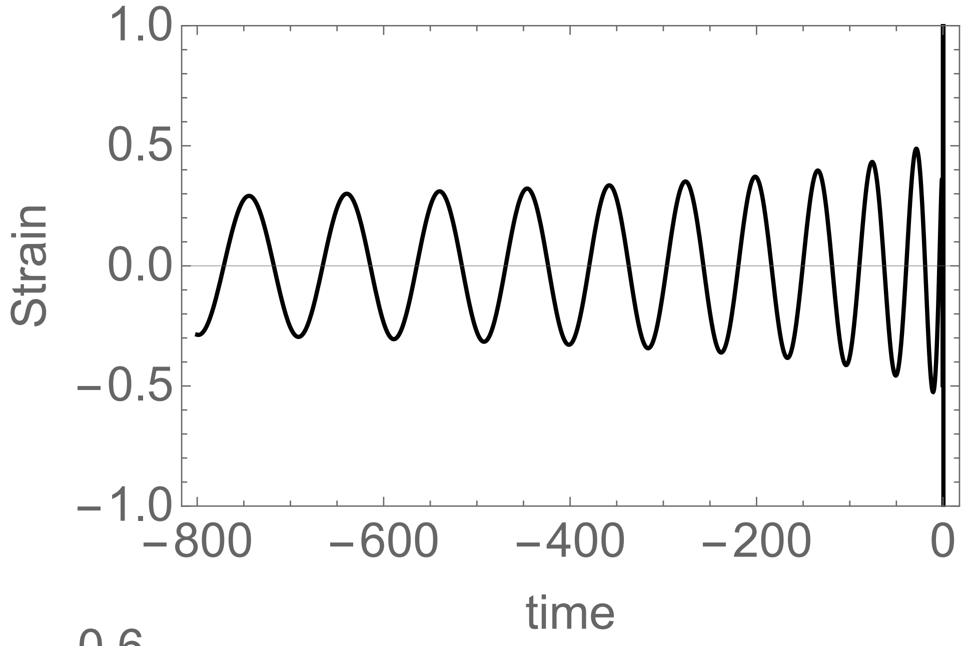
Image: Gravitational wave strain from simulation of GW150914

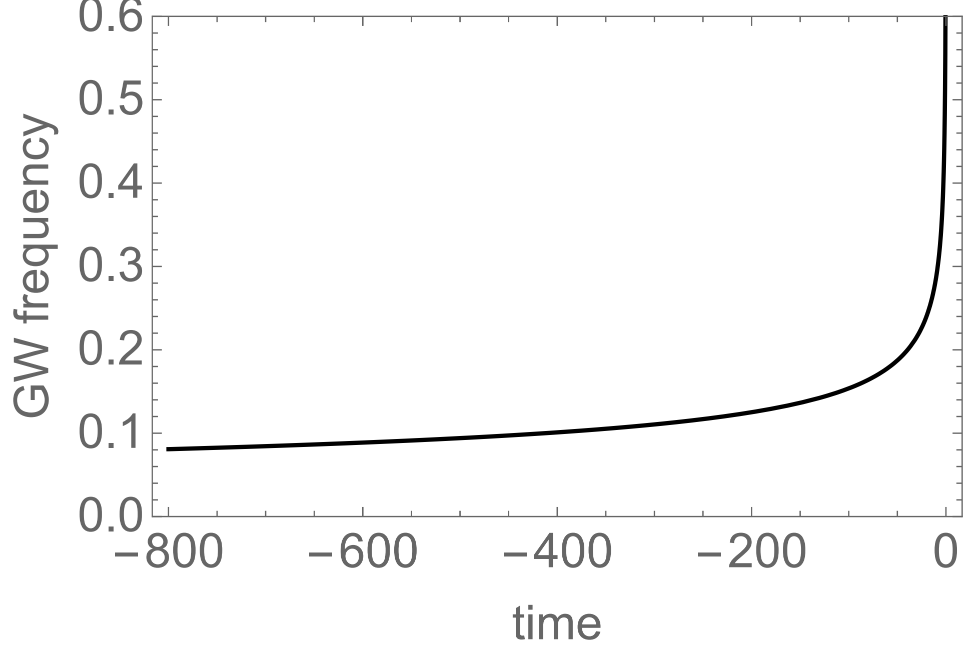
What does a BBH waveform look like?
1. Early inspiral

- **post-Newtonian** gives the waveform when **v/c << 1**
  - $h = A(t)\, e^{i\,\Phi(t)}$
  - Eventually **blows up**, as

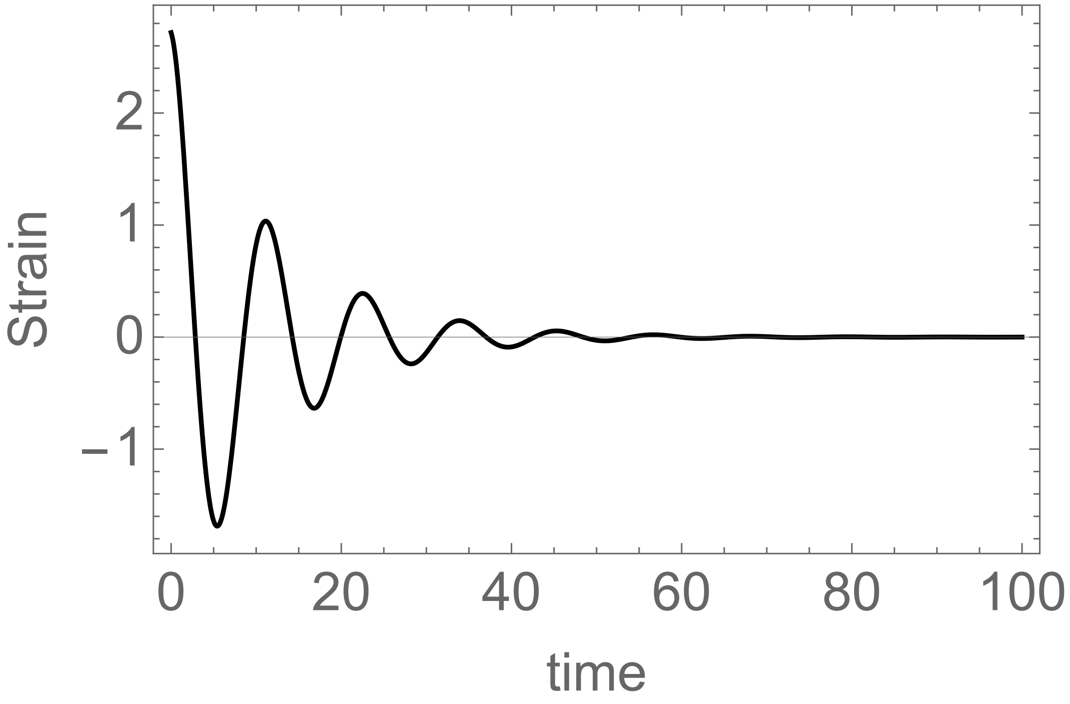
    $v \sim r\,\Phi'$

    is no longer small close to merger





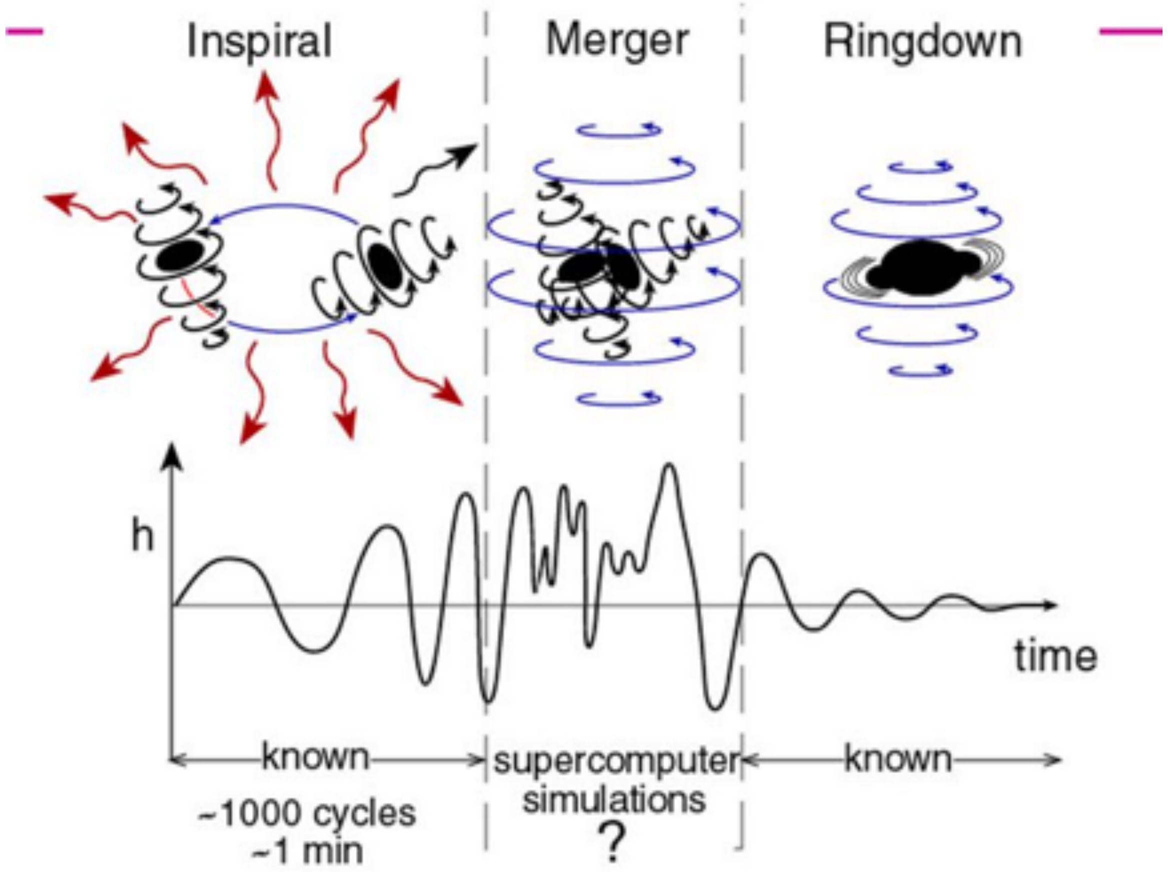
What does a BBH waveform look like?
2. Post-merger



