Inspiraling binaries of compact objects

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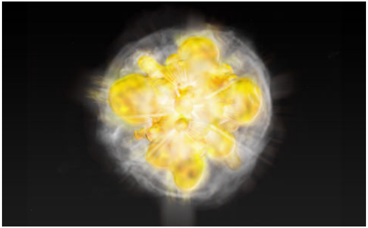
Gravitational sources in GR



Matter (or energy) sources affect the geometry of spacetime

$\Rightarrow$ Motions of asymmetric sources can generate dynamical deformations

- Spacetime description:
  - curved "surface" of dimension 4
  - coordinate system arbitrary (4 gauge degrees of freedom)
  - 4d "distance" of the form $(\mathrm{d}s)^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$
  - Riemann tensor $R^{\lambda}{}_{\rho\mu\nu}$ = intrinsic curvature responsible for tidal effects

$$R_{0i0j} \propto \partial_{ij}\left(\frac{Gm}{rc^2}\right)$$

- Source description (in the generic coordinate grid x^μ)
  - energy density: T^{00}
  - momentum density (or energy flux): T^{0i}
  - stress tensor: T^{ij}

Gravitational waves

Linear analysis near Minkowski metric $\eta_{\mu\nu}$ [Einstein 1918]: $g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$

Equation for the gravitational field

$$\partial_\nu \left(\delta g^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} \delta g^\lambda_{\ \lambda} \right) = 0 \quad \text{(gauge conditions removing 4 d.o.f.)}$$

$$\Box \left(\delta g^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} \delta g^\lambda_{\ \lambda} \right) = -\frac{16\pi G}{c^4} T^{\mu\nu}$$

$$\Rightarrow$$

Einstein quadrupole formula (amplitude)

$$h^{\rm rad}_{ij}(\boldsymbol{X}, T) = \frac{2G}{Rc^4} \left(P_{ijkl}(\boldsymbol{N}) \ddot{Q}_{kl}(T - R/c) + \mathcal{O}\left(\frac{1}{c}\right) \right) + \mathcal{O}\left(\frac{1}{R^2}\right)$$

Using the first non-linear gravitational terms in the rhs of $\Box$ equation:

Einstein quadrupole formula (flux)

$$\frac{d}{du} E_{\rm N}(u) = -\frac{G}{5c^5} \dddot{Q}_{ij}(u) \dddot{Q}_{ij}(u)$$

Beyond the linear theory

First nonlinear solutions... first doubts! [Eintein & Rosen 1937]

- Are the singularities of the cylindrical wave solution problematic?
- Should point masses following geodesics radiate? What about EP?
- What is meant by gravitational waves?

Answers found in the 60's-70's:

- Einstein-Rosen singularities are coordinate related
- Radiation is a non-local property
  $\hookrightarrow$ to be studied w.r.t. the faraway observer "at rest"
- True dynamical degrees of freedom of the gravitational field
  $\hookrightarrow$
  - Transport E, $\boldsymbol{p}$, $\boldsymbol{L}$ over large distances w.r.t. curvature radius
  - Transport information via the "vacuum" part of the curvature: initial discontinuity propagates along GW "rays"

2 d.o.f. h_+, $h_\times \sim \Theta\Theta$, $\Theta\Phi$ components of $h_{ij}^{\rm rad}$ in spherical coordinates

Binary systems as GW sources

Typical signal amplitude

$$h \sim \epsilon \frac{Gmv^2}{Rc^4}$$

Characteristics of a good GW source:

- $\dfrac{Gm}{Rc^2}$ not too small: must be abundant or massive enough
- ϵ not too small: must be asymmetric
- v/c comparable to 1: must be relativistic

$\hookrightarrow$ for bound systems: $\dfrac{Gm}{rc^2} \sim 1 \rightarrow$ strong field

Best candidates

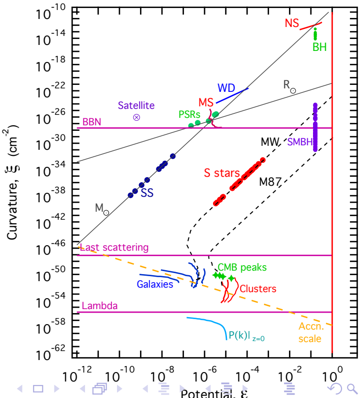
Binary systems of compact objects

Such binaries may be composed of:

- Neutron stars (LIGO, Virgo)
- Stellar-mass black holes (LIGO, Virgo, even LISA for the heavy ones)
- Supermassive black holes (LISA, PTA)

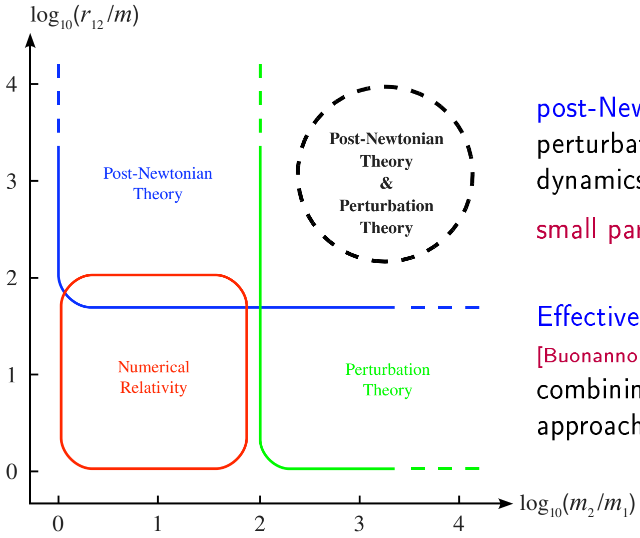
Interest of modeling the dynamics accurately

