Secular evolution of stellar cluster @GC

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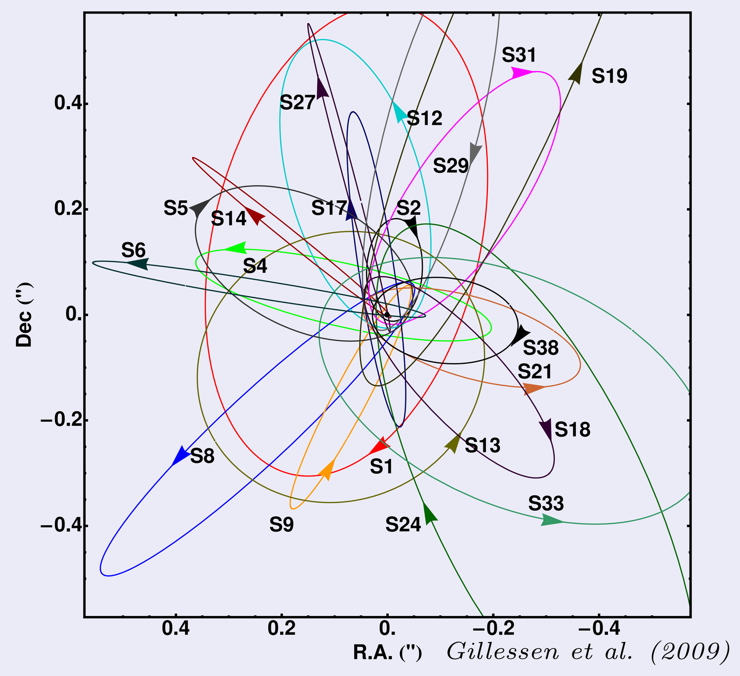
IAP

1 The case of quasi-Keplerian systems

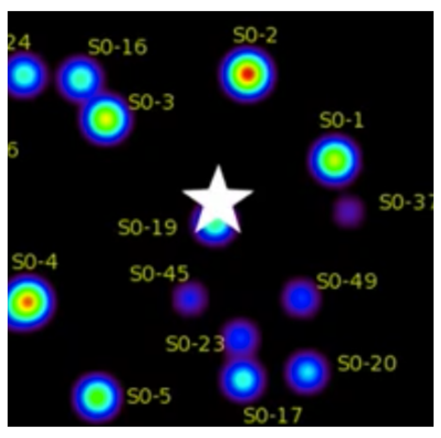
- Describe the secular evolution driven by **finite−N effects** for a **quasi-Keplerian system**
  - inhomogeneous
  - dynamically degenerate
  - stable
  - self-gravitating
  - discrete
- **How efficiently are BHs fed?**



Gillessen et al. (2009)

- Some references:
  - *Rauch, Tremaine (1996)*: Resonant relaxation
  - *Meritt et al. (2011)*: Schwarzschild barrier
  - *Bar-Or, Alexander (2014, 2016)*: η−formalism
  - *Sridhar, Touma (2016)*: Gilbert's method for Landau
  - *Fouvry, Pichon, Magorrian (2016)*: BBGKY approach

2 Quasi-Keplerian systems



- BH dominates the dynamics: $\varepsilon = M_\star / M_\bullet \ll 1$
  $\Longrightarrow$ Keplerian orbits are **closed**.
- **Dynamical degeneracy**: $\forall \boldsymbol{J}, \boldsymbol{n}\cdot\boldsymbol{\Omega}_{\rm Kep}(\boldsymbol{J}) = 0$.
  $\Longrightarrow$ **Delaunay variables**

$$\boldsymbol{J} = \big(\underbrace{I = J_r + L}_{\text{Fast } J^{\rm f}}, \underbrace{L, L_z}_{\text{Slow } \boldsymbol{J}^{\rm s}}\big) \;;\; \boldsymbol{\theta} = \big(\underbrace{\theta^{\rm f}}_{\text{Kep. phase}}, \underbrace{\boldsymbol{\theta}^{\rm s}}_{\text{Int. of motion}}\big)$$

