Cluster of galaxies

The weak lensing method

Basic ideas
Shift-Jacobian
Shear-moments relation
From shear to convergence
Some applications
Practical problems
Strong-lensing/weak-lensing combination
The weak lensing regime:

- Larger distances
- Small distortion

Einstein ring

Cluster

Strong distortion of source image

Weak lensing regime:
- Larger distances
- Small distortion
The weak lensing regime

A round source (no field) is transformed to an ellipse (first order).

Additionally, the source position is shifted: \( \vec{r}_0 \neq \vec{r}_w \).
The weak lensing regime

The lens equation: \[ \vec{r}_S = \vec{r} - \vec{\nabla} \phi \]

In the weak-field regime we make a local approximation of the lens equation near the source position \( \vec{r}_0 \):

Local coordinates in the source plane: \( \vec{r}_S = \vec{r}_0 + \delta \vec{r}_S \)
Local coordinates in the lens plane: \( \vec{r} = \vec{r}_0 + \delta \vec{r} \)

We will expand the potential near the source center \( \vec{r}_0 \):

\[ \phi \simeq \phi(\vec{r}_0 + \delta \vec{r}) \quad \text{to order 2} \quad \text{with:} \quad \vec{r}_S = \vec{r}_0 + \delta \vec{r}_S \]

\[ \vec{r}_S = \vec{r} - \vec{\nabla} \phi \]
Unlensed source

Image center position shift (small)

Source image elliptical distortion
Going back to the lens equation:

\[
\begin{align*}
\vec{r}_S &= x - \phi_x = -\phi_1 + x_0 + dx \\
y_S &= y - \phi_y = -\phi_2 + y_0 + dy
\end{align*}
\]

First order effect: global shift

\[
\vec{r} = \vec{r}_0 + \delta \vec{r}
\]

\[
\vec{\nabla} \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}
\]

\[
\vec{r}_S = -\vec{\nabla} \phi + \vec{r}_0 + \delta \vec{r}
\]
\[ \vec{r}_s = -\vec{\nabla}\phi + \vec{r}_0 + \delta \vec{r} \quad \text{with} \quad \vec{r}_s = \vec{r}_0 + \delta \vec{r}_s \]

\[ \delta \vec{r}_s = -\vec{\nabla}\phi + \delta \vec{r} \]

The shift affect the distribution on objects around the lens.
\[ \phi(\vec{r}_0 + \delta \vec{r}) = \phi_0 + \nabla \phi \cdot \delta \vec{r} + \frac{1}{2} \phi_{11} dx^2 + \phi_{12} dx dy + \frac{1}{2} \phi_{22} dy^2 \quad \text{with:} \quad \phi_{ij} = \left[ \frac{\delta^2 \phi}{\delta x_i \delta x_j} \right]_{\vec{r}=\vec{r}_0} \]

Going back to the lens equation:

\[ x_S = x - \phi_x = -\phi_1 + x_w + dx - \phi_{11} dx - \phi_{12} dy \]
\[ y_S = y - \phi_y = -\phi_2 + y_w + dy - \phi_{22} dy - \phi_{12} dx \]

\[ M = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{12} & \phi_{22} \end{pmatrix} \]

\[ \vec{r}_S = -\nabla \phi + \vec{r}_0 + \delta \vec{r} - M \delta \vec{r} \]
\[ \vec{r}_S = -\vec{\nabla} \phi + \vec{r}_0 + \delta \vec{r} - M \delta \vec{r} \quad \text{with} \quad \vec{r}_S = \vec{r}_0 + \delta \vec{r}_S \]

\[ \delta \vec{r}_S = -\vec{\nabla} \phi + \delta \vec{r} - M \delta \vec{r} \]

We introduce the centered (shift free) coordinate \( \delta \vec{u} = \delta \vec{r} - \vec{q}_0 \)

here \( \vec{q}_0 \) is the shift at order 2

\[ \delta \vec{r}_S = -\vec{\nabla} \phi + \vec{q}_0 - M \vec{q}_0 + \delta \vec{u} - M \delta \vec{u} \]

No shift must be \( = 0 \)
Introducing \( J = \begin{pmatrix} 1 - \phi_{11} & -\phi_{12} \\ -\phi_{12} & 1 - \phi_{22} \end{pmatrix} \)

\[
\delta \vec{r}_S = - \vec{\nabla} \phi + \vec{q}_0 - M \vec{q}_0 + \delta \vec{u} - M \delta \vec{u} = - \vec{\nabla} \phi + J \vec{q}_0 + J \delta \vec{u}
\]

