Dark Matters Meeting
to celebrate Joe Silk's 75$^{\text{th}}$ birthday

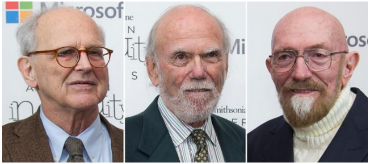
GRAVITATIONAL WAVES & THEORY

Luc Blanchet

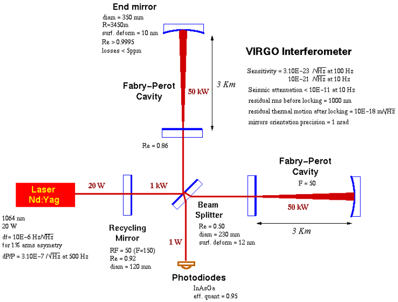
Gravitation et Cosmologie ($\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$)
Institut d'Astrophysique de Paris

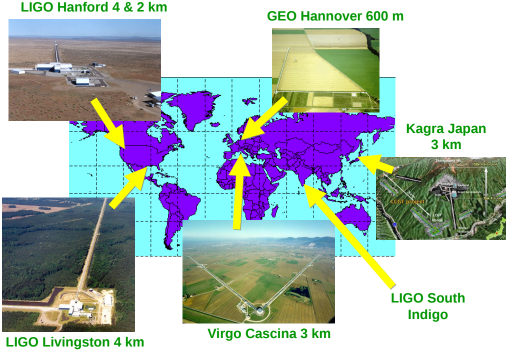
13 décembre 2017

World-wide network of gravitational wave detectors

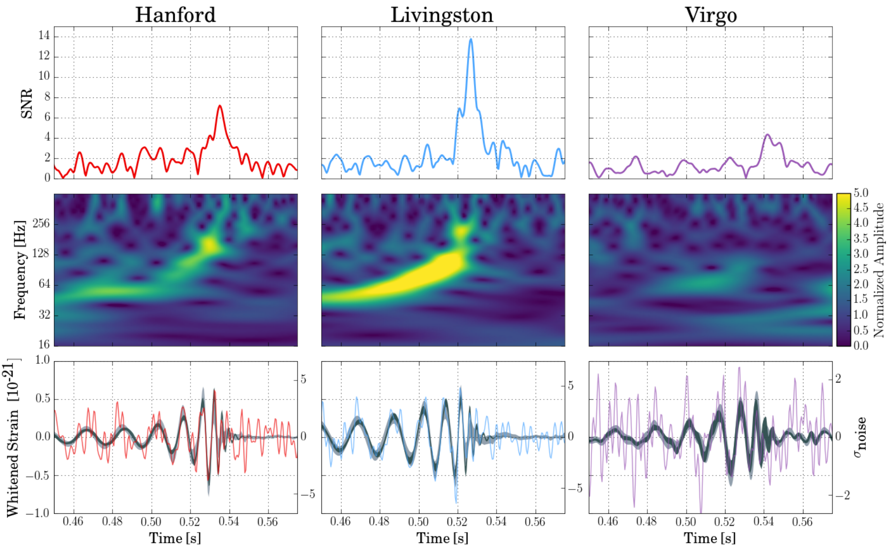


[Rainer Weiss, Barry Barish & Kip Thorne, Nobel prize 2017]

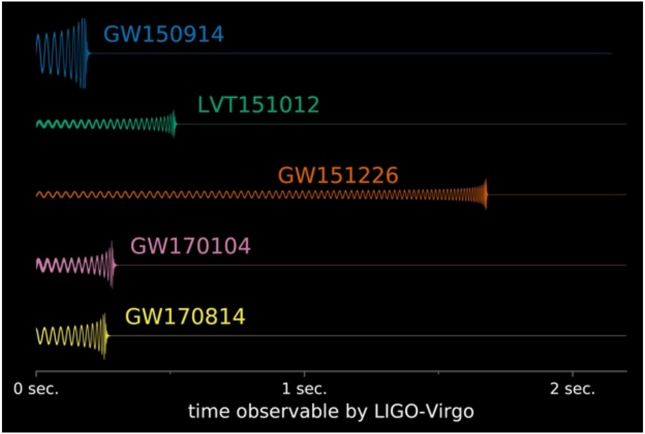




Binary black-hole event GW170814 [LIGO/Virgo collaboration 2017]

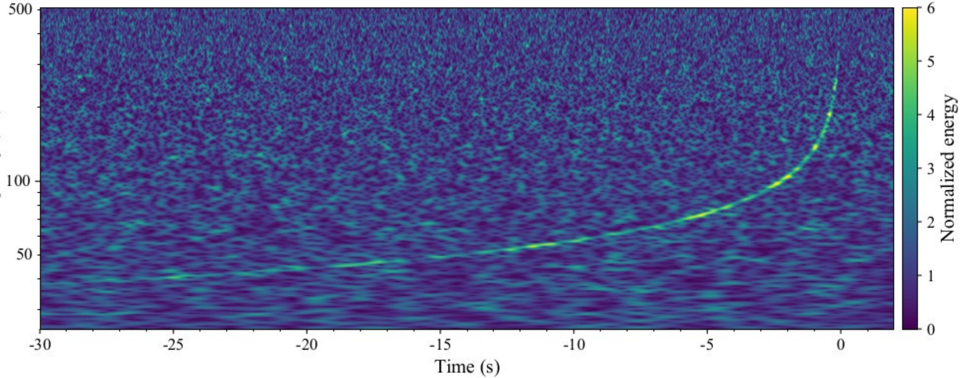


Binary black-hole events [LIGO/Virgo collaboration 2016, 2017]



- For BH binaries the detectors are mostly sensitive to the merger phase and a few cycles are observed before coalescence
- For NS binaries the detectors will be sensitive to the inspiral phase prior the merger and thousands of cycles are observable

