Light-cone Averaging in Cosmology

A Gauge Invariant Approach

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Backreaction: where do we stand ?
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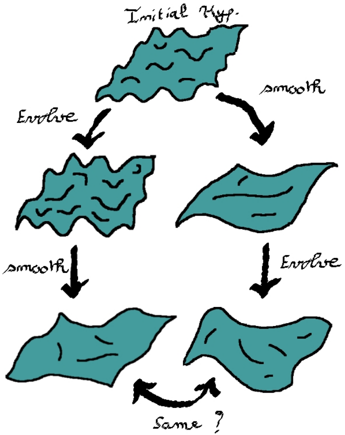
Outline

References :

- M. Gasperini, G. Marozzi, F.N., G. Veneziano – JCAP07 (2011) 008 (arXiv:1104.1167)
- I. Ben-Dayan, M. Gasperini, G. Marozzi, F.N., G. Veneziano – To appear soon...

Introduction

Importance of averaging



The evolution of an inhomogeneous spacetime after averaging differs from the evolution of its averaged spacetime.

Few questions:

- Can smoothing of structure contribute to an acceleration term (DE)?
  Is there an effect from small scales to large scales ?
  $\Rightarrow$ Nice way out of the coincidence problem? (T. Buchert)
- Consequences on cosmological parameters? (C. Clarkson, J. Larena)
  Negligible for a physical reason?
- What is the scale of homogeneity in the Universe? $100Mpc$?
- Fitting Problem : What is the best-fit FLRW model to a given lumpy Universe?
- How Einstein field equations transform after a coarse-graining procedure? How do we average vectors and tensors?

The necessity to work on the light-cone

Observations are made on the light-cone.
$\Rightarrow$ we need a formulation of a (gauge invariant) averaging procedure in terms of null hypersurfaces.

Few attempts have been done in the past (up to my knowledge).

We can find, in the literature:

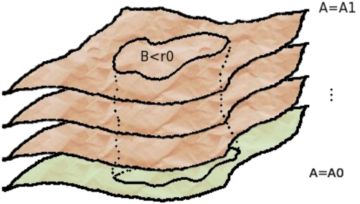
- A. Coley : definition of a Raychaudhuri equation for a null scalar ($\hat{\theta} = \frac{1}{2} k^a{}_{;a}$) , gr-qc/0905.2442 .
- S. Räsänen : relations for the redshift, the expansion rate, and the angular distance close to statistically isotropic and homogeneous universes , astro-ph/1107.1176, 0912.3370 .

Averaging on spatial hypersurfaces

Averaging on a spatial hypersurface $\Sigma(A)$

Foliation : $A(x) - A_0 = 0$
with A a timelike scalar

$$n_\mu = -\frac{\partial_\mu A}{\sqrt{-\partial_\mu A \partial^\mu A}}$$
$$n_\mu n^\mu = -1$$



Averaged scalars :

$$\langle S \rangle_{A_0} = \frac{I(S, A_0)}{I(1, A_0)} \ ,$$

B a scalar with spacelike gradient

$$\text{with} \quad : \ I(S, A_0) = \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} \sqrt{-(\partial A)^2} \delta(A(x) - A_0) \Theta(r_0 - B(x)) S(x) \ .$$

We also find

$$\partial_{A_0} \langle S \rangle_{A_0} = \left\langle \frac{n_\mu \partial^\mu S}{[-(\partial A)^2]^{1/2}} \right\rangle_{A_0} + \left\langle \frac{S\Theta}{[-(\partial A)^2]^{1/2}} \right\rangle_{A_0} - \langle S \rangle_{A_0} \left\langle \frac{\Theta}{[-(\partial A)^2]^{1/2}} \right\rangle_{A_0} ,$$

In ADM coordinates : $ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$,
A is homogeneous, $\Theta \equiv \nabla_\mu n^\mu = N^{-1} \partial_0 \ln(det(g_{ij}))$, $N^i = 0 \Rightarrow$ Buchert's relations.