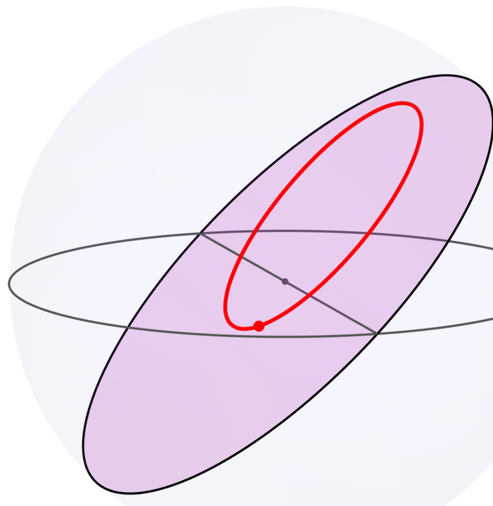
$$\boldsymbol{\Omega}_{\rm Kep} = (\Omega_{\rm Kep}, 0, 0)\,.$$

- Orbits characterised by **wires' coordinates**

$$\boldsymbol{\mathcal{E}} = (\boldsymbol{J}, \boldsymbol{\theta}^{\rm s})\,.$$

- System **phase-mixed** w.r.t. the Kep. phase

$$F(\boldsymbol{J}, \boldsymbol{\theta}) \simeq \overline{F}(\boldsymbol{\mathcal{E}})\,.$$



- Keplerian wires **precess** in $\boldsymbol{\theta}^{\rm s}$

$$\boldsymbol{\Omega}^{\rm s} = \frac{\partial \overline{\Phi}_{\rm prec}}{\partial \boldsymbol{J}^{\rm s}} = \frac{\partial\big[\overline{\Phi}_{\rm self} + \overline{\Phi}_{\rm rel} + \overline{\Phi}_{\rm ext}\big]}{\partial \boldsymbol{J}^{\rm s}}\,.$$

Disc has mass — SMBH relativistic correction

3 BBGKY Hierarchy

truncation @ 3 pt function

$\mathcal{R} = (\boldsymbol{J}, \boldsymbol{\theta}^{\rm s})$

Actions+slow angle

- **BBGKY−n=1 equation**

$$\frac{\partial \overline{F}}{\partial \tau} + [\overline{F}, \overline{\Phi} + \overline{\Phi}_{\rm a}] + \frac{1}{N}\int {\rm d}\mathcal{R}_2\, [\overline{C}(\mathcal{R}_1, \mathcal{R}_2), \overline{U}_{12}]_{(1)} = 0\,.$$

- **BBGKY−n=2 equation**

$$\frac{1}{2}\frac{\partial C}{\partial \tau} + [\overline{C}(\mathcal{R}_1, \mathcal{R}_2), \overline{\Phi}(\mathcal{R}_1) + \overline{\Phi}_{\rm a}(\mathcal{R}_1)]_{(1)} + \frac{[\overline{F}(\mathcal{R}_1)\overline{F}(\mathcal{R}_2), \overline{U}_{12}]_{(1)}}{(2\pi)^{d-k}}$$
$$+ \int {\rm d}\mathcal{R}_3\, \overline{C}(\mathcal{R}_2, \mathcal{R}_3)\, [\overline{F}(\mathcal{R}_1), \overline{U}_{13}]_{(1)} + (1 \leftrightarrow 2) = 0\,. \qquad (50)$$

using averaging over fast angle

$$\overline{F}(\boldsymbol{J}, \boldsymbol{\theta}^{\rm s}) = \int \frac{{\rm d}\boldsymbol{\theta}^{\rm d}}{(2\pi)^{d-k}}\, F(\boldsymbol{J}, \boldsymbol{\theta}^{\rm s}, \boldsymbol{\theta}^{\rm d})\,.$$

