Lensing in cosmology

First case: lensing of a quasar by a galaxy

The Einstein cross: QSO 2337+0305

Distances in cosmology

Images and caustics in the isothermal potential

The mass-sheet degeneracy

Time delays

Microlensing variability
This time the complexity will increase
We will see multiple caustics merging
The lens equation in cosmology

In the weak field limit and for small deviations, the lens equation is still valid if we use the cosmological angular distances

\[ D_I \theta = d \rightarrow D_I = \frac{d}{\theta} \]

angular distances

In the weak field limit and for small deviations, the lens equation is still valid if we use the cosmological angular distances.

Distances in cosmology

Comoving distance: 

\[ D_C = \frac{c}{H_0} \int \frac{dz}{E(z)} \quad H(z) = H_0 E(z) \]

Comoving angular distance: 

\[ D_M = \begin{cases} 
\frac{1}{2} \sqrt{-1} \sinh \left( \frac{1}{2} K D_C \right) & \text{for } K > 0 \\
D_C & \text{for } K = 0 \\
-\frac{1}{2} \sqrt{-1} \sin \left( \frac{1}{2} K D_C \right) & \text{for } K < 0
\end{cases} \]

Curvature: \( K \)

Curvature density parameter:

\[ \Omega_K = -\left( \frac{c}{H_0} \right)^2 K \]

Angular distance: 

\[ D_A = \frac{D_M}{1+z} \]

Do not subtract angular distances: use comoving angular distance then normalize using redshift.
An interesting cosmological situation
The Einstein cross: QSO 2337+0305

A distant quasar source: $z=1.695$
(light travel time: 9.846 Gyr)

A nearby galactic lens: $z=0.0395$
(light travel time: 0.540 Gyr)

(discovered by John Huchra in 1985)
The elliptical lens

The Einstein cross

QSO 2237+0305 (HST)
A simple model: elliptical isothermal potential

\[ \phi = \sqrt{(1 - \eta)x^2 + (1 + \eta)y^2} \quad \text{for small ellipticity} \quad \phi \approx r \left( 1 - \frac{\eta}{2} \cos 2\theta \right) \]

The lens equation:

\[ \vec{r}_s = \vec{r} - \vec{\nabla} \phi \]

With:

\[ dr = r - 1 \]

\[ \vec{r}_s = (dr + \frac{\eta}{2} \cos 2\theta) \quad \vec{u}_r - \eta \sin 2\theta \quad \vec{u}_\theta \]

(to first order in \( \eta \))
A simple model: elliptical isothermal potential

Circular source with impact parameter $\vec{r}_0$ and radius $R_0$

\[ \vec{r}_s = (dr + \frac{\eta}{2} \cos 2\theta) \; \vec{u}_r - \eta \sin 2\theta \; \vec{u}_\theta \]

\[ \vec{r}_0 = (x_0, y_0) \quad dr = \frac{\eta}{2} \cos 2\theta - x_0 \cos \theta - y_0 \sin \theta \pm \sqrt{R_0^2 - (\eta \sin 2\theta - x_0 \sin \theta + y_0 \cos \theta)^2} \]

\[ \vec{r}_s = \vec{R}_s + \vec{r}_0 \quad ; \quad |\vec{R}_s| = R_0 \]
Radial position of the images

\[ dr = -\frac{\eta}{2} \cos 2\theta - x_0 \cos \theta - y_0 \sin \theta \]
\[ dr = \frac{\eta}{2} \cos 2\theta - x_0 \cos \theta - y_0 \sin \theta \pm \sqrt{R_0^2 - (\eta \sin 2\theta - x_0 \sin \theta + y_0 \cos \theta)^2} \]

Image forms if: \[ |df_0| = |\eta \sin 2\theta - x_0 \sin \theta + y_0 \cos \theta| < R_0 \]

Here represented for: \[ x_0 = 0 \quad ; \quad y_0 = 0 \quad ; \quad df_0 = |\eta \sin 2\theta| \]
Source at center of elliptical lens,

\[ dr = \frac{\eta}{2} \cos 2\theta \pm \sqrt{R_0^2 - (\eta \sin 2\theta)^2} \]

