A more rigorous derivation

- Drop weak-gravity assumption = assume geodesic motion in strongly curved spacetime when calculating source (ie quadrupole, octupole etc)

$$\bar{H}^{\mu\nu} \equiv \eta^{\mu\nu} - (-g)^{1/2} g^{\mu\nu} \qquad \partial_\beta \bar{H}^{\alpha\beta} = 0$$ harmonic gauge

$$\Rightarrow \quad \Box_{\rm flat} \bar{H}^{\alpha\beta} = -16\pi\tau^{\alpha\beta} \qquad \tau^{\alpha\beta} = (-g)T^{\alpha\beta} + (16\pi)^{-1}\Lambda^{\alpha\beta}$$

Full Einstein equations!

$$\Lambda^{\alpha\beta} = 16\pi(-g)t^{\alpha\beta}_{\rm LL} + (\bar{H}^{\alpha\mu}{}_{,\nu}\bar{H}^{\beta\nu}{}_{,\mu} - \bar{H}^{\alpha\beta}{}_{,\mu\nu}\bar{H}^{\mu\nu})$$

$$16\pi(-g)t^{\alpha\beta}_{\rm LL} \equiv g_{\lambda\mu}g^{\nu\rho}\bar{H}^{\alpha\lambda}_{,\nu}\bar{H}^{\beta\mu}_{,\rho}$$
$$+\frac{1}{2}g_{\lambda\mu}g^{\alpha\beta}\bar{H}^{\lambda\nu}_{,\rho}\bar{H}^{\rho\mu}_{,\nu} - 2g_{\mu\nu}g^{\lambda(\alpha}\bar{H}^{\beta)\nu}_{,\rho}\bar{H}^{\rho\mu}_{,\lambda}$$
$$+\frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau})\bar{H}^{\nu\tau}_{,\lambda}\bar{H}^{\rho\sigma}_{,\mu}$$

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harmonic gauge

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Full Einstein equations!

$$\Lambda^{\alpha\beta} = 16\pi(-g) t^{\alpha\beta}_{\text{LL}} + (\bar{H}^{\alpha\mu}{}_{,\nu}\, \bar{H}^{\beta\nu}{}_{,\mu} - \bar{H}^{\alpha\beta}{}_{,\mu\nu}\, \bar{H}^{\mu\nu})$$

$$16\pi(-g) t^{\alpha\beta}_{\text{LL}} \equiv g_{\lambda\mu} g^{\nu\rho} \bar{H}^{\alpha\lambda}_{,\nu} \bar{H}^{\beta\mu}_{,\rho}$$
$$+ \frac{1}{2} g_{\lambda\mu} g^{\alpha\beta} \bar{H}^{\lambda\nu}_{,\rho} \bar{H}^{\rho\mu}_{,\nu} - 2 g_{\mu\nu} g^{\lambda(\alpha} \bar{H}^{\beta)\nu}_{,\rho} \bar{H}^{\rho\mu}_{,\lambda}$$
$$+ \frac{1}{8} (2 g^{\alpha\lambda} g^{\beta\mu} - g^{\alpha\beta} g^{\lambda\mu})(2 g_{\nu\rho} g_{\sigma\tau} - g_{\rho\sigma} g_{\nu\tau}) \bar{H}^{\nu\tau}_{,\lambda} \bar{H}^{\rho\sigma}_{,\mu}$$

From gauge condition,

$$\tau^{\alpha\beta}{}_{,\beta} = 0$$

= geodesic motion in curved metric *g*

A more rigorous derivation

- Following same procedure as before, we re-obtain Green, quadrupole formula but with T replaced by τ and

$$\bar{H}^{\mu\nu} \approx \bar{h}^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2} h \eta^{\mu\nu}$$

$$g_{00} = -1 - 2\frac{\phi}{c^2} + O(1/c^4) \qquad \tau^{00} = T^{00}\left(1 + O(1/c^2)\right)$$

$$g_{0i} = O(1/c^3) \qquad \tau^{0i} = T^{0i}\left(1 + O(1/c^2)\right)$$

$$g_{ij} = \left(1 - 2\frac{\phi}{c^2}\right)\delta_{ij} + O(1/c^4) \qquad \tau^{ij} = \left(T^{ij} + \frac{1}{4\pi G}\left(\partial^i\phi\partial^j\phi - \frac{1}{2}\delta^{ij}\partial_k\phi\partial^k\phi\right)\right)\left(1 + O(1/c^2)\right)$$

- So Green formula formula gets corrected, but quadrupole formula is NOT
- The extra terms in the Green formula account for the factor 2 discrepancy with the quadrupole formula found for a circular, Keplerian binary