Effective scale factor:

$$\frac{1}{\tilde{a}}\frac{\partial \tilde{a}}{\partial A_0} \equiv \frac{1}{3I(1,A_0)}\frac{\partial I(1,A_0)}{\partial A_0} = \frac{1}{3}\left\langle \frac{\Theta}{[-(\partial A)^2]^{1/2}} \right\rangle_{A_0}$$

Generalized Friedmann's equations:

$$\left(\frac{1}{\tilde{a}}\frac{\partial \tilde{a}}{\partial A_0}\right)^2 = \frac{8\pi G}{3}\left\langle \frac{\epsilon}{[-(\partial A)^2]} \right\rangle_{A_0} - \frac{1}{6}\left\langle \frac{\mathcal{R}_S}{[-(\partial A)^2]} \right\rangle_{A_0} - \frac{1}{9}\left[\left\langle \frac{\Theta^2}{[-(\partial A)^2]} \right\rangle_{A_0} - \left\langle \frac{\Theta}{[-(\partial A)^2]^{1/2}} \right\rangle_{A_0}^2\right] + \frac{1}{3}\left\langle \frac{\sigma^2}{[-(\partial A)^2]} \right\rangle_{A_0} ,$$

$$-\frac{1}{\tilde{a}}\frac{\partial^2 \tilde{a}}{\partial {A_0}^2} = \frac{4\pi G}{3}\left\langle \frac{\epsilon + 3\pi}{[-(\partial A)^2]} \right\rangle_{A_0} - \frac{1}{3}\left\langle \frac{\nabla^\nu(n^\mu \nabla_\mu n_\nu)}{[-(\partial A)^2]} \right\rangle_{A_0} + \frac{1}{6}\left\langle \frac{\partial_\mu A \partial^\mu [(\partial A)^2]\Theta}{[-(\partial A)^2]^{5/2}} \right\rangle_{A_0} - \frac{2}{9}\left[\left\langle \frac{\Theta^2}{[-(\partial A)^2]} \right\rangle_{A_0} - \left\langle \frac{\Theta}{[-(\partial A)^2]^{1/2}} \right\rangle_{A_0}^2\right] + \frac{2}{3}\left\langle \frac{\sigma^2}{[-(\partial A)^2]} \right\rangle_{A_0} .$$

with ϵ and π the ADM energy density and pressure.

M. Gasperini, G. Marozzi, G. Veneziano, 0912.3244/gr-qc

Averaging on the light-cone

M. Gasperini, G. Marozzi, F.N., G. Veneziano – JCAP07 (2011) 008 (arXiv:1104.1167)

(Gauge invariant) Light-cone Averaging

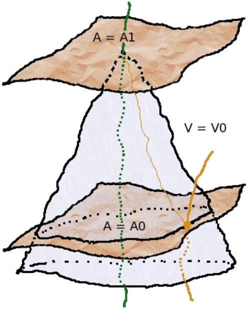
Naïvely: Should $A(x)$ be upgraded to a null scalar ?
No ! $\partial_\mu A \partial^\mu A = 0 \Rightarrow$ undefined !

Correct way: The integration of $S(x)$ over the volume is naturally

$$I(S;-;A_0,V_0) = \int_{\mathcal{M}_4} d^4x \sqrt{-g}\ \Theta(V_0-V)\Theta(A-A_0)S(x) \quad , \quad \partial_\mu V \partial^\mu V = 0 \quad .$$

Ex.: variation of the volume averages along the flow lines normal to $\Sigma(A)$:

$$\begin{aligned} &\int_{\mathcal{M}_4} d^4x\sqrt{-g}\ \Theta(V_0-V)\Theta(A-A_0)\nabla^\mu n_\mu \\ = \quad & -\int_{\mathcal{M}_4} d^4x\sqrt{-g}\,\Theta(V_0-V)\delta(A-A_0)\sqrt{-\partial_\mu A\partial^\mu A} \\ & + \int_{\mathcal{M}_4} d^4x\sqrt{-g}\,\delta(V_0-V)\Theta(A-A_0)\frac{-\partial_\mu V\partial^\mu A}{\sqrt{-\partial_\nu A\partial^\nu A}} \ . \end{aligned}$$

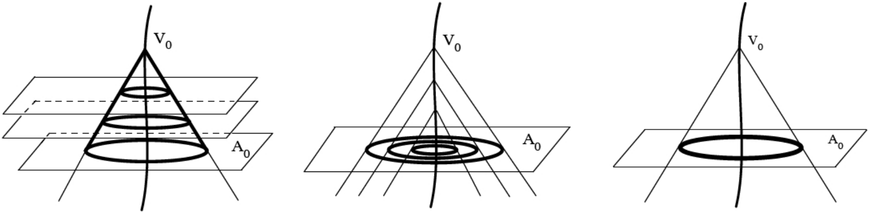


we thus define...