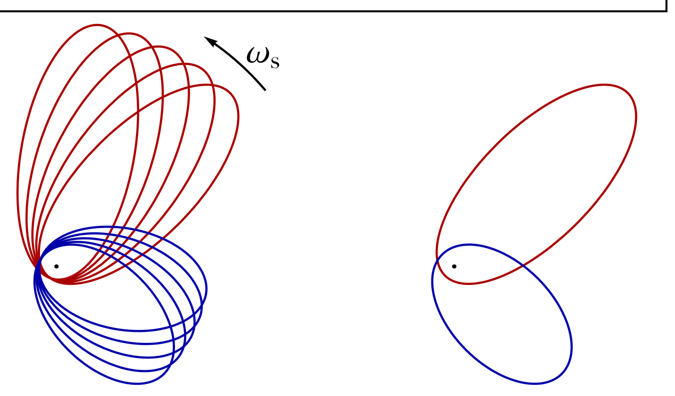
Bogoliubov's synchronization hypothesis

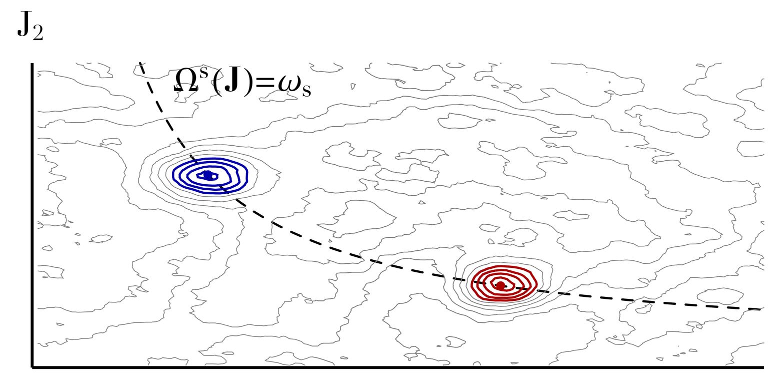
4 The degenerate Balescu-Lenard equation

- **The master equation of resonant relaxation**

$$\frac{\partial \overline{F}(\boldsymbol{J}, \tau)}{\partial \tau} = \frac{1}{N}\frac{\partial}{\partial \boldsymbol{J}_1^{\rm s}}\cdot\Bigg[\sum_{\boldsymbol{m}_1^{\rm s}, \boldsymbol{m}_2^{\rm s}} \boldsymbol{m}_1^{\rm s} \int {\rm d}\boldsymbol{J}_2\, \frac{\delta_{\rm D}(\boldsymbol{m}_1^{\rm s}\cdot\boldsymbol{\Omega}_1^{\rm s} - \boldsymbol{m}_2^{\rm s}\cdot\boldsymbol{\Omega}_2^{\rm s})}{|\mathcal{D}_{\boldsymbol{m}_1^{\rm s}, \boldsymbol{m}_2^{\rm s}}(\boldsymbol{J}_1, \boldsymbol{J}_2, \boldsymbol{m}_1^{\rm s}\cdot\boldsymbol{\Omega}_1^{\rm s})|^2} \times \Big[\boldsymbol{m}_1^{\rm s}\cdot\frac{\partial}{\partial \boldsymbol{J}_1^{\rm s}} - \boldsymbol{m}_2^{\rm s}\cdot\frac{\partial}{\partial \boldsymbol{J}_2^{\rm s}}\Big]\, \overline{F}(\boldsymbol{J}_1, \tau)\, \overline{F}(\boldsymbol{J}_2, \tau)\Bigg]\,.$$

- Some properties:
  - $\overline{F}(\boldsymbol{J}, \tau)$: **Orbital distorsion.**
  - $\partial\tau$: $\tau = t\, M_\star/M_\bullet$, BH dominance.
  - $1/N$: $1/N$ **resonant relaxation.**
  - $\partial/\partial \boldsymbol{J}_1^{\rm s}\cdot$: **Adiabatic conservation.**
  - $\delta_{\rm D}$: **Resonance on precessions.**
  - $1/\mathcal{D}_{\boldsymbol{m}_1^{\rm s}, \boldsymbol{m}_2^{\rm s}}$: **Self-gravity.**