- To prove the quadrupole formula in the self-gravitating regime
  - important issue after the discovery of the Hulse-Taylor pulsar
    $\hookrightarrow$ needed to confirm the first indirect detection of GW
  - problem subject to controversy in the early 80's
- To understand the cardinal 2-body problem in general relativity
- To help extracting information from observational data
  $\hookrightarrow$ relevant to build waveform templates for the LIGO/Virgo/LISA DA



$\Rightarrow$ will allow to explore a regime with much stronger fields than in binary pulsars

Approximation techniques



post-Newtonian approximation: perturbative derivation of the dynamics

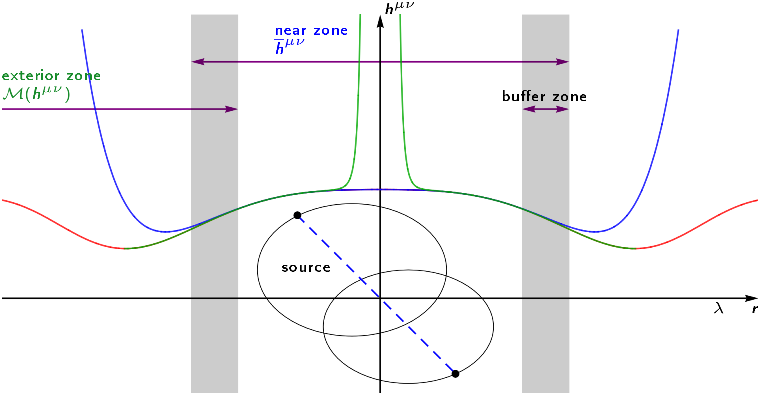
small parameter: $v^2/c^2 = 1\text{PN}$

Effective-One-Body techniques [Buonanno & Damour 1999] permits combining outputs from the 3 approaches

Validity of the post-Newtonian regime

- Small velocities: $\max v \ll c$
- Size of matter source $D \ll \lambda$

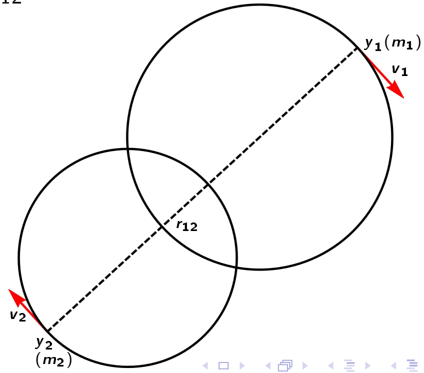
- Restriction to a spacetime domain of $(\boldsymbol{x}, t)$ with
  - $|\boldsymbol{x}| \ll \lambda \quad \longrightarrow \quad$ near zone
  - $t \in$ time domain



Case of compact binaries in its late stages

- Weak field:
  $$\varepsilon \sim \frac{Gm}{r_{12}c^2} \sim \frac{v^2}{c^2} \ll 1$$
  but: starts to be large enough so that radiation reaction effects show up
  $\rightarrow$ non-gravitational external forces then negligible
- Large separation: $\dfrac{R_A}{r_{12}} \sim \dfrac{Gm_A}{r_{12}c^2} \sim \varepsilon$

very clean system

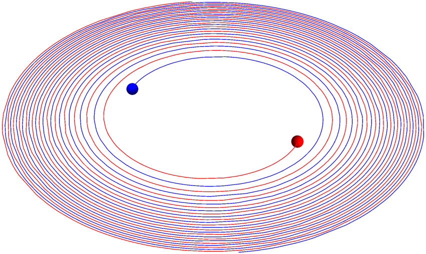


Newtonian model of the inspiraling phase

- Leading order quadrupolar flux $\equiv$ Newtonian order

balance equations for E, $\boldsymbol{J}$ $\Rightarrow \begin{cases} \bullet\ e \searrow 0 \text{ for isolated binaries} \\ \bullet\ E \text{ and } r_{12} \searrow \text{ at a rate} \sim \epsilon^{5/2} \end{cases}$

$= 2.5\text{PN order}$



$$\omega^2 = \frac{Gm}{r_{12}^3}\left[1 + \left(\frac{Gm}{r_{12}c^2}\right)(\dots) + \dots\right]$$

$$E = -\frac{\mu c^2 x}{2}\left[1 + x(\dots) + \dots\right]$$

$$\text{with } x = \left(\frac{Gm\omega}{c^3}\right)^{2/3}$$

- For circular orbits: $E = E(x)$ and $\mathcal{F} = \mathcal{F}(x)$ are gauge invariant
- Convergence at the formal ISCO:
  - slow for $m_2/m_1 \to 0$
  - seemingly better for $m_1 \sim m_2$