Images when: \( \sin 2\theta < R_0 \)

4 images

Images centers: \( \sin 2\theta = 0 \)

Source near center Of elliptical lens
Caustics for the isothermal potential

\[ \phi = \sqrt{(1 - \eta)x^2 + (1 + \eta)y^2} \approx r \left(1 - \frac{\eta}{2} \cos 2\theta\right) \]

\[ x_s = x - \frac{\partial \phi}{\partial x} \]

\[ y_s = y - \frac{\partial \phi}{\partial y} \]

\[ J = \frac{\partial x_s}{\partial x} \frac{\partial y_s}{\partial y} - \frac{\partial x_s}{\partial y} \frac{\partial y_s}{\partial x} \approx \frac{r - 1}{r} - \frac{3 \cos 2\theta}{2r} \eta \]

To first order in \( \eta \)

Critical lines: \( J = 0 \) \( \rightarrow \) \( r = 1 + \frac{3}{2} \eta \cos 2\theta \)
Caustics:

\[
\begin{align*}
xs &= \left(\frac{3}{2}\cos \theta + \frac{1}{2}\cos 3\theta\right) \eta \\
ys &= \left(-\frac{3}{2}\sin \theta + \frac{1}{2}\sin 3\theta\right) \eta
\end{align*}
\]

The amplitude of the caustics diagram is: \(2 \eta\)

We transform the equation for the critical lines to the source plane by using the lens equation.
Image equation

\[ dr = \frac{\eta}{2} \cos 2\theta - x_0 \cos \theta \pm \sqrt{R_0^2 - df_0^2} \]

\[ df_0 = \eta \sin 2\theta - x_0 \sin \theta + y_0 \cos \theta \]

For \( x_0 = 2\eta, y_0 = 0 \)

\( (df_0)_\theta = 0 \quad ; \quad (df_0)_{\theta, \theta} = 0 \)

Cusp caustic = order 3
Image formation $\rightarrow |df_0| < R_0$; $R_0$ source radius
Image equation: 

\[ dr = \frac{\eta}{2} \cos 2\theta - x_0 \cos \theta - y_0 \sin(\theta) \pm \sqrt{R_0^2 - d_{f_0}^2} \]

For \( x_0 \approx 0.7 \eta, \ y_0 \approx 0.7 \eta \)

\( (d_{f_0}), \theta = 0 \)

Fold caustic = order 2

\[ df_0 = \eta \sin 2\theta - x_0 \sin \theta + y_0 \cos \theta \]
Image formation $\rightarrow |df_0|<R_0$ ; $R_0$ source radius
The mass sheet degeneracy

Let introduce a new surface density \( \tilde{\kappa} \)

It relates to the initial surface density \( \kappa \) by:

\[
\kappa = (1 - \lambda) \tilde{\kappa} + \lambda
\]

With: \( \kappa = \frac{1}{2} \Delta \phi \) and: \( \tilde{\kappa} = \frac{1}{2} \Delta \tilde{\phi} \)

\[
\phi = (1 - \lambda) \tilde{\phi} + \frac{1}{2} \lambda \left( x^2 + y^2 \right)
\]

(take laplacian and check it is working)
\[ \kappa = (1 - \lambda) \tilde{\kappa} + \lambda \]  

\[ \phi = (1 - \lambda) \tilde{\phi} + \frac{1}{2} \lambda (x^2 + y^2) \]  

The lens equation:

\[ \begin{align*}
    x_s &= x - \frac{\partial \phi}{\partial x} = (1 - \lambda) \left( x - \frac{\partial \tilde{\phi}}{\partial x} \right) = (1 - \lambda) \tilde{x}_s \\
    y_s &= y - \frac{\partial \phi}{\partial y} = (1 - \lambda) \left( y - \frac{\partial \tilde{\phi}}{\partial y} \right) = (1 - \lambda) \tilde{y}_s
\end{align*} \]

The lens equation with the new surface density \( \tilde{\kappa} \) is equivalent to the former lens equation. If we re-scale the source coordinates, the two equations are equivalent.