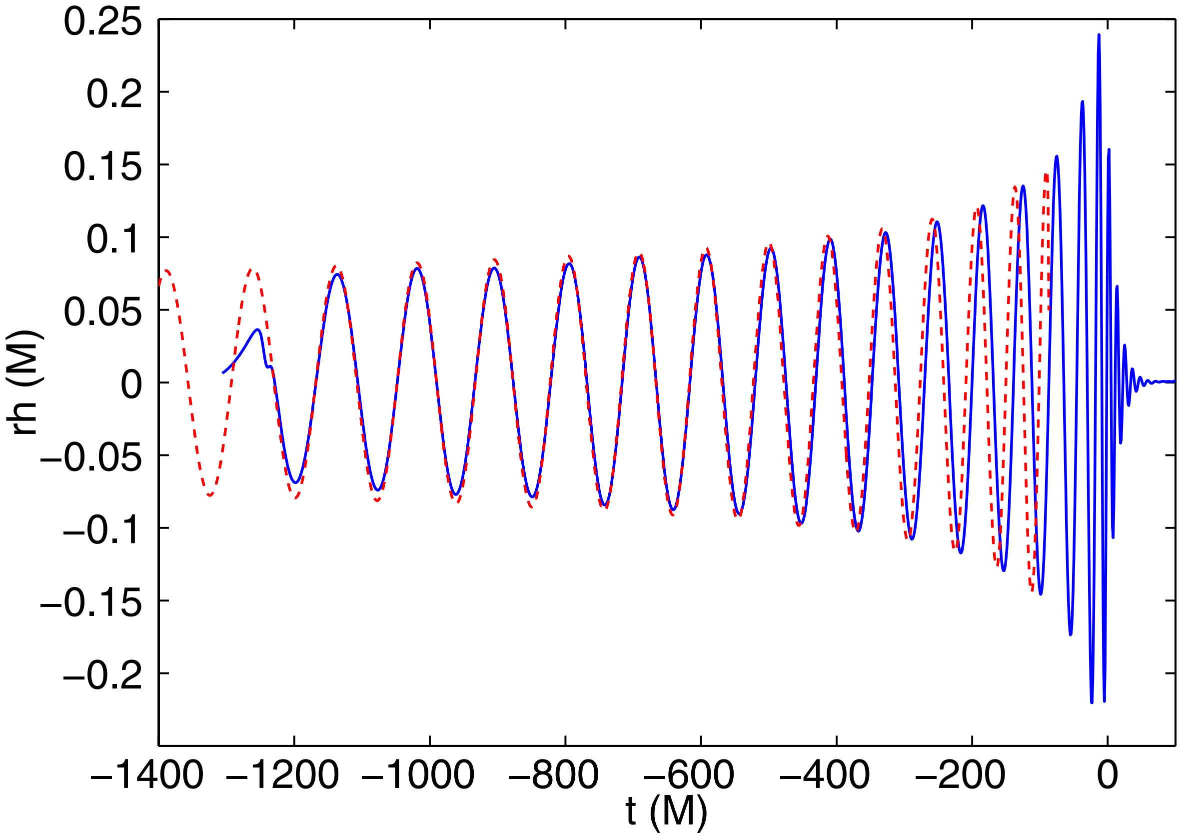
- After merger: **perturbed Kerr BH**
- Linear perturbation theory predicts **quasi-normal ringdown**:
    - $A e^{i (\omega t - t/\tau)}$ where ω and τ depend on **mass** and **spin**
- A (complex) is unknown; **need NR**
- **Final mass and spin** as function of initial masses and spins is unknown: **need NR**

What does a BBH waveform look like?
3. Complete waveform

- Before 2005: Kip Thorne's sketch



- After 2005: Numerical Relativity (e.g. Baker et al. 2007)



and with NR?

- **One configuration** (D, m_1/m_2, S_1, S_2) at a time.
- Simulations take from **days to weeks to months**, depending on number of **orbits**, **m_1/m_2** and **S**
- Scale-invariance of BBH
- Need **fast** model for GW detection and parameter estimation
- Use small (<1000) number of currently-known NR waveforms to (i) test and (ii) extend fast **approximate waveform models**.

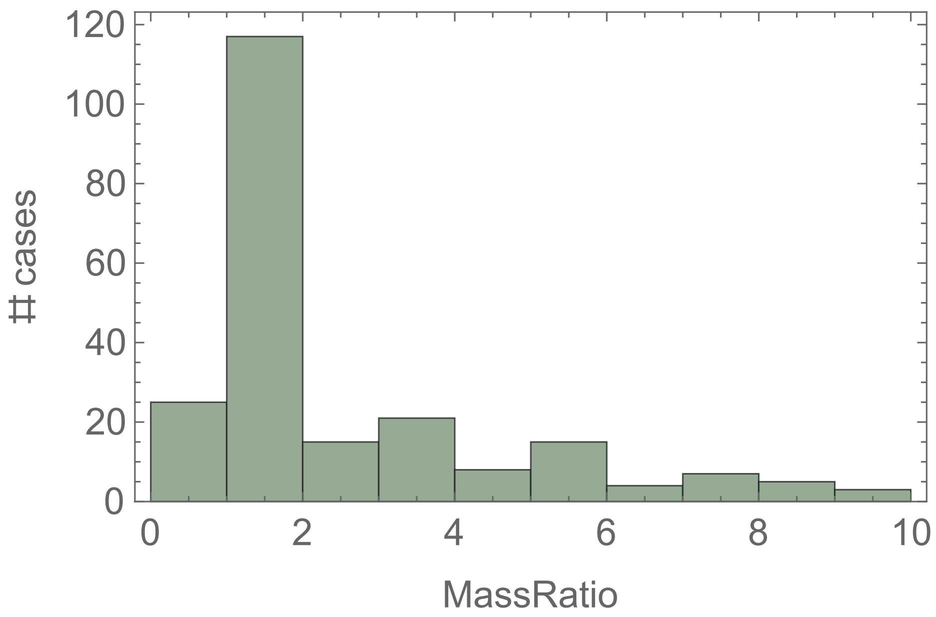
Waveform catalogues

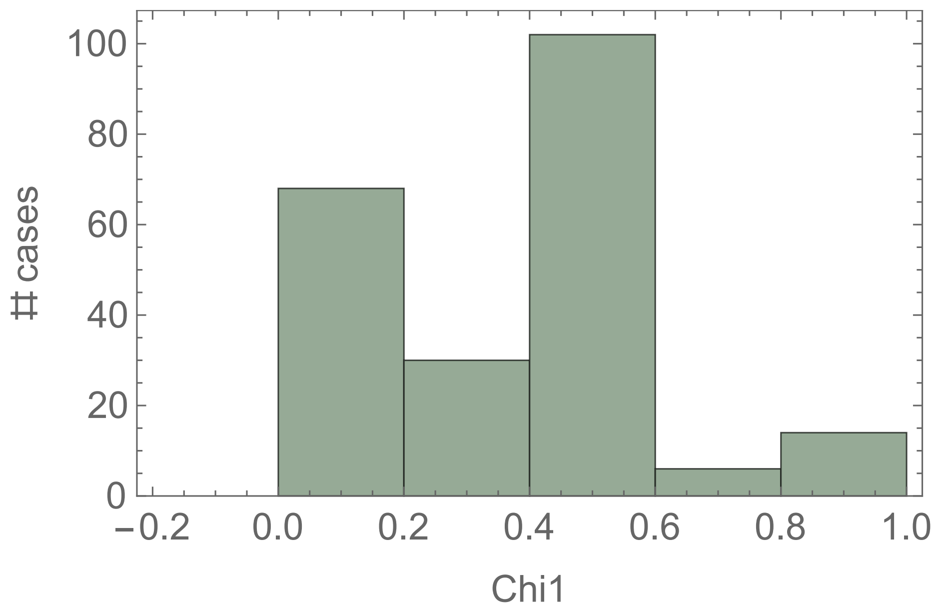
- Collections of **~100s** of waveforms from different BBH configurations
- Configurations parametrised by q, χ_1, χ_2, ω_0 (# orbits)
- Multiple numerical **resolutions**

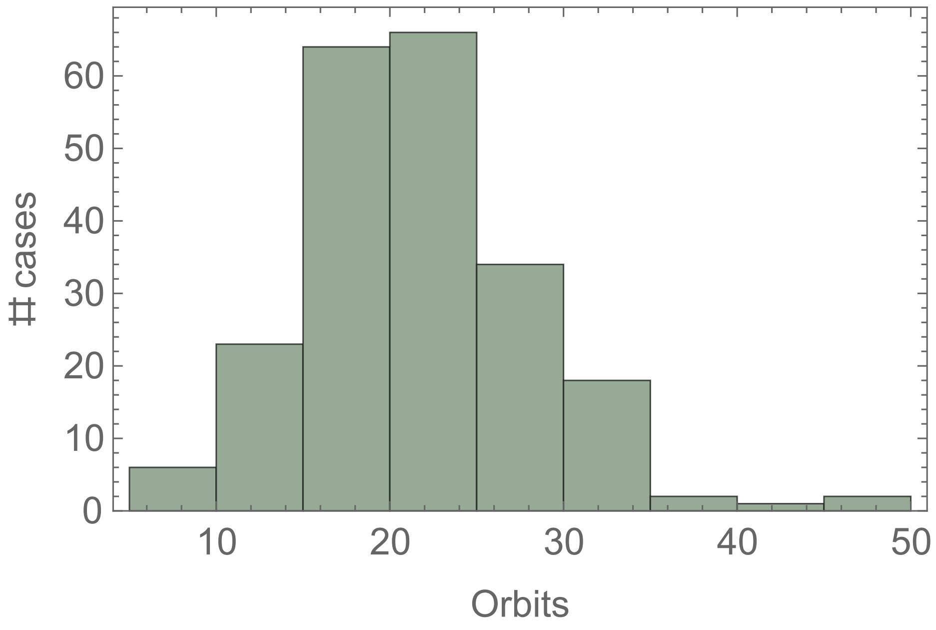
Waveform catalogues

- **SXS**: black-holes.org/waveforms
    - 220 configurations
    - Described in arxiv:1605.03204
- **Georgia Tech**: einstein.gatech.edu/catalog
    - 452 configurations
    - Described in arXiv:1304.6077
- Other groups have internal **private** catalogues