Possible approaches

- PN Iteration Scheme in Harmonic coordinates: PNISH
  - French flavored: effective $T_{\rm pp}^{\mu\nu}$ + dim reg + asymptotic matching
    - initiated at IHP and Meudon in 1981 with Damour & Deruelle
    - Blanchet-Damour-Iyer generation in the mid 80's formalism
    - calculation of 2.5PN quantities achieved in the mid 90's
    - 3PN in the early 2000's

    $\hookrightarrow$ Blanchet, Damour, Iyer, Faye, Le Tiec, Marsat, Bohé, Bernard, ...
  - American flavored: perfect fluid + splitting of volume integrals
    $\hookrightarrow$ Will, Wiseman, Kidder, Pati, ...
- Method à la Einstein-Infeld-Hoffmann (strong-field region avoidance)
  $\hookrightarrow$ Futamase, Itho, Asada
- Effective Field Theory approach in harmonic coordinates: EFT
  $\hookrightarrow$ Goldberger, Rothstein, Porto, Ross, Foffa, Sturani, Kol, Smolkin, Levi, ...
- Hamiltonian approach: ADM
  $\hookrightarrow$ Schäfer, Jaranowski, Damour, Steinhoff, Hergt, Hartung, ...

PN iteration in harmonic coordinates

Metric perturbation: $h^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} - \eta^{\mu\nu}$

Harmonic gauge equations

$$\partial_\nu h^{\mu\nu} = 0 \qquad \text{(gauge conditions)}$$

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu} \equiv \frac{16\pi G}{c^4}|g|T^{\mu\nu} + \Lambda^{\mu\nu}(\partial h, \partial h)$$

$h^{\mu\nu}$ searched in the form $\displaystyle\sum_{m\geq m_0(\mu,\nu)} c^{-m} h^{\mu\nu}_{[m]}$

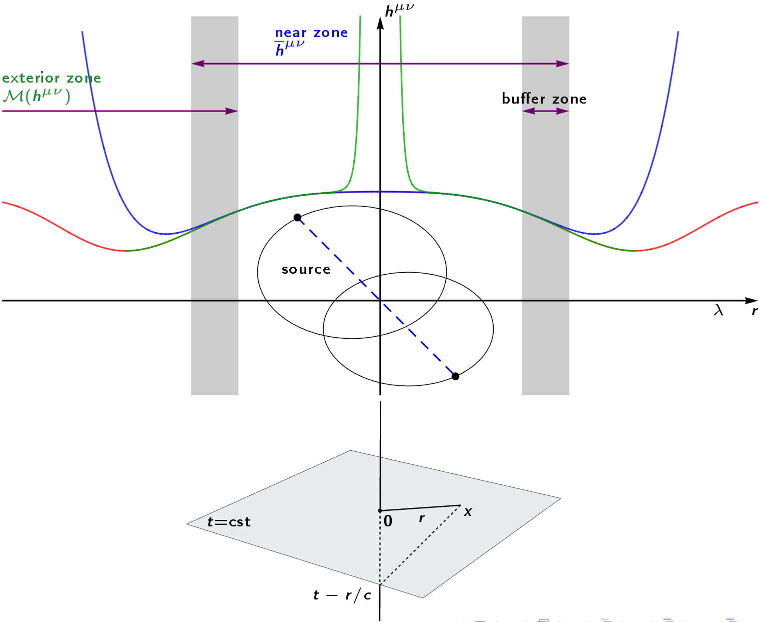
- Assume that previous orders $h^{\mu\nu}_{[m']}$ are known
- Solution for $h^{\mu\nu}_{[m]}$

$$h^{\mu\nu}_{[m]} = 16\pi G\left\{\Box_{\mathbf{R}}^{-1}\left[\overline{\tau}^{\mu\nu}(\overline{h}^{\alpha\beta})\right] + \sum_{\ell\geq 0}\overline{\partial_L\left(\frac{R_L^{\mu\nu}(t-r/c) - R_L^{\mu\nu}(t+r/c)}{r}\right)}\right\}_{[m-4]}$$

$$R_L^{\mu\nu} = R^{\mu\nu}_{i_1\ldots i_\ell}[\mathcal{M}(h^{\alpha\beta})]$$

- Go to the next order

Field multipole expansion outside the source



Blanchet-Damour-Iyer multipole moments

Outside the source:

$$\Box h^{\mu\nu} = \Lambda^{\mu\nu} \quad \text{(Relaxed EE)} \qquad \partial_\nu h^{\mu\nu} = 0 \quad \text{(Gauge Cond)}$$

- At linear order: $\Lambda^{\mu\nu} \longrightarrow 0$
- No-incoming wave solution of (REE): $h^{\mu\nu} = \sum_{\ell \geq 0} \partial_L \left(\frac{\mathcal{H}_L^{\mu\nu}(t - r/c)}{r} \right)$

  retarded solution of (REE) + (Gauge Cond):

$$h^{\mu\nu} = h^{\mu\nu}[\underbrace{I_L, J_L,}_{\text{source moments}} \underbrace{W_L, X_L, Y_L, Z_L}_{\text{gauge moments}}]$$

- General expression for I_L, J_L found by asymptotic matching

$$I_L(t_r) = \underset{L}{\mathrm{STF}} \int \mathrm{d}^3x \; \overbrace{x^L}^{x^{i_1} \ldots x^{i_\ell}} \left(\frac{\mathcal{T}^{00} + \mathcal{T}^{ii}}{c^2} \right)_{t = t_r} + \mathcal{O}\left(\frac{1}{c}\right)$$