An example: a binary system

- Binary with total mass *M*, reduced mass *μ*, separation *R*, orbital frequency *Ω*; orbit lies in xy plane
- Consider GWs along z axis at distance r

$$h_{ij}^{\mathrm{TT}} = h \times \begin{bmatrix} \cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & -\cos 2\Omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$h = \frac{4\mu\Omega^2 R^2}{r} = \frac{4\mu M^{2/3}\Omega^{2/3}}{r}$$

$$h \simeq 10^{-21}\left(\frac{M}{2\,M_\odot}\right)^{5/3}\left(\frac{1\,\mathrm{hour}}{P}\right)^{2/3}\left(\frac{1\,\mathrm{kiloparsec}}{r}\right)$$

$$\simeq 10^{-22}\left(\frac{M}{2.8\,M_\odot}\right)^{5/3}\left(\frac{0.01\,\mathrm{sec}}{P}\right)^{2/3}\left(\frac{100\,\mathrm{Megaparsecs}}{r}\right)$$

vs $h_{\mathrm{Sun}} \sim G\,M_{\mathrm{sun}}/(R_{\mathrm{sun}}\,c^2)$ ~2 x 10^{-6}

Generalizing the quadrupole formula

- Why? Approximate because based on slow-motion, weak gravity approximations
- Drop slow-motion approximation = include mass octupole, current quadrupole and higher order terms

$$\bar{h}^{jk} = \frac{2}{r}\left[\ddot{I}^{jk} - 2n_i\ddot{S}^{ijk} + n_i\dddot{M}^{ijk}\right]_{t'=t-r},$$

current quadrupole

$$I^{jk}(t') = \int x'^j x'^k T^{00}(t',\mathbf{x}')d^3x'$$

mass quadrupole

$$S^{ijk}(t') = \int x'^j x'^k T^{0i}(t',\mathbf{x}')d^3x',$$

$$M^{ijk}(t') = \int x'^i x'^j x'^k T^{00}(t',\mathbf{x}')d^3x'.$$

mass octupole

$$\bar{h}^{jk}(t,\mathbf{x}) = \frac{2}{r}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int\left[\left(T^{00} - 2T^{0l}n_l + T^{lm}n_l n_m\right)x'^j x'^k\right]_{t'=t-|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x',$$

all multipole moments (Press 1977)

A potentially complicated waveform structure

Quadrupole (or quadrupole + octupole + higher moments) formula + geodesic motion is often decent approximation,

eg for particle around Kerr BH (”kludge” waveforms)