This is known as the mass-sheet degeneracy (adding a constant density) and leads to a re-scaling of both lens and source coordinates.
Time delays

Basic idea: The path of light for each image is different
Consequence: a time delay between the images

Refsdal (1964)
In practice the source: quasar is variable. Thus time delays can be observed.
The time delay

\[ \tau = \left(\frac{1+z_L}{c}\right) \frac{D_L D_S}{D_{LS}} \left( \frac{1}{2}(\vec{\theta} - \vec{\beta})^2 - \psi(\vec{\theta}) \right) \]

\[ d_I = \left(\frac{c}{H_0}\right)^{-1} D_I \]

\[ D_I \propto D_C = \frac{c}{H_0} \int \frac{dz}{E(z)} \]

The first thing to note is that the time delay is proportional to: \( H_0^{-1} \)

Thus measuring the time delay is direct measurement of \( H_0 \)
In practice what we measure is the differential time delay between the images

\[ \tau = \frac{(1+z_L)}{H_0} \frac{d_L d_S}{d_{LS}} \left( \frac{1}{2} (\vec{\theta} - \vec{\beta})^2 - \psi(\vec{\theta}) \right) \rightarrow \tau(\theta, \beta) = T_d \left( \frac{1}{2} (\vec{\theta} - \vec{\beta})^2 - \psi(\vec{\theta}) \right) \]

\[ \Delta \tau_{A,B} = \tau(\theta_A, \beta) - \tau(\theta_B, \beta) = T_d \left( \frac{1}{2} (\vec{\theta}_A - \vec{\beta})^2 - \psi(\vec{\theta}_A) - \frac{1}{2} (\vec{\theta}_B - \vec{\beta})^2 + \psi(\vec{\theta}_B) \right) \]

This is clearly model dependent: one needs to estimate the potential

For a singular isothermal sphere: \[ \Delta \tau_{A,B} \propto \left( R_A^2 - R_B^2 \right) \]

Kochaneck & Schechter (2004)
First it is nothing really new...

If we minimize the time delay with respect to $\vec{\theta}$

We obtain the lens equation:

$$\tau = \left( \frac{1 + z_L}{H_0} \right) \frac{d_L}{d_{LS}} \left( 1 + \frac{1}{2} (\vec{\theta} - \vec{\beta})^2 - \psi(\vec{\theta}) \right)$$

$$\vec{\beta} = \vec{\theta} - \vec{\nabla} \phi$$

The formulation: time delay or lens equation Are seen as equivalent
The physical interpretation of the time delay

\[ \tau = \frac{(1 + z_L)}{H_0} \frac{d_L d_S}{d_{LS}} \left( \frac{1}{2} (\vec{\theta} - \vec{\beta})^2 - \psi(\vec{\theta}) \right) \]

- Geometric delay
- Gravitational delay (the Shapiro delay),
\( \delta_1 = a_1 - D_{LS} \approx \frac{1}{2} \frac{b^2}{D_{LS}} \quad \delta_2 = a_2 - D_L \approx \frac{1}{2} \frac{b^2}{D_L} \)

\( b = D_L(\theta - \beta) \)

\( \delta = \delta_1 + \delta_2 = \frac{D_L D_S}{D_{LS}} (\theta - \beta)^2 \)

\( d_I = \left( \frac{c}{H_0} \right)^{-1} D_I \quad \tau = \frac{\delta}{c} \quad \tau = \frac{(1 + z_L)}{H_0} \frac{d_L}{d_{LS}} \frac{1}{2} \left( \theta - \beta \right)^2 - \psi(\bar{\theta}) \)

With the appropriate scale factor we recover the geometric time delay
The Shapiro time delay

First predicted in 1964 by Irwin Shapiro

For a nearly static and weak field

The time delay due to the gravitational field

is directly proportional to the Newtonian potential

\[
\tau = \frac{(1+z_L)}{H_0} \frac{d_L d_S}{d_{LS}} \left( \frac{1}{2} (\vec{\theta} - \vec{\beta})^2 - \psi(\vec{\theta}) \right)
\]
How do time delay look like in practice?