Post-Minkowskian iteration

$h^{\mu\nu}$ searched in the form $\sum_{n\geq 1} G^n h^{\mu\nu}_{(n)}$

- Solution of (REE) assuming past stationarity

$$p^{\mu\nu}_{(n)} = \underbrace{\mathrm{FP}\Box^{-1}_{\mathrm{R}}}_{\text{appropriate regularized retarted integral}} \Lambda^{\mu\nu}_{(n)}$$

- Solution for (REE) + (Gauge Cond) built as $h^{\mu\nu}_{(n)} = p^{\mu\nu}_{(n)} + q^{\mu\nu}_{(n)}$

$$\Box q^{\mu\nu}_{(n)} = 0 \quad \text{with} \quad \partial_\nu q^{\mu\nu}_{(n)} = -\partial_\nu p^{\mu\nu}_{(n)} \quad \Rightarrow \quad q^{\mu\nu}_{(n)}$$

General solution

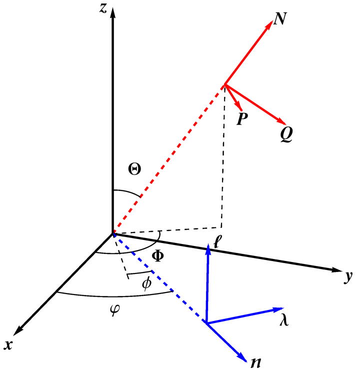
$$h^{\mu\nu}_{(n)} = p^{\mu\nu}_{(n)} + q^{\mu\nu}_{(n)}$$

Gravitational waves

First term of the multipole expansion of the form

$$h_{ij}^{\mathrm{rad}}(\mathbf{X}, T) = \frac{4G}{c^4 R} \sum_{\ell=2}^{+\infty} \frac{1}{c^{\ell-2}\ell!} \Big\{ N_{L-2} U_{ijL-2} - \frac{2\ell}{c(2\ell+1)} N_{aL-2}\epsilon_{ab(i} V_{j)bL-2} \Big\}^{\mathrm{TT}} (T - R/c)$$

Generalizes the quadrupole formula



Radiative moments

Link between the various multipole moment sets

$$\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{I_L, J_L, \ldots} \arrow[r, "\text{link S} \to \text{Can}"'] & \boxed{M_L, S_L} \arrow[r, "\text{link Can} \to \text{R}"'] & \boxed{U_L, V_L} \arrow[d] \\
\text{source} \arrow[u] & & \text{waveform}
\end{tikzcd}$$

comparison between $h^{\mu\nu}_{\text{ext}}$ and h^{TT}_{ij} + coord. transf.

Link Can $\longrightarrow$ R

$$U_L[I,J,\ldots] = U_L^{\text{inst}}[M,S] + U_L^{\text{tail}}[M,S] + U_L^{\text{tail-tail}}[M,S] + U_L^{\text{mem}}[M,S] + \ldots$$
$$V_L[I,J,\ldots] = V_L^{\text{inst}}[M,S] + V_L^{\text{tail}}[M,S] + V_L^{\text{tail-tail}}[M,S] + V_L^{\text{mem}}[M,S] + \ldots$$

- instantaneous terms: function of $\partial_t^k M_L(T_R)$, $\partial_t^k S_L(T_R)$
- tail terms: depend weakly on the source past history
- memory terms: depend strongly on the source past history

Recent results: radiative moments

All links for 3.5PN waveforms have been computed [F., Blanchet, Iyer (2014)]
Likewise for the quartic tail of tail of tail [Marchand, Blanchet, F. (2016)]