Definitions and illustration

$$
\begin{aligned}
I(1;V_0;A_0) &= \int_{\mathcal{M}_4} d^4x\sqrt{-g}\,\delta(V_0-V)\Theta(A-A_0)\frac{|\partial_\mu V\partial^\mu A|}{\sqrt{-\partial_\nu A\partial^\nu A}} \to \langle S\rangle^{A_0}_{V_0}, \\
I(1;A_0;V_0) &= \int_{\mathcal{M}_4} d^4x\sqrt{-g}\,\Theta(V_0-V)\delta(A-A_0)\sqrt{-\partial_\mu A\partial^\mu A} \to \langle S\rangle^{V_0}_{A_0}\ , \\
I(1;V_0,A_0;-) &= \int_{\mathcal{M}_4} d^4x\sqrt{-g}\ \delta(V_0-V)\delta(A-A_0)|\partial_\mu V\partial^\mu A| \to \langle S\rangle_{V_0,A_0}
\end{aligned}
$$

These integrals are gauge invariant, invariant under reparam. $A \to \tilde{A}(A)$ and $V \to \tilde{V}(V)$, invariant under general coordinate transformations (GCT).



$\langle S\rangle^{A_0}_{V_0}$, $\langle S\rangle^{V_0}_{A_0}$, $\langle S\rangle_{V_0,A_0}$

Buchert-Ehlers commutation rules

We define $k_\mu \equiv \partial_\mu V$ (null vector) and make use of identities like

$$k^\mu \partial_\mu \delta(V - V_0) = 0 \ , \ \ \delta'(A - A_0) = \frac{k^\mu \partial_\mu \delta(A - A_0)}{k^\nu \partial_\nu A} \ , \ \ \delta'(V - V_0) = \frac{\partial^\mu A \partial_\mu \delta(V - V_0)}{k^\nu \partial_\nu A} \ .$$

Thus we find

$$\frac{\partial}{\partial A_0} \langle S \rangle_{V_0, A_0} = \left\langle \frac{k \cdot \partial S}{k \cdot \partial A} \right\rangle_{V_0, A_0} + \left\langle \frac{\nabla \cdot k}{k \cdot \partial A} S \right\rangle_{V_0, A_0} - \left\langle \frac{\nabla \cdot k}{k \cdot \partial A} \right\rangle_{V_0, A_0} \langle S \rangle_{V_0, A_0} \ ,$$

$$\begin{aligned} \frac{\partial}{\partial V_0} \langle S \rangle_{V_0, A_0} = & \left\langle \frac{\partial A \cdot \partial S}{k \cdot \partial A} \right\rangle_{V_0, A_0} - \left\langle k \cdot \partial S \frac{(\partial A)^2}{(k \cdot \partial A)^2} \right\rangle_{V_0, A_0} \\ & + \left\langle \left[\Box A - \nabla_\mu \left(k^\mu \frac{(\partial A)^2}{k \cdot \partial A} \right) \right] \frac{S}{k \cdot \partial A} \right\rangle_{V_0, A_0} \\ & - \left\langle \left[\Box A - \nabla_\mu \left(k^\mu \frac{(\partial A)^2}{k \cdot \partial A} \right) \right] \frac{1}{k \cdot \partial A} \right\rangle_{V_0, A_0} \langle S \rangle_{V_0, A_0} \ , \end{aligned}$$

$$\frac{\partial}{\partial A_0}\langle S\rangle^{A_0}_{V_0} = \left\langle \frac{k\cdot\partial S}{k\cdot\partial A}\right\rangle^{A_0}_{V_0} + \left\langle \left[\nabla\cdot k - \frac{1}{2}\frac{k^\mu\partial_\mu\left((\partial A)^2\right)}{(\partial A)^2}\right]\frac{S}{k\cdot\partial A}\right\rangle^{A_0}_{V_0} - \left\langle \left[\nabla\cdot k - \frac{1}{2}\frac{k^\mu\partial_\mu\left((\partial A)^2\right)}{(\partial A)^2}\right]\frac{1}{k\cdot\partial A}\right\rangle^{A_0}_{V_0}\langle S\rangle^{A_0}_{V_0} \ ,$$

