Non-Singular Cosmological Models

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Motivation

- Inflation works very well.... but we haven't detected the inflaton field.
- The big-bang model suffers from the singularity problem.
- Bounce models can be viewed as an extension to inflation.
- Reformulation of some cosmological questions....
  ...as the homogeneity or the particle horizon.
- Bounces can be used as laboratories for quantum gravity theories.
- It's not difficult to violate the singularity Theorems...
  - Self-interacting scalar-Field + curvature.
  - Non-linear electrodynamics.
  - Quantum effects.

Quantization of Minisuperspace models

- Dirac quantization procedure

$$\mathcal{H}(\hat{p}^{\mu}, \hat{q}_{\mu})\Psi(q) = 0 \quad .$$

The quantities $\hat{p}^{\mu}, \hat{q}_{\mu}$ are the phase space operators related to the homogeneous degrees of freedom of the model. Usually this equation can be written as

$$-\frac{1}{2}f_{\rho\sigma}(q_{\mu})\frac{\partial\Psi(q)}{\partial q_{\rho}\partial q_{\sigma}} + U(q_{\mu})\Psi(q) = 0 \quad , \tag{1}$$

where $f_{\rho\sigma}(q_{\mu})$ is the minisuperspace DeWitt metric of the model. Writing Ψ in polar form, $\Psi = \mathcal{R}\exp(i\mathcal{S})$, and substituting it into (1), we obtain the following equations:

$$\frac{1}{2}f_{\rho\sigma}(q_{\mu})\frac{\partial\mathcal{S}}{\partial q_{\rho}}\frac{\partial\mathcal{S}}{\partial q_{\sigma}} + U(q_{\mu}) + Q(q_{\mu}) = 0 \quad ,$$

$$f_{\rho\sigma}(q_{\mu})\frac{\partial}{\partial q_{\rho}}\left(\mathcal{R}^2\frac{\partial\mathcal{S}}{\partial q_{\sigma}}\right) = 0 \quad ,$$

where ,

$$Q(q_{\mu}) \equiv -\frac{1}{2\mathcal{R}}f_{\rho\sigma}\frac{\partial^2\mathcal{R}}{\partial q_{\rho}\partial q_{\sigma}}$$

is called the quantum potential.

Quantization of Minisuperspace models

$$p^\rho = \frac{\partial \mathcal{S}}{\partial q_\rho},$$

where the momenta are related to the velocities in the usual way:

$$p^\rho = f^{\rho\sigma}\frac{1}{N}\frac{\partial q_\sigma}{\partial t}.$$

To obtain the quantum trajectories we have to solve the following system of first order differential equations, called the guidance relations:

$$\frac{\partial \mathcal{S}}{\partial q_\rho} = f^{\rho\sigma}\frac{1}{N}\dot{q}_\sigma. \tag{2}$$

Eqs.(2) are invariant under time reparametrization. Hence, even at the quantum level, different choices of $N(t)$ yield the same space-time geometry for a given non-classical solution $q_\alpha(t)$.

Classical Minisuperspace models

The total GR Hamiltonian can be expressed as

$$H_T \doteq \int dt\, d^3x \left(N\,\mathcal{H}_0 + N_i\,\mathcal{H}^i + \lambda\,P + \lambda_i\,P^i \right)$$

where, $\mathcal{H}_0 \doteq \mathcal{G}_{ijkl}\,\pi^{ij}\,\pi^{kl} - h^{1/2}\,{}^3\mathcal{R}$, and $\mathcal{H}^i \doteq -2\,\pi^{ij}_{\;\;;j}$.
If we consider a homogeneous and isotropic space-time given by a metric of the form

$$ds^2 = -N^2\,dt^2 + a^2\,(t)\;\gamma_{ij}\,dx^i\,dx^j \quad , \quad \gamma_{ij}\,dx^i\,dx^j = \frac{dr^2}{1-\mathcal{K}\,r^2} + r^2\,d\Omega^2$$

$$\boxed{H_T = N\left(-\frac{P_a^2}{24a} - 6\,a\,\mathcal{K}\right)} \quad , \quad P_a = -12\frac{a\dot{a}}{N}$$