$$
\begin{aligned}
U^{\text{inst}}_{ijkl}(T_R) &= M^{(4)}_{ijkl} + \frac{G}{c^3}\left[-\frac{21}{5}M^{(5)}_{\langle ij}M_{kl\rangle} - \frac{63}{5}M^{(4)}_{\langle ij}M^{(1)}_{kl\rangle} - \frac{102}{5}M^{(3)}_{\langle ij}M^{(2)}_{kl\rangle}\right] \\
&+ \frac{G}{c^5}\left[\frac{7}{55}M_{a\langle i}M^{(7)}_{jkl\rangle a} + \frac{7}{55}M^{(1)}_{a\langle i}M^{(6)}_{jkl\rangle a} + \frac{1}{25}M^{(2)}_{a\langle i}M^{(5)}_{jkl\rangle a} - \frac{28}{11}M^{(3)}_{a\langle i}M^{(4)}_{jkl\rangle a} - \frac{273}{55}M^{(4)}_{a\langle i}M^{(3)}_{jkl\rangle a}\right. \\
&\quad - \frac{203}{55}M^{(5)}_{a\langle i}M^{(2)}_{jkl\rangle a} - \frac{49}{55}M^{(6)}_{a\langle i}M^{(1)}_{jkl\rangle a} + \frac{14}{275}M^{(7)}_{a\langle i}M_{jkl\rangle a} + \frac{14}{33}M_{a\langle ij}M^{(7)}_{kl\rangle a} + \frac{37}{33}M^{(1)}_{a\langle ij}M^{(6)}_{kl\rangle a} \\
&\quad + \frac{9}{11}M^{(2)}_{a\langle ij}M^{(5)}_{kl\rangle a} + \frac{8}{33}M^{(3)}_{a\langle ij}M^{(4)}_{kl\rangle a} + \frac{9}{5}S_{\langle i}S^{(5)}_{jkl\rangle} + \frac{16}{5}S_{\langle ij}S^{(5)}_{kl\rangle} + \frac{48}{5}S^{(1)}_{\langle ij}S^{(4)}_{kl\rangle} + \frac{32}{5}S^{(2)}_{\langle ij}S^{(3)}_{kl\rangle} \\
&\quad + \varepsilon_{ab\langle i}\left(-\frac{3}{5}M_{j\underline{a}}S^{(6)}_{kl\rangle b} - \frac{63}{25}M^{(1)}_{j\underline{a}}S^{(5)}_{kl\rangle b} + \frac{3}{5}M^{(2)}_{j\underline{a}}S^{(4)}_{kl\rangle b} + \frac{18}{5}M^{(3)}_{j\underline{a}}S^{(3)}_{kl\rangle b} + \frac{9}{5}M^{(4)}_{j\underline{a}}S^{(2)}_{kl\rangle b}\right. \\
&\qquad + \frac{3}{5}M^{(5)}_{j\underline{a}}S^{(1)}_{kl\rangle b} + \frac{3}{25}M^{(6)}_{j\underline{a}}S_{kl\rangle b} - \frac{8}{15}S_{j\underline{a}}M^{(6)}_{kl\rangle b} - \frac{24}{25}S^{(1)}_{j\underline{a}}M^{(5)}_{kl\rangle b} - \frac{8}{5}S^{(2)}_{j\underline{a}}M^{(4)}_{kl\rangle b} \\
&\qquad \left.\left. + \frac{16}{3}S^{(3)}_{j\underline{a}}M^{(3)}_{kl\rangle b} + \frac{72}{5}S^{(4)}_{j\underline{a}}M^{(2)}_{kl\rangle b} + \frac{56}{5}S^{(5)}_{j\underline{a}}M^{(1)}_{kl\rangle b} + \frac{232}{75}S^{(6)}_{j\underline{a}}M_{kl\rangle b} + \frac{29}{75}M^{(6)}_{jkl\rangle a}S_b\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
U^{\text{tail3}}_{ij}(T_R) &= \frac{G^3 M^3}{c^9}\int_0^{+\infty} \mathrm{d}\tau\, M^{(6)}_{ij}(T_R-\tau)\left[\frac{4}{3}\ln^3\left(\frac{c\tau}{2b_0}\right) + \frac{11}{3}\ln^2\left(\frac{c\tau}{2b_0}\right) + \frac{124627}{11025}\ln\left(\frac{c\tau}{2b_0}\right)\right. \\
&\quad \left. - \frac{428}{105}\ln\left(\frac{c\tau}{2b_0}\right)\ln\left(\frac{c\tau}{2r_0}\right) - \frac{1177}{315}\ln\left(\frac{c\tau}{2r_0}\right) + \frac{129268}{33075} + \frac{428}{315}\pi^2\right] + \mathcal{O}\left(\frac{1}{c^{12}}\right)
\end{aligned}
$$

$\hookrightarrow$ allowed to obtain the 4.5PN terms in $\mathcal{F}$

Recent results: partial high order amplitudes

Binaries of non-spinning objects

What we know

	PN orders	
quantities	circular	eccentric
EOM	4PN	3.5PN
E, $\boldsymbol{J}$	4PN	3.5PN
E, $\boldsymbol{J}$ flux	3.5PN	3PN
$\phi(t)$	3.5PN	3PN
$h_{+,\times}$	3PN	1PN

Recent partial results for $\mathcal{F}$

- Circular case (no spin)
  4.5PN contributions (without 4PN)
  [Marchand, Blanchet, F. (2016)]
  I_{ij} at 4PN in progress [Marchand's thesis]

Recent partial results for $h_{+,\times}$

- Eccentric case (no spin)
  inst. part of the waveform at 3PN
  [Chandra Mishra, Arun, Iyer (2015)]
- Circular case at 3.5PN:
  1. mode (2,2)
     [F., Marsat, Blanchet, Iyer (2012)]
  2. mode (3,3), (3,1)
     [F., Blanchet, Iyer (2014)]

Works in progress...

Equations of motion and conserved quantities

- Compute the EOM
  - from a reduced action S, à la Fokker: $S_{\text{red}} = S[h^{\alpha\beta}_{(\text{REE})}[\boldsymbol{y}_A], \boldsymbol{y}_B]$

$$\left(\frac{\delta S}{\delta y_A^i}\right)_{h=h_{(\text{REE})}[\boldsymbol{y}_A]} = 0 \quad \& \quad h_{(\text{REE})}[\boldsymbol{y}_A] \text{ built with } \Box^{-1}_{\text{sym.}} \quad \Leftrightarrow \quad \frac{\delta S_{\text{red}}}{\delta y_A^i} = 0$$

  - from the conservation of some $T^{\mu\nu}_{\text{effective}}$
- Deduce E, $\boldsymbol{J}$, ..., in the COM frame
- Compute the relevant U_L, V_L in COM
- Deduce:
  - the polarization amplitudes
  - the orbital phase for circular orbits

$$\frac{\mathrm{d}E(\omega)}{\mathrm{d}t} = -\mathcal{F}(\omega) \qquad \frac{\mathrm{d}\phi(\omega)}{\mathrm{d}t} = \omega$$

description of the dynamics of *extended* bodies in terms of:

- Worldline density: $n(x) = \int \mathrm{d}\lambda \frac{\delta^4(x^\mu - y^\mu)}{\sqrt{-g}}$ = covariant $\delta^3(\boldsymbol{x} - \boldsymbol{y})$
- Effective linear momentum p_μ and spin $S_{\mu\nu}$
- Effective quadrupole moment: $J^{\mu\nu\rho\sigma}$ $\rightarrow$ encode spin/tidal induced quadrupoles
- Effective octupole moment: $J^{\lambda\mu\nu\rho\sigma}$

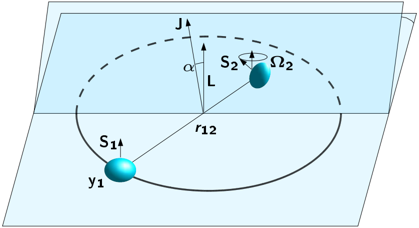
[Bailey, Israel (1975); Dixon (70's); Steinhoff, Pueztfeld (2009); Marsat (2015)]

$$T^{\mu\nu} = n\left[p^{(\mu}u^{\nu)}c + \frac{c^2}{3}R^{(\mu}{}_{\lambda\rho\sigma}J^{\nu)\lambda\rho\sigma} + \frac{c^2}{6}\nabla^\lambda R^{(\mu}{}_{\tau\rho\sigma}J_\lambda{}^{\nu)\tau\rho\sigma} + \frac{c^2}{12}\nabla^{(\mu}R_{\lambda\tau\rho\sigma}J^{\nu)\lambda\tau\rho\sigma}\right]$$
$$+ \nabla_\rho\left\{n\left[u^{(\mu}c\,S^{\nu)\rho} - \frac{c^2}{6}R^{(\mu}{}_{\tau\lambda\sigma}J^{|\rho|\nu)\tau\lambda\sigma} - \frac{c^2}{3}R^{(\mu}{}_{\tau\lambda\sigma}J^{\nu)\rho\tau\lambda\sigma} + \frac{c^2}{3}R^{\rho}{}_{\tau\lambda\sigma}J^{(\mu\nu)\tau\lambda\sigma}\right]\right\}$$
$$- \frac{2c^2}{3}\nabla_\rho\nabla_\sigma\left\{nJ^{\rho(\mu\nu)\sigma}\right\} + \frac{c^2}{3}\nabla_\lambda\nabla_\rho\nabla_\sigma\left\{nJ^{\sigma\rho(\mu\nu)\lambda}\right\} + \ldots$$

Orbital phase for quasi-circular orbits I

Notations:

- $\boldsymbol{S}$: total spin
- $\boldsymbol{\Sigma}$: antisymmetric spin
- m: total mass
- $\nu = \mu/m$: symmetric mass ratio



- Resonance effects ignored
- Absorption effects ignored
- Quasi-circular motion

$$\phi(x) = -\frac{x^{-5/2}}{32\nu}\left[1 + x\varphi_{\rm NS} + x^{3/2}\frac{\varphi_{\rm SO}}{Gm^2} + x^2\frac{\varphi_{\rm SS}}{G^2m^4} + x^{7/2}\frac{\varphi_{\rm SSS}}{G^3m^6} + x^5\varphi_{\rm T} + \ldots\right]$$

Orbital phase for quasi-circular orbits II

$$\phi(x) = -\frac{x^{-5/2}}{32\nu}\left[1 + x\varphi_{\rm NS} + x^{3/2}\frac{\varphi_{\rm SO}}{Gm^2} + x^2\frac{\varphi_{\rm SS}}{G^2m^4} + x^{7/2}\frac{\varphi_{\rm SSS}}{G^3m^6} + x^5\varphi_{\rm T} + \ldots\right]$$

$$\begin{aligned}
\varphi_{\rm SO} = & \frac{235}{6}S_\ell + \frac{125}{8}\delta\Sigma_\ell + x\ln x\left[\left(-\frac{554345}{2016} - \frac{55}{8}\nu\right)S_\ell + \left(-\frac{41745}{448} + \frac{15}{8}\nu\right)\frac{\delta m}{m}\Sigma_\ell\right] \\
& + x^{3/2}\left[\frac{940\pi}{3}S_\ell + \frac{745\pi}{6}\frac{\delta m}{m}\Sigma_\ell\right] \\
& + x^2\left[\left(-\frac{8980424995}{6096384} + \frac{6586595}{6048}\nu - \frac{305}{288}\nu^2\right)S_\ell + \left(-\frac{170978035}{387072} + \frac{2876425}{5376}\nu + \frac{4735}{1152}\nu^2\right)\frac{\delta m}{m}\Sigma_\ell\right] \\
& + x^{5/2}\left[\left(\frac{2388425\pi}{3024} - \frac{9925\pi}{36}\nu\right)S_\ell + \left(\frac{3237995\pi}{12096} - \frac{258245\pi}{2016}\nu\right)\frac{\delta m}{m}\Sigma_\ell\right] + \mathcal{O}\left(\frac{1}{c^6}\right)
\end{aligned}$$