SXS catalogue parameter space







- Axes of parameter space covered well

- Corners not so well

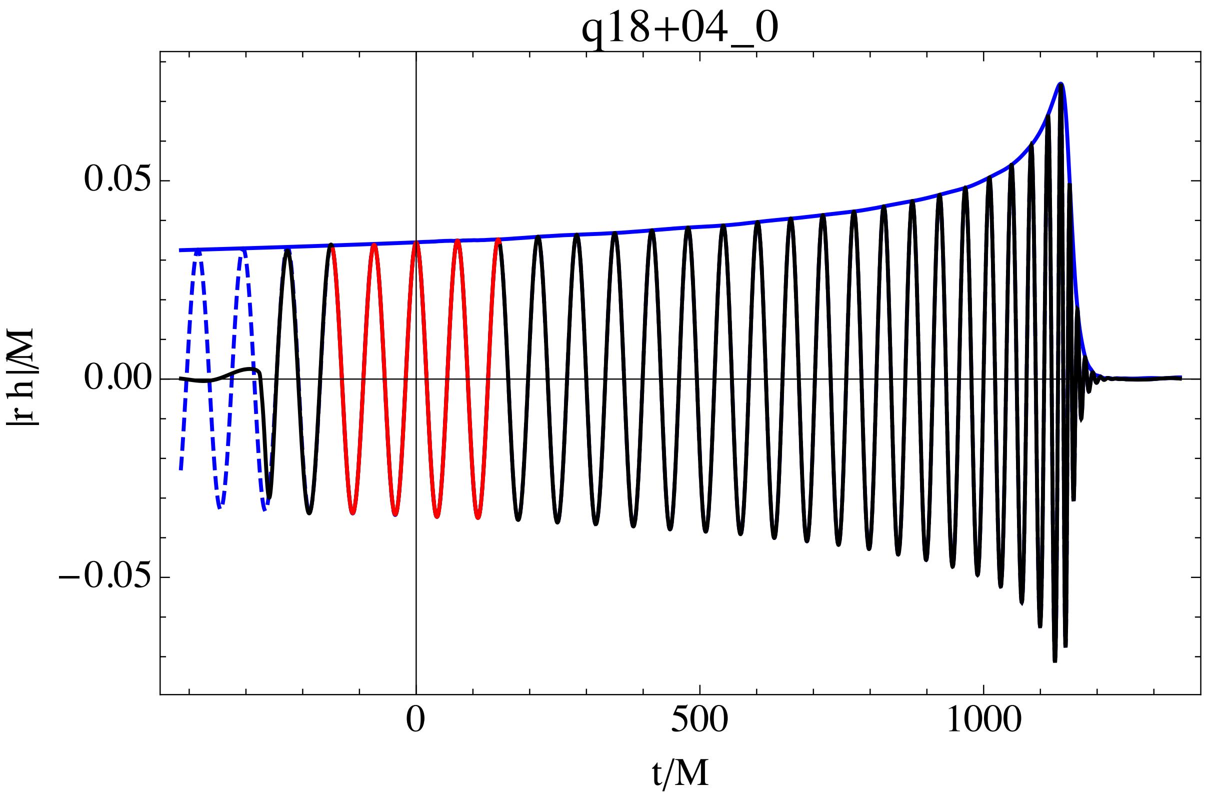
Comparisons with PN

- Post-Newtonian expansion valid when binary is **far separated**
- NR **very expensive** for large separations
- **How late** can we trust PN?
  - For LIGO, need waveform model for the system under consideration **valid over the sensitive band**
  - High mass: low frequency: early inspiral out of band. For very high mass, **only need NR**.
  - Better PN models of the inspiral give you good models for **lower masses**.
- Early PN+BBH comparisons: PN works **surprisingly late**

Combining NR and PN

- NR **too expensive** for low mass systems
- Waveform modelling: combining PN and NR to make a "complete" waveform model
- Three main approaches:
    - **Hybrid**: blend early inspiral PN and NR late inspiral and merger
    - **Phenomenological**: PN for inspiral, functions with unkown coefficients for the merger; fit coefficients from NR simulations
    - **Effective-One-Body**: full inspiral-merger-ringdown model from ODEs
- **NR** essential for all

Hybrid waveforms



Husa et al. 2015

- Given an NR waveform for **$0 < t < t_{final}$**, add a PN waveform for **$t < 0$**
- Subtleties:
  - Blend the two **in a region** to avoid discontinuity
  - What **PN parameters** correspond to the NR t = 0? Matching.
  - Hybrid waveform **error will grow** as t -> -Infinity
- If all under control, get a waveform **much longer than NR**
- Still only have **one** waveform

Effective-One-Body models

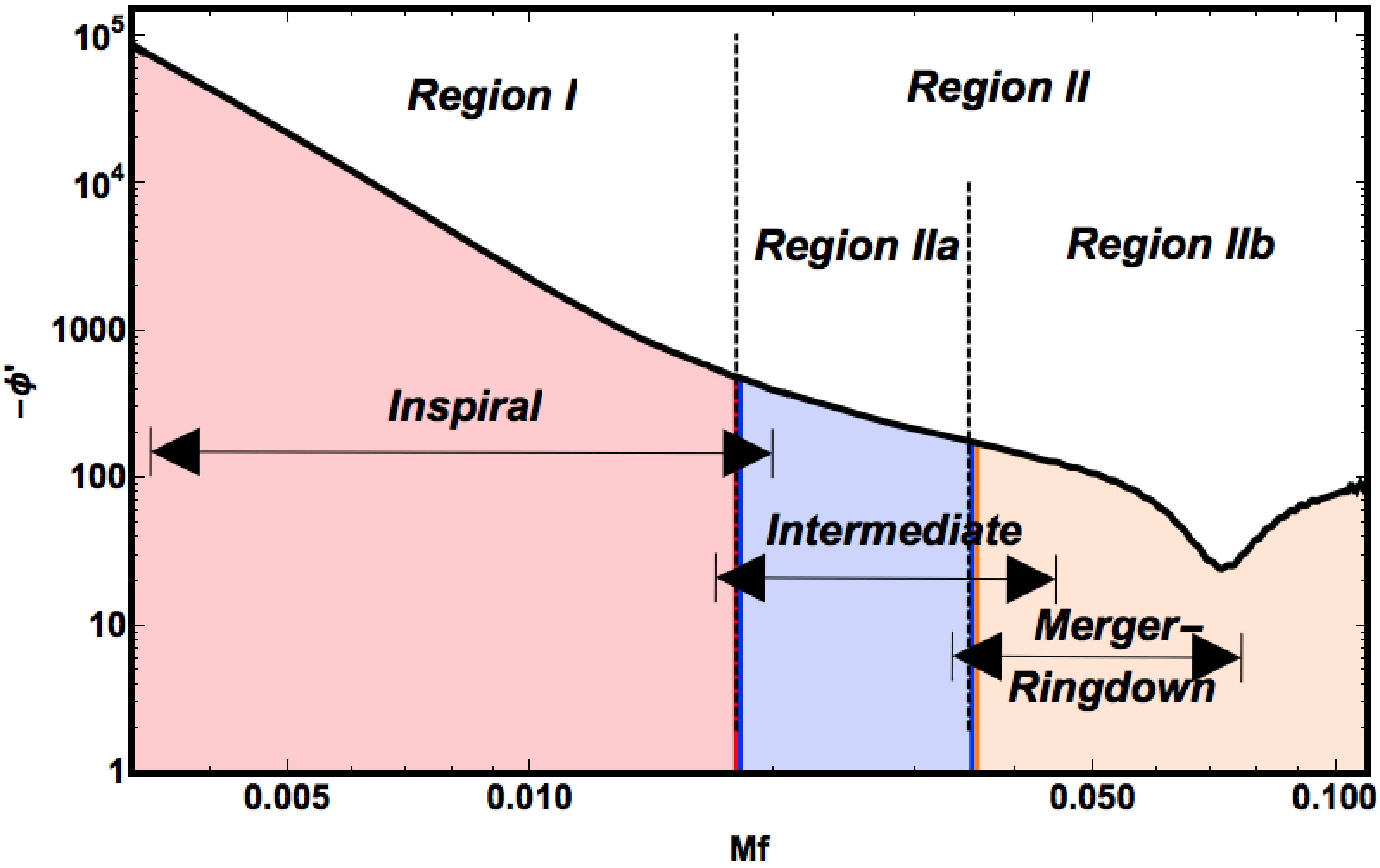
- See Stas' talk next

A frequency-domain phenomenological waveform model: PhenomD

1. Collect a large number of **NR waveforms**
2. **Hybridise** with SEOB, uncalibrated
3. Split into **three** regions: inspiral, intermediate, merger-ringdown
4. In each region, look at the waveforms for essential features *in the* **frequency domain** (where LIGO lives).
5. Add **phenomenological terms** to the base model in each region with **undetermined parameters**
6. Fit the parameters to the **SEOBv2-NR hybrids** (fit to a subset, check with the rest)

PhenomD regions

Phase derivative



PhenomD: Region I (PN, Inspiral)

- Mf < 0.018
- PN stationary phase approximation (TaylorF2)
  +
  4 higher order PN terms
  fitted to hybrids (SEOBv2 + NR)

$$\phi_{\mathrm{TF2}} = 2\pi f t_c - \varphi_c - \pi/4 + \frac{3}{128\,\eta}(\pi f M)^{-5/3} \sum_{i=0}^{7} \varphi_i(\Xi)(\pi f M)^{i/3}$$

$$\phi_{\mathrm{Ins}} = \phi_{\mathrm{TF2}}(Mf;\Xi) + \frac{1}{\eta}\left(\sigma_0 + \sigma_1 f + \frac{3}{4}\sigma_2 f^{4/3} + \frac{3}{5}\sigma_3 f^{5/3} + \frac{1}{2}\sigma_4 f^2\right)$$

