Global gravitational instability of FLRW backgrounds

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w/ X. Roy, T. Buchert, S. Carloni, **CQG 28,** 165004 (2011)

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Backreaction: where do we stand ? – 22 - 23 November 2011 / IAP

Foliation of space-time

- Hypotheses:
  - fluid without pressure and vorticity $\rightarrow p = 0,\ w_{ij} = 0$
  - hypersurfaces orthogonal to the fluid flow $\rightarrow u^\mu \perp \Sigma_t$
  - Lagrangian picture $\rightarrow \alpha = 1,\ \beta^i = 0$
- Space-time metric:

$$g_{\alpha\beta} dX^\alpha dX^\beta = -dt^2 + h_{ij} dX^i dX^j$$

- Constraint & evolution equations:

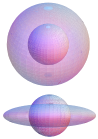
$$\begin{gathered}
{}^3\mathcal{R} - K^a{}_b K^b{}_a + K^2 = 16\pi G\,\varrho + 2\Lambda \\
\mathcal{D}_a K^a{}_i - \mathcal{D}_i K = 0 \\
\partial_t h_{ij} = -2K_{ij} \\
\partial_t K_{ij} = {}^3\mathcal{R}_{ij} - 2K^a{}_i K_{aj} + K K_{ij} + 4\pi G h_{ij}\varrho - \Lambda h_{ij}
\end{gathered}$$

Kinematical reformulation

- Decomposition of the covariant derivative of the fluid four-velocity:

$$\Theta_{ij} \equiv \nabla_i u_j = \frac{1}{3}\Theta h_{ij} + \sigma_{ij}$$

Θ_{ij}	expansion rate tensor
Θ	expansion rate
σ_{ij}	shear rate tensor



- Identification:

$$K_{ij} \equiv -h^{\alpha}{}_{i}h^{\beta}{}_{j}\nabla_\alpha u_\beta = -\Theta_{ij}$$

$$K \equiv -\Theta$$

- Reformulation:

$$\frac{1}{3}\Theta^2 = 8\pi G\varrho - \frac{1}{2}{}^3\mathcal{R} + \sigma^2 + \Lambda$$

$$\partial_t\Theta = -4\pi G\varrho - 2\sigma^2 - \frac{1}{3}\Theta^2 + \Lambda$$

$$\partial_t h_{ij} = 2\Theta_{ij}$$

Averaging procedure

- Average of a scalar field ψ over a compact spatial domain D:

$$\left\langle \psi(t, X^i) \right\rangle_D \equiv \frac{1}{V_D} \int_D \psi(t, X^i) \sqrt{h(t, X^i)}\, d^3X$$

V_D volume of the domain $\left(V_D \equiv \int_D \sqrt{h(t, X^i)}\, d^3X \right)$

h determinant of h_{ij}

- Non-commutation rule between spatial averaging and evolution:

$$\partial_t \langle \psi \rangle_D - \langle \partial_t \psi \rangle_D = \langle \Theta \rangle_D \langle \psi \rangle_D - \langle \Theta \psi \rangle_D$$

- Definition of a dimensionless effective scale factor:

$$a_D \equiv \left(\frac{V_D}{V_{D_i}} \right)^{1/3}$$

$$\langle \Theta \rangle_D = \frac{\partial_t V_D}{V_D} \quad \Rightarrow \quad \langle \Theta \rangle_D = 3 \frac{\partial_t a_D}{a_D}$$

Global evolution of a domain

- Buchert's equations:

$$3\left(\frac{\dot{a}_D}{a_D}\right)^2 = 8\pi G\left\langle\varrho\right\rangle_D - \frac{1}{2}\left({}^3\mathcal{R}_D + \mathcal{Q}_D\right) + \Lambda$$

$$3\frac{\ddot{a}_D}{a_D} = -4\pi G\left\langle\varrho\right\rangle_D + \mathcal{Q}_D + \Lambda$$

$$\left\langle\varrho\right\rangle_D^{\bullet} + 3\frac{\dot{a}_D}{a_D}\left\langle\varrho\right\rangle_D = 0$$

- Averaged scalar curvature: $${}^3\mathcal{R}_D \equiv \left\langle{}^3\mathcal{R}\right\rangle_D$$

- Kinematical back-reaction: $$\mathcal{Q}_D \equiv \frac{2}{3}\left\langle\left(\Theta - \left\langle\Theta\right\rangle_D\right)^2\right\rangle_D - 2\left\langle\sigma^2\right\rangle_D$$