In Schutz's formalism for perfect fluids, the main idea is to use the thermodynamics potentials to describe its four-velocity. Suppose we have a thermodynamic fluid with equation of state $p = p\,(\mu, s)$, the Lagrangian is to be taken as

$$L = -\int d^3x\,\sqrt{-g}\,(\mathcal{R} - 16\pi p) \quad .$$

Classical Minisuperspace models

Following the Dirac's procedure for degenerate theories and applying canonical transformation we can put the hamiltonian in the simple form

$$H_{mat} = N\frac{P_\phi}{a^{3w}}.$$

Where we had assumed a equation of state for the perfect fluid $p = w\rho$, with w constant. Thus the Hamiltonian of the system is

$$H = N\mathcal{H} = N\left(-\frac{P_a^2}{24a} - 6\,a\,\mathcal{K} + \sum_k \frac{P_{\phi_k}}{a^{3w_k}}\right)$$

Dust plus Radiation Classical model

The Hamiltonian of a system composed of dust plus radiation is

$$H = N\mathcal{H} = N\left(-\frac{P_a^2}{24a} - 6\,a\,\mathcal{K} + \frac{P_T}{a} + P_\phi\right)$$

The ϕ field is associated with dust and the radiation with the T field. The equation of motion are given by:

$$\begin{aligned}
\dot{\phi} &= \{\phi, H\} = N \qquad \dot{P}_\phi = 0 \\
\dot{T} &= \{T, H\} = \tfrac{N}{a} \qquad \dot{P}_T = 0 \\
\dot{a} &= \{a, H\} = -\tfrac{N}{12a} P_a \\
\delta N &= 0 \rightarrow H = 0 \Rightarrow \qquad \frac{P_a^2}{24a} = -6\mathcal{K}a + \frac{P_T}{a} + P_\phi
\end{aligned}$$

Combining theses equations we find the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = N^2\left[-\frac{\mathcal{K}}{a^2} + \frac{1}{6}\left(\frac{P_T}{a^4} + \frac{P_\phi}{a^3}\right)\right]$$

The importance of this result is the identification of the conjugate momenta with the total content of dust and radiation in the universe.

$$P_\phi = 16\pi G a^3 \rho_m \qquad P_T = 16\pi G a^4 \rho_r$$

Dust plus Radiation Quantum model

Quantization in the conformal gauge,

$$N = a \longrightarrow \dot{T} = 1 \quad .$$

The scale factor is define only in the half-line, which means that the hamiltonian is in general non-hermitian. So if one requires unitary evolution, the Hilbert sub-space must be compose of function satisfying the condition:

$$\int_{-\infty}^{\infty} d\phi \left[\frac{\partial \xi^* (a, \phi)}{\partial a} \psi (a, \phi) \right]_{a=0} = \int_{-\infty}^{\infty} d\phi \left[\frac{\partial \psi (a, \phi)}{\partial a} \xi^* (a, \phi) \right]_{a=0}$$

For practical purpose is enough to calculate the norm of the wave function and check its dependence with time. Using the co-ordinate basis the dynamical equation is written as

$$i \frac{\partial}{\partial \eta} \psi (a, \phi, \eta) = \left(-\frac{1}{2m} \frac{\partial^2}{\partial a^2} + \frac{m}{2} \mathcal{K} a^2 + ia \frac{\partial}{\partial \phi} \right) \psi (a, \phi, \eta) \qquad (3)$$

There are formal solutions for all three cases $\mathcal{K} = 0, \pm 1$[1].

[1] N. Pinto-Neto, E. Sergio Santini, and F.T. Falciano, Phys.Lett.A **344**, 131 (2005)

Eingenstates of total matter content.

These wavefunction are eingenstates of the operator $\hat{p}_\phi|\psi\rangle = p_\phi|\psi\rangle$, and since $\left[\hat{H}, \hat{p}_\phi\right] = 0$ this property is preserve by time evolution.

$$\Psi\left(a, \phi, \eta\right) = \psi\left(a, \eta\right)\, \exp\{i\, p_\phi\, \phi\}$$

$$i\frac{\partial\psi\left(a, \eta\right)}{\partial\eta} = \left(-\frac{1}{2m}\frac{\partial^2}{\partial a^2} + \frac{m}{2}\mathcal{K}a^2 - p_\phi\, a\right)\psi\left(a, \eta\right)$$

The Restriction over the Hilbert space now reads

$$\left.\frac{\partial\psi\left(a, t\right)}{\partial a}\right|_{a=0} = \alpha\ \psi\left(a, t\right)|_{a=0} \qquad \text{with } \alpha \in \Re$$

A solution $\psi(a, \eta)$ can be obtained from a initial wave function $\psi_0(a)$ using the propagator of a forced harmonic oscillator $K(2, 1) \equiv K(\eta_2, a_2; \eta_1, a_1)$.

$$K(2, 1) = \sqrt{\frac{mw}{2i\pi\sin\left(w\eta\right)}}\, e^{iS_{cl}}$$

Eingenstates of total matter content ($\mathcal{K}=0$).