[Bohé, Marsat, Blanchet (2013); Marsat, Bohé, Blanchet, Buonanno (2014)]

ORBITAL PHASE FOR QUASI-CIRCULAR ORBITS II

$$\phi(x) = -\frac{x^{-5/2}}{32\nu}\left[1 + x\varphi_{\rm NS} + x^{3/2}\frac{\varphi_{\rm SO}}{Gm^2} + x^2\frac{\varphi_{\rm SS}}{G^2 m^4} + x^{7/2}\frac{\varphi_{\rm SSS}}{G^3 m^6} + x^5\varphi_{\rm T} + \ldots\right]$$

$$\begin{aligned}\varphi_{\rm SS} = {} & S_\ell^2\left(-25\kappa_+ - 50\right) + S_\ell\Sigma_\ell\left(-25\delta\kappa_+ - 50\delta + 25\kappa_-\right) + \Sigma_\ell^2\left(\frac{25\delta\kappa_-}{2} - \frac{25\kappa_+}{2} - \frac{5}{16} + \nu\left(25\kappa_+ + 50\right)\right)\\ & + x\left[S_\ell^2\left(\left(\frac{2215\delta\kappa_-}{48} + \frac{15635\kappa_+}{84} - \frac{31075}{126}\right) + \nu\left(30\kappa_+ + 60\right)\right)\right.\\ & \quad + S_\ell\Sigma_\ell\left(\left(\frac{47035\delta\kappa_+}{336} - \frac{9775\delta}{42} - \frac{47035\kappa_-}{336}\right) + \nu\left(30\delta\kappa_+ + 60\delta - \frac{2575\kappa_-}{12}\right)\right)\\ & \quad + \Sigma_\ell^2\left(\left(-\frac{47035\delta\kappa_-}{672} + \frac{47035\kappa_+}{672} - \frac{410825}{2688}\right)\right.\\ & \qquad \left.\left. + \nu\left(-\frac{2935\delta\kappa_-}{48} - \frac{4415\kappa_+}{56} + \frac{23535}{112}\right) + \nu^2\left(-30\kappa_+ - 60\right)\right)\right] + \mathcal{O}\left(\frac{1}{c^3}\right)\end{aligned}$$

[Bohé, F., Marsat, Porter (2015)]

For black holes: $\kappa_+ = 2$

$\kappa_- = 0$

For neutron stars: $\kappa_A = 4\text{–}8$

Orbital phase for quasi-circular orbits II

$$\phi(x) = -\frac{x^{-5/2}}{32\nu}\left[1 + x\varphi_{\rm NS} + x^{3/2}\frac{\varphi_{\rm SO}}{Gm^2} + x^2\frac{\varphi_{\rm SS}}{G^2m^4} + x^{7/2}\frac{\varphi_{\rm SSS}}{G^3m^6} + x^5\varphi_{\rm T} + \ldots\right]$$

$$\begin{aligned}\varphi_{\rm SSS} = {} & S_\ell^3\left(\frac{185\kappa_+}{2} - 55\lambda_+ + 515\right) + S_\ell^2\Sigma_\ell\left(\frac{1105\delta\kappa_+}{8} - \frac{165\delta\lambda_+}{2} + \frac{3085\delta}{4} - \frac{4205\kappa_-}{24} + \frac{165\lambda_-}{2}\right) \\ & + S_\ell\Sigma_\ell^2\left(-\frac{2095\delta\kappa_-}{12} + \frac{165\delta\lambda_-}{2} + \frac{2095\kappa_+}{12} - \frac{165\lambda_+}{2} + \frac{24815}{96} + \nu\left(-275\kappa_+ + 165\lambda_+ - 1540\right)\right) \\ & + \Sigma_\ell^3\left(\frac{385\delta\kappa_+}{6} - \frac{55\delta\lambda_+}{2} + \frac{55\delta}{64} - \frac{385\kappa_-}{6} + \frac{55\lambda_-}{2}\right. \\ & \left.\left. + \nu\left(-\frac{365\delta\kappa_+}{8} + \frac{55\delta\lambda_+}{2} - \frac{1025\delta}{4} + \frac{4175\kappa_-}{24} - \frac{165\lambda_-}{2}\right)\right)\right) + \mathcal{O}\left(\frac{1}{c^2}\right).\end{aligned}$$

[Marsat (2015)]

For black holes: $\lambda_+ = 2$
$\lambda_- = 0$

Orbital phase for quasi-circular orbits II

$$\phi(x) = -\frac{x^{-5/2}}{32\nu}\left[1 + x\varphi_{\rm NS} + x^{3/2}\frac{\varphi_{\rm SO}}{Gm^2} + x^2\frac{\varphi_{\rm SS}}{G^2m^4} + x^{7/2}\frac{\varphi_{\rm SSS}}{G^3m^6} + x^5\varphi_{\rm T} + \ldots\right]$$

$$\varphi_{\rm T} = 12k_1^{(2)}\frac{R_1c^2}{Gm}\left(1 + 12\frac{X_2}{\underset{m_1/m}{X_1}}\right)\left[1 + x\frac{5(3179 - 919X_1 - 2286X_1^2 + 260X_1^3)}{672(12 - 11X_1)} - \pi x^{3/2} + \mathcal{O}\left(\frac{1}{c^4}\right)\right] + 1 \leftrightarrow 2$$