Global evolution of a domain

- Buchert's equations:

$$
3\left(\frac{\dot{a}_D}{a_D}\right)^2 = 8\pi G\left\langle\varrho\right\rangle_D - 3\frac{k_{D_i}}{a_D^2} - \frac{1}{2}\left(\mathcal{W}_D + \mathcal{Q}_D\right) + \Lambda
$$

$$
3\frac{\ddot{a}_D}{a_D} = -4\pi G\left\langle\varrho\right\rangle_D + \mathcal{Q}_D + \Lambda
$$

$$
\left\langle\varrho\right\rangle_D^{\bullet} + 3\frac{\dot{a}_D}{a_D}\left\langle\varrho\right\rangle_D = 0
$$

- Curvature back-reaction:

$$
{}^3\mathcal{R}_D \equiv \left\langle{}^3\mathcal{R}\right\rangle_D \equiv 6k_{D_i}/a_D^2 + \mathcal{W}_D
$$

- Kinematical back-reaction:

$$
\mathcal{Q}_D \equiv \frac{2}{3}\left\langle\left(\Theta - \left\langle\Theta\right\rangle_D\right)^2\right\rangle_D - 2\left\langle\sigma^2\right\rangle_D
$$

Averaged inhomogeneous cosmology

- Addressing the : *nature of dark components*
- Buchert's equations:

$$3\left(\frac{\dot{a}_D}{a_D}\right)^2 = 8\pi G \langle \varrho \rangle_D - 3\frac{k_{D_i}}{a_D^2} - \frac{1}{2}\left(\mathcal{W}_D + \mathcal{Q}_D\right) + \Lambda$$

$$3\frac{\ddot{a}_D}{a_D} = -4\pi G \langle \varrho \rangle_D + \mathcal{Q}_D + \Lambda$$

$$\langle \varrho \rangle\dot{}_D + 3\frac{\dot{a}_D}{a_D}\langle \varrho \rangle_D = 0$$

$$\dot{\mathcal{Q}}_D + 6\frac{\dot{a}_D}{a_D}\mathcal{Q}_D + \dot{\mathcal{W}}_D + 2\frac{\dot{a}_D}{a_D}\mathcal{W}_D = 0$$

- $\mathcal{Q}_D > 0 \Rightarrow$ effective dark energy
  $\mathcal{Q}_D < 0 \Rightarrow$ effective cosmological dark matter

Stability ?

- Issues:
  - *Do FLRW backgrounds provide a correct approximation to describe the dynamics of the averaged space-time?*
  - *Are FLRW backgrounds globally gravitationally stable within the class of solutions of averaged inhomogeneous cosmologies?*

- 'Global gravitational instability of FLRW backgrounds—interpreting the dark sectors'. **CQG 28**, 165004 (2011)

  X. Roy, T. Buchert, S. Carloni and N. Obadia

Structure of Friedmann's equation

- Power-law structure:

$$3\left(\frac{\dot{a}_D}{a_D}\right)^2 = \frac{8\pi G \langle\varrho\rangle_{D_i}}{a_D^3} - 3\frac{k_{D_i}}{a_D^2} + \frac{\Lambda}{a_D^0} - \frac{1}{2}\left(\mathcal{W}_D + \mathcal{Q}_D\right)$$

- Integrability condition:

$$\dot{\mathcal{Q}}_D + 6\frac{\dot{a}_D}{a_D}\mathcal{Q}_D + \dot{\mathcal{W}}_D + 2\frac{\dot{a}_D}{a_D}\mathcal{W}_D = 0\,.$$

Structure of Friedmann's equation

- Power-law structure:

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- Integrability condition:

$$a_D\,\frac{d\,\mathcal{Q}_D}{d\,a_D} + 6\mathcal{Q}_D + a_D\,\frac{d\,\mathcal{W}_D}{d\,a_D} + 2\mathcal{W}_D = 0 \;\Rightarrow\; \mathcal{Q}_D \propto a_D^n\,,\; \mathcal{W}_D \propto a_D^p$$

Perturbative results

- Li & Schwarz, **PRD 76**, 083011(2007): Onset of cosmological backreaction

$$ds^2 = -dt^2 + a(t)^2 \left((1 - 2\Psi)\ \delta_{ij}\ +\ D_{ij}\Xi \right)\ dxi\, dx^j$$

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$$\langle \mathcal{R} \rangle_D = \frac{\text{Order 1}}{a_D^2} + \frac{\text{Order 2}}{a_D} \Leftrightarrow \mathcal{W}_D = \frac{\text{Order 2}}{a_D} \,,\; \mathcal{Q}_D = \frac{\text{Order 2}}{a_D}$$