PhenomD: Region IIa (NR, intermediate)

- **Connect** the phase between Region I and Region IIb via this form for the phase derivative:

$$\eta\,\phi'_{\mathrm{Int}} = \beta_1 + \beta_2 f^{-1} + \beta_3 f^{-4}$$

- Fit a **4th order polynomial** for the amplitude

PhenomD: Region IIb (Merger-ringdown)

- In time domain, simple model for merger-ringdown might be **exponentially-damped sine wave**:

$$h(t) = e^{2\pi(i f_{\mathrm{RD}} t - f_{\mathrm{damp}}|t|)}$$

- Fourier transform is **Lorentzian**:

$$\tilde{h}(\omega) = -\frac{1}{\pi}\frac{f_{\mathrm{damp}}}{(f - f_{RD})^2 + f_{\mathrm{damp}}^2}$$

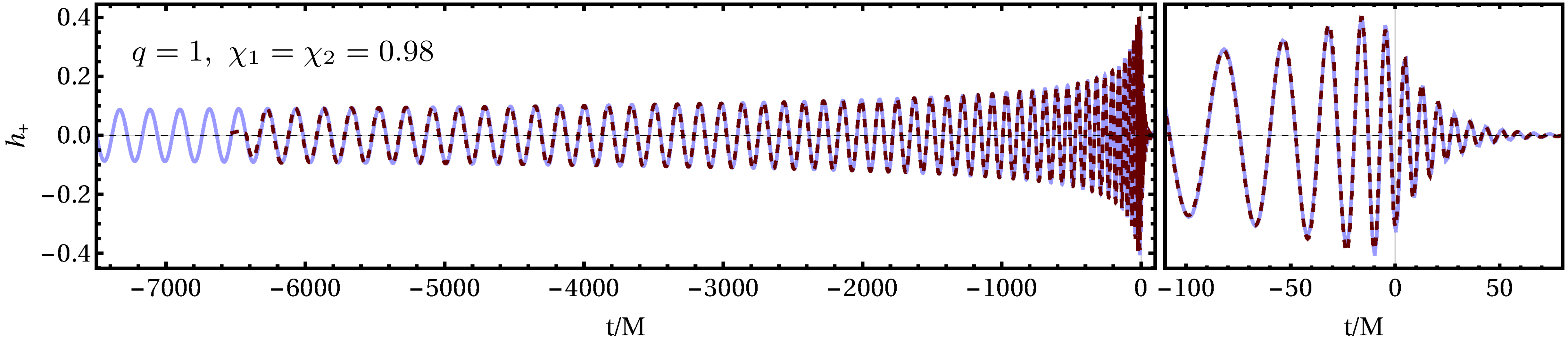
- However, has **wrong high-f falloff** (f^{-2} instead of exponential), so multiply Lorentzian by **exponential**:

$$\frac{A_{\mathrm{MR}}}{A_0} = \gamma_1 \frac{\gamma_3 f_{\mathrm{damp}}}{(f - f_{\mathrm{RD}})^2 + (\gamma_3 f_{\mathrm{damp}})^2} e^{-\frac{\gamma_2 (f - f_{\mathrm{RD}})}{\gamma_3 f_{\mathrm{damp}}}}$$

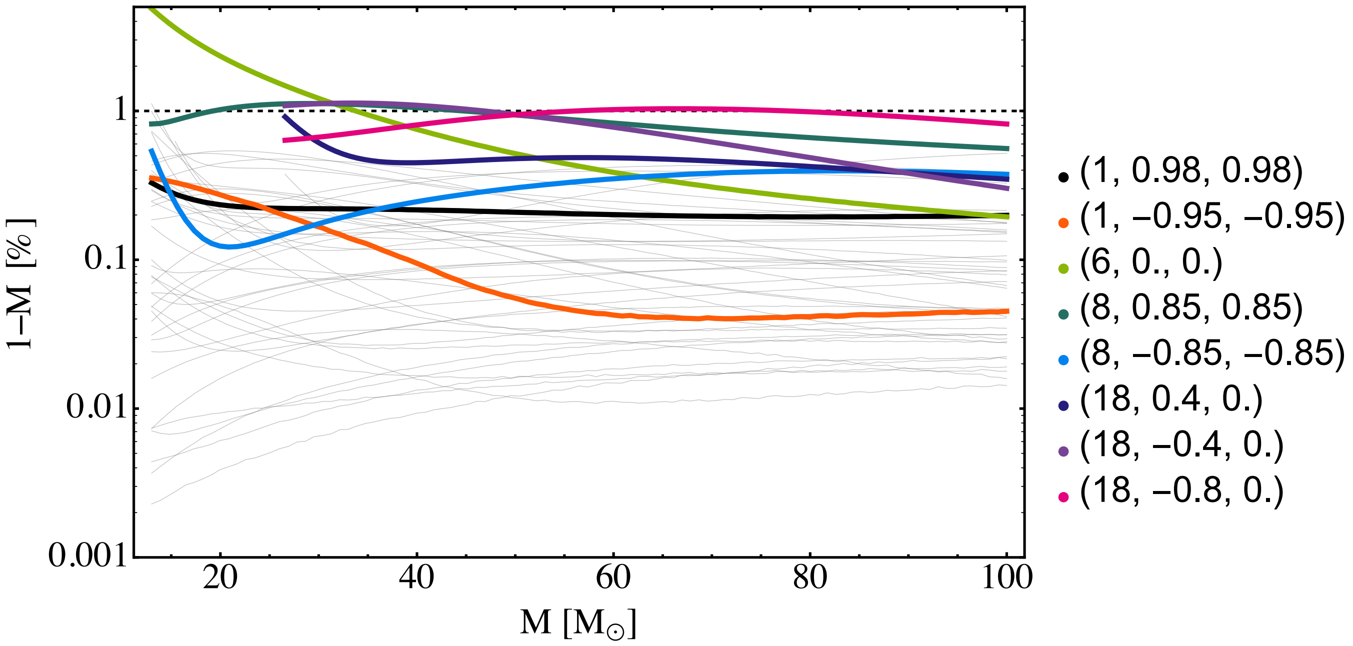
PhenomD: Combining the regions

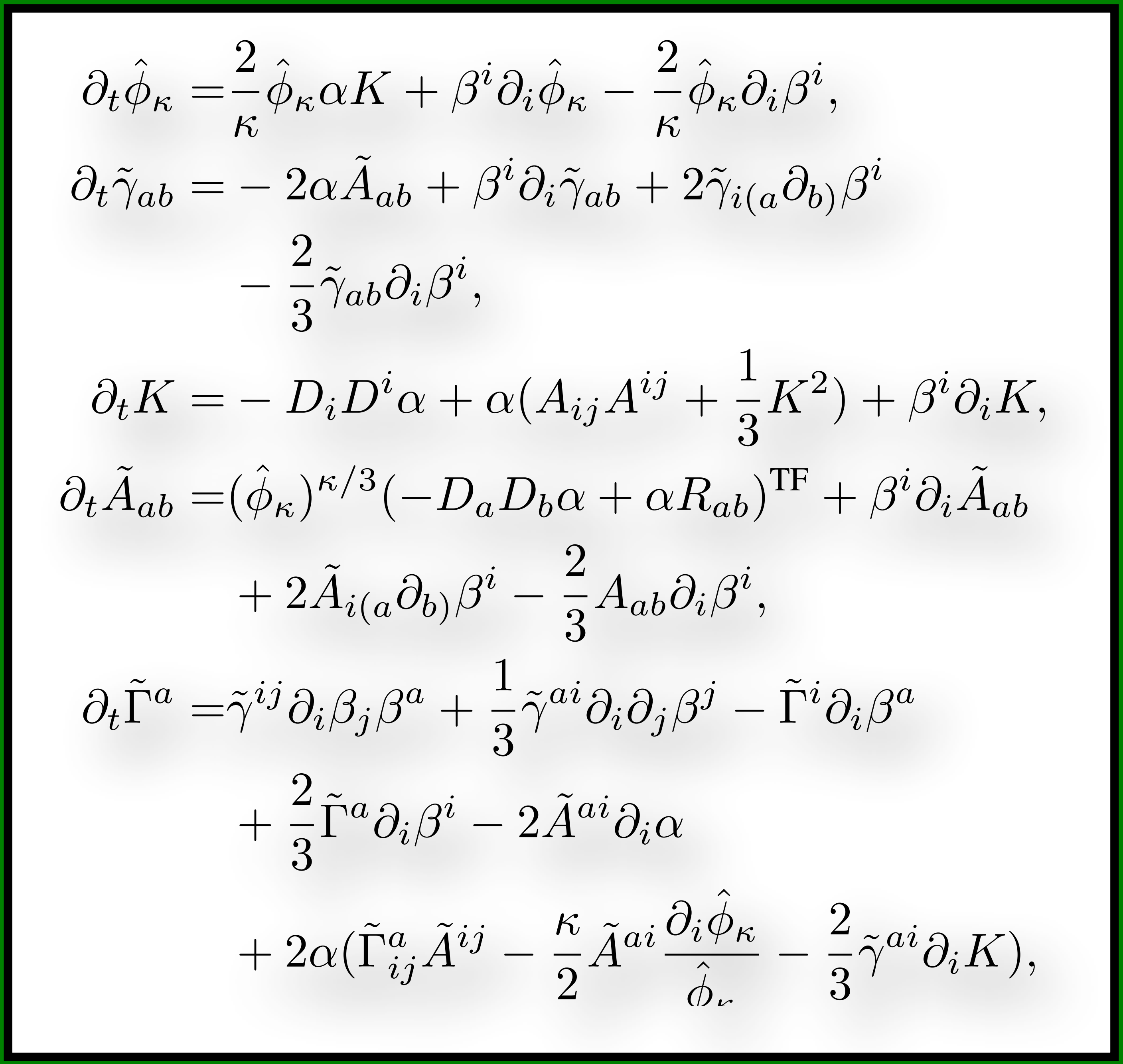
- Models are **C^1** (first derivative continuous) across region boundaries
- Regions combined with a simple **step function**
- Unknown parameters **fitted** against a **subset** of the SEOBv2+NR hybrids
- Fit parameters depend on physical parameters (q, S_1, S_2)
- Single **effective spin** approximation
- Model quality **tested against remaining hybrids**
- Ready to use in LALSimulation (**open source**)
- Used for LIGO **parameter estimation** results

How good is the PhenomD model?