Restricting to the flat case ($\mathcal{K}=0$)
If we define the initial wave function as

$$\psi_0(a_1) = \left(\frac{8\sigma}{\pi}\right)^{1/4} \exp\left\{-\sigma a_1^2\right\}$$

Then,

$$\psi(a,\eta) = \int_{-\infty}^{\infty} da_1\, K(2,1)\psi_0(a_1) = \mathcal{R}e^{i\mathcal{S}}$$

and following the guidance relations,

$$a' = -\frac{4\sigma\eta}{m^2+4\sigma^2\eta^2}\,a + \frac{m^2+2\sigma\eta^2}{m(m^2+4\sigma^2\eta^2)}\,p_\phi\eta$$

With solution,

$$\boxed{a\left(\eta\right) = C_0\sqrt{m^2+4\sigma^2\eta^2} + \frac{p_\phi}{2m}\eta^2}$$

Superpositions of total dust mass eigenstates

Still for the flat case, we assume plane wave solution for ϕ and then make a gaussian superposition to construct a square-integrabel wave function.

$$\Psi\,(a,\phi,\eta) = \psi_{p_\phi}\,(a,\eta)\;\exp\{-i\,\phi\,p_\phi\}$$

The initial condition is a even function of a

$$\psi_{p_\phi}\,(a,0) = \left(\frac{8\sigma}{\pi}\right)^{1/4} \exp\left\{-\,(\sigma + i\,q)\,a^2\right\}\;,$$

where σ and $q \in \Re$ and $\sigma > 0$. Taking the gaussian superposition,

$$\Psi\,(a,\phi,\eta) = \int dp_\phi\;\exp^{-\gamma(p_\phi - p_0)^2}\,\psi_{p_\phi}\,(a,\eta)\;\exp\left\{-i\,\phi\,p_\phi\right\} \tag{4}$$

we find,

$$\int_0^\infty da\int_{-\infty}^\infty d\phi\,\|\Psi\|^2 = \sqrt{\frac{8\pi^3}{\gamma}}\left(1 + \frac{1}{\sqrt{\pi}}\,erf\left(\frac{p_0\eta^2}{2m}\right)\right).$$

Superpositions of total dust mass eigenstates

Recalling the guidance relations, the trajectories can be computed by solving the given systems of equation

$$a' = \frac{2}{m}\left(\frac{\Im\left(A\right)}{4\nu} + \frac{m}{2\mu\eta}\left(\mu - m^2 + 2qm\eta\right)\right) a + \frac{\Im\left(B\right)}{4\nu}\phi$$

$$P_\phi = \left(\frac{\Im\left(C\right)}{2\nu}\phi + \frac{\Im\left(B\right)}{4\nu}a\right)$$

$$\phi' = a$$

Where,

$$\mu = 4\left(\sigma^2 + q^2\right)\eta^2 - 4qm\eta + m^2$$

$$\nu = \left(\gamma + \frac{\sigma\eta^4}{4\mu}\right)^2 + \frac{\eta^6}{(24m\mu)^2}\left(\mu + 3m^2 - 6qm\eta\right)^2$$

$$A = \left[\frac{m\sigma\eta^2}{\mu} + i\frac{\eta}{2\mu}\left(\mu + m^2 - 2qm\eta\right)\right]^2\left[\gamma + \frac{\sigma\eta^4}{4\mu} - i\frac{\eta^3}{24m\mu}\left(\mu + 3m^2 - 6qm\eta\right)\right]$$