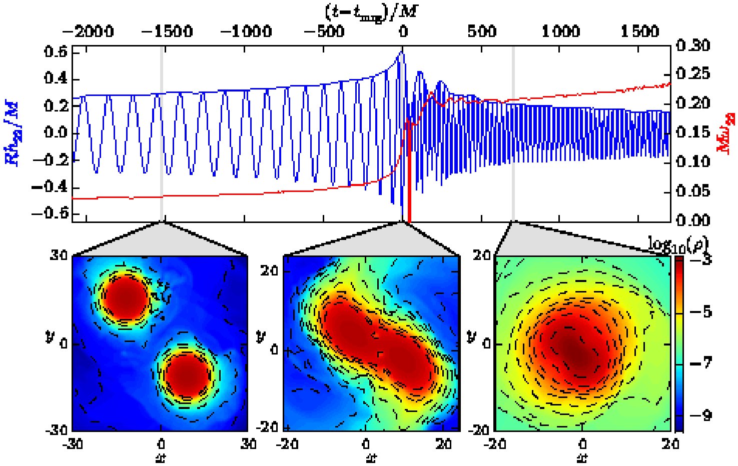
Binary neutron star event GW170817 [LIGO/Virgo 2017]



- The signal is observed during $\sim 100\,\mathrm{s}$ and ~ 3000 cycles and is the loudest gravitational-wave signal yet observed with a combined SNR of 32.4
- The chirp mass is accurately measured to $\mathcal{M} = \mu^{3/5} M^{2/5} = 1.98\, M_{\odot}$
- The advent of multi-messenger Astronomy with the concomitant discovery of a short gamma ray burst and an optical kilonova

Post-merger waveform of neutron star binaries

[Shibata *et al.*, Rezzolla *et al.* 1990-2010s]



100 years of gravitational radiation [Einstein 1916]

348 DOC. 32 INTEGRATION OF FIELD EQUATIONS

688 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. Einstein.

Bei der Behandlung der meisten speziellen (nicht prinzipiellen) Probleme auf dem Gebiete der Gravitationstheorie kann man sich damit begnügen, die $g_{\mu\nu}$ in erster Näherung zu berechnen. Dabei bedient man sich mit Vorteil der imaginären Zeitvariable $x_4 = it$ aus denselben Gründen wie in der speziellen Relativitätstheorie. Unter »erster Näherung« ist dabei verstanden, daß die durch die Gleichung

$$\boxed{g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}} \qquad (1)$$

⟸ small perturbation of Minkowski's metric

definierten Größen $\gamma_{\mu\nu}$, welche linearen orthogonalen Transformationen gegenüber Tensorcharakter besitzen, gegen 1 als kleine Größen behandelt werden können, deren Quadrate und Produkte gegen die ersten Potenzen vernachlässigt werden dürfen. Dabei ist $\delta_{\mu\nu} = 1$ bzw. $\delta_{\mu\nu} = 0$, je nachdem $\mu = \nu$ oder $\mu \neq \nu$.

Wir werden zeigen, daß diese $\gamma_{\mu\nu}$ in analoger Weise berechnet werden können wie die retardierten Potentiale der Elektrodynamik.

GW solutions in metric theories of gravity

1. Small perturbation of the metric around flat space-time

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| \ll 1$$

2. Restrict attention to theories admitting GW solutions propagating at the speed of light: $c_g = 1$. Far from the sources the waves are planar, hence

$$\Box h_{\mu\nu} = 0 \quad \Longleftrightarrow \quad h_{\mu\nu} = h_{\mu\nu}(t-z)$$

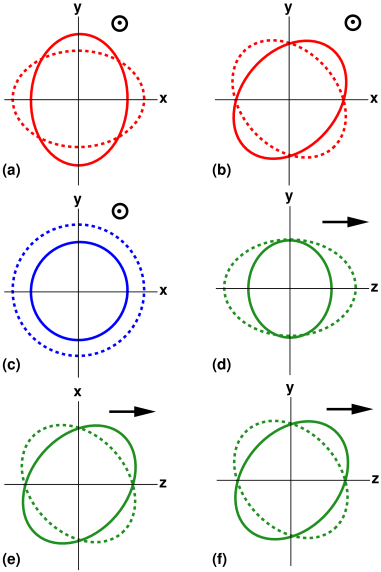
3. From the linearized Bianchi's identity obtain

$$\boxed{\Box R_{\mu\nu\rho\sigma} = 0 \quad \Longleftrightarrow \quad R_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}(t-z)}$$

   showing that GWs have an invariant, coordinate-independent meaning

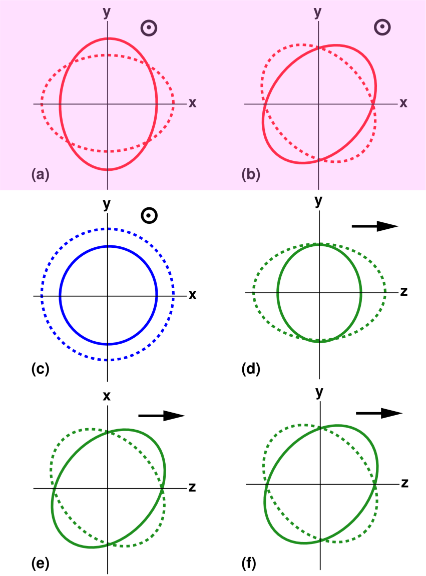
4. The six components R_{0i0j} (where $i,j = x,y,z$) represent six independent components (polarization modes)
5. In GR $R_{\mu\nu} = 0$ hence there are only two independent polarization modes