- Excellent agreement with NR





$$
\begin{aligned}
\partial_t \hat{\phi}_\kappa =& \frac{2}{\kappa}\hat{\phi}_\kappa \alpha K + \beta^i \partial_i \hat{\phi}_\kappa - \frac{2}{\kappa}\hat{\phi}_\kappa \partial_i \beta^i,\\
\partial_t \tilde{\gamma}_{ab} =& -2\alpha \tilde{A}_{ab} + \beta^i\partial_i\tilde{\gamma}_{ab} + 2\tilde{\gamma}_{i(a}\partial_{b)}\beta^i\\
& -\frac{2}{3}\tilde{\gamma}_{ab}\partial_i\beta^i,\\
\partial_t K =& -D_iD^i\alpha + \alpha(A_{ij}A^{ij} + \frac{1}{3}K^2) + \beta^i\partial_i K,\\
\partial_t \tilde{A}_{ab} =& (\hat{\phi}_\kappa)^{\kappa/3}(-D_aD_b\alpha + \alpha R_{ab})^{\mathrm{TF}} + \beta^i\partial_i\tilde{A}_{ab}\\
& + 2\tilde{A}_{i(a}\partial_{b)}\beta^i - \frac{2}{3}A_{ab}\partial_i\beta^i,\\
\partial_t\tilde{\Gamma}^a =& \tilde{\gamma}^{ij}\partial_i\beta_j\beta^a + \frac{1}{3}\tilde{\gamma}^{ai}\partial_i\partial_j\beta^j - \tilde{\Gamma}^i\partial_i\beta^a\\
& + \frac{2}{3}\tilde{\Gamma}^a\partial_i\beta^i - 2\tilde{A}^{ai}\partial_i\alpha\\
& + 2\alpha(\tilde{\Gamma}^a_{ij}\tilde{A}^{ij} - \frac{\kappa}{2}\tilde{A}^{ai}\frac{\partial_i\hat{\phi}_\kappa}{\hat{\phi}_\kappa} - \frac{2}{3}\tilde{\gamma}^{ai}\partial_i K),
\end{aligned}
$$

3. Recent NR work

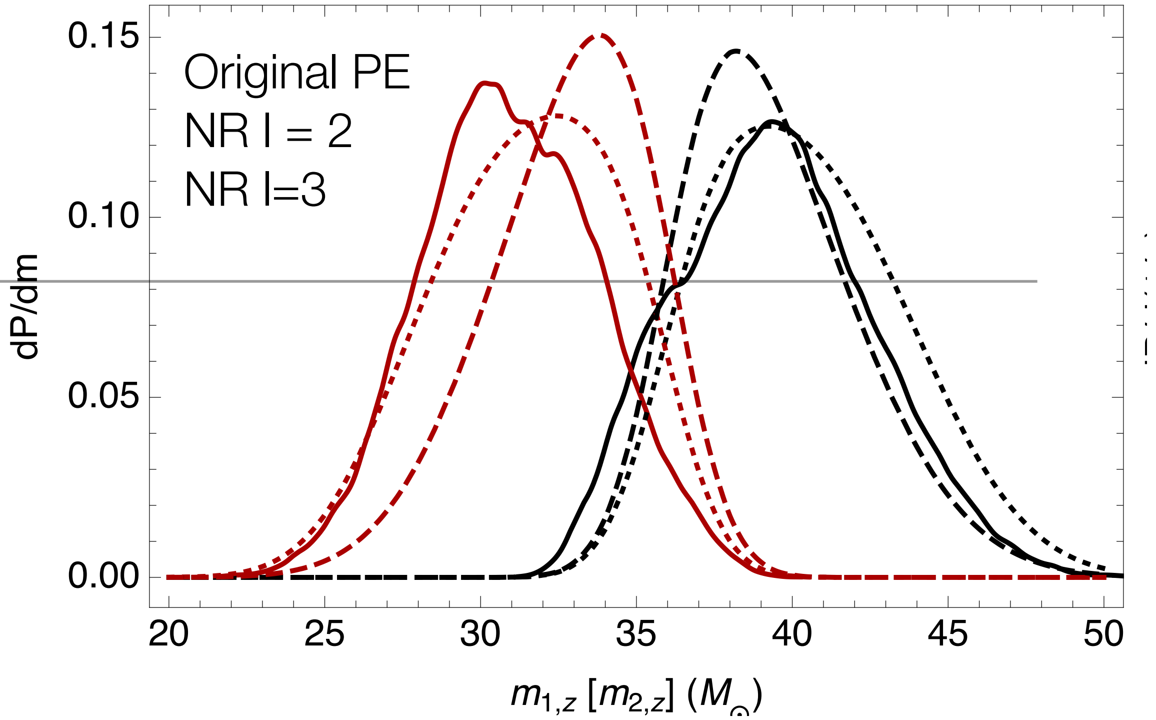
Image: The BSSN formulation of the Einstein equations

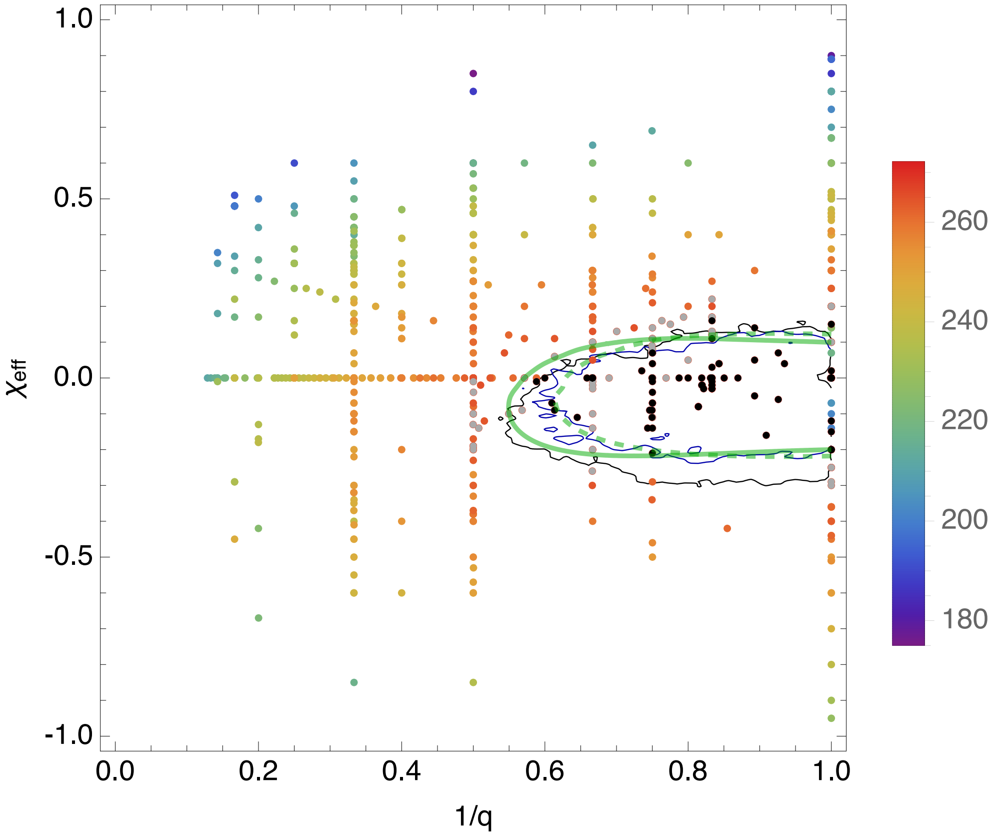
Waveform systematics

- How good are the **approximate waveform models** in the region of GW150914?
- Always an error; will bias the parameters measured by LIGO
- How big is the bias? Larger than the LIGO **noise error bars**?
- Michael Pürrer talk at APS in April:
    - Use an NR waveform as **injected** LIGO data and measure parameters of this waveform using approximate waveform models.
    - Do you recover the true parameters of the NR waveform within the noise error bars?
    - Summary: nothing to worry about so far

Directly comparing

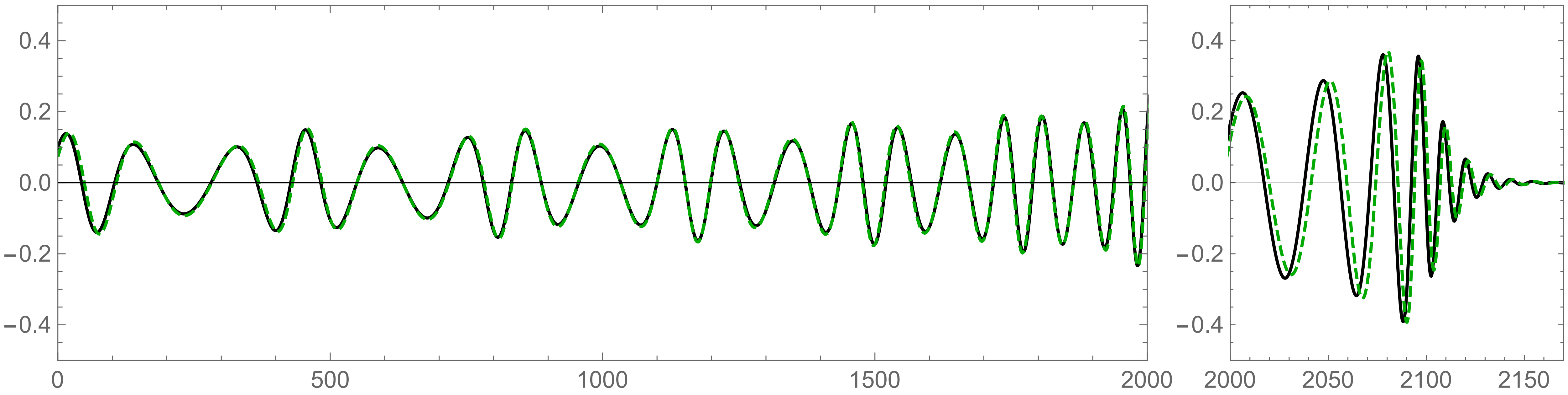
- GW150914 is high mass: 6 orbits in the LIGO band
- Many NR waveforms available which are **entirely in-band** at this mass
- Compare the LIGO data with all **available NR waveforms**
- Interpolate the **likelihood** between available points in parameter space
- Similar results to PE from **approximate waveform models**
- See LIGO and Virgo collaborations, Abbott et al., *Directly comparing GW150914 with numerical solutions of Einstein's equations for binary black hole coalescence*, http://arxiv.org/abs/arXiv:1606.01262