5 Physical origin of Schwarzschild barrier

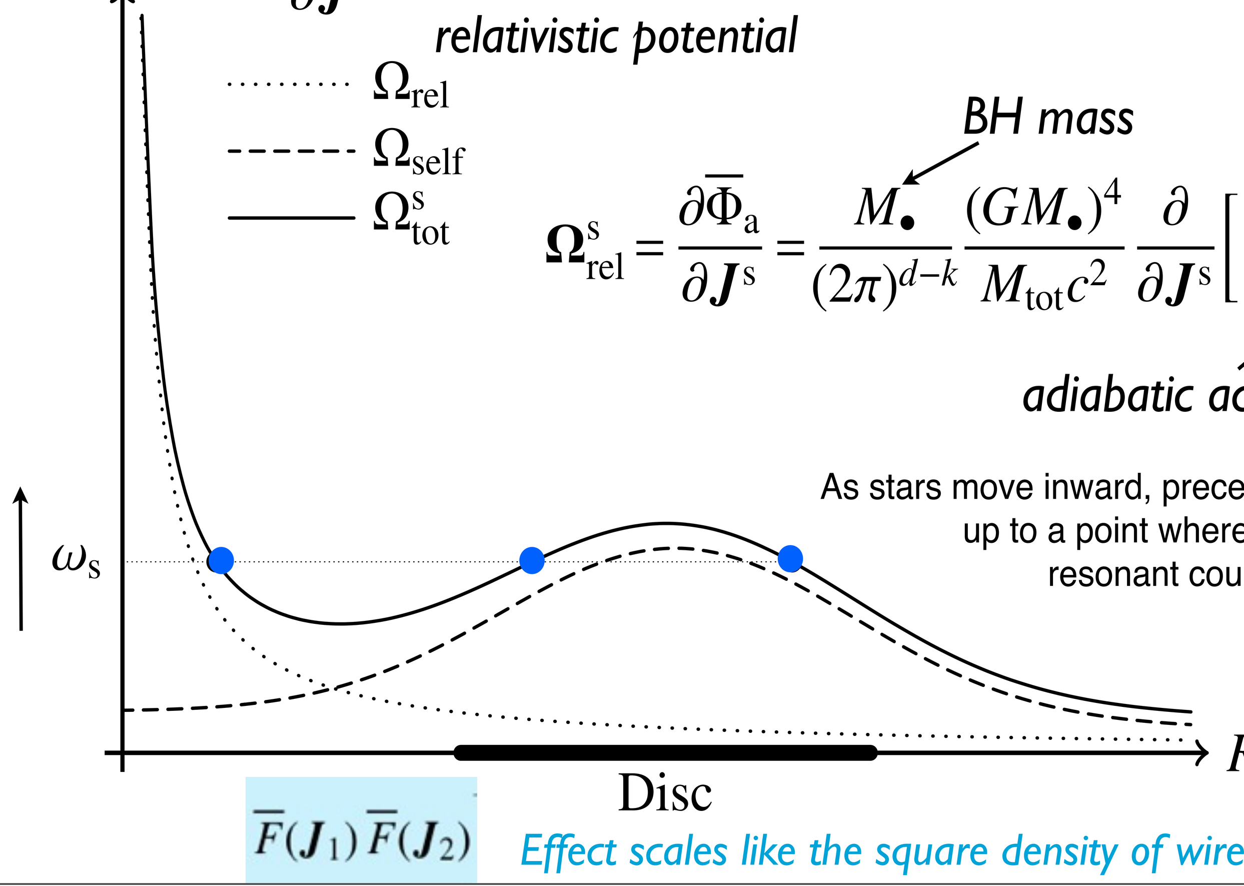
One PN and 1.5PN relativistic correction