No shift \( \Rightarrow \vec{q}_0 = J^{-1} \vec{\nabla} \phi \approx \vec{\nabla} \phi \)

Finally in the centered coordinates: \( \Rightarrow \delta \vec{r}_S = J \delta \vec{u} \)
The effect of the second term is a distortion of the source. An initially round source is transformed to an elliptical one. $J$ is the Jacobian matrix.
shift of position $\simeq -\vec{\nabla}\phi$

A round source becomes an ellipse

Image distortion

$J = \begin{pmatrix}
1 - \phi_{11} & -\phi_{12} \\
-\phi_{12} & 1 - \phi_{22}
\end{pmatrix}$

distortion matrix
Re-writing the Jacobian by introducing the convergence $\kappa$

\[
J = \begin{pmatrix}
1 - \phi_{11} & -\phi_{12} \\
-\phi_{12} & 1 - \phi_{22}
\end{pmatrix}
= \begin{pmatrix}
1 - \kappa - \gamma_1 & -\gamma_2 \\
-\gamma_2 & 1 - \kappa + \gamma_1
\end{pmatrix}
= (1 - \kappa) \begin{pmatrix}
1 - g_1 & -g_2 \\
-g_2 & 1 + g_1
\end{pmatrix}
\]

\[
\kappa = \frac{1}{2}(\phi_{11} + \phi_{22}) \quad \gamma_1 = \frac{1}{2}(\phi_{11} - \phi_{22}) \quad \gamma_2 = \phi_{12}
\]

\[
g_i = \frac{\gamma_i}{(1 - \kappa)} \quad \text{(reduced shear)}
\]
What is observable

\[ J = \begin{pmatrix} 1 - \phi_{11} & -\phi_{12} \\ -\phi_{12} & 1 - \phi_{22} \end{pmatrix} = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix} = (1 - \kappa) \begin{pmatrix} 1 - g_1 & -g_2 \\ -g_2 & 1 + g_1 \end{pmatrix} \]

We don’t observe the effect of convergence
It represents an absolute unknown scale

We observe only the reduced shear

\[ g_i = \frac{\gamma_i}{1 - \kappa} \approx \gamma_i \]

Reduced shear and shear equivalent in the weak lensing regime
\[ J = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix} \]

**Effect of convergence**

**Effect of shear**
The second order moments

\[ Q_{ij} = \int \Sigma (\vec{r}) x_i x_j d^2 x \]

And the associated matrix

\[ Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{pmatrix} \]
Transforming the second order moment matrix

\[ Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{pmatrix} \]

\[ Q = Q_0 \begin{pmatrix} 1 - \alpha & \beta \\ \beta & 1 + \alpha \end{pmatrix} \]

\[ \alpha = \frac{Q_{22} - Q_{11}}{Q_{11} + Q_{22}} \]

\[ \beta = \frac{2Q_{21}}{Q_{11} + Q_{22}} \]

\[ (\alpha, \beta) \text{ are directly related with } (g_1, g_2) \]

Must look like the observable part of \( J \)

\[ J_0 = \begin{pmatrix} 1 - g_1 & g_2 \\ g_2 & 1 + g_1 \end{pmatrix} \]
The second order moments are associated with an equivalent elliptical contour and thus a quadratic form

As a consequence it is useful to represent the effect of shear in terms of quadratic forms and their associated matrix
How does the shear transformation affect the ellipticities of galaxies?

\[
\alpha = \frac{Q_{22} - Q_{11}}{Q_{11} + Q_{22}} \quad \beta = \frac{2Q_{21}}{Q_{11} + Q_{22}}
\]

Let say we have some initial value for these 2 parameters. Then we apply a shear transformation

\[
X \text{ represents the coordinate system} \quad Y \equiv J_0 X \quad J_0 = \begin{pmatrix} 1-g_1 & g_2 \\ g_2 & 1+g_1 \end{pmatrix}
\]

\[
X = \begin{pmatrix} x \\ y \end{pmatrix}
\]
The effect of the transformation is to transform the elliptical contour represented by the second order moments into another elliptical contour.