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- Buchert, **CQG 28**, 164007 (2011): Relativistic Zeldovitch Approximation

$$^{\mathrm{RZA}}g_{ij}(t,X^k) = a^2(t)\left\{G_{ij} + \xi(t)\left(G_{aj}\mathcal{P}^a{}_i + G_{ib}\mathcal{P}^b{}_j\right) + \xi^2(t)G_{ab}\mathcal{P}^a{}_i\mathcal{P}^b{}_j\right\}$$

Perturbative results

- Li & Schwarz, **PRD 76**, 083011(2007): Onset of cosmological backreaction

$$\langle\mathcal{R}\rangle_D = \frac{\text{Order 1}}{a_D^2} + \frac{\text{Order 2}}{a_D} \;\Leftrightarrow\; \mathcal{W}_D = \frac{\text{Order 2}}{a_D}\ ,\ \mathcal{Q}_D = \frac{\text{Order 2}}{a_D}$$

- Buchert, **CQG 28**, 164007 (2011): Relativistic Zeldovitch Approximation

$$^{\mathrm{RZA}}\mathcal{Q}_{\mathcal{D}} = \frac{\dot{\xi}^2\left(\mathcal{Q}^{\mathrm{initial}}_{\mathcal{C}_D} + \xi\gamma_2 + \xi^2\gamma_3\right)}{\left(1+\xi\langle \mathrm{I_i}\rangle_{\mathcal{C}_\mathcal{D}} + \xi^2\langle \mathrm{II_i}\rangle_{\mathcal{C}_\mathcal{D}} + \xi^3\langle \mathrm{III_i}\rangle_{\mathcal{C}_\mathcal{D}}\right)^2} \propto \frac{\mathcal{Q}^{\mathrm{initial}}_{\mathcal{C}_D}}{a}$$
$$\gamma_2 := 6\langle \mathrm{III_i}\rangle_{\mathcal{C}_\mathcal{D}} - \frac{2}{3}\langle \mathrm{II_i}\rangle_{\mathcal{C}_\mathcal{D}}\langle \mathrm{I_i}\rangle_{\mathcal{C}_\mathcal{D}}\ ,\ \gamma_3 := 2\langle \mathrm{I_i}\rangle_{\mathcal{C}_\mathcal{D}}\langle \mathrm{III_i}\rangle_{\mathcal{C}_\mathcal{D}} - \frac{2}{3}\langle \mathrm{II_i}\rangle^2_{\mathcal{C}_\mathcal{D}}\ .$$

Observations ?

- Larena, Alimi, Buchert, Kunz & Corasaniti, **PRD 76**, 083011(2009):

$$ {}^4\mathbf{g}^{\mathcal{D}} = -dt^2 + L_{H_0}^2\, a_{\mathcal{D}}^2 \left(\frac{dr^2}{1-\kappa_{\mathcal{D}}(t)r^2} + d\Omega^2 \right) $$

$$ \langle \mathcal{R} \rangle_D = \frac{\kappa_{\mathcal{D}}(t)\ \left| \langle \mathcal{R}_0 \rangle_D \right| a_{\mathcal{D}_0}^2}{a_{\mathcal{D}}^2(t)} $$

$$ H_{\mathcal{D}}^2(a_{\mathcal{D}}) = H_{\mathcal{D}}^2 \left(\Omega_m^{\mathcal{D}} a_{\mathcal{D}}^{-3} + \Omega_X^{\mathcal{D}} a_{\mathcal{D}}^{n} \right) $$

$$ \kappa_{\mathcal{D}}(a_{\mathcal{D}}) = -sgn\left((n+6)\Omega_X^{\mathcal{D}_0} \right) a_{\mathcal{D}}^{(n+2)} $$

- SNe, Acoustic peaks, ..

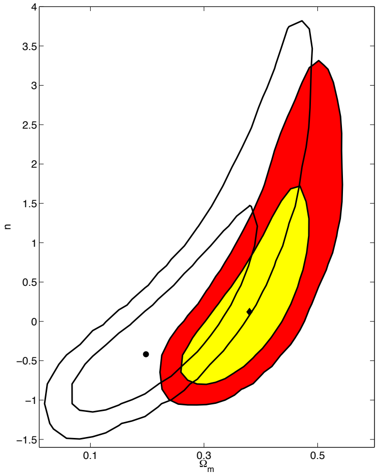


Figure 2. Supernovae and CMB constraints in the $(\Omega_m^{\mathcal{D}_0},\mathrm{n})$ plane for the averaged effective model with zero Friedmannian curvature (filled ellipses) and for a standard flat FLRW model with a quintessence field with constant equation of state $w = -(n+3)/3$ (black ellipses). The disk and diamond represent the absolute best–fit models respectively for the standard FLRW model and the averaged effective model.