Problem with time delay estimations

Some practical examples of light curves of images for a variety of lenses
Problem with time delay estimations

The time delay is model dependent

Any model of the potential or surface density is affected by the mass-sheet degeneracy

It is essential to find a method to deal with the mass-sheet degeneracy

Treu & Koopmans (2002) propose to use stellar kinematics

Keeton & Zabludoff (2004) use the environment of the lens (galaxy counts, weak lensing)
Some practical examples of light curves of images for a variety of lenses

A short review from the literature
JVAS B1422+231
PG 1115+080
RX J0911+0551
<table>
<thead>
<tr>
<th>System</th>
<th>Our Results</th>
<th>Published Values</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>JVAS B0218+357</td>
<td>$\Delta t_{AB} = 9.9^{+4.0}<em>{-0.9}$ or $\Delta t</em>{AB} = 11.8 \pm 2.3$</td>
<td>$\Delta t_{AB} = 10.1^{+1.6}_{-1.6}$</td>
<td>Cohen et al. (2000)</td>
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<tr>
<td></td>
<td></td>
<td>$\Delta t_{BA} = 12 \pm 3$</td>
<td>Corbett et al. (1996)</td>
</tr>
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<td></td>
<td></td>
<td>$\Delta t_{BA} = 10.5 \pm 0.4$</td>
<td>Biggs et al. (1999)</td>
</tr>
<tr>
<td>SBS 0909+523</td>
<td>unreliable</td>
<td>$\Delta t_{BA} = 49 \pm 6$</td>
<td>Goicoechea et al. (2008)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Delta t_{BA} = 45^{+11}_{-6}$</td>
<td>Ullán et al. (2006)</td>
</tr>
<tr>
<td>RX J0911+0551</td>
<td>2 solutions; $\Delta t_{BA} \sim 146$ or $\sim 157$</td>
<td>$\Delta t_{BA} = 150 \pm 6$</td>
<td>Burud (2001)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Delta t_{BA} = 146 \pm 4$</td>
<td>Hjorth et al. (2002)</td>
</tr>
<tr>
<td>FBQS J0951+2635</td>
<td>unreliable</td>
<td>$\Delta t_{AB} = 16 \pm 2$</td>
<td>Jakobsson et al. (2005)</td>
</tr>
<tr>
<td>HE 1104-1805</td>
<td>impossible to distinguish but identical within error bars</td>
<td>$\Delta t_{BA} = 152^{+2.8}_{-3.0}$</td>
<td>PoinDEXeter et al. (2007)</td>
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<td></td>
<td></td>
<td>$\Delta t_{BA} = 161 \pm 7$</td>
<td>Ofek &amp; Maoz (2003)</td>
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<td></td>
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<td>$\Delta t_{BA} = 157 \pm 10$</td>
<td>Wyrzykowski et al. (2003)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Delta t_{BA} = 162.2^{+6.3}_{-5.9}$</td>
<td>Morgan et al. (2008a)</td>
</tr>
<tr>
<td>PG 1115+080</td>
<td>dependent on method</td>
<td>$\Delta t_{CA} \sim 9.4$</td>
<td>Schechter et al. (1997)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Delta t_{CB} = 23.7 \pm 3.4$</td>
<td>Schechter et al. (1997)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Delta t_{CB} = 25.0^{+3.3}_{-3.8}$</td>
<td>Barkana (1997)</td>
</tr>
<tr>
<td>JVAS B1422+231</td>
<td>contradictory results between methods: BAC or CAB?</td>
<td>$\Delta t_{BA} = 1.5 \pm 1.4$</td>
<td>Patnaik &amp; Narasimha (2001)</td>
</tr>
<tr>
<td></td>
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<td>$\Delta t_{AC} = 7.6 \pm 2.5$</td>
<td></td>
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<td></td>
<td>$\Delta t_{BC} = 8.2 \pm 2.0$</td>
<td></td>
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<tr>
<td>SBS 1520+530</td>
<td>$\Delta t_{AB} = 125.8 \pm 2.1$</td>
<td>$\Delta t_{AB} = 130 \pm 3$</td>
<td>Burud et al. (2002c)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Delta t_{AB} = 130.5 \pm 2.9$</td>
<td>Gayullina et al. (2005b)</td>
</tr>
<tr>
<td>CLASS B1600+434</td>
<td>$\Delta t_{AB} = 47.8 \pm 1.2$</td>
<td>$\Delta t_{AB} = 51 \pm 4$</td>
<td>Burud et al. (2000)</td>
</tr>
<tr>
<td>CLASS B1608+656</td>
<td>$\Delta t_{BA} = 31.6 \pm 1.5$</td>
<td>$\Delta t_{BA} = 31.5^{+2.1}_{-1}$</td>
<td>Fassnacht et al. (2002)</td>
</tr>
<tr>
<td></td>
<td>$\Delta t_{BC} = 35.7 \pm 1.4$</td>
<td>$\Delta t_{BC} = 36.0 \pm 1.5$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\Delta t_{BD} = 77.5 \pm 2.2$</td>
<td>$\Delta t_{BD} = 77.0^{+2.1}_{-2}$</td>
<td></td>
</tr>
<tr>
<td>HE 2149-2745</td>
<td>unreliable</td>
<td>$\Delta t_{AB} = 103 \pm 12$</td>
<td>Burud et al. (2002a)</td>
</tr>
</tbody>
</table>