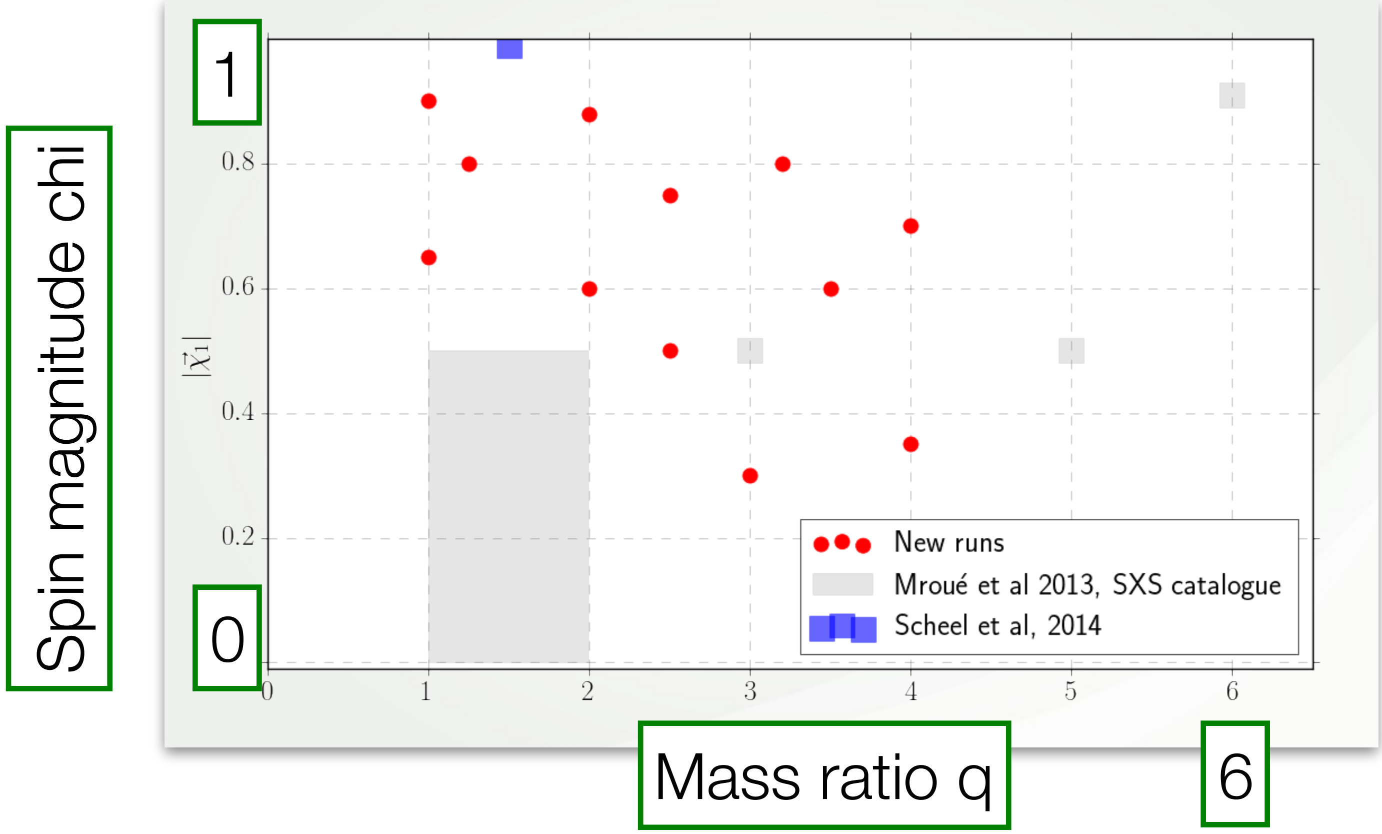
Eccentricity

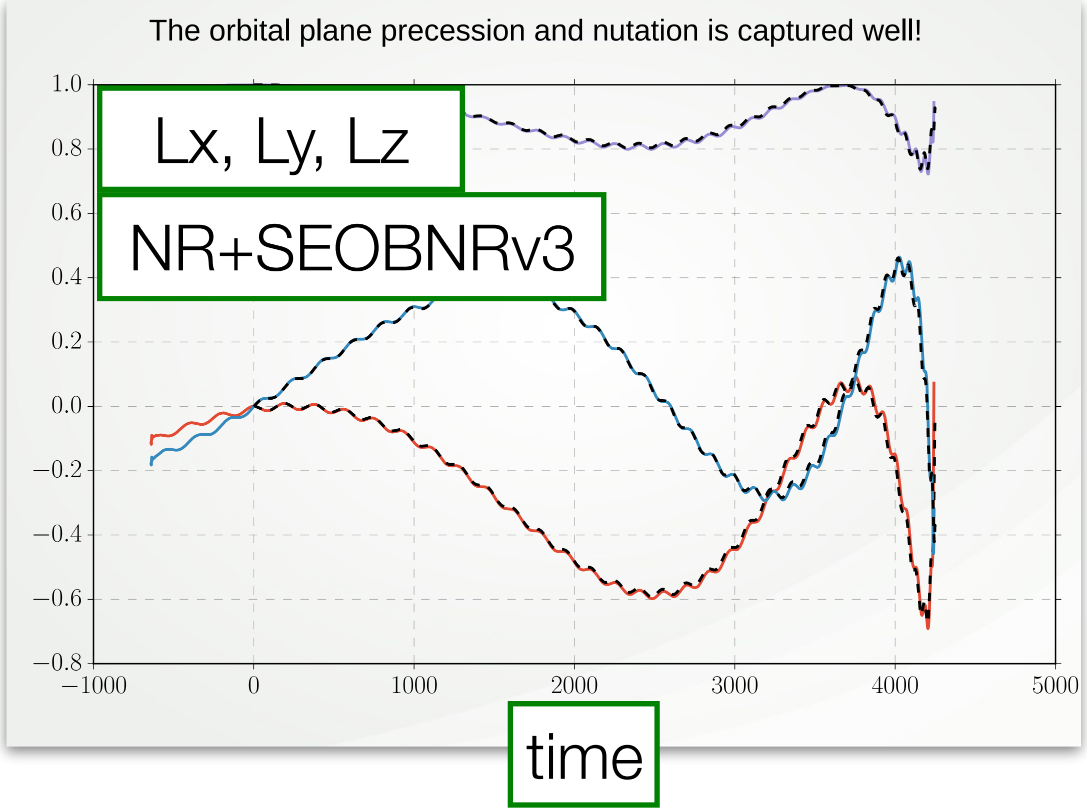
- Eccentric binaries **circularise** (Peters 1964): e ~ 0 well before merger.
- Measure/bound eccentricity of **GW events** such as GW150914?
- Need **eccentric waveform model**
- Use **post-Newtonian** and **Numerical Relativity**
- Only need late inspiral+merger; e.g. **last 5 orbits** for GW150914
- Eccentric PN inspiral + NR **circular merger**
- IH talk at GR21 - paper soon!



Precessing BBH parameter space coverage

- ~120 new precessing waveforms run by AEI, CITA with SpEC
- Extend range of parameter space in mass ratio and spin
- Several spin angles for each (q, chi) combination