Scaling Solutions (1/2)

- Closure condition:

$$(n+6)\,\mathcal{Q}_{D_i} a_D^n + (p+2)\,\mathcal{W}_{D_i} a_D^p = 0$$

n, p	scaling parameters
$\cdot_i$	initial value

- $n \neq p : (n = -6, p = -2) \Rightarrow$

$$\mathcal{Q}_D = \mathcal{Q}_{D_i} a_D^{-6}$$
$$\mathcal{W}_D = \mathcal{W}_{D_i} a_D^{-2}$$

- $n = p \Rightarrow$

$$(n+2)\,\mathcal{W}_D = -(n+6)\,\mathcal{Q}_D$$

Note: solutions $(n = -6, p = -2)$ and $n = p = -6$ physically equivalent

Scaling Solutions (2/2)

- Global contributions of inhomogeneities:

$$X_D \equiv \mathcal{Q}_D + \mathcal{W}_D$$

$$X_D = -\frac{4}{n+2}\mathcal{Q}_D = \frac{4}{n+6}\mathcal{W}_D$$

- Properties of X_D:
  - departure from FLRW
    - $\rightarrow$ FLRW background $\equiv X_D = 0$
    - $\rightarrow$ instability sectors $\equiv X_D \neq 0$
  - effective dark components
    - $\rightarrow$ dark components $\in$ instability sectors

Autonomous system (1/2)

- Hubble functional: $$H_D \equiv \frac{\partial_t a_D}{a_D}$$
- Dimensionless effective cosmological parameters:

$$\Omega_m^D \equiv \frac{8\pi G}{3H_D^2} \langle \varrho \rangle_D$$

$$\Omega_k^D \equiv -\frac{k_{D_i}}{a_D^2 \, H_D^2}$$

$$\Omega_X^D \equiv -\frac{X_D}{6H_D^2}$$

- Volume deceleration: $$q_D \equiv -1 - \frac{\partial_t H_D}{H_D^2}$$

Autonomous system (2/2)

- Dynamical system:
  - constraint equations:

$$\Omega_m^{\mathcal{D}} - (n+2)\,\Omega_X^{\mathcal{D}} = 2\,q_{\mathcal{D}}$$
$$\Omega_m^{\mathcal{D}} + \Omega_k^{\mathcal{D}} + \Omega_X^{\mathcal{D}} = 1$$

  - evolution equations:

$${\Omega_m^{\mathcal{D}}}' = \Omega_m^{\mathcal{D}} \left(\Omega_m^{\mathcal{D}} - (n+2)\,\Omega_X^{\mathcal{D}} - 1 \right)$$
$${\Omega_X^{\mathcal{D}}}' = \Omega_X^{\mathcal{D}} \left(\Omega_m^{\mathcal{D}} - (n+2)\,\Omega_X^{\mathcal{D}} + n + 2 \right)$$

$$\cdot' \equiv \partial \cdot / \partial_{N_D} \quad \text{with} \quad N_D \equiv \ln a_D$$

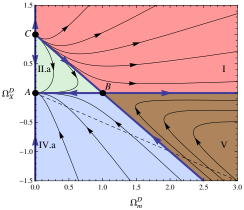
$\Rightarrow$ determines the orbit of a cosmological state $(\Omega_m^{\mathcal{D}}, \Omega_X^{\mathcal{D}}, \Omega_k^{\mathcal{D}}, n)$ in the corresponding phase space