$$\begin{aligned}\frac{\partial}{\partial V_0}\langle S\rangle^{A_0}_{V_0} = & \left\langle \frac{\partial A\cdot\partial S}{k\cdot\partial A}\right\rangle^{A_0}_{V_0} - \left\langle k\cdot\partial S\frac{(\partial A)^2}{(k\cdot\partial A)^2}\right\rangle^{A_0}_{V_0} \\ & + \left\langle \left[\Box A - \partial^\mu A\partial_\mu \ln\left((\partial A)^2\right)\right]\frac{S}{k\cdot\partial A}\right\rangle^{A_0}_{V_0} \\ & - \left\langle \left[\nabla\cdot k\frac{(\partial A)^2}{(k\cdot\partial A)^2} + \frac{1}{2}k^\mu\partial_\mu\left(\frac{(\partial A)^2}{(k\cdot\partial A)^2}\right)\right] S\right\rangle^{A_0}_{V_0} \\ & - \left\langle \left[\Box A - \partial^\mu A\partial_\mu \ln\left((\partial A)^2\right)\right]\frac{1}{k\cdot\partial A}\right\rangle^{A_0}_{V_0}\langle S\rangle^{A_0}_{V_0} \\ & + \left\langle \left[\nabla\cdot k\frac{(\partial A)^2}{(k\cdot\partial A)^2} + \frac{1}{2}k^\mu\partial_\mu\left(\frac{(\partial A)^2}{(k\cdot\partial A)^2}\right)\right]\right\rangle^{A_0}_{V_0}\langle S\rangle^{A_0}_{V_0} \ .\end{aligned}$$

What are the applications?

Applications – General considerations

We introduce an adapted system of coordinates : *geodesic light-cone* (GLC) *coordinates* ($\sim$ gauge fixing of *observational coordinates*), defined by :

$$ds^2 = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\theta^a - U^a dw)(d\theta^b - U^b dw) \quad ; \quad a, b = 1, 2$$

(6 arbitrary functions : Υ, U^a, γ_{ab})

Notice that:

- w is a null coordinate : $\partial_\mu w \, \partial^\mu w = 0$,
- $\partial_\mu \tau$ defines a geodesic flow : $(\partial^\nu \tau) \nabla_\nu (\partial_\mu \tau) \equiv 0$,
- an observer defined by constant τ spacelike hyp. is in geodesic motion.

FLRW case:
$w = \eta + r$, $\tau = t$, $\Upsilon = a(t)$, $U^a = 0$, $\gamma_{ab} d\theta^a d\theta^b = a^2(t) r^2 (d\theta^2 + \sin^2\theta d\phi^2)$

I – Simplified Buchert-Ehlers rules

In these new coordinates, the velocity of a generic observer n^μ satisfies:

$$k_\mu n^\mu = \frac{1}{\Upsilon} \frac{\partial_\tau A}{\sqrt{-(\partial A)^2}} \quad , \quad k_\mu \equiv \partial_\mu w$$

Geodesic observer $\Rightarrow A = A(\tau)$ and we can always set $A = \tau \Rightarrow \; k_\mu n^\mu = \Upsilon^{-1}$.

The covariant divergence of k^μ takes the form: $\nabla_\mu k^\mu = -\frac{1}{\Upsilon} \partial_\tau \left(\ln \sqrt{|\gamma|} \right)$.

Buchert-Ehlers rules on the light-cone are then (simplified to)

$$\begin{aligned}
\partial_{\tau_0} \langle S \rangle = \langle \partial_\tau S \rangle \quad &+ \quad \left\langle S \partial_\tau \ln \sqrt{|\gamma|} \right\rangle - \left\langle \partial_\tau \ln \sqrt{|\gamma|} \right\rangle \langle S \rangle \ , \\
\partial_{w_0} \langle S \rangle = \langle \partial_w S + U^a \partial_a S \rangle \quad &+ \quad \left[\left\langle S \partial_w \ln \sqrt{|\gamma|} \right\rangle + \left\langle S \left[\partial_a U^a + U^a \partial_a \ln \sqrt{|\gamma|} \right] \right\rangle \right] \\
&- \quad \left[\left\langle \partial_w \ln \sqrt{|\gamma|} \right\rangle + \left\langle \partial_a U^a + U^a \partial_a \ln \sqrt{|\gamma|} \right] \right\rangle \langle S \rangle \ ,
\end{aligned}$$

where $\langle \ldots \rangle$ can be replaced either by $\langle \ldots \rangle_{w_0, \tau_0}$ or $\langle \ldots \rangle^{\tau_0}_{w_0}$ in each formula.