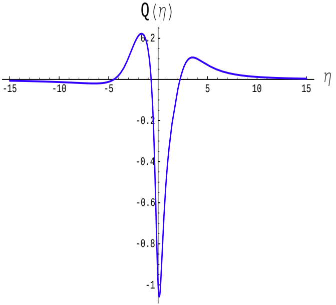
$$B = -2\,i\left[\frac{m\sigma\eta^2}{\mu} + i\frac{\eta}{2\mu}\left(\mu + m^2 - 2qm\eta\right)\right]\left[\gamma + \frac{\sigma\eta^4}{4\mu} - i\frac{\eta^3}{24m\mu}\left(\mu + 3m^2 - 6qm\eta\right)\right]$$

$$C = -\left[\gamma + \frac{\sigma\eta^4}{4\mu} - i\frac{\eta^3}{24m\mu}\left(\mu + 3m^2 - 6qm\eta\right)\right]$$

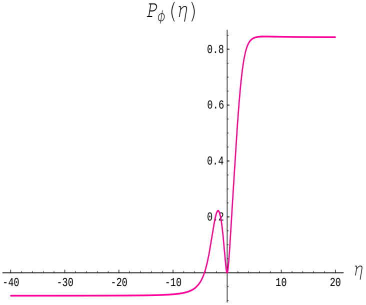
Superpositions of total dust mass eigenstates

Cannot be solve analiticaly, so we integrated numerically with the choice of $a(0) = 1$.

The Quantum Potential as a function of time



Total amount of matter content



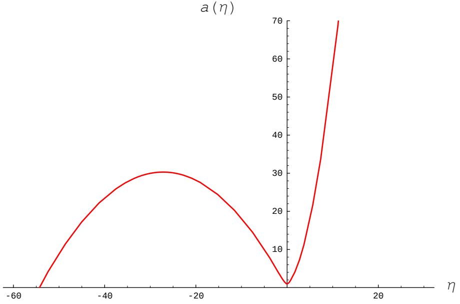
Superpositions of total dust mass eigenstates

Recalling the Friedmann equation,

$$\left(\frac{a'}{a^2}\right)^2 = \frac{1}{6}\left(\frac{P_T}{a^4} + \frac{P_\phi}{a^3}\right)$$

- the universe starts from a classical singularity with exotic matter $(\rho < 0)$
- quantum effects avoid the collapse and transform exotic matter into matter $(\rho > 0)$.
- the universe expands classical as $\eta + \eta^2$.

Evolution of the scale-factor



Free Scalar-Field Model

Matter content described by stiff matter. The total lagrangian in natural units reads

$$L = \sqrt{-g}\left[\frac{\mathcal{R}}{6l^2} - \frac{1}{2}\phi_{;\mu}\phi^{;\mu}\right] \quad ,$$

We can simplify the hamiltonian by defining $\alpha \equiv \ln(a)$, obtaining

$$H = \frac{N}{2\exp(3\alpha)}\left[-p_\alpha^2 + p_\phi^2 - \mathcal{K}\exp(4\alpha)\right] \quad , \tag{5}$$

$$p_\alpha = -\frac{e^{3\alpha}\dot{\alpha}}{N} \ , \ p_\phi = \frac{e^{3\alpha}\dot{\phi}}{N} \quad .$$

But we should keep in mind that $a_{phys} = la/\sqrt{2}$, where V is the total volume divided by a^3 of the spacelike hypersurfaces.

Classical Behavior

p_ϕ is a constant of motion which we will call $\bar{k}$. The classical solutions are, in the gauge $N = 1$ (cosmic time):

1) For $\mathcal{K} = 0$:

$$\phi = \pm\alpha + c_1 \ , \ a = e^{\alpha} = 3\bar{k}\tau^{1/3} \ , \ \phi = \frac{\ln(\tau)}{3} + c_2 \quad .$$

2) For $\mathcal{K} = 1$:

$$a = e^{\alpha} = \frac{\bar{k}}{\cosh(2\phi - c_1)} \ , \qquad \dot{\phi} = e^{-3\alpha}\bar{k} \quad .$$

3) For $\mathcal{K} = -1$:

$$a = e^{\alpha} = \frac{\bar{k}}{|\sinh(2\phi - c_1)|} \ , \qquad \dot{\phi} = e^{-3\alpha}\bar{k} \quad .$$

These solutions describe universes contracting forever to or expanding forever from a singularity. Near the singularity, all solutions behave as in the flat case. There is no inflation. Hence, in all models there is at least one singularity and no inflationary phase, as it should be for a classical stiff matter fluid.