An elliptical contour is represented by the quadratic form associated with the moments:

\[ q = X^T Q X \]

Introducing \( Y = J_0 X \) \( Q \) transforms to:

\[ q_s = X^T J_o^T Q J_o X \]

Thus \( Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{pmatrix} \) Transforms to:

\[ Q_s = J_o^T Q J_o \]
With

\[ J_o = \begin{pmatrix} 1 - g_1 & g_2 \\ g_2 & 1 + g_1 \end{pmatrix} \]

To first order in \( g_i \) and for a circular contour \( Q_s = J_o^T J_o \)

\[ \alpha = \frac{Q_{22} - Q_{11}}{Q_{11} + Q_{22}} = g_1 \quad \beta = \frac{2Q_{21}}{Q_{11} + Q_{22}} = g_2 \]
For a general non-circular contours

$$Q_s = J_o^T Q J_o$$

Applying a shear transformation leads to:

$$\alpha_s = \alpha + \mu g_1 \quad \beta_s = \beta + \mu g_2 \quad \text{With: } \mu \approx 1$$

$$\mu \approx 1 - \langle e \rangle^2 \quad \text{With: } \langle e \rangle \approx 0.25 \quad \Rightarrow \quad \mu \approx 1$$

Kaiser & Squires (1993)
By averaging on a distribution of randomly oriented sources

\[
\langle \alpha \rangle = \langle \frac{Q_{22} - Q_{11}}{Q_{11} + Q_{22}} \rangle = 0
\]

\[
\langle \beta \rangle = \langle \frac{2Q_{12}}{Q_{11} + Q_{22}} \rangle = 0
\]

We have:

\[
\langle \alpha_s \rangle \simeq g_1 \quad \langle \beta_s \rangle \simeq g_2
\]

In practice some complications may occur in the statistics of orientations and ellipticities
Using complex ellipticity and shear

Defining the complex ellipticity:

\[ \epsilon = \frac{Q_{11} - Q_{22} + 2\, iQ_{12}}{Q_{11} + Q_{22} + 2\sqrt{Q_{11}Q_{22} - Q_{12}^2}} \]

\[ \epsilon^* = \frac{\epsilon + g}{1 + g^*} \epsilon \approx \epsilon + g \]

\[ g = g_1 + i\, g_2 \approx \gamma_1 + i\, \gamma_2 \]

In the weak lensing regime

\[ g^* : \text{Complex conjugate} \]

On average the statistical mean of the complex ellipticity for unlensed sources should be zero

Bartelmann & Schneider (2001)
Once the shear is known and a shear map is obtained

The simplest approach is to fit a model for the potential

This model must reproduce the shear map through a least-square minimization
Making a general model free map
Estimating the convergence from the shear

\[ \gamma_1 = \frac{1}{2} (\phi_{11} - \phi_{22}) \quad ; \quad \gamma_2 = \phi_{12} \quad ; \quad \gamma = \gamma_1 + I \gamma_2 \]

With:

\[ \phi = \frac{1}{\pi} \int \kappa(\vec{u}) \log(|\vec{r} - \vec{u}|) d^2 \vec{u} \]

Then:

\[ \gamma(\vec{r}) = \frac{1}{\pi} \int \kappa(\vec{u}) \chi(\vec{r} - \vec{u}) d^2 \vec{u} \quad \text{and} \quad \chi(\vec{r}) = \frac{x^2 - y^2 - 2 I xy}{|r|^4} \]
Estimating the convergence from the shear

The integral:

\[ \gamma(\vec{r}) = \frac{1}{\pi} \int \kappa(\vec{u}) \chi(\vec{r} - \vec{u}) d^2 \vec{u} \]

Has an inversion formula (see Kaiser & Squires 1993)

\[ \kappa(\vec{r}) - \kappa_0 = \frac{1}{\pi} \int \gamma(\vec{u}) \chi^*(\vec{r} - \vec{u}) d^2 \vec{u} \]

Thus basically the convergence is obtained by convolving the shear with a kernel.
First we build a shear map

The moments: $Q_{ij}$ are estimated from the data by using background galaxies

The shear is estimated from the moments: $\epsilon = \frac{Q_{11} - Q_{22} + 2IQ_{12}}{Q_{11} + Q_{22} + 2\sqrt{Q_{11}Q_{22} - Q_{12}^2}} \approx \epsilon_0 + g$
The convergence (projected surface density) is obtained

\[ \kappa(\vec{r}) - \kappa_0 = \frac{1}{\pi} \int y(\vec{u}) \chi^*(\vec{r} - \vec{u}) d^2\vec{u} \]
There are many approaches to reconstruct the convergence from the shear.

