How to detect GWs

Measure proper (i.e. physical) distance between two free-falling test masses

- Each test mass follows geodesics

Consider flat spacetime + GW (TT perturbation)

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{d x^\alpha}{d\tau} \frac{d x^\beta}{d\tau} = 0$$

$$\frac{d^2 x^i}{dt^2} = -(\Gamma^i_{tt} + 2\Gamma^i_{tj} v^j + \Gamma^i_{jk} v^j v^k) + v^i (\Gamma^t_{tt} + 2\Gamma^t_{tj} v^j + \Gamma^t_{jk} v^j v^k)$$

$$v^i = d x^i / d t \ll 1 \quad \Rightarrow \quad \frac{d^2 x^i}{dt^2} + {\Gamma^i}_{tt} = 0 \qquad {\Gamma^i}_{tt} = \frac{1}{2}\left(2\partial_t h^{TT}_{jt} - \partial_j h^{TT}_{tt}\right) = 0$$

- Coordinate positions of test massive unaffected, but how about proper distance?

How to detect GWs

GW in z direction, test masses at x=0, y=0 and x=L_c, y=0

$$L = \int_0^{L_c} dx \sqrt{g_{xx}} = \int_0^{L_c} dx \sqrt{1 + h_{xx}^{\mathrm{TT}}(t, z=0)}$$

$$\simeq L_c \left[1 + \frac{1}{2} h_{xx}^{\mathrm{TT}}(t, z=0)\right]$$

$$\frac{\delta L}{L} \simeq \frac{1}{2} h_{xx}^{\mathrm{TT}}(t, z=0)$$

Measurable effect!

A better derivation

- Locally flat coordinates $$ds^2 = -dt^2 + d\mathbf{x}^2 + O\left(\frac{\mathbf{x}^2}{\mathcal{R}^2}\right)$$
- Geodesics at $x^i = 0$ and $x^i = L^i(t)$
- Proper distance is $\sqrt{L^i L_i}$ up to errors $\sim h L^2/\lambda^2 \ll 1$
(NB *L* is Fabry-Perot cavity length for terrestrial detectors)
- Separation vector L^μ between two geodesics obeys geodesic deviation equation $$\frac{D^2 L^\mu}{d\tau^2} = R^\mu{}_{\alpha\beta\gamma} u^\alpha u^\beta L^\gamma$$
- With $u^\mu = \delta^\mu_t$ and $L^\mu = (0, L^i)$ $$\frac{d^2 L^i(t)}{dt^2} = -R_{itjt}(t, \mathbf{0}) L^j(t)$$

A better derivation

$$R_{itjt} = -\frac{1}{2}\ddot{h}_{ij}^{\mathrm{TT}} + \Phi_{,ij} + \dot{\Xi}_{(i,j)} - \frac{1}{2}\ddot{\Theta}\delta_{ij}$$

$$\Theta \sim \Phi \sim \frac{\text{mass}}{r} \qquad \Xi \sim \frac{\text{linear momentum}}{r}$$

→ Only TT piece (=GW) contributes far from the source!

More formally, let's show this from field equations on the board

$$\begin{aligned} \nabla^2\Theta &= -8\pi\rho\,, \\ \nabla^2\Phi &= 4\pi\left(\rho + 3P - 3\dot{S}\right) \\ \nabla^2\Xi_i &= -16\pi S_i\,, \end{aligned}$$

$$\begin{aligned} \nabla^2 S &= \dot{\rho}\,, \\ \nabla^2\sigma &= -\frac{3}{2}P + \frac{3}{2}\dot{S}\,, \\ \nabla^2\sigma_i &= 2\dot{S}_i\,. \end{aligned}$$

$$\begin{aligned} T_{tt} &= \rho\,, \\ T_{ti} &= S_i + \partial_i S\,, \\ T_{ij} &= P\delta_{ij} + \sigma_{ij} + \partial_{(i}\sigma_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\sigma, \end{aligned}$$

A back-of-the-envelope derivation of the quadrupole formula

Moments of mass and current distributions:

$$M_0 \equiv \int \rho \, d^3x = M$$

$$S_1 \equiv \int \rho v_j \, x_k \, \epsilon_{ijk} \, d^3x = S_i$$

$$M_1 \equiv \int \rho \, x_i \, d^3x = M \, L_i$$

$$M_2 \equiv \int \rho \, x_i \, x_j \, d^3x = M \, L_{ij}$$

$$h \sim \frac{G}{c^2} \frac{M_0}{r}$$

Conservation of mass

$$h \sim \frac{G}{c^3} \frac{d}{dt} \frac{M_1}{r} \sim \frac{G}{c^3} \frac{P}{r}$$