Compilation For 11 systems

Eulaers (2012)
Grillo et al. (2018)

Grillo et al. (2020)

$H_0 \approx 73 \text{ km/s/Mpc}$

Planck estimate

$H_0 \approx 67.4 \text{ km/s/Mpc}$
Why do we observe un-correlated variability of the images of QSO 2337+0305?

Typical time scale of variations
~ a few months to years

Typical Einstein radius crossing time
For a solar mass star in the galaxy
A few hundred days

Wozniak etal. (2000)
What is going on?

The deflection angle is perturbed by the field of the local stars.
Typical Numbers

The main galaxy: \( M \approx 10^{10} \text{ solar mass} \); \( R_E \propto \sqrt{M} \) \( \rightarrow \) \( R_E \approx 30 \text{ Kpc} \)

Solar mass star: \( R_E \approx \sqrt{10^{-10} \times 30 \text{ kpc}} \approx 0.3 \text{ pc} \)

Density in the solar neighborhood: \( 0.08 \text{ solar mass/pc}^3 \)

Projected density in the solar neighborhood: \( 0.08 \times \text{scale height} \approx 0.08 \times 150 \approx 12 \text{ solar mass/pc}^2 \)

Mean distance between stars: \( \sqrt{\frac{1}{12}} \approx 0.29 \text{ pc} \)

Perturbation by stars very likely
Use ray tracing to reconstruction the amplification map
And the local caustics due to the stars

Local equations: total field=field of the galaxy+sum of the field of the local stars

\[
\phi = \sqrt{(1 - \eta)x^2 + (1 + \eta)y^2 + \sum \mu_i \log(|\vec{r} - \vec{r}_i|)}
\]

\[
\vec{r}_S = \vec{r} - \nabla \phi
\]

Ray-tracing and amplification maps / caustics reconstruction

\[
\mu_i = \frac{m_i}{M_0}
\]

ratio of the mass of the star \( m_i \) to the mass of the galaxy \( M_0 \)
Practical result
In practice we observe a trajectory of the quasar in this map.

The larger the source, the stronger is the attenuation of the peaks.

Light curve
The structure of the source (quasar) as inferred from caustic crossing (Finite source size effect)

Shalyapin et al. (2002)

Best model

standard accretion disk around a supermassive black hole

90% of the light is emitted by a region with size less than: $1.2 \times 10^{-2} \, pc$