$$\boldsymbol{\Omega}^{\rm s} = \frac{\partial[\overline{\Phi} + \overline{\Phi}_{\rm a}]}{\partial \boldsymbol{J}^{\rm s}}$$

relativistic potential

$$\boldsymbol{\Omega}^{\rm s}_{\rm rel} = \frac{\partial \overline{\Phi}_{\rm a}}{\partial \boldsymbol{J}^{\rm s}} = \frac{M_\bullet}{(2\pi)^{d-k}}\frac{(GM_\bullet)^4}{M_{\rm tot}c^2}\frac{\partial}{\partial \boldsymbol{J}^{\rm s}}\Big[-\frac{3}{I^3 L} + \frac{2GM_\bullet}{c}\frac{sL_z}{I^3 L^3}\Big]\,.$$

BH mass — *BH spin* — *adiabatic action*



As stars move inward, precession frequencies increase up to a point where it essentially prevents any resonant coupling with the disc's region.

$\overline{F}(\boldsymbol{J}_1)\,\overline{F}(\boldsymbol{J}_2)$ *Effect scales like the square density of wires*

6 Individual stochastic diffusion

- Self-consistent diffusion of the **system as a whole**
  $\Longrightarrow$ **Anisotropic Balescu-Lenard equation**

$$\frac{\partial \overline{F}}{\partial \tau} = \frac{\partial}{\partial \boldsymbol{J}^{\rm s}}\cdot\Big[\boldsymbol{A}(\boldsymbol{J}, \tau)\,\overline{F}(\boldsymbol{J}, \tau) + \boldsymbol{D}(\boldsymbol{J}, \tau)\cdot\frac{\partial \overline{F}}{\partial \boldsymbol{J}^{\rm s}}\Big]\,.$$

$\boldsymbol{A}(\overline{F})$ drift vector, $\boldsymbol{D}(\overline{F})$ diffusion tensor.

- **Individual dynamics** of a wire at position $\boldsymbol{\mathcal{J}}(\tau)$
  $\Longrightarrow$ **Stochastic Langevin equation** - (Risken (1996))

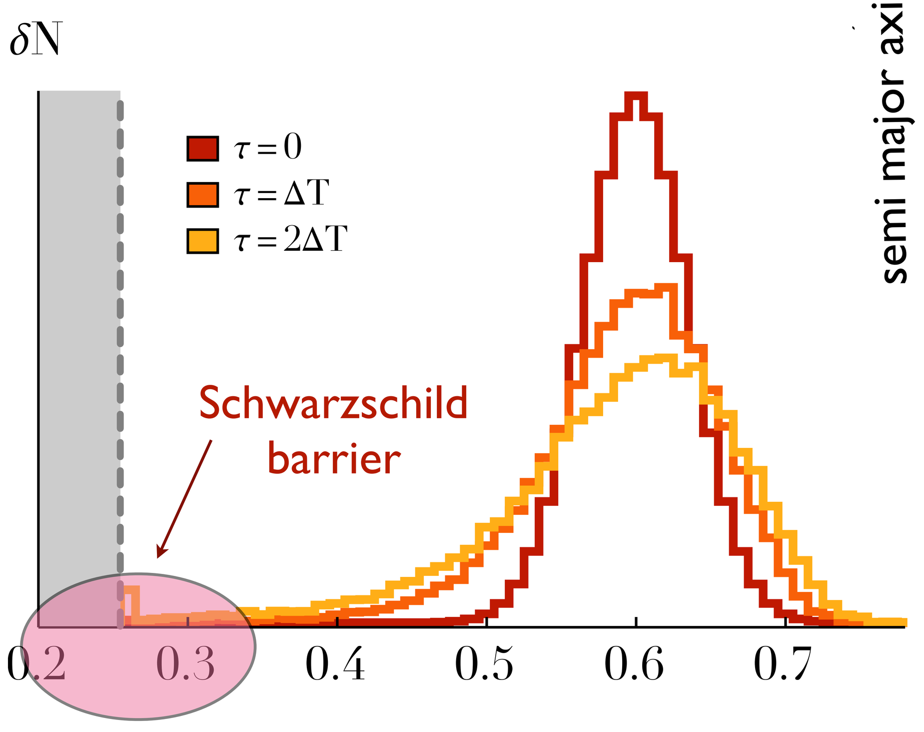
$$\frac{{\rm d}\boldsymbol{\mathcal{J}}}{{\rm d}\tau} = \boldsymbol{h}(\boldsymbol{\mathcal{J}}, \tau) + \boldsymbol{g}(\boldsymbol{\mathcal{J}}, \tau)\cdot\boldsymbol{\Gamma}(\tau)\,. \qquad \text{"Ito" Process}$$