Quantum minisuperspace model

The operator version of Eq. (5), with the factor ordering which makes it covariant through field redefinitions, reads

$$\frac{1}{2e^{3\alpha}}\left(-\frac{\partial^2\Psi}{\partial\alpha^2}+\frac{\partial^2\Psi}{\partial\phi^2}+\mathcal{K}e^{4\alpha}\Psi\right)=0 \quad , \tag{6}$$

Applying the Bohmian quantization procedure to the wave function $\Psi=\mathcal{R}e^{i\mathcal{S}}$ we find the quantum potential

$$Q(\alpha,\phi)=\frac{1}{\mathcal{R}}\left[\frac{\partial^2\mathcal{R}}{\partial\alpha^2}-\frac{\partial^2\mathcal{R}}{\partial\phi^2}\right] \quad . \tag{7}$$

and the guidance relations,

$$\frac{\partial\mathcal{S}}{\partial\alpha}=-\frac{e^{3\alpha}\dot{\alpha}}{N} \qquad , \qquad \frac{\partial\mathcal{S}}{\partial\phi}=\frac{e^{3\alpha}\dot{\phi}}{N} \quad . \tag{8}$$

There are formal solutions for all three cases $\mathcal{K}=0,\pm1$[2].

[2] FelipeT. Falciano, N. Pinto-Neto, and E. Sergio Santini, **gr-qc/07071088** accepted for publication in PRD

Quantum minisuperspace model

For $\mathcal{K} = 0$:
In this case the general solution is $\Psi(\alpha, \phi) = F(\alpha + \phi) + G(\alpha - \phi)$, where F and G are arbitrary functions. which can be written as Fourier transforms as

$$\Psi(\alpha, \phi) = \int dk U(k) e^{ik(\alpha+\phi)} + \int dk V(k) e^{ik(\alpha-\phi)} \quad , \tag{9}$$

In Ref.[3,4] were made gaussian superpositions of these solutions with the choice $U(k) = V(\pm k) = A(k)$, with $A(k)$ given by

$$A(k) = \exp\left[-\frac{(k - \sqrt{2}d)^2}{\sigma^2} \right] \quad , \tag{10}$$

with $\sigma > 0$, presenting bouncing non-singular solutions, oscillating universes and expanding singular models.

[3] R. Colistete Jr., J. C. Fabris, and N. Pinto-Neto, PRD **62**, 083507 (2000).
[4] N Pinto-Neto and E. Sergio Santini, Phys. Lett. **A 315**, 36 (2003).

Generalized gaussian superpositions

Generalizing the parameter σ^2 in (10) by a complex number:

$$A(k) = \exp\left[-\frac{(k-\sqrt{2}d)^2}{\sigma^2 + i4h}\right] \quad , \tag{11}$$

and choosing $U(k) = V(k) = A(k)$, we obtain the solution $\Psi = \mathcal{R}e^{i\,\mathcal{S}}$, with:

$$\begin{aligned}
\mathcal{R} &= \sqrt{2\pi}\ \sqrt[4]{\sigma^4 + 16h^2}\ e^{-\frac{\sigma^2}{8}(\alpha^2+\phi^2)}\sqrt{\cosh\left(\frac{\sigma^2\phi\alpha}{2}\right) + \cos[2\phi(h\alpha - d)]} \\
\mathcal{S} &= d\alpha - \frac{h}{2}(\alpha^2 + \phi^2) + \arctan\left\{\tanh\left(\frac{\sigma^2\alpha\phi}{4}\right)\tan[\phi(h\alpha - d)]\right\} \\
&\quad + \arctan\left(\sqrt{\frac{\sqrt{\sigma^4+16h^2} - \sigma^2}{\sqrt{\sigma^4+16h^2} + \sigma^2}}\right) \quad ,
\end{aligned}$$