II – Redshift drift (RSD)

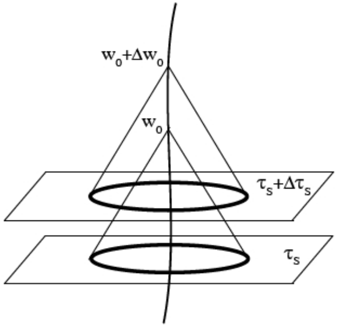
Definition

The RSD is the rate of change of the redshift of a given source with respect to the observer's proper time.

Assuming S and O are in geodesic motion with negligible peculiar velocities, the RSD in a FLRW Universe during proper time Δt_0 is

$$\frac{\Delta z}{\Delta t_0} = (1+z)H_0 - H_s = \frac{\dot{a}_0 - \dot{a}_s}{a_s} \; ,$$

with H_0 and H_s the respective Hubble rates.



Remark: Do not use standard candles.

Υ_0 indep. of $\theta^a_0 \Rightarrow \; \Delta(1+z) = \frac{\partial(1+z)}{\partial w_0}\Delta w_0 + \frac{\partial(1+z)}{\partial \tau_0}\Delta\tau_0 + \frac{\partial(1+z)}{\partial \tau_s}\Delta\tau_s + \frac{\partial(1+z)}{\partial \theta^a_s}\Delta\theta^a_s$.

For a geodesic observer: $n_\mu = -\partial_\mu \tau$ and $\dot{x}^\mu \sim n^\mu = (1/\Upsilon, 1, U^a/\Upsilon)$
$\Rightarrow \; \Delta\tau = \Upsilon \Delta w, \, \Delta\theta^a = U^a \Delta w \quad \Rightarrow \quad \Delta w_0 = \Delta\tau_0/\Upsilon_0 = \Delta\tau_s/\Upsilon_s$

So using $\widetilde{H}_0 \equiv \frac{1}{\Upsilon_0}\frac{\partial \Upsilon_0}{\partial \tau_0}$ we get

$$\frac{\Delta z}{\Delta \tau_0} = (1+z)\widetilde{H}_0 + \frac{1}{\Upsilon_0}\frac{\partial}{\partial w_0}(1+z) + \frac{\partial}{\partial \tau_s}\ln(1+z) + \frac{U^a_s}{\Upsilon_0}\frac{\partial}{\partial \theta^a_s}(1+z) \; .$$

Averaging RSD over the 2-sphere embedded in the LC ($w = w_0$,$\tau = \tau_s$) gives

$$\begin{aligned}
\frac{\langle \Delta z \rangle_{w_0,\tau_s}}{\Delta\tau_0} &= \langle 1+z \rangle_{w_0,\tau_s} \widetilde{H}_0 + \frac{1}{\Upsilon_0}\partial_{w_0}\langle 1+z\rangle_{w_0,\tau_s} + \partial_{\tau_s}\langle \ln(1+z)\rangle_{w_0,\tau_s} + Q_w(z) + Q_\tau(z) \; , \\
Q_\tau(z) &= -\left\langle \ln(1+z)\partial_\tau \ln\sqrt{|\gamma|}\right\rangle_{w_0,\tau_s} + \langle \ln(1+z)\rangle_{w_0,\tau_s}\left\langle \partial_\tau \ln\sqrt{|\gamma|}\right\rangle_{w_0,\tau_s} \; , \\
Q_w(z) &= \frac{1}{\Upsilon_0}\left\{ -\left\langle (1+z)\partial_w \ln\sqrt{|\gamma|}\right\rangle_{w_0,\tau_s} - \left\langle (1+z)\left[\partial_a U^a + U^a \partial_a \ln\sqrt{|\gamma|}\right]\right\rangle_{w_0,\tau_s} \right. \\
&\quad \left. + \langle 1+z\rangle_{w_0,\tau_s}\left[\left\langle \partial_w \ln\sqrt{|\gamma|}\right\rangle_{w_0,\tau_s} + \left\langle \partial_a U^a + U^a \partial_a \ln\sqrt{|\gamma|}\right\rangle_{w_0,\tau_s}\right]\right\} \; .
\end{aligned}$$