Conservation of linear momentum

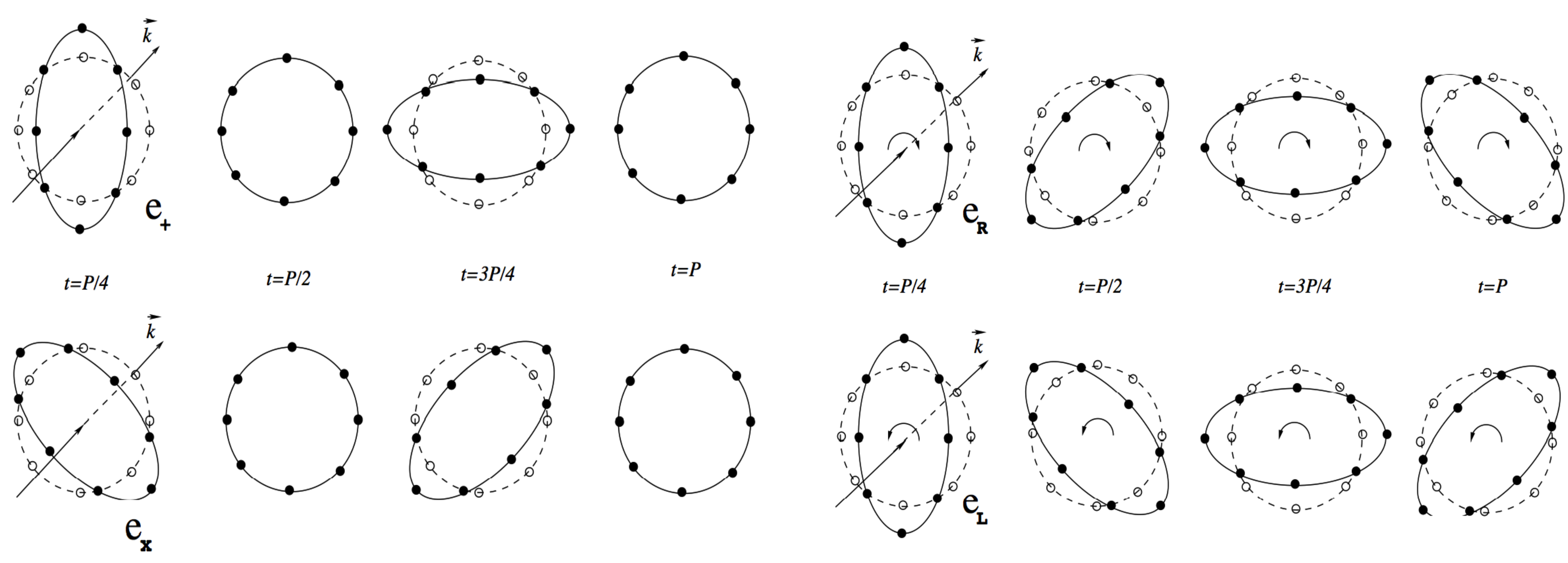
$$h \sim \frac{G}{c^4} \frac{d}{dt} \frac{S_1}{r}$$

$$h \sim \frac{G}{c^4} \frac{d^2}{dt^2} \frac{M_2}{r}$$

Conservation of angular momentum

Geometrical meaning of h_+ and $h_\times$

Figures from Rezzolla's notes



$$\mathbf{e}_+ \equiv \vec{e}_x \otimes \vec{e}_x - \vec{e}_y \otimes \vec{e}_y$$

$$\mathbf{e}_\times \equiv \vec{e}_x \otimes \vec{e}_x + \vec{e}_y \otimes \vec{e}_y$$

$$\mathbf{e}_{\rm R} \equiv \frac{\mathbf{e}_+ + i\mathbf{e}_\times}{\sqrt{2}} \qquad \mathbf{e}_{\rm L} \equiv \frac{\mathbf{e}_+ - i\mathbf{e}_\times}{\sqrt{2}}$$

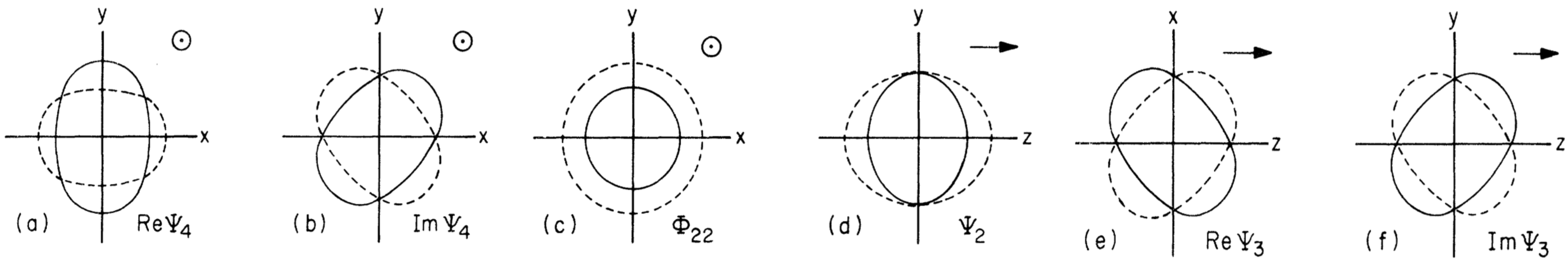
Beyond GR: more polarizations?