Generalized gaussian superpositions

In the gauge $N = e^{3\alpha}$, the solution yields a planar system given by:

$$\alpha' = (h\alpha - d) - \frac{1}{4}\frac{\sigma^2\phi\sin[2\phi(h\alpha - d)] + 4h\phi\sinh\left(\frac{\sigma^2\phi\alpha}{2}\right)}{\cosh\left(\frac{\sigma^2\phi\alpha}{2}\right) + \cos[2\phi(h\alpha - d)]} =: f(\alpha,\phi), \qquad (12)$$

and

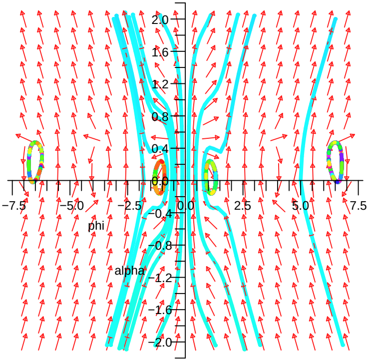
$$\phi' = -h\phi + \frac{1}{4}\frac{\sigma^2\alpha\sin[2\phi(h\alpha - d)] + 4(h\alpha - d)\sinh\left(\frac{\sigma^2\phi\alpha}{2}\right)}{\cosh\left(\frac{\sigma^2\phi\alpha}{2}\right) + \cos[2\phi(h\alpha - d)]} =: g(\alpha,\phi). \qquad (13)$$

Due to the symmetries

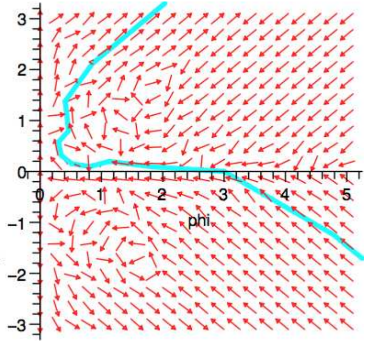
$$\begin{array}{lcl} f(\alpha,\phi;h,d) = f(-\alpha,-\phi;-h,d) & , & g(\alpha,-\phi;h,d) = g(-\alpha,-\phi;-h,d), \\ f(\alpha,\phi;h,d) = -f(-\alpha,\phi;h,-d) & , & g(\alpha,\phi;h,d) = g(-\alpha,\phi;h,-d), \end{array}$$

$h \to -h$ inversion around the origin with time reversion, while $d \to -d$ reflexion in the ϕ axis. Hence, one can make definite choices of sign for these parameters without loss of generality.

Generalized gaussian superpositions



Field plot for the direction of the geometrical tangent to the trajectories, for the values $\sigma^2 = 2, h = 1/8$, and $d = -1$.



Field plot for the direction of the geometrical tangent to the trajectories, for the values $\sigma^2 = 2, h = 0.5$, and $d = -1$.

Non-Singular Inflationary Universe

There is an interesting case when $\sigma^2 = 0$, hence $A(k) = \exp\left[i\frac{(k-\sqrt{2}d)^2}{4h}\right]$. Then the wave function reduces to

$$\Psi(u,v) = 2\sqrt{\pi|h|}\left[\exp i\left(-hu^2 + \sqrt{2}du + \frac{\pi}{4}\right) + \exp i\left(-hv^2 + \sqrt{2}dv + \frac{\pi}{4}\right)\right] \quad .$$

Its norm is given by $R = 4\sqrt{\pi|h|}\cos[\phi(h\alpha - d)]$, yielding the quantum potential

$$Q = (h\alpha - d)^2 - h^2\phi^2 \quad . \tag{14}$$

The guidance relations given by (12) and (13) now reduce to

$$\alpha' = h\alpha - d \qquad , \qquad \phi' = -h\phi \quad . \tag{15}$$

The only critical point $(\phi = 0, \alpha = \frac{d}{h})$ is a saddle point and, as it is well known, it represents an unstable equilibrium. Note that there are two regions of different signs of $\dot{\alpha}$ separated by the line $\alpha = d/h$.

Non-Singular Inflationary Universe

The analytical solutions read

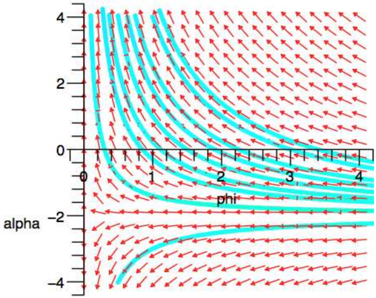
$$a = e^{\alpha} = e^{d/h}\exp(\alpha_0\, e^{ht}) \quad \text{and} \quad \phi = \phi_0\, e^{-ht} \quad ,$$