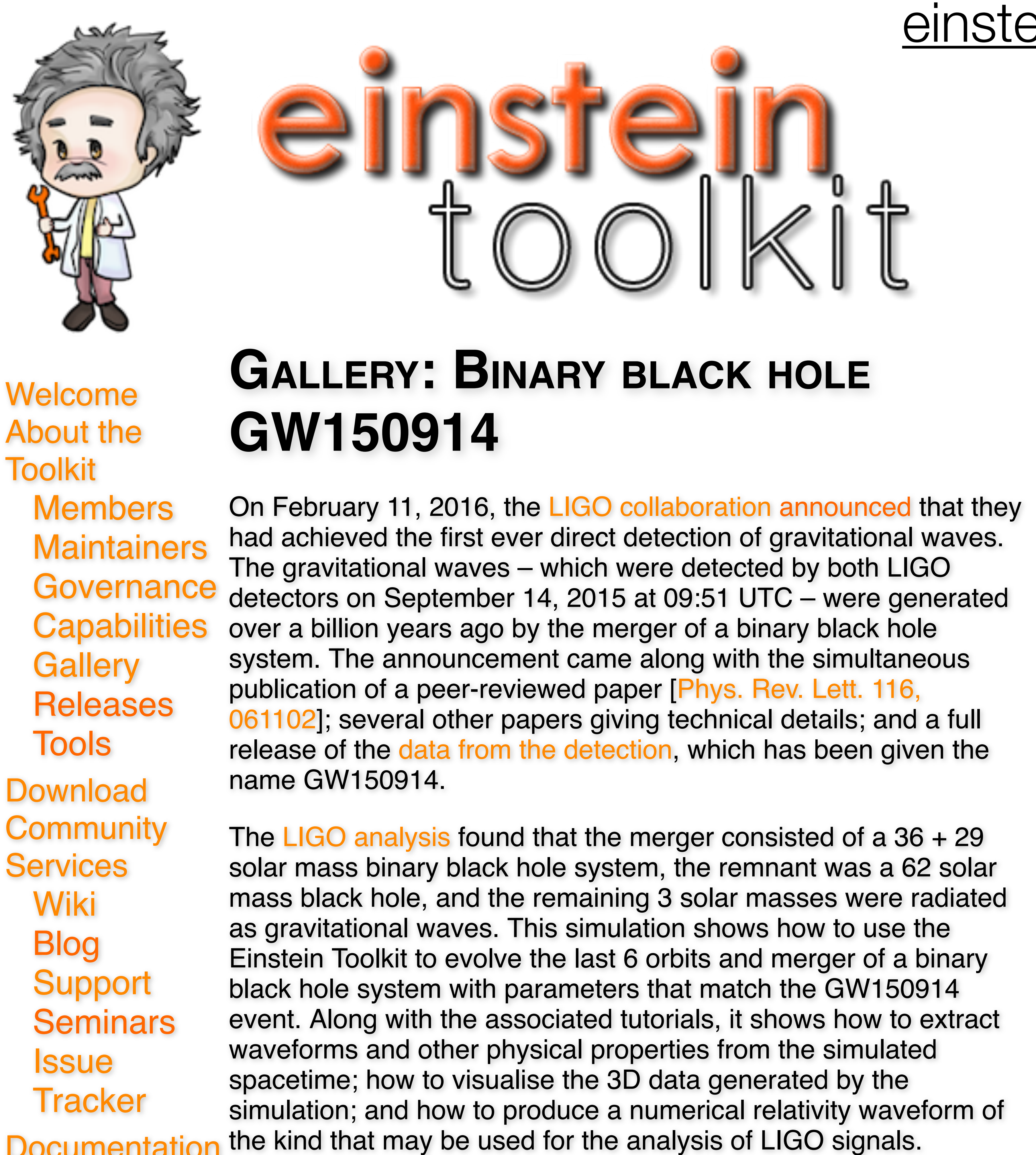



The orbital plane precession and nutation is captured well!

Open source Numerical Relativity

- **Cactus** framework: open source, developed by **Ed Seidel**'s group at the Albert Einstein Institute in the late 90s
- Foundation of most NR codes today
- **Einstein Toolkit** is an entirely open source set of NR codes based around Cactus
- See einsteintoolkit.org/about/gallery for examples
- **GW150914 example** coming soon, including fully open parameter file, instructions, and **tutorials** for analysis and visualisation [Wardell, IH, Bentivegna]
- Simulate GW150914 on ~100 cores in a few days **yourself**!

einsteintoolkit.org
(soon)



GALLERY: BINARY BLACK HOLE GW150914

On February 11, 2016, the LIGO collaboration announced that they had achieved the first ever direct detection of gravitational waves. The gravitational waves – which were detected by both LIGO detectors on September 14, 2015 at 09:51 UTC – were generated over a billion years ago by the merger of a binary black hole system. The announcement came along with the simultaneous publication of a peer-reviewed paper [Phys. Rev. Lett. 116, 061102]; several other papers giving technical details; and a full release of the data from the detection, which has been given the name GW150914.

The LIGO analysis found that the merger consisted of a 36 + 29 solar mass binary black hole system, the remnant was a 62 solar mass black hole, and the remaining 3 solar masses were radiated as gravitational waves. This simulation shows how to use the Einstein Toolkit to evolve the last 6 orbits and merger of a binary black hole system with parameters that match the GW150914 event. Along with the associated tutorials, it shows how to extract waveforms and other physical properties from the simulated spacetime; how to visualise the 3D data generated by the simulation; and how to produce a numerical relativity waveform of the kind that may be used for the analysis of LIGO signals.

Physical parameters

Initial separation D	10 M
Mass ratio $q = m_1/m_2$	36/29 ~ 1.24
Spin $\chi_1 = a_1/m_1$	0.31
Spin $\chi_2 = a_2/m_2$	-0.46

Physical properties

Number of orbits	6
Time to merger	899 M
Mass of final BH	0.95 M
Spin of final BH (dimensionless)	0.69

Computational details

Parameter file	GW150914.rpar
Thornlist	GW150914.th (ET_2015_11 release thornlist with Llama multi-block code added)
Submission command	simfactory/bin/sim create-submit GW150914_28 --define N 28 --parfile repos/GW150914/ParameterFiles/GW150914.rpar --procs 120 --walltime 24:00:00
Total memory	98 GB
Run time	5.6 days on 120 cores (Intel(R) Xeon(R) CPU X5650 @ 2.66GHz)
Cost	16108 core hours

TUTORIALS

- Compile and run: Compile the code and run the simulation
- VisIt: Visualise the data using VisIt
- SimulationTools tutorials: these can be run with Mathematica, or can be viewed interactively with the free Wolfram CDF Player. Download a zip file of all SimulationTools tutorials, or download them individually below.

SIMULATION DATA

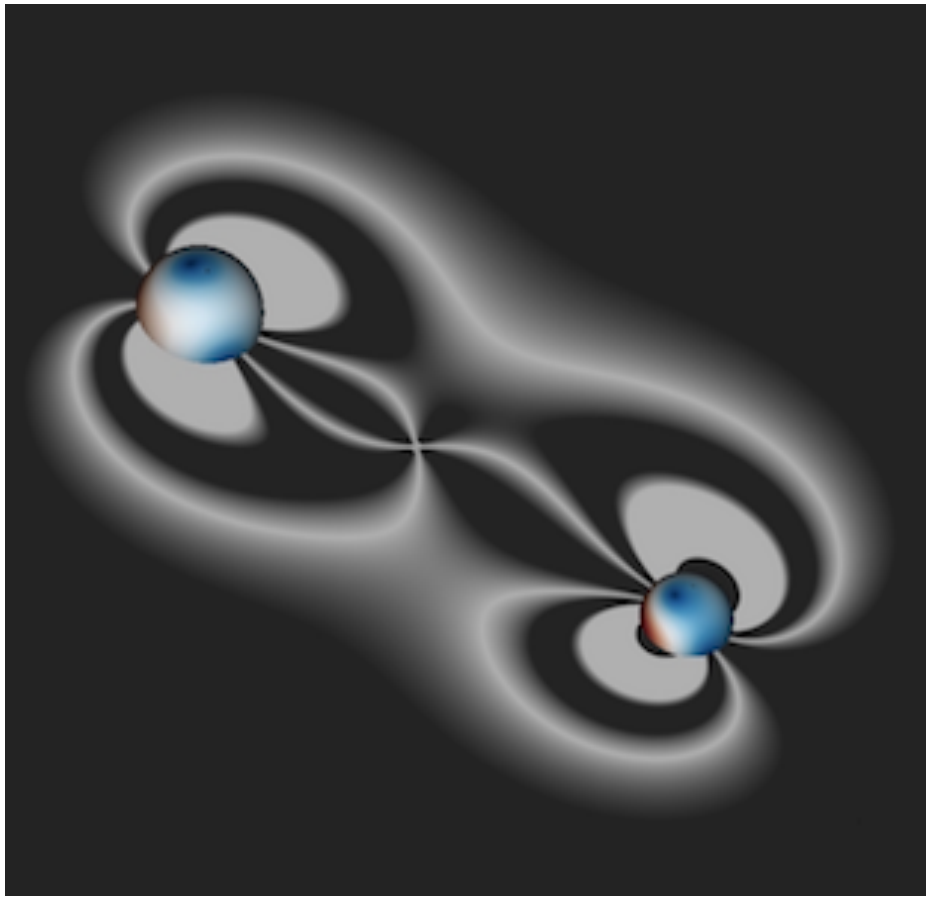
Lightweight simulation data with only a small number of iterations of 3D output is available for download from Zenodo:

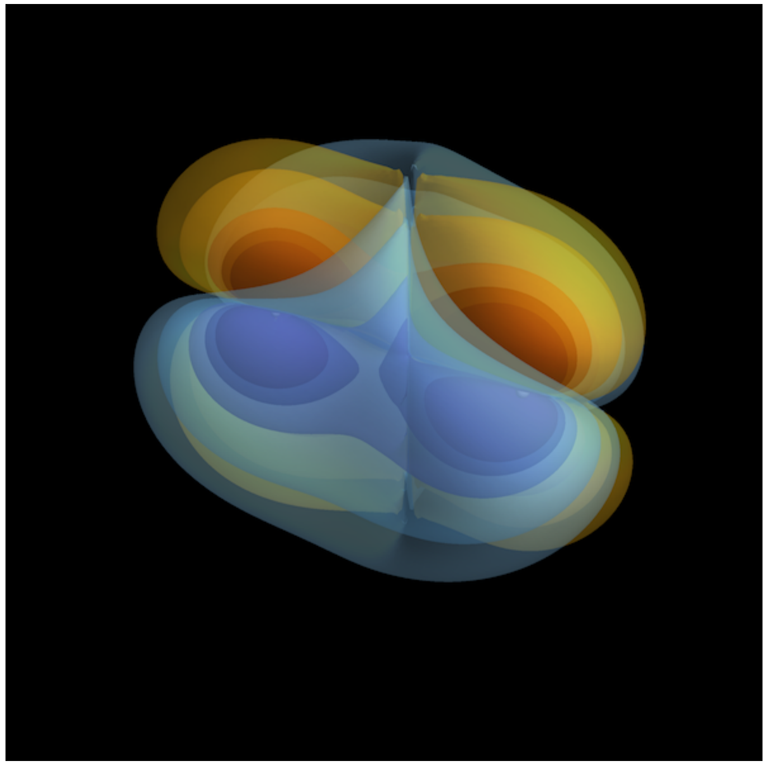
DOI 10.5281/zenodo.60213

The full simulation comprises several terabytes of data and can be made available upon request.