GW polarization modes in metric theories of gravity



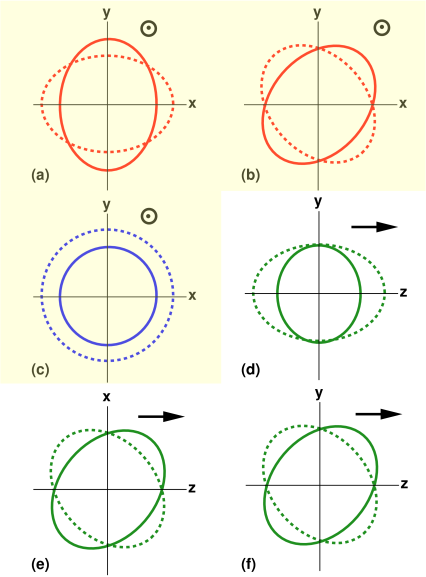
- General Relativity
- Scalar-Tensor theory
- Massive Gravity theory

GW polarization modes in metric theories of gravity



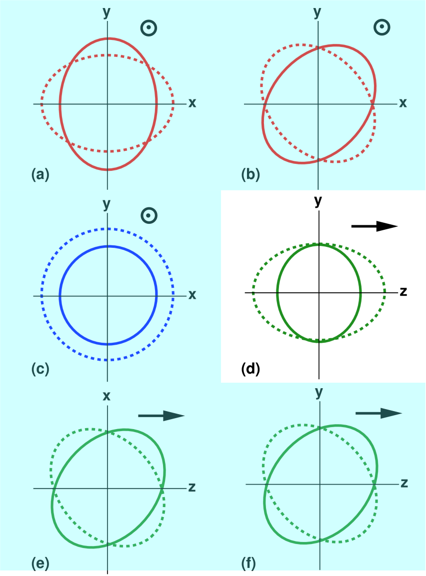
- General Relativity
- Scalar-Tensor theory
- Massive Gravity theory

GW polarization modes in metric theories of gravity



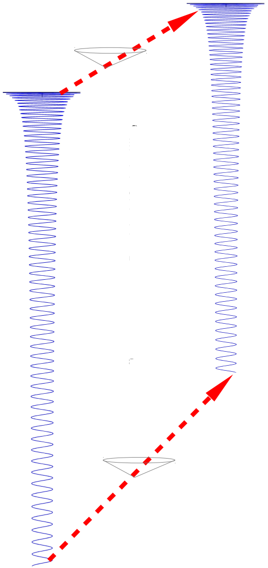
- General Relativity
- Scalar-Tensor theory
- Massive Gravity theory

GW polarization modes in metric theories of gravity



- General Relativity
- Scalar-Tensor theory
- Massive Gravity theory

Bounding the mass of the graviton [Will 1998]



1. Dispersion relation for a massive graviton

$$\boxed{\frac{v_\mathrm{g}^2}{c^2} = 1 - \frac{m_\mathrm{g}^2 c^4}{E_\mathrm{g}^2}}$$

2. The frequency of GW sweeps from low to high frequency during the inspiral and the speed of GW varies from lower to higher (close to c) speed at the end

3. The constraint is [LIGO/Virgo collaboration 2016]

$$\boxed{m_\mathrm{g} \lesssim 10^{-22}\,\mathrm{eV}}$$

Dark energy after GW170817 [Bettoni *et al.* 2017; Creminelli & Vernizzi 2017]

1. The observed time delay between GW170817 and the GRB constrains

$$|c_{\mathsf{g}} - c_{\mathsf{em}}| \lesssim 10^{-15} c$$

2. Consider models of dark energy and modified gravity characterized by a single scalar degree of freedom (Horndeski theory)

$$\begin{aligned} L &= G_2(\phi, X) + G_3(\phi, X)\Box\phi + G_4(\phi, X)R \\ &- 2G_{4,X}(\phi, X)\left(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu}\right) + G_5(\phi, X)E^{\mu\nu}\phi_{\mu\nu} \\ &+ \frac{1}{3}G_{5,X}(\phi, X)\left(\Box\phi^3 - 3\Box\phi\phi^{\mu\nu}\phi_{\mu\nu} + 2\phi^{\mu\nu}\phi_{\mu\rho}\phi^{\rho}_{\nu}\right) \end{aligned}$$

3. Imposing the speed of GWs to be one $($*i.e.* $c_{\mathsf{g}} \equiv c_{\mathsf{T}} = 1)$ drastically reduces the space of allowed theories

$$\boxed{L_{c_T=1} = G_2(\phi, X) + G_3(\phi, X)\Box\phi + B_4(\phi)R}$$

where the third term simply recovers the standard conformal coupling

Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1947]

$$4\pi R^2 \bar{\mathfrak{J}} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{\mathfrak{J}}_{\mu\nu}^2 - \frac{1}{3}\left(\sum_{\mu} \dddot{\mathfrak{J}}_{\mu\mu}\right)^2\right].$$

1. Einstein quadrupole formula

$$\boxed{\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\mathsf{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}}$$

2. Amplitude quadrupole formula

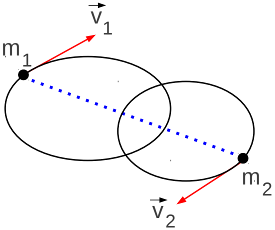
$$\boxed{h_{ij}^{\mathsf{TT}} = \frac{2G}{c^4 D}\left\{\frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2}\left(t-\frac{D}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\mathsf{TT}} + \mathcal{O}\left(\frac{1}{D^2}\right)}$$