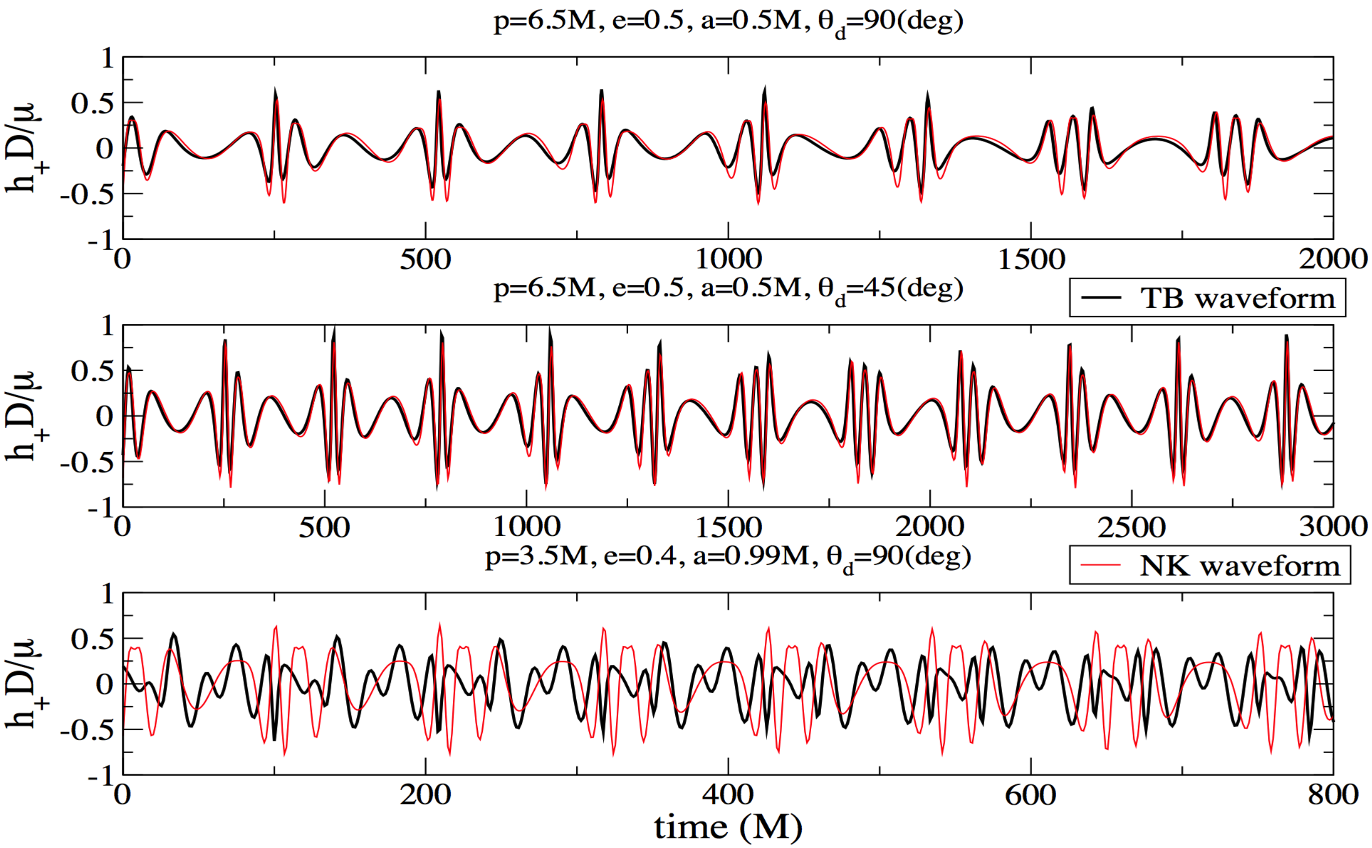
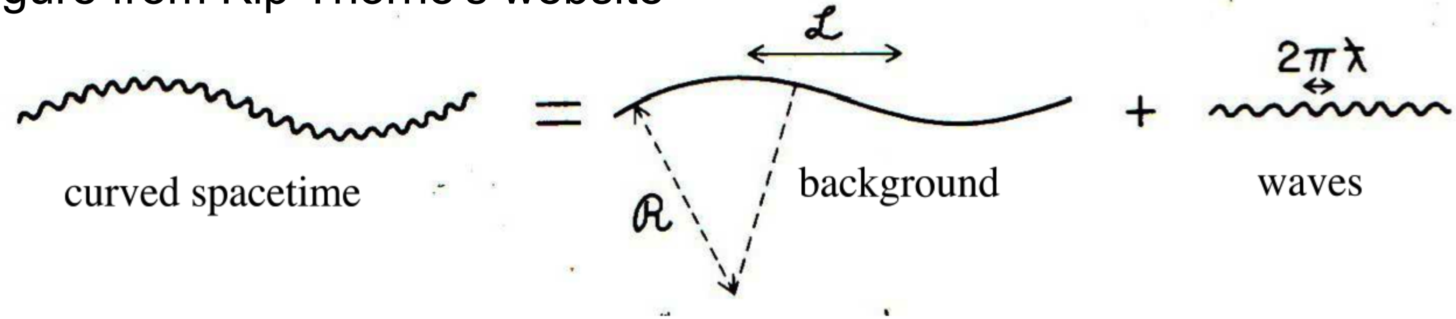


Figure from Babak et al Phys. Rev. D 75, 024005 (2007)

The stress energy tensor of GWs

Figure from Kip Thorne's website



geometric-optics regime

$$g^{\mathrm{B}}_{\alpha\beta} \quad \equiv \quad \langle g_{\alpha\beta} \rangle \qquad g_{\alpha\beta} = g^{\mathrm{B}}_{\alpha\beta} + \varepsilon h_{\alpha\beta} + \varepsilon^2 j_{\alpha\beta} + O(\varepsilon^3)$$

$$\begin{aligned} 0 &= G_{\alpha\beta} \\ &= G_{\alpha\beta}[g^{\mathrm{B}}_{cd}] + \varepsilon G^{(1)}_{\alpha\beta}[h_{cd}; g^{\mathrm{B}}_{ef}] + \varepsilon^2 G^{(1)}_{\alpha\beta}[j_{cd}; g^{\mathrm{B}}_{ef}] + \varepsilon^2 G^{(2)}_{\alpha\beta}[h_{cd}; g^{\mathrm{B}}_{ef}] \\ &\quad + O(\varepsilon^3) \, . \end{aligned}$$

$$G_{\alpha\beta}[g^{\mathrm{B}}_{cd}] = 0, \qquad G^{(1)}_{\alpha\beta}[h_{cd}; g^{\mathrm{B}}_{ef}] = 0 \, , \qquad G^{(1)}_{\alpha\beta}[j_{cd}; g^{\mathrm{B}}_{ef}] = -G^{(2)}_{\alpha\beta}[h_{cd}; g^{\mathrm{B}}_{ef}] \, .$$