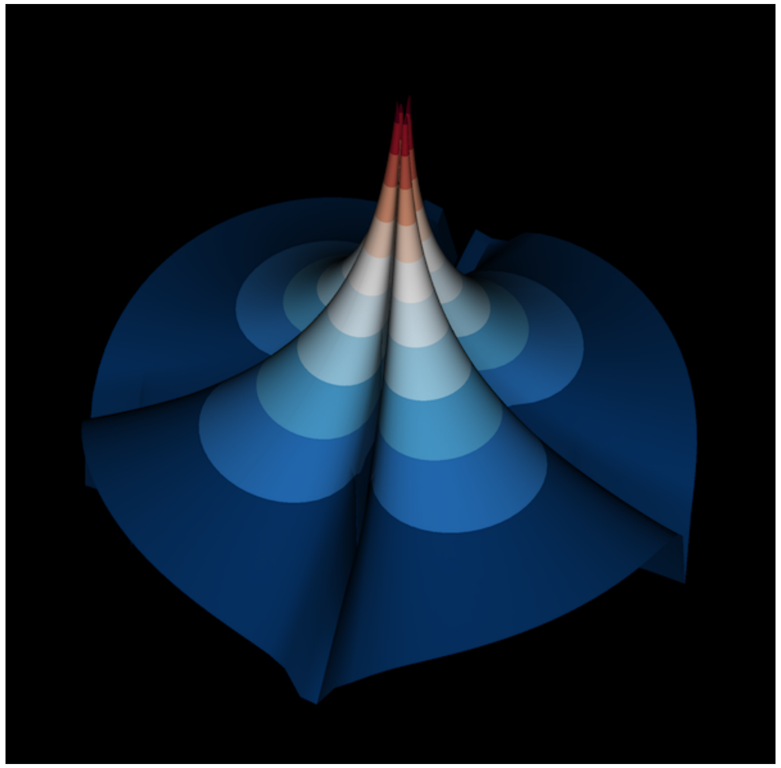
IMAGES AND MOVIES

Horizons



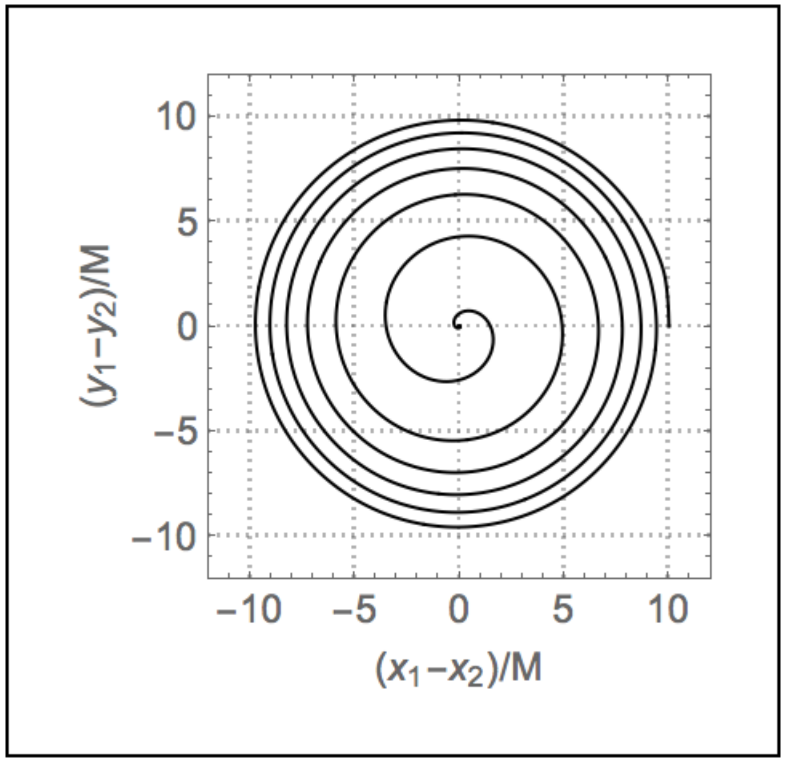


The real part of Ψ_4, the component of the Riemann tensor representing outgoing gravitational radiation.



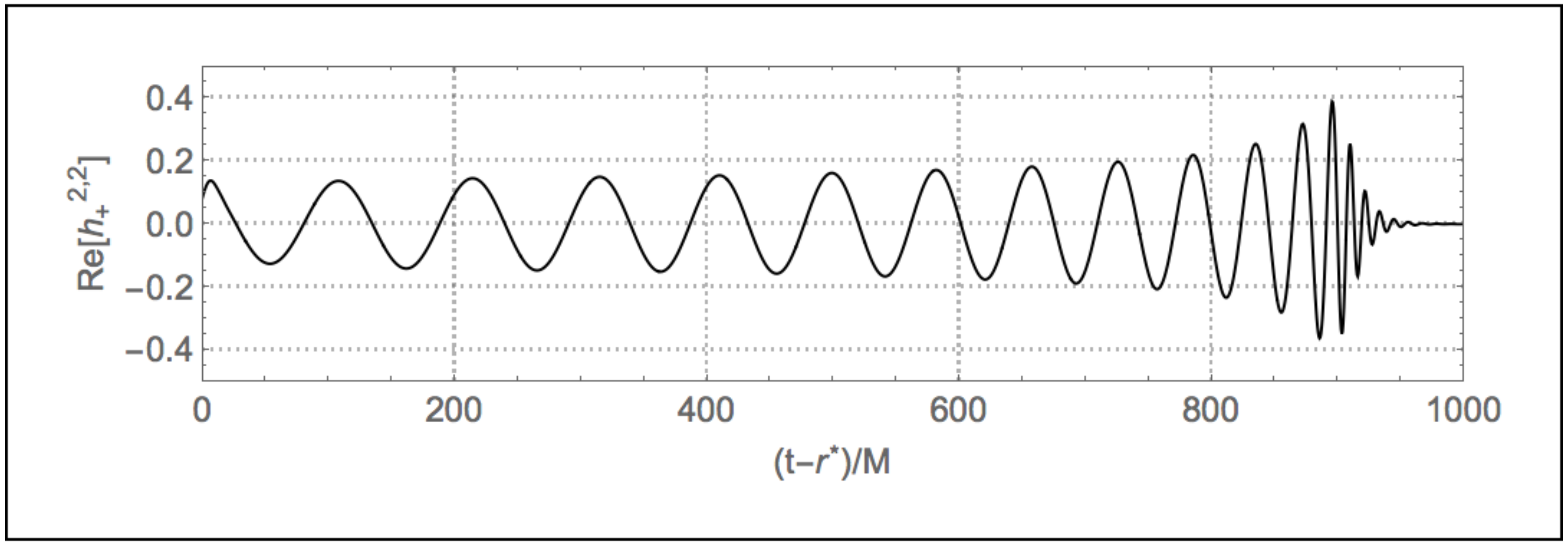
Elevation plot of the magnitude of Ψ_4 on the equatorial plane at $t = 0$.

Horizon coordinate trajectories

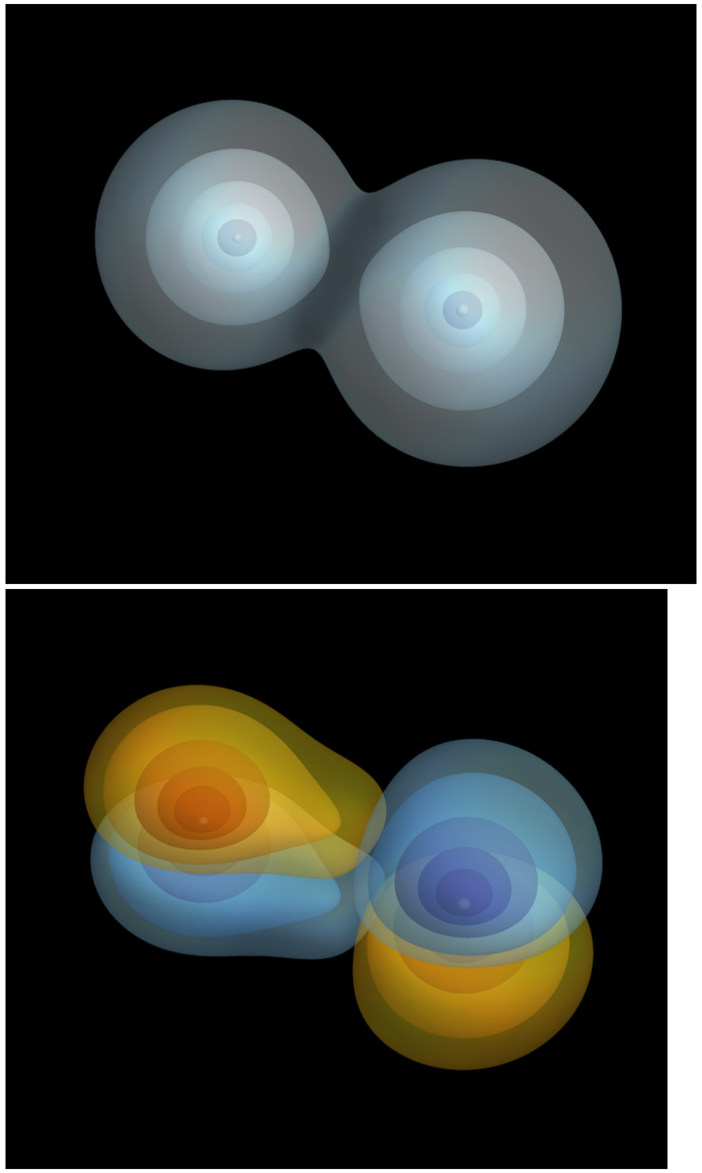


Coordinate tracks of the centroids of the apparent horizons showing inspiral of the binary due to emission of energy and angular momentum in gravitational waves

Gravitational waveform



Curvature scalars



Scalar curvature invariants computed from the Riemann tensor, R_{abcd}, and its dual, ${}^*R_{abcd}$. Left: the Kretschmann scalar, $R_{abcd}R^{abcd}$. Right: the Chern-Pontryagin scalar, $R_{abcd}{}^*R^{abcd}$.

Thank you!

- Useful information:
    - Book: **Introduction to 3+1 Numerical Relativity** (Miguel Alcubierre)
    - SXS waveform **catalogue**: black-holes.org/waveforms
    - einsteintoolkit.org/about/gallery
    - **PhenomD** papers: 1508.07250 and 1508.07253