3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$\boxed{F_i^{\mathsf{reac}} = -\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{v}{c}\right)^7}$$

which is a 2.5PN $\sim (v/c)^5$ effect in the source's equations of motion

Application to compact binaries [Peters & Mathews 1963; Peters 1964]



$$\begin{cases} a \text{ semi-major axis of relative orbit} \\ e \text{ eccentricity of relative orbit} \\ \omega = \frac{2\pi}{P} \text{ orbital frequency} \end{cases}$$

$$M = m_1 + m_2$$
$$\mu = \frac{m_1 m_2}{M}$$

$$\boxed{\nu = \frac{\mu}{M} \qquad 0 < \nu \leqslant \frac{1}{4}}$$

Averaged energy and angular momentum balance equations

$$\langle \frac{\mathrm{d}E}{\mathrm{d}t} \rangle = -\langle \mathcal{F}^{\mathsf{GW}} \rangle \qquad \langle \frac{\mathrm{d}J_i}{\mathrm{d}t} \rangle = -\langle \mathcal{G}_i^{\mathsf{GW}} \rangle$$

are applied to a Keplerian orbit (using Kepler's law $GM = \omega^2 a^3$)

$$\boxed{\begin{aligned} \langle \frac{\mathrm{d}P}{\mathrm{d}t} \rangle &= -\frac{192\pi}{5c^5} \nu \left(\frac{2\pi G M}{P} \right)^{5/3} \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}} \\ \langle \frac{\mathrm{d}e}{\mathrm{d}t} \rangle &= -\frac{608\pi}{15c^5} \nu \frac{e}{P} \left(\frac{2\pi G M}{P} \right)^{5/3} \frac{1 + \frac{121}{304}e^2}{(1-e^2)^{5/2}} \end{aligned}}$$

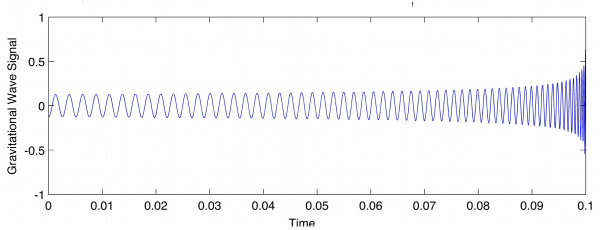
Orbital phase evolution of compact binaries

[Dyson 1969; Esposito & Harrison 1975; Wagoner 1975]

1. Compact binaries are circularized when they enter the detector's bandwidth
2. The amplitude and phase evolution follow an adiabatic chirp in time

$$\boxed{\begin{aligned} a(t) &= \left(\frac{256}{5}\frac{G^3M^3\nu}{c^5}(t_c - t)\right)^{1/4} \\ \phi(t) &= \phi_c - \frac{1}{32\nu}\left(\frac{256}{5}\frac{c^3\nu}{GM}(t_c - t)\right)^{5/8} \end{aligned}}$$

3. The amplitude and orbital frequency diverge at the instant of coalescence t_c since the approximation breaks down



Why inspiralling binaries require high PN modelling

[Caltech "3mn paper" 1992; Blanchet & Schäfer 1993]

VOLUME 70, NUMBER 20 PHYSICAL REVIEW LETTERS 17 MAY 1993

The Last Three Minutes: Issues in Gravitational-Wave Measurements of Coalescing Compact Binaries

Curt Cutler,(1) Theocharis A. Apostolatos,(1) Lars Bildsten,(1) Lee Samuel Finn,(2) Eanna E. Flanagan,(1) Daniel Kennefick,(1) Dragoljub M. Markovic,(1) Amos Ori,(1) Eric Poisson,(1) Gerald Jay Sussman,(1),(a) and Kip S. Thorne(1)

(1) *Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125*
(2) *Department of Physics and Astronomy, Northwestern University, Evanston, Illinois 60208*

(Received 24 August 1992)

Gravitational-wave interferometers are expected to monitor the last three minutes of inspiral and final coalescence of neutron star and black hole binaries at distances approaching cosmological, where the event rate may be many per year. Because the binary's accumulated orbital phase can be measured to a fractional accuracy $\ll 10^{-3}$ and relativistic effects are large, the wave forms will be far more complex and carry more information than has been expected. Improved wave form modeling is needed as a foundation for extracting the waves' information, but is not necessary for wave detection.

PACS numbers: 04.30.+x, 04.80.+z, 97.60.Jd, 97.60.Lf

A network of gravitational-wave interferometers (the American LIGO [1], the French/Italian VIRGO [2], and possibly others) is expected to be operating by the end of the 1990s. The most promising waves for this network are from the inspiral and coalescence of neutron star (NS) and black hole (BH) binaries [3, 4], with an estimated event rate of $\sim$ (3/year)[distance/(200 Mpc)]^3 [5]. This Letter reports initial results of a new research effort that

as the signal sweeps through the interferometers' band, their overlap integral will be strongly reduced. This sensitivity to phase does *not* mean that accurate templates are needed in searches for the waves (see below). However, once the waves have been found, and if accurate templates are in hand, then from the orbital phasing one can infer each of the system's parameters λ_i to an accuracy of order the change $\Delta\lambda_i$ which alters by unity the

Why inspiralling binaries require high PN modelling

[Caltech "3mn paper" 1992; Blanchet & Schäfer 1993]

A network of gravitational-wave interferometers (the American LIGO [1], the French/Italian VIRGO [2], and possibly others) is expected to be operating by the end of the 1990s. The most promising waves for this network are from the inspiral and coalescence of neutron star (NS) and black hole (BH) binaries [3, 4], with an estimated event rate of $\sim$ (3/year)[distance/(200 Mpc)]^3 [5]. This Letter reports initial results of a new research effort that is changing our understanding of these waves; further details will be given in the authors' papers cited in the references.

A binary's inspiral and coalescence will produce two gravitational wave forms, one for each polarization. By cross correlating the outputs of three or more interferometers, both wave forms can be monitored and the source's direction can be determined to within ~ 1 degree [4, 6].

We shall divide each wave form into two parts: the *inspiral wave form*, emitted before tidal distortions become noticeable ($\lesssim 10$ cycles before complete disruption or merger [7, 8]), and the *coalescence wave form*, emitted during distortion, disruption, and/or merger.