Similar decomposition of Riemann tensor in vacuum via Newman-Penrose scalars

$$\Psi_2(u) = -\tfrac{1}{6} R_{z0z0}(u)\,,$$
$$\Psi_3(u) = -\tfrac{1}{2} R_{x0z0} + \tfrac{1}{2} i\, R_{y0z0}\,,$$
$$\Psi_4(u) = -R_{x0x0} + R_{y0y0} + 2i\, R_{x0y0}\,,$$
$$\Phi_{22}(u) = -R_{x0x0} - R_{y0y0}\,.$$

$$\Psi_2(u) \quad (s=0), \qquad \Phi_{22}(u) \quad (s=0)$$
$$\Psi_3(u) \quad (s=\pm 1), \qquad \Psi_4(u) \quad (s=\pm 2)$$

Figures from Eardley, Lee and Lightman 1973



e.g. Dipolar emission if equivalence principle is violated (Brans-Dicke, scalar tensor theories, etc)

$$h \sim \frac{1}{R}\frac{d}{dt}(m_{\mathrm{GW},1}\mathbf{x}_1 + m_{\mathrm{GW},2}\mathbf{x}_2) \sim \frac{\eta m}{R}\mathbf{v}\left(\frac{m_{\mathrm{GW},1}}{m_{\mathrm{I},1}} - \frac{m_{\mathrm{GW},2}}{m_{\mathrm{I},2}}\right)$$

A real detector: frequencies and not distances

- Detectors are laser interferometers
- Photons acculumulate phase change $\delta\phi = 2\pi\,\delta L/\lambda$ when proper distance between "mirrors" change

  Analysis valid for $L \ll \lambda$
- More in general, we can integrate photon geodesics between mirrors; photon frequency will change due to GW and produce phase change

$$\frac{\Delta\nu}{\nu} = \frac{1}{2}(1+\cos\theta)\Psi(t) - \cos\theta\,\Psi(t+\tau(1-\cos\theta)/2) - \frac{1}{2}(1-\cos\theta)\Psi(t+\tau),$$

$$\cos\theta \equiv \boldsymbol{\sigma}\cdot\boldsymbol{n}, \qquad \Delta\Phi = 2\pi\int_0^t \Delta\nu(t')dt'$$

Estabrook & Wahlquist 1975

$$\Psi(t) \equiv \frac{h_{ij}^{\mathrm{TT}}\sigma^i\sigma^j}{\sin^2\theta}$$

$\tau =$ round-trip laser travel time between mirrors

$\boldsymbol{\sigma}$ and $\boldsymbol{n} =$ propagation directions of laser and GW

The detector transfer function

- Exercise: from the shift in the laser frequency, show that a monochromatic GW with frequency *f* propagating orthogonally to the detector causes an effective change in each detector arm, given by

$$\delta L = h_+ \frac{L}{2} \frac{\sin(\pi f \tau)}{\pi f \tau} = h_+ \frac{L}{2} \frac{\sin(\pi f/\Lambda)}{\pi f/\Lambda}$$

- This transfer function *T(f)* explains why the sensitivity of GW interferometers worsens linearly with *f* at high frequencies

The detector response

At least when *L*<< *λ* (i.e. *T(f)* ~ *1*) an interferometer measures

$$h(t) = \frac{1}{2}(h_{ij}u^i u^j - h_{ij}v^i v^j)$$

$$= D^{ij}h_{ij}(t) = F_+ h_+(t) + F_\times h_\times(t)$$

$$D^{ij} = \frac{1}{2}\left(u^i u^j - v^i v^j\right)$$ Detector tensor

F_+ , $F_\times$ Beam pattern functions

Pattern functions

Exercise: derive pattern functions for detectors at 90 and 60 degrees, and plot them

$$F_+^{(90^\circ)} = \frac{1}{2}\left(1 + \cos^2\theta\right)\cos 2\phi \cos 2\psi - \cos\theta \sin 2\phi \sin 2\psi,$$

$$F_\times^{(90^\circ)} = \frac{1}{2}\left(1 + \cos^2\theta\right)\cos 2\phi \sin 2\psi + \cos\theta \sin 2\phi \cos 2\psi.$$

$$F_+^{(60^\circ)} = \frac{\sqrt{3}}{2}\left[\frac{1}{2}(1 + \cos^2\theta)\cos 2\phi \cos 2\psi - \cos\theta \sin 2\phi \sin 2\psi\right],$$

$$F_\times^{(60^\circ)} = \frac{\sqrt{3}}{2}\left[\frac{1}{2}(1 + \cos^2\theta)\cos 2\phi \sin 2\psi + \cos\theta \sin 2\phi \cos 2\psi\right].$$