$\boldsymbol{h}$ and $\boldsymbol{g}$ vector and tensor, and $\boldsymbol{\Gamma}$ stochastic Langevin forces.
$\Longrightarrow$ **Dual equation**, whose ensemble average gives back BL.

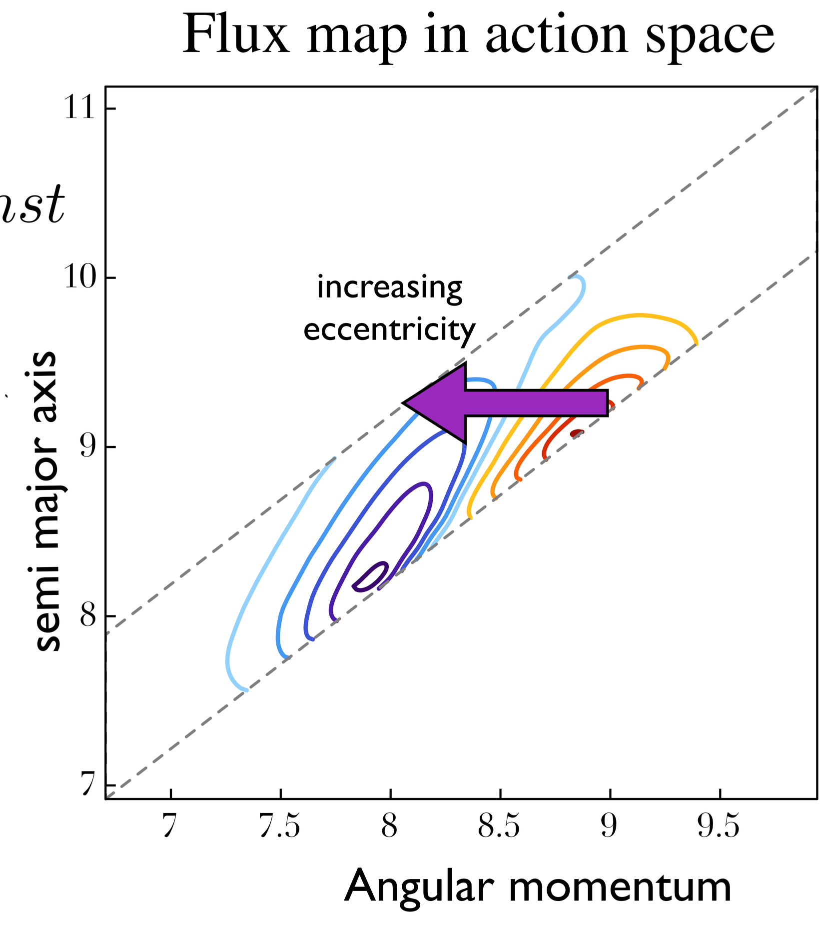
- In the Langevin's rewriting, **particles are dressed orbits.**
  $\Longrightarrow$ Huge gains in timesteps for integration.

7 Stochastic diffusion

Resonant relaxation drives the disc to a configuration of ***lower*** angular momentum @ fixed semi major axis $J_r + L = const$



Flux map in action space



Multi species ?

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