Fixed Points

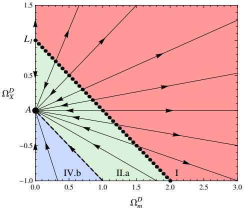
	Coordinates $(\Omega_m^{\mathcal{D}}, \Omega_X^{\mathcal{D}})$	Scale factor	Stability {eigenvalues}
point $\mathcal{A}$	$(0,0)$	$a_{\mathcal{D}}(t) = H_{\mathcal{D}_i}\,(t - t_i)\,+\,1$	$n < -2$, attractor $n > -2$, saddle $\{-1, n+2\}$
point $\mathcal{B}$	$(1,0)$	$a_{\mathcal{D}}(t) = \left(\frac{3}{2} H_{\mathcal{D}_i}\,(t - t_i)\,+\,1\right)^{\frac{2}{3}}$	$n < -3$, saddle $n > -3$, repeller $\{1, n+3\}$
point $\mathcal{C}$	$(0,1)$	$a_{\mathcal{D}}(t) = \left(-\frac{n}{2} H_{\mathcal{D}_i}\,(t - t_i)\,+\,1\right)^{-\frac{2}{n}} \quad (n \neq 0)$ $a_{\mathcal{D}}(t) = \exp(H_{\mathcal{D}_i}\,(t - t_i)) \quad (n = 0)$	$n < -3$, repeller $-3 < n < -2$, saddle $n > -2$, attractor $\{-n-3, -n-2\}$
line $\mathcal{L}_1$	$\Omega_m^{\mathcal{D}} + \Omega_X^{\mathcal{D}} = 1$	$a_{\mathcal{D}}(t) = \left(\frac{3}{2} H_{\mathcal{D}_i}\,(t - t_i)\,+\,1\right)^{2/3}$	$(n = -3)$
line $\mathcal{L}_2$	$\Omega_m^{\mathcal{D}} = 0$	$a_{\mathcal{D}}(t) = H_{\mathcal{D}_i}\,(t - t_i)\,+\,1$	$(n = -2)$

Phase space diagrams

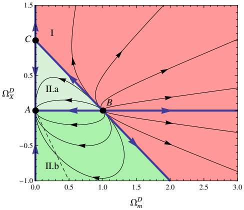
$n < -3$



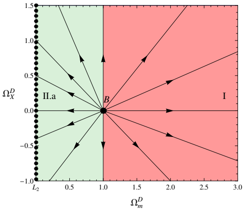
$n = -3$



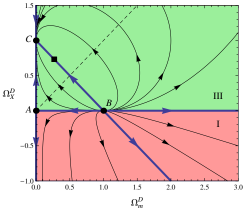
$-3 < n < -2$



$n = -2$

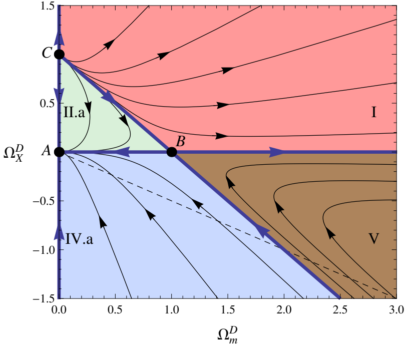


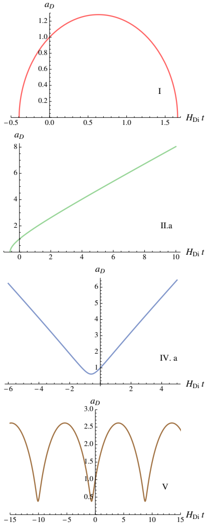
$n > -2$



Orbits for $n < -3$

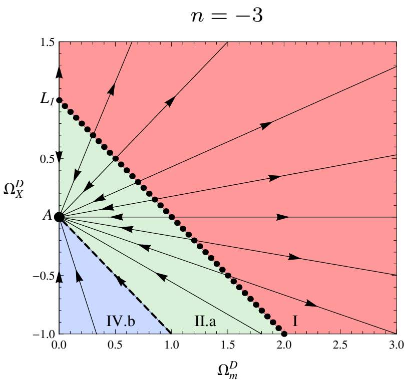
$n = -4$

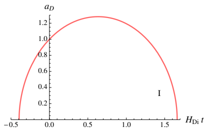


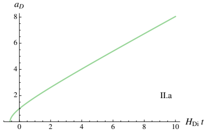


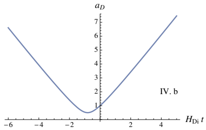
- arrows $\equiv$ increasing N_D
- dots $\equiv$ fixed points
- thick straight lines (dark blue) $\equiv$ invariant lines
- dashed line $\equiv$ vanishing q_D (acceleration below, deceleration above)