Not necessary by using the inversion formula we just presented.

Since the shear is related to the convergence by a convolution, Fourier methods are natural.

But other means like, for instance, maximum entropy may be also used to recover the convergence.

\[
\gamma(\vec{r}) = \frac{1}{\pi} \int \kappa(\vec{u}) \chi(\vec{r} - \vec{u}) d^2 \vec{u} ; \quad \chi(\vec{r}) = \frac{x^2 - y^2 - 2I_{xy}}{|r|^4}
\]
Note: note all sources around the cluster are at the same distance

We must have spectroscopy and redshift data

1) to eliminate foreground objects (galaxies closer to us than the lens)
2) to estimate the distances background sources (galaxies behind the lens)

When no spectroscopy is available photometric redshifts are used instead to estimate

The redshifts

\[ \vec{\beta} = \vec{\theta} - \frac{D_{LS}}{D_L} \hat{\alpha} \rightarrow \vec{r}_S = \vec{r} - \frac{D_{LS}}{D_S D_L} \hat{\alpha} = \theta - \vec{\nabla} \phi \]

Thus \( \phi \) must be re-scaled as a function of \( D_S \)
Example of shear maps and cluster surface density reconstruction from the literature.

Reconstructions for 3 clusters of galaxies:

- Cl1358+62 (1998)
- MS1054-03 (2000)
- Abell 1689 (2007)
Light distribution in the cluster Cl1358+62

Shear map (equivalent elliptical distortion)

Hoekstra (1998)
Light distribution in the cluster Cl1358+62

Surface density reconstruction

Hoekstra (1998)
Light distribution in MS1054-03

Shear map (vector representation)

Hoekstra (2000)
Hoekstra (2000)
Light distribution in the cluster Abell 1689

Shear Map (vector representation)

Oguri et al. (2007) – obtained with the Subaru telescope
Surface density reconstruction
Oguri et al. (2007)

Comparison of contours
Light: red
Weak lensing: blue
Important results from weak lensing: the bullet cluster

The distribution of baryons and DM are different
Practical problems with the estimation of moments

\[ Q_{ij} = \int \Sigma (\vec{r}) x_i x_j d^2 x \rightarrow \text{is quickly dominated by noise out of the galaxy} \]

Generic problem: the integral does not converge...
Practical problems with the estimation of moments

Generic problem: the integral does not converge…
→ solution use some weight function

Second problem: in practice the data are convolved with the PSF

Solution: actual moments are the sum of the galaxy moments +PSF moments

Problem: the PSF may not have converging moments
Solution to the problem of PSF non converging moments:

We estimate the associated quadratic form in another way

For instance fit some generic quadratic function to the data

\[ F\left(a_0 x^2 + a_1 x y + a_2 y^2 \right) \]

Method: convolve the generic function with the PSF and then fit the parameters of the quadratic form

In practice we may use Gaussians (F is an exponential) or any other functions
Strong lensing in clusters of galaxies

Reconstruction for parametric potential model
Or general description by the singular perturbative method

For Abell 1689

Halkola et al. 2006
identified 107 multiples images
And 32 image systems
Strong lensing in clusters of galaxies

For Abell 1689, Halkola et al. 2006

Could reproduce all the images systems by assigning NSIE or NFW dark matter halo’s to individual galaxies
Strong lensing in clusters of galaxies

Important asset of lensing in clusters:
several sources with different Redshifts (additional constraints)

When combined with weak lensing data
The mass-sheet degeneracy may be broken

See Bradac et al. (2004)
An example of combined strong+weak lensing data

Oguri et al. (2007)
Subaru data

NFW profile
Send me your questions or demand on specific part of the course. Let me know if you will attend the course at IAP.

alard@iap.fr or christophe.alard@gmail.com

(send before March 1rst)

On March 21rst

I will make a short summary of the course
I will then pose some problems with simple solutions
We will also look at some simple numerical applications and propose some basic programming.

On March 28th

The last session of the course will be dedicated to a discussions on numerical methods and applications to real data. This will be also the time to answer The last questions.