The stress energy tensor of GWs

Average Einstein equations on scale >> *λ* and << *L*

$$\Delta j_{\alpha\beta} = j_{\alpha\beta} - \langle j_{\alpha\beta} \rangle$$

$$G^{(1)}_{\alpha\beta}[\langle j_{cd} \rangle; g^{\rm B}_{ef}] = -\langle G^{(2)}_{\alpha\beta}[h_{cd}; g^{\rm B}_{ef}] \rangle$$

$$G^{(1)}_{\alpha\beta}[\Delta j_{cd}] = -G^{(2)}_{\alpha\beta}[h_{cd}; g^{\rm B}_{ef}] + \langle G^{(2)}_{\alpha\beta}[h_{cd}; g^{\rm B}_{ef}] \rangle$$

$$G_{\alpha\beta}[g^{\rm B}_{cd} + \varepsilon^2 \langle j_{cd} \rangle] = 8\pi G \, T^{\rm GW,eff}_{\alpha\beta} + O(\varepsilon^3) \qquad T^{\rm GW,eff}_{\alpha\beta} = -\frac{1}{8\pi G} \langle G^{(2)}_{\alpha\beta}[h_{cd}; g^{\rm B}_{ef}] \rangle$$

Commuting derivatives and using *λ* << *L*

$$T^{\rm GW,eff}_{\alpha\beta} = \frac{1}{32\pi G} \left\langle \nabla^{\rm B}_{\alpha} h^{\rm TT}_{\rho\sigma} \, \nabla^{\rm B}_{\beta} h^{\rho\sigma}_{\rm TT} \right\rangle$$

The GW luminosity

Quadrupole formula + GW stress energy tensor

$$L_{\text{mass quadrupole}} \equiv \frac{1}{5}\frac{G}{c^5}\langle\, \dddot{\boldsymbol{I}} \,\rangle^2 = \frac{1}{5}\frac{G}{c^5}\langle \dddot{I}_{jk}\dddot{I}_{jk}\rangle^2$$

$$\dddot{I}_{jk} \sim \frac{(\text{mass of the system in motion}) \times (\text{size of the system})^2}{(\text{time scale})^3} \sim \frac{MR^2}{\tau^3} \sim \frac{Mv^2}{\tau}$$

$$L_{\text{mass quadrupole}} \sim \frac{G}{c^5}\frac{Mv^2}{\tau} \qquad G/c^5 \sim 10^{-59}$$

(in CGS units)

Conversion of any type of energy into GWs is inefficient, unless large masses and/or v ~ c

Propagation of GWs

GW propagating in z-direction

$$\Box h_{ij}^{\mathrm{TT}} = 0 \quad \Rightarrow \quad h_{ij}^{\mathrm{TT}} = h_{ij}^{\mathrm{TT}}(t-z)$$

$$\partial_z h_{zj}^{\mathrm{TT}} = 0 \quad \Rightarrow \quad h_{zj}^{\mathrm{TT}} = 0$$

$$h_{ii}^{TT} = 0 \quad \Rightarrow \quad \begin{aligned} h_{xx}^{\mathrm{TT}} &= -h_{yy}^{\mathrm{TT}} \equiv h_+(t-z)\,; \\ h_{xy}^{\mathrm{TT}} &= h_{yx}^{\mathrm{TT}} \equiv h_\times(t-z)\,. \end{aligned}$$

Propagation of GWs

$$\boldsymbol{h}^{\mathrm{TT}} = h^{+}(t-z)\boldsymbol{e}^{+} + h^{\times}(t-z)\boldsymbol{e}^{\times}$$

$$\boldsymbol{e}^{+} \equiv \boldsymbol{e}_x \otimes \boldsymbol{e}_x - \boldsymbol{e}_y \otimes \boldsymbol{e}_y \,,$$
$$\boldsymbol{e}^{\times} \equiv \boldsymbol{e}_x \otimes \boldsymbol{e}_y + \boldsymbol{e}_y \otimes \boldsymbol{e}_x \,.$$

Linear polarization

$$h^{+}(t-z) = h(t-z)\cos 2\lambda$$
$$h^{\times}(t-z) = h(t-z)\sin 2\lambda$$

Circular polarization

$$h^{\times}(t-z) = \pm i h(t-z)$$
$$h^{+}(t-z) = h(t-z)$$

Elliptic polarization=
other phase differences

Binary with masses m_1 and m_2, separation R, orbital frequency Ω, distance r;

θ = angle between orbital angular momentum and direction to observer
(θ = 0 or 180 deg: face-on; θ = 90: edge on)

$$h^{+} = \frac{2m_1m_2}{rR}(1+\cos^2\theta)\cos[2\Omega(t-r)+2\Delta\phi]\,,$$
$$h^{\times} = -\frac{4m_1m_2}{rR}\cos\theta\sin[2\Omega(t-r)+2\Delta\phi]\,,$$