Orbits for $n = -3$



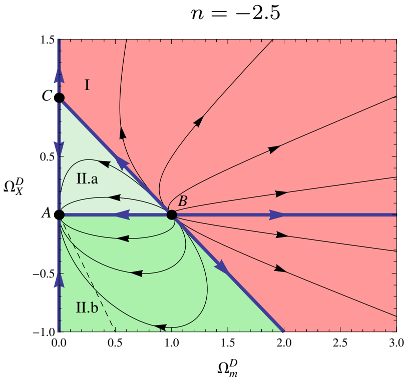


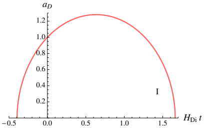


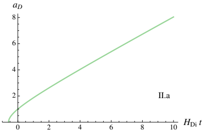


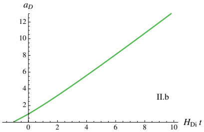
- arrows $\equiv$ increasing N_D
- dots $\equiv$ fixed points
- dashed line $\equiv$ vanishing q_D *and* orbit (acceleration below, deceleration above)

Orbits for $-3 < n < -2$



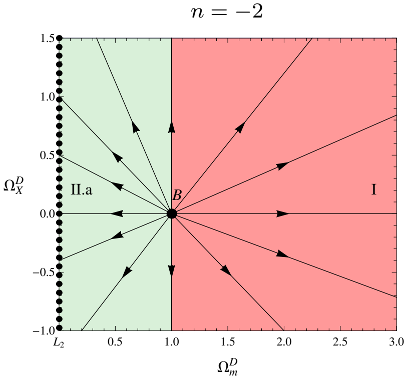


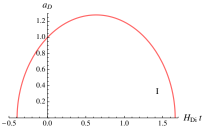


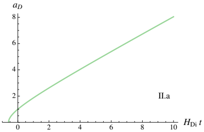


- arrows $\equiv$ increasing N_D
- dots $\equiv$ fixed points
- thick straight lines (dark blue) $\equiv$ invariant lines
- dashed line $\equiv$ vanishing q_D
  (acceleration below, deceleration above)

Orbits for $n = -2$



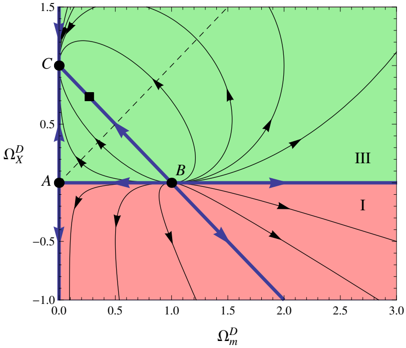


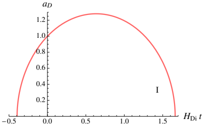


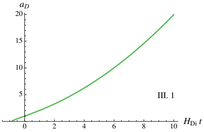
- arrows $\equiv$ increasing N_D
- dots $\equiv$ fixed points
- thick straight lines (dark blue) $\equiv$ invariant lines
- dashed line $\equiv$ vanishing q_D
  (coincides with $\Omega_m^D = 0$)

Orbits for $n > -2$

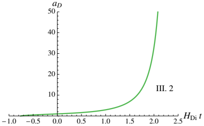
$n = -1$







$n \leq 0$



$n > 0$

- arrows $\equiv$ increasing N_D
- dots $\equiv$ fixed points
- thick straight lines (dark blue) $\equiv$ invariant lines
- dashed line $\equiv$ vanishing q_D
  (acceleration above, deceleration below)

Conclusion

Does a Friedmann-like state, lying on the line $\Omega_X^{\mathcal{D}} = 0$, converge on average to the same or another Friedmann-like state when subjected to perturbations?

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- Stability of FLRW backgrounds:

  i) $n < -2$ *and* $k_{D_i} < 0 \rightarrow$ *attractor* $\equiv$ *Milne state*

  ii) $n < -3$ *and* $k_{D_i} = 0 \rightarrow$ *attractor* $\equiv$ *Einstein–de Sitter state*

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Does a Friedmann-like state, lying on the line $\Omega_X^{\mathcal{D}} = 0$, converge on average to the same or another Friedmann-like state when subjected to perturbations?

- Stability of FLRW backgrounds:

  i) $n < -2$ *and* $k_{D_i} < 0 \rightarrow$ *attractor* $\equiv$ *Milne state*

  ii) $n < -3$ *and* $k_{D_i} = 0 \rightarrow$ *attractor* $\equiv$ *Einstein–de Sitter state*

- Instability of FLRW backgrounds:

  i) $n < -2$ *and* $k_{D_i} > 0$ ("*DM*" *for* $n < -3$ & $k_{D_i} > 0$)

  ii) $-3 \leq n < -2$ *and* $k_{D_i} = 0$

  iii) $n > -2$ ("*DE*" *for* $n > -2$ & $\Omega_X^{\mathcal{D}} > 0$)