III – Luminosity distance – classically (See Ido's Talk)

A photon with momentum k_μ emitted by a source S and received by the observer O is redshifted with a factor z with

$$1 + z = \frac{(k_\mu n^\mu)_s}{(k_\nu n^\nu)_0} = \frac{\Upsilon(w_0, \tau_0, \theta^a_0)}{\Upsilon(w_0, \tau_s, \theta^a_s)} \equiv \frac{\Upsilon_0}{\Upsilon_s} \ .$$

The (well known) relation between the *luminosity distance* d_L and the *angular distance* as seen from the observer is

$$d_L = (1+z) d_o = (1+z)^2 d_s \quad , \quad d_s^2 = \frac{d\mathcal{A}}{d\Omega} \stackrel{\text{FLRW}}{=} \ r a(t_s) \ .$$

We can write

$$d_s^2(w = w_0, \tau_s, \theta^a) = \lim_{\rho \to 0} \ \rho^2 \frac{\sqrt{|\gamma_s|}}{\sqrt{|\gamma(\rho)|}} \quad ,$$

where ρ is the proper radius of an infinitesimal sphere centered around the observer, $w = w_0$ defines the past light-cone connecting source and observer, and $\tau = \tau_s$ defines the spacelike hypersurface normal to n^μ at the source.

III – Luminosity distance – averaged (See Ido's Talk)

Averaging d_L and $(1+z)$ on the 2-sphere embedded in the LC gives:

$$\langle d_L \rangle_{w_0,\tau_s} = \frac{\int d^2\theta \sqrt{|\gamma(w_0,\tau_s,\theta^a)|}\left[\Upsilon^2(w_0,\tau_0,\theta^a)/\Upsilon^2(w_0,\tau_s,\theta^a)\right] d_s(w_0,\tau_s,\theta^a)}{\int d^2\theta \sqrt{|\gamma(w_0,\tau_s,\theta^a)|}} ,$$

$$\langle 1+z \rangle_{w_0,\tau_s} = \frac{\int d^2\theta \sqrt{|\gamma(w_0,\tau_s,\theta^a)|}\;\left[\Upsilon(w_0,\tau_0,\theta^a)/\Upsilon(w_0,\tau_s,\theta^a)\right]}{\int d^2\theta \sqrt{|\gamma(w_0,\tau_s,\theta^a)|}} .$$

$\Rightarrow$ we can evaluate how the $d_L(z)$ relation is affected by inhomogeneities.

Considering averages on const-z hypersurfaces ($\Rightarrow$ measuring geodesic obs. with velocity $u^\mu \neq$ obs. defined by the flow lines of $\Sigma(A)$, $A = k_\mu u^\mu$), we get

$$\langle d_L \rangle_{w_0,z} = (1+z)^2 \frac{\int d^2\theta \sqrt{|\gamma(w_0,\tau(z,w_0,\theta^a),\theta^a)|} d_s(w_0,\tau(z,w_0,\theta^a),\theta^a)}{\int d^2\theta \sqrt{|\gamma(w_0,\tau(z,w_0,\theta^a),\theta^a)|}} ,$$

where $\tau(z,w_0,\theta^a)$ is the solution of $\frac{\Upsilon(w_0,\tau,\theta^a)}{\Upsilon_0} = \frac{1}{1+z}$.

$\Rightarrow$ gives back $d_L = (1+z)^2 d_s$ for an homogeneous Universe.

Conclusion

What we can say:

- Light-cone averaging is doable in a gauge invariant way.
- It can be applied to study usual relations in the inhomogeneous context, e.g.: Buchert-Ehlers relations, $d_L(z)$, RSD, ...

Questions up to now:

- What are the new effects involved by light-cone averaging?
- How big is the effect on the observables ?

On-going work: See Ido's Talk

- Studing non-perturbative models of Universe is hard!
- So we consider a perturbed Universe in our GLC metric...
  ... and compute the luminosity-distance relation up to second order with power spectrum taken from data fits.

Thank you!

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