As the binary, driven by gravitational radiation reaction, spirals together, its *inspiral wave form* sweeps upward in frequency f (it "chirps"). The interferometers will observe the last several thousand cycles of inspiral (from $f \sim 10$ Hz to ~ 1000 Hz), followed by coalescence.

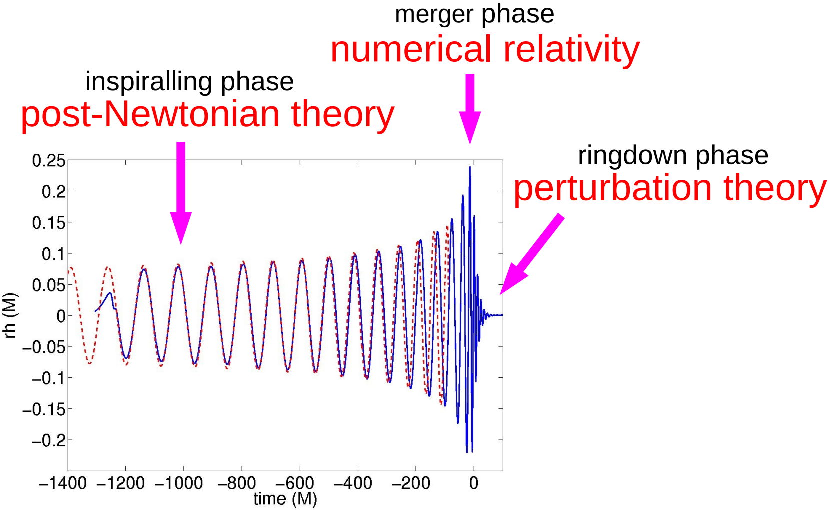
Theoretical calculations of the wave forms are generally made using the post-Newtonian (PN) approximation to general relativity. Previous calculations have focused as the signal sweeps through the interferometers' band, their overlap integral will be strongly reduced. This sensitivity to phase does *not* mean that accurate templates are needed in searches for the waves (see below). However, once the waves have been found, and if accurate templates are in hand, then from the orbital phasing one can infer each of the system's parameters λ_i to an accuracy of order the change $\Delta\lambda_i$ which alters by unity the number of cycles $\mathcal{N}_{\rm cyc}$ spent in the interferometers' band.

We shall assume (as almost always is the case) that the binary's orbit has been circularized by radiation reaction [10]. Then the only parameters λ_i that can significantly influence the inspiral template's phasing are the bodies' masses, vectorial spin angular momenta, and spin-induced quadrupole moments (which we shall ignore because, even for huge spins, they produce orbital phase shifts no larger than ~ 1 [8]). More specifically, the number of cycles spent in a logarithmic interval of frequency, $d\mathcal{N}_{\rm cyc}/d\ln f = (1/2\pi)(d\Phi/d\ln f)$, is

$$\frac{d\mathcal{N}_{\rm cyc}}{d\ln f} = \frac{5}{96\pi}\frac{1}{\mu M^{2/3}(\pi f)^{5/3}}\left\{1 + \left(\frac{743}{336} + \frac{11}{4}\frac{\mu}{M}\right)x \right. \\ \left. -[4\pi + \text{S.O.}]x^{1.5} + [\text{S.S.}]x^2 + O(x^{2.5})\right\}. \qquad (1)$$

Here M is the binary's total mass, μ its reduced mass, and $x \equiv (\pi M f)^{2/3} \simeq M/D$ the PN expansion parameter (with D the bodies' separation and $c = G = 1$). The PN correction [$O(x)$ term] is from [13]. In the P$^{1.5}$N correction [$O(x^{1.5})$ term], the 4π is created by the waves' in-

The gravitational chirp of compact binaries



The GW templates of compact binaries

1. In principle, the templates are obtained by matching together:
   - A high-order 3.5PN waveform for the inspiral [Blanchet *et al.* 1998, 2002, 2004]
   - A highly accurate numerical waveform for the merger and ringdown [Pretorius 2005; Baker *et al.* 2006; Campanelli *et al.* 2006]
2. In practice, for black hole binaries (such as GW150914), effective methods that interpolate between the PN and NR play a key role in the data analysis
   - Hybrid inspiral-merger-ringdown (IMR) waveforms [Ajith *et al.* 2011] are constructed by matching the PN and NR waveforms in a time interval through an intermediate phenomenological phase
   - Effective-one-body (EOB) waveforms [Buonanno & Damour 1998] are based on resummation techniques extending the domain of validity of the PN approximation beyond the inspiral phase
3. In the case of neutron star binaries (such as GW170817), the masses are smaller and the templates are entirely based on the 3.5PN waveform

Einstein field equations as a "Problème bien posé"

- Start with the GR action for the metric $g_{\mu\nu}$ with the matter term

$$\boxed{S_{\text{GR}} = \underbrace{\frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\,R}_{\text{Einstein-Hilbert action}} + \underbrace{S_{\text{m}}[g_{\mu\nu},\Psi]}_{\text{matter fields}}}$$

- Add the harmonic coordinates gauge-fixing term (where $\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$)

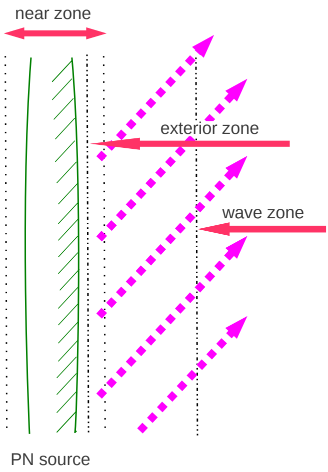
$$\boxed{S_{\text{GR}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\left(\sqrt{-g}\,R \underbrace{-\frac{1}{2}\mathfrak{g}_{\alpha\beta}\partial_\mu\mathfrak{g}^{\alpha\mu}\partial_\nu\mathfrak{g}^{\beta\nu}}_{\text{gauge-fixing term}}\right) + S_{\text{m}}}$$