[Vines, Flanagan, Hinderer (2011); Damour, Nagar, Villain (2012)]

4PN dynamics: system description

Action of the full system

$$S = \frac{c^3}{16\pi G}\int \mathrm{d}^4x \left[\sqrt{-g}\left(\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} - \Gamma^\rho_{\mu\nu}\Gamma^\lambda_{\rho\lambda}\right) - \underbrace{\frac{1}{2}g_{\mu\nu}\Gamma^\mu\Gamma^\nu}_{\text{gauge fixing term}}\right]$$

$$- \sum_A m_A c^2 \int \mathrm{d}t \sqrt{-(g_{\mu\nu})_A v_A^\mu v_A^\nu}$$

- Gravitational Fokker Lagrangian put under the form

$$L_g = \frac{c^4}{32\pi G}\int \mathrm{d}^3\mathbf{x}\left[\frac{1}{2}h_{\mu\nu}\Box h^{\mu\nu} - \frac{1}{4}h\Box h + h\partial h\partial h + \ldots\right]$$

$h^{\mu\nu} = h^{\mu\nu}[y_A]$ truncated at order $1/c^6$

- Matching formula

$$L_g^{\text{red}} = \underbrace{\mathop{\mathrm{FP}}_{B=0}\int \mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B \overline{\mathcal{L}}_g^{\text{red}}}_{\text{near zone PN}} + \underbrace{\cancel{\mathop{\mathrm{FP}}_{B=0}\int \mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B \mathcal{M}(\mathcal{L}_g^{\text{red}})}}_{\text{exterior zone}\to 5.5\text{PN}}$$

Homogeneous part of the solution $\overline{h}^{\mu\nu}$

- Has the form

$$\sum_{\ell\geq 0} \partial_L \left(\frac{R_L^{\mu\nu}(t-r/c) - R_L^{\mu\nu}(t+r/c)}{r} \right)$$

- Contains the reaction force of the tail waves

effect non-local in time with a conservative part

- First contributes to the action at 4PN

$$S_{\mathrm{tail}} = \frac{Gm}{5c^8} \mathrm{Pf}_{2s_0/c} \int\int \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} I_{ij}^{(3)}(t) I_{ij}^{(3)}(t')$$

s_0 must be of the form $r_0\, e^{-\alpha}$

4PN DYNAMICS: RESULTS

EOM obtained in 3 steps [Bernard, Blanchet, Bohé, F., Marsat (2016,2017)]:

1. Explicit integration of $\overline{\mathcal{L}}_g$ over $\mathbb{R}^3$
   $\hookleftarrow$ point particle divergences treated with dim reg
2. Elimination of $\ddot{y}_A$, $\dddot{y}_A$, ... from the Lagrangian
   $\hookrightarrow$ the result
   - does not depend of r_0
   - is manifestly Poincaré invariant (at 4PN)
3. Unexpected presence of IR-regularization dependent terms
   $\hookrightarrow$ parametrized by a second unknown parameter ☹
4. Calculation the 2 unknown parameters
   $\hookleftarrow$ energy and periastron advance compared with their expressions at first order in m_2/m_1 (self-force calculations)

Agreement with the 4PN Hamiltonian of [Damour, Jaranowski, Schäfer (2014)]

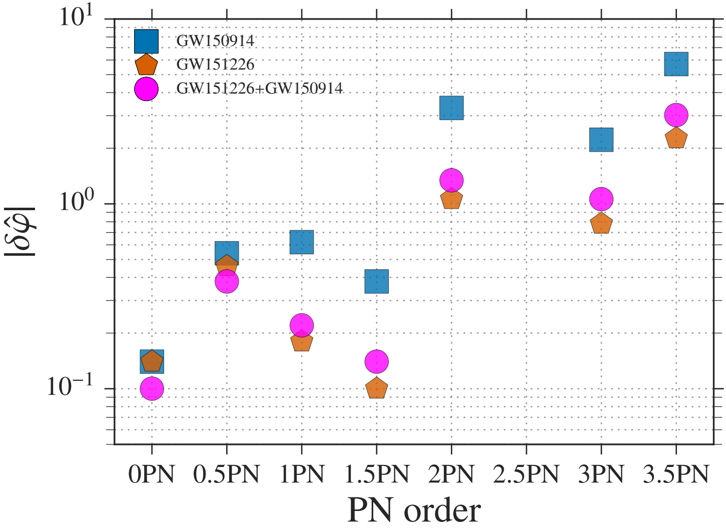
What to do next?

For non-spinning binaries: increase the accuracy to meet NR

- Clarify the 2 unknown parameters at 4PN; compute $\mathcal{F}$
- Complete the computation of $h_{+,\times}$ at 3.5PN
- Go beyond quasi-equilibrium configurations (resonances) see Newtonian dynamical tides [Flanagan, Hinderer (2008); Chakrabarti, Delsate, Steinhoff (2013)]

For spinning binaries

- Investigate the dynamics beyond quasi-circular motions [Klein's work]
  $\hookrightarrow$ crucial to investigate the possible biases in LIGO/Virgo DA
- Take absorption effects into account
  $\hookrightarrow$ fairly small but present
- Compute all 4PN spin contributions to the phase
- Obtain the 3.5PN amplitude



Thanks for your attention