The time parameter t is related to cosmic time τ through $\tau = \int dt e^{3\alpha(t)}$
$\Rightarrow \tau - \tau_0 = \mathrm{Ei}(3\alpha_0\, e^{ht})/h$, where $\mathrm{Ei}(x)$ is the exponential-integral function.
These solutions represent ever expanding or contracting non-singular models, depending on the sign of h. For $h > 0$, the Hubble and deceleration parameters $\dot{a}/a$ and $\ddot{a}/a$ read (a dot denotes a derivative in cosmic time τ)

$$\frac{\dot{a}}{a} = \frac{\alpha_0 h e^{ht}}{a^3} \quad ,$$

$$\frac{\ddot{a}}{a} = \frac{\alpha_0 h^2}{a^6}\, e^{ht}(1 - 2\alpha_0\, e^{ht}) \quad ,$$

$$\mathcal{R} = -\frac{6\alpha_0 h^2}{a^6}\, e^{ht}(1 - \alpha_0\, e^{ht}) \quad .$$



Non-Singular Inflationary Universe

There are three important phases in this model.

- For $t << 0$ the Universe expands acceleratelly from its minimum size $a_0 = e^{d/h}$ (remember that for the physical scale factor one has $a_0^{\rm phys} = le^{d/h}/\sqrt{2V}$), which occurs in the infinity past $t \rightarrow -\infty$ when the curvature is null but increasing while scale factor grows. The scalar field is very large in that phase.
- For $t >> 0$ the Universe expands decelerately, the scale factor is immensely big, the scalar field becomes negligible and the curvature approaches zero again. The transition occurs when $ht_{\rm tran} = -\ln(2\alpha_0)$.
- Around $ht = 0$ one has

$$a \approx e^{\alpha_0 + d/h}[1 + \alpha_0 ht + (\alpha_0 h^2 + \alpha_0^2 h^2)t^2/2! + ...]. \quad (16)$$

If $\alpha_0 >> 1$ (and hence $t > t_{\rm tran}$, which means in the deceleration phase), one can write $a \approx e^{\alpha_0 + d/h}\exp(\alpha_0 ht)$. In that case, from $\tau = \int dta^3(t)$, one obtains that $a \propto (\tau - \tau_0)^{1/3}$ and $\phi' \propto 1/\tau \propto 1/a^3$, as in the classical regime.

Perturbations in a Quantum minisuperspace model

The model is described by a perfect fluid with equation of state $p = w\,\epsilon$. The Hamiltonian can be written as [5] [6]

$$H = N\left[H_0^{(0)} + H_0^{(2)}\right] + \Lambda_N P_N + \int d^3x \phi\, \pi_\psi + \int d^3x \Lambda_\phi \pi_\phi$$

$$H_0^{(0)} \equiv -\frac{l_{Pl}^2 P_a^2}{4aV} + \frac{P_T}{a^{3w}}$$

$$H_0^{(2)} \equiv \frac{1}{2a^3} + \int d^3x \pi^2 + \frac{aw}{2}\int d^3x v^{,i} v_{,i}$$

The Bardeen potential is related to v by $\Phi^{,i}{}_{,i} = -\frac{3\sqrt{(w+1)\epsilon_0}}{2\sqrt{w}}\, l_{Pl}^2 a \left(\frac{v}{a}\right)'$.

Dirac quantization of the wave function
$\Psi\left[N, a, \phi(x^i), \psi(x^i), v(x^i), T\right] \implies \Psi\left[a, v(x^i), T\right]$

$$\frac{\partial}{\partial N}\Psi = \frac{\delta}{\delta\phi}\Psi = \frac{\delta}{\delta\psi}\Psi = \left(H_0^{(0)} + H_0^{(2)}\right)\Psi = 0$$

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[5] P. Peter, E. Pinho, and N. Pinto-Neto, JCAP **07**, 014.

[6] E. J. C. Pinho, and N. Pinto-Neto, hep-th/0610192.