- Obtain a well-posed system of equations [Choquet-Bruhat 1952]

$$\boxed{\begin{aligned} &\mathfrak{g}^{\mu\nu}\partial^2_{\mu\nu}\mathfrak{g}^{\alpha\beta} = \frac{16\pi G}{c^4}|g|T^{\alpha\beta} + \overbrace{\Sigma^{\alpha\beta}[\mathfrak{g},\partial\mathfrak{g}]}^{\text{non-linear source term}} \\ &\underbrace{\partial_\mu\mathfrak{g}^{\alpha\mu} = 0}_{\text{harmonic-gauge condition}} \end{aligned}}$$

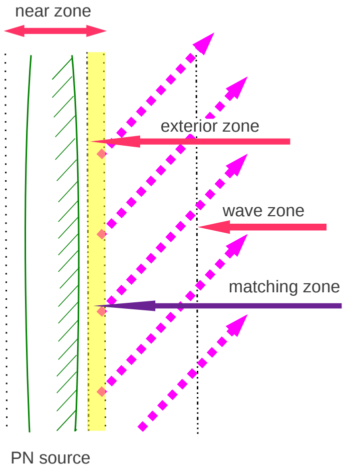
The MPM-PN formalism [Blanchet-Damour-Iyer 1980s; Blanchet 1995, 1998]

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



The MPM-PN formalism [Blanchet-Damour-Iyer 1980s; Blanchet 1995, 1998]

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



$$\underbrace{\overline{\mathcal{M}(h^{\alpha\beta})} = \mathcal{M}(\bar{h}^{\alpha\beta})}_{\text{matching equation}}$$

The MPM-PN formalism [Blanchet-Damour-Iyer 1980s; Blanchet 1995, 1998]

1. Radiative multipole moments observed at infinity from the source ($\mathcal{J}^+$)

$$\boxed{U_L(T - R/c)\,, \qquad V_L(T - R/c)}$$

2. Source multipole moments describe a specific matter system

$$\boxed{I_L(t)\,, \qquad J_L(t)}$$

- The relations between the radiative moments and the source moments are obtained by the MPM algorithm
- The expressions of the source moments in terms of the source parameters follow from the matching to the PN source
- The radiation reaction effects in the PN solution are also obtained

The source multipole moments [Blanchet 1995, 1998]

$$\boxed{\begin{aligned}
\mathrm{I}_L(t) &= \mathrm{PF}\int \mathrm{d}^3\mathbf{x}\int_{-1}^{1}\mathrm{d}z\biggl\{\delta_\ell\hat{x}_L\Sigma - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)}\delta_{\ell+1}\hat{x}_{iL}\Sigma_i^{(1)} \\
&\qquad\qquad + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)}\delta_{\ell+2}\hat{x}_{ijL}\Sigma_{ij}^{(2)}\biggr\}\bigg|_{(\mathbf{x},t+zr/c)} \\
\mathrm{J}_L(t) &= \mathrm{PF}\int \mathrm{d}^3\mathbf{x}\int_{-1}^{1}\mathrm{d}z\,\epsilon_{ab\langle i_\ell}\biggl\{\delta_\ell\hat{x}_{L-1\rangle a}\Sigma_b \\
&\qquad\qquad - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)}\delta_{\ell+1}\hat{x}_{L-1\rangle ac}\Sigma_{bc}^{(1)}\biggr\}\bigg|_{(\mathbf{x},t+zr/c)}
\end{aligned}}$$

- Σ, Σ_i and Σ_{ij} are the matter currents defined from the PN expansion of the components of the source's stress-energy pseudo tensor
- The FP procedure means the Hadamard "Partie Finie" and plays the role of an IR regularization in the multipole moments

The radiative quadrupole moment [Marchand, Blanchet & Faye 2016]

$$U_{ij}(t) = I^{(2)}_{ij}(t) + \underbrace{\frac{GM}{c^3}\int_0^{+\infty} \mathrm{d}\tau\, I^{(4)}_{ij}(t-\tau)\left[2\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{6}\right]}_{\text{1.5PN tail integral}}$$

$$+\frac{G}{c^5}\left\{\underbrace{-\frac{2}{7}\int_0^{+\infty}\mathrm{d}\tau\, I^{(3)}_{a<i}I^{(3)}_{j>a}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms}\right\}$$

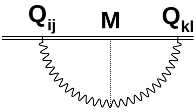
$$+\underbrace{\frac{G^2M^2}{c^6}\int_0^{+\infty}\mathrm{d}\tau\, I^{(5)}_{ij}(t-\tau)\left[2\ln^2\left(\frac{\tau}{2\tau_0}\right)+\frac{57}{35}\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{124627}{22050}\right]}_{\text{3PN tail-of-tail integral}}$$

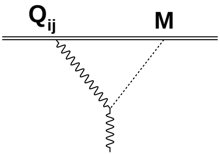
$$+\underbrace{\frac{G^3M^3}{c^9}\int_0^{+\infty}\mathrm{d}\tau\, I^{(6)}_{ij}(t-\tau)\left[\frac{4}{3}\ln^3\left(\frac{\tau}{2\tau_0}\right)+\cdots+\frac{129268}{33075}+\frac{428}{315}\pi^2\right]}_{\text{4.5PN tail-of-tail-of-tail integral}}$$

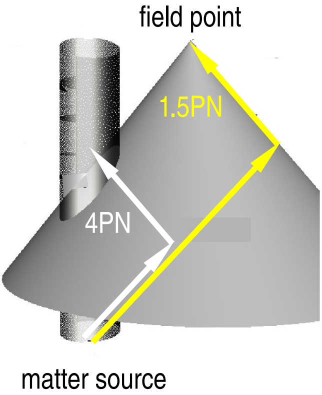
$$+\mathcal{O}\left(\frac{1}{c^{10}}\right)$$

The gravitational wave tail effect

[Blanchet & Damour 1988; Blanchet 1993, 1997; Foffa & Sturani 2011; Galley *et al.* 2016]







- In the near zone (4PN effect)

$$\boxed{S^{\text{tail}} = \frac{G^2 M}{5c^8} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|}\, I_{ij}^{(3)}(t)\, I_{ij}^{(3)}(t')}$$

- In the far zone (1.5PN effect)

$$\boxed{h_{ij}^{\text{tail}} = \frac{4G}{c^4 r}\frac{GM}{c^3}\int_{-\infty}^{t} \mathrm{d}t'\, I_{ij}^{(4)}(t')\ln\left(\frac{t-t'}{\tau_0}\right)}$$

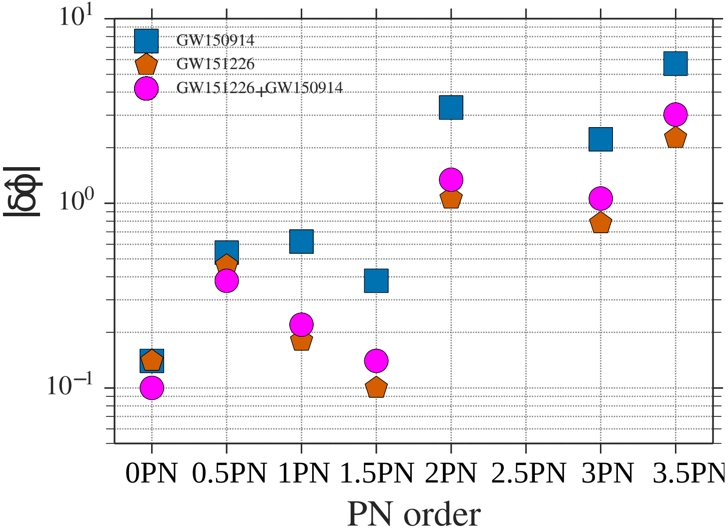
Phasing formula of inspiralling compact binaries

[BDIWW 1995; B 1996, 1998; BIJ 2002, BFIJ 2002; BDEI 2006]

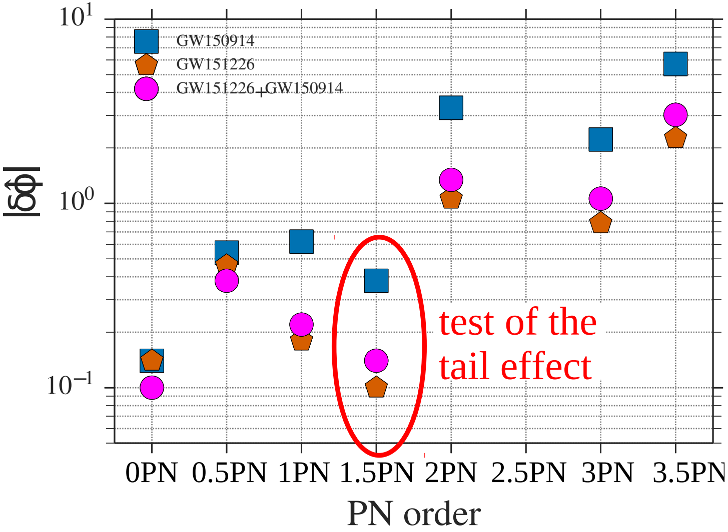
$$\begin{aligned}\phi(\omega) = \phi_0 - \frac{1}{32\nu}\left(\frac{GM\omega}{c^3}\right)^{-5/3} \Bigg\{ 1 & \\ & + \overbrace{\left(\frac{3715}{1008}+\frac{55}{12}\nu\right)\left(\frac{GM\omega}{c^3}\right)^{2/3}}^{\text{1PN}} \\ & - \overbrace{10\pi\left(\frac{GM\omega}{c^3}\right)}^{\text{1.5PN (tail)}} \\ & + \overbrace{\left(\frac{15293365}{1016064}+\frac{27145}{1008}\nu+\frac{3085}{144}\nu^2\right)\left(\frac{GM\omega}{c^3}\right)^{4/3}}^{\text{2PN}} + \cdots \Bigg\}\end{aligned}$$

The phase evolution is currently known up to 3.5PN order

Measurement of PN parameters [LIGO/Virgo 2016]

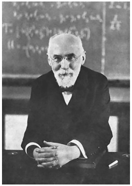


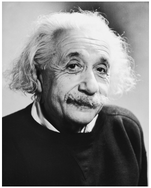
Measuring GW tails [Blanchet & Sathyaprakash 1994, 1995]



The 1PN equations of motion

[Lorentz & Droste 1917; Einstein, Infeld & Hoffmann 1938]





$$\boxed{\begin{aligned}\frac{\mathrm{d}^2\boldsymbol{r}_A}{\mathrm{d}t^2} = -\sum_{B\neq A}\frac{Gm_B}{r_{AB}^2}\boldsymbol{n}_{AB}\Bigg[1 - 4\sum_{C\neq A}\frac{Gm_C}{c^2 r_{AC}} - \sum_{D\neq B}\frac{Gm_D}{c^2 r_{BD}}\left(1 - \frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{BD}}{r_{BD}^2}\right) \\ + \frac{1}{c^2}\left(\boldsymbol{v}_A^2 + 2\boldsymbol{v}_B^2 - 4\boldsymbol{v}_A\cdot\boldsymbol{v}_B - \frac{3}{2}(\boldsymbol{v}_B\cdot\boldsymbol{n}_{AB})^2\right)\Bigg] \\ + \sum_{B\neq A}\frac{Gm_B}{c^2 r_{AB}^2}\boldsymbol{v}_{AB}[\boldsymbol{n}_{AB}\cdot(3\boldsymbol{v}_B - 4\boldsymbol{v}_A)] - \frac{7}{2}\sum_{B\neq A}\sum_{D\neq B}\frac{G^2 m_B m_D}{c^2 r_{AB} r_{BD}^3}\boldsymbol{n}_{BD}\end{aligned}}$$