Perturbations with no back-reaction

With the ansatz $\Psi\left[a, v, T\right] = \Psi_{(0)}\left(a, T\right)\Psi_{(2)}\left(a, T, v\right]$, the system decouples in two equations

$$i\frac{\partial\Psi_{(0)}(a,T)}{\partial T} = \frac{1}{4}\frac{\partial^2\Psi_{(0)}(a,T)}{\partial\chi^2} \quad \text{, where we defined } \chi = \frac{2}{3}(1-w)^{-1}a^{3(1-w)/2}$$

$$i\frac{\partial}{\partial T}\Psi_{(2)}(v,T) = \left(-\frac{a^{(3w-1)/2}}{2} + \int d^3x\frac{\delta^2}{\delta v^2} + \frac{wa^{(3w+1)}}{2}\int d^3x v^{,i}v_{,i}\right)\Psi_{(2)}(v,T)$$

Initial condition:

$$\Psi_{(0)}(\chi) = \left(\frac{8}{T_0\pi}\right)^{1/4} e^{\left(-\frac{\chi^2}{T_0}\right)} \implies a(T) = a_0\left[1+\left(\frac{T}{T_0}\right)^2\right]^{1/3(1-w)}$$

Using this solution to perform a canonical transformation and changing to the Heisenberg picture,

$$v'' - w\, v^{,i}{}_{,i} - \frac{a''}{a}v = 0 \qquad \text{where a } ' \text{ means derivative with respect to conformal time.}$$

Far from the bounce the normal modes should satisfy the equation

$$v'' + \left[w\,k^2 + \frac{2(3w-1)}{(1+3w)^2\eta^2}\right]v = 0 \tag{17}$$

Perturbations with no back-reaction

Taking the formal expansion

$$\frac{v}{a} \approx A_1 \left[1 - wk^2 \int \frac{d\bar{\eta}}{a^2} \int^{\bar{\eta}} a^2 \, d\bar{\bar{\eta}}\right] + A_2 \left[\int \frac{d\bar{\eta}}{a^2} - wk^2 \int \frac{d\bar{\eta}}{a^2} \int^{\bar{\eta}} a^2 \, d\bar{\bar{\eta}} \int^{\bar{\bar{\eta}}} \frac{d\bar{\bar{\bar{\eta}}}}{a^2}\right] + \mathcal{O}(k^4)$$

The leading order in the far past can be written as

$$\frac{v}{a} \sim \begin{cases} A_1 - A_2 \, T_0 \, a_0^{3(w-1)} \, \frac{T_0}{T} & \text{in the far past} \\ \left(A_1 + \pi \, a_0^{3(w-1)} \, T_0 \, A_2\right) + \frac{T_0}{T} \, a_0^{3(w-1)} \, T_0 \, A_2 & \text{in the far future} \end{cases}$$

The k dependence can be derived by matching solution of eq.(17). The power spectrum $\mathcal{P}_\Phi \sim k^3 |\Phi|^2$ goes as

$$\mathcal{P}_\Phi \sim k^{n_s - 1} \qquad \text{with} \quad n_s = 1 + \frac{12w}{1+3w}$$

For the tensor perturbations [7]

$$\mathcal{P}_h \sim k^{n_T} \qquad \text{with} \quad n_T = \frac{12w}{1+3w}$$

[7] P. Peter, E. Pinho, and N. Pinto-Neto, Phys. Rev. D **73**, 104017 (2006).

Perturbations with no back-reaction

The curvature scale at the bounce

$$L_0 \equiv T_0\, a_0^3 \;\propto\; \frac{1}{\sqrt{R_0}}\ ,\ \text{where } R_0 \text{ is the scalar curvature.}$$

Constraining the amplitude of scalar perturbations $A_s^2 = 2.08 \times 10^{-10}$, and the spectral index $n_s \lesssim 1.01$ we find that $L_0 \gtrsim 1500 l_{Pl}$. There is another constraint that comes from the total mass of the universe today, $P_T = -\frac{\partial S}{\partial T} = 10^{60}\, M_{pl}$, evaluated for $T/T_0 \gg 1$ yields

$$\frac{4a_0^3}{9\,T_0^2} > 10^{60} \quad \Longrightarrow \quad a_0 > 10^{22} \ \text{very unprobable!!!}$$

Remedy this problem by a dislocated wave function.[8]

$$\Psi_0 = \left(\frac{8}{\pi\, T_0}\right)^{1/4} \left[e^{-\frac{(\chi-q)^2}{T_0}} + e^{-\frac{(\chi+q)^2}{T_0}} \right] \quad .$$

[8] Work in progress P. Peter, N. Pinto-Neto, and F. T. Falciano

Summary

1. Quantum effects can avoid the singularity.
2. There is the possibility of matter/ exotic matter creation and annihilation.
3. A free scalar field can describe a non-singular inflationary model.
4. Perturbation Theory is well establish and can reproduce a scale-invariant spectrum.