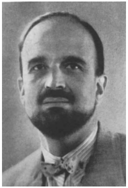
4PN: state-of-the-art on equations of motion

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3}+\frac{4G^2m_2^2}{r_{12}^3}+\cdots\right]n_{12}^i+\cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}}$$
$$+\underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}}+\underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\\text{radiation reaction}}}+\underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}}+\underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\\text{radiation reaction}}}+\underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\\text{conservative \& radiation tail}}}+\mathcal{O}\left(\frac{1}{c^9}\right)$$

3PN
- [Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab] — ADM Hamiltonian
- [Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002] — Harmonic EOM
- [Itoh & Futamase 2003; Itoh 2004] — Surface integral method
- [Foffa & Sturani 2011] — Effective field theory

4PN
- [Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014] — ADM Hamiltonian
- [Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017abc] — Fokker Lagrangian
- [Foffa & Sturani 2012, 2013] (partial results) — Effective field theory

Fokker action of N particles [Fokker 1929]



1. Gauge-fixed Einstein-Hilbert action for N point particles

$$S_{\text{g.f.}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\Big[R \underbrace{-\frac{1}{2}g_{\mu\nu}\Gamma^\mu\Gamma^\nu}_{\text{Gauge-fixing term}}\Big]$$

$$-\sum_A \underbrace{m_A c^2 \int \mathrm{d}t\,\sqrt{-(g_{\mu\nu})_A\, v_A^\mu v_A^\nu/c^2}}_{N\text{ point particles}}$$

2. Fokker action is obtained by inserting an explicit PN solution of the Einstein field equations

$$g_{\mu\nu}(\mathbf{x},t) \longrightarrow \overline{g}_{\mu\nu}(\mathbf{x};\boldsymbol{y}_B(t),\boldsymbol{v}_B(t),\cdots)$$

3. The PN equations of motion of the N particles (self-gravitating system) are

$$\boxed{\frac{\delta S_{\mathsf{F}}}{\delta \boldsymbol{y}_A} \equiv \frac{\partial L_{\mathsf{F}}}{\partial \boldsymbol{y}_A} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathsf{F}}}{\partial \boldsymbol{v}_A}\right) + \cdots = 0}$$

4. The Fokker action is equivalent to the effective action used by the EFT

Problem of the IR divergences

1. The tail effect implies the appearance of IR divergences in the Fokker action at the 4PN order
2. Our initial calculation of the Fokker action was based on the Hadamard regularization to treat the IR divergences (FP procedure when $B \to 0$)
3. However computing the conserved energy and periastron advance for circular orbits we found it does not agree with GSF calculations
4. The problem was due to the HR and conjectured that a different IR regularization would give (modulo shifts)

$$L = L^{\text{HR}} + \underbrace{\frac{G^4 m\, m_1^2 m_2^2}{c^8 r_{12}^4}\Big(\delta_1 (n_{12} v_{12})^2 + \delta_2 v_{12}^2\Big)}_{\text{two ambiguity parameters } \delta_1 \text{ and } \delta_2}$$

5. Matching with GSF results for the energy and periastron advance uniquely fixes the two ambiguity parameters and we are in complete agreement with the results from the Hamiltonian formalism [DJS]

Dimensional regularization of the IR divergences

- The Hadamard regularization of IR divergences reads

$$I_{\mathcal{R}}^{\mathsf{HR}} = \mathop{\mathrm{FP}}_{B=0} \int_{r>\mathcal{R}} \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B F(\mathbf{x})$$

- The corresponding dimensional regularization reads

$$I_{\mathcal{R}}^{\mathsf{DR}} = \int_{r>\mathcal{R}} \frac{\mathrm{d}^d\mathbf{x}}{\ell_0^{d-3}} F^{(d)}(\mathbf{x})$$

- The difference between the two regularization is of the type ($\varepsilon = d-3$)

$$\boxed{\mathcal{D}I = \sum_q \left[\underbrace{\frac{1}{(q-1)\varepsilon}}_{\text{IR pole}} - \ln\left(\frac{r_0}{\ell_0}\right)\right] \int \mathrm{d}\Omega_{2+\varepsilon}\, \varphi_{3,q}^{(\varepsilon)}(\mathbf{n}) + \mathcal{O}(\varepsilon)}$$

Ambiguity-free completion of the 4PN EOM

[Marchand, Bernard, Blanchet & Faye 2017]

1. The tail effect contains a UV pole which cancels the IR pole coming from the instantaneous part of the action

$$\boxed{S_{\mathsf{F}}^{\mathsf{tail}} = \frac{2G^2M}{5c^8}\int_{-\infty}^{+\infty} Q_{ij}^{(3)}(t)\int_0^{+\infty}\mathrm{d}\tau\left[\ln\left(\frac{c\sqrt{\bar{q}}\,\tau}{2\ell_0}\right)\underbrace{-\frac{1}{2\varepsilon}}_{\text{UV pole}}+\frac{41}{60}\right]Q_{ij}^{(4)}(t-\tau)}$$

2. For the tail effect we are in complete agreement with the EFT calculation based on a single Feynman diagram [Galley, Leibovich, Porto & Ross 2011]
3. Adding up all contributions we obtain the conjectured form of the ambiguity terms with the correct values of the ambiguity parameters δ_1 and δ_2
4. The lack of a consistent matching in the ADM Hamiltonian formalism [DJS] forces this formalism to be plagued by one ambiguity parameter