Perturbation theory in Bianchi I space-times

Predictions from an anisotropic inflationary era

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Outline

Motivations

- A sufficient period of accelerated expansion solves the problem of **flatness** and **isotropy** in the hot big-bang.
- Inflation also explains the nearly **scale-invariant spectrum** but assuming perturbations around a homogenous, isotropic and flat, that is Friedmann-Lemaître (FL) space-time.
- It would be more satisfactory to get all these predictions in a consistent way.

Publications

- We have studied perturbations around a homogenous flat but **anisotropic** space-time: Bianchi I
- T.S. Pereira, C. Pitrou, J.-P. Uzan JCAP **09** 006 (2007) [arXiv:0707.0736].
- C. Pitrou, T.S. Pereira, J.-P. Uzan [arXiv:0801.3596].

Metric

- $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \sum_{i=1}^{3} X_i^2(t)\,\left(\mathrm{d}x^i\right)^2$.
- $\mathrm{d}s^2 = S^2\left[-\mathrm{d}\eta^2 + \gamma_{ij}(\eta)\mathrm{d}x^i\mathrm{d}x^j\right]$
- $S \equiv [X_1(t)X_2(t)X_3(t)]^{1/3}, \quad S\mathrm{d}\eta = \mathrm{d}t$

Matter content

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \left(\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi + V\right)g_{\mu\nu}$$

Anisotropy: shear

- $\sigma_{ij} \equiv \frac{1}{2}\gamma'_{ij}$
- $\gamma \equiv \gamma^{ij}\sigma_{ij} = 0$
- $\sigma^2 \equiv \sigma_{ij}\sigma^{ij}$
- $x^2 \equiv \sigma^2/(6\mathcal{H}^2)$

Friedmann equations

- $\mathcal{H}^2 = \frac{\kappa}{3}\left[\frac{1}{2}\varphi'^2 + V(\varphi)S^2\right] + \frac{\sigma^2}{6}$,
- $\mathcal{H}' = -\frac{\kappa}{3}[\varphi'^2 - V(\varphi)S^2] - \frac{\sigma^2}{3}$
- $(\sigma^i_j)' = -2\mathcal{H}\sigma^i_j$
- $\varphi'' + 2\mathcal{H}\varphi' + S^2 V_\varphi = 0$ (Klein-Gordon)
- $\kappa = 8\pi G, \quad \mathcal{H} = S'/S$

Main features

- σ is like a massless scalar field.
- $\sigma \sim 1/S^2$
- Klein-Gordon is the same!
- Positive energy condition is $x < 1$.

General solutions

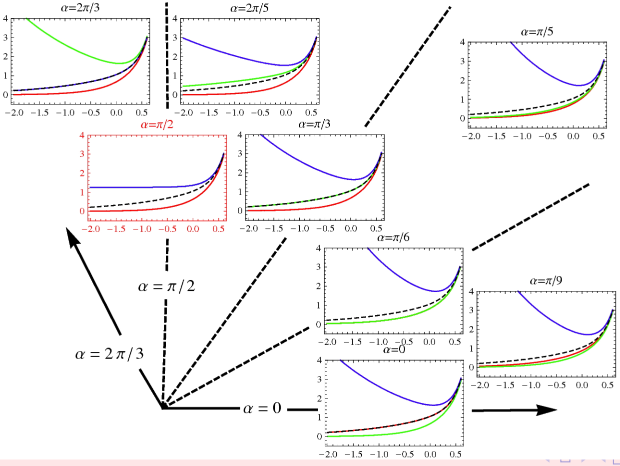
Adapted coordinates

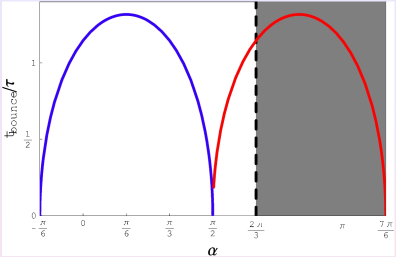
- In a basis where σ_{ij} is diagonal
  $\gamma_{ij} = \exp\left[2\beta_i(t)\right]\delta_{ij} = a_i^2\delta_{ij}$
- Directional scale factors and Hubble parameters
  $X_i = S a_i, \quad h_i = \frac{X_i'}{X_i} = \mathcal{H} + \beta_i'$

Pure Cosmological constant: single parameter α

- $S(t) = S_* \left[\sinh\left(t/\tau_*\right)\right]^{1/3}, \qquad \tau_*^{-1} = \sqrt{3\kappa V}$
- $X_i = S_* \left[\sinh\left(\frac{t}{\tau_*}\right)\right]^{1/3} \left[\tanh\left(\frac{t}{2\tau_*}\right)\right]^{\frac{2}{3}\sin\alpha_i}$
- $\alpha_i = \alpha + i\frac{2\pi}{3}$

Directional scale factors in function of α





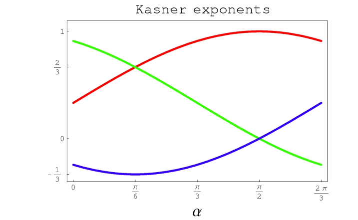
Particular cases with two scale factors being equal

- $\alpha = \pi/2$: Bouncing time at the singularity
- $\alpha = \pi/6$

Kasner exponents

$$\mathrm{d}s^2_{\mathrm{Kasner}} = -\mathrm{d}t^2 + S_*^2 \sum_{i=1}^{3} \left(\frac{t}{2\tau_*}\right)^{2p_i} (\mathrm{d}x^i)^2$$

$$p_i(\alpha) = \frac{2}{3}\sin\alpha_i + \frac{1}{3}$$

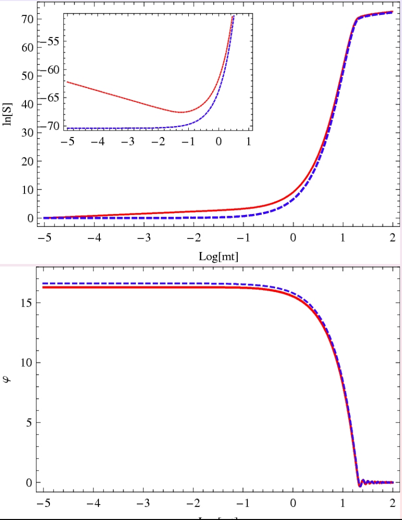


Invariants of the metric

- $R = \frac{4}{3\tau_*^2}$
- $R_{\mu\nu}R^{\mu\nu} = \frac{4}{9\tau_*^2}$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{1}{27\tau_*^2}\left\{8 + \frac{4}{\cosh^4\left[t/(2\tau_*)\right]}\right. \\ \left. + \frac{32\cosh\left(t/\tau_*\right)}{\sinh^4\left(t/\tau_*\right)}\left[3\cos(\alpha)^2\sin(\alpha) - \sin(\alpha)^3 + 1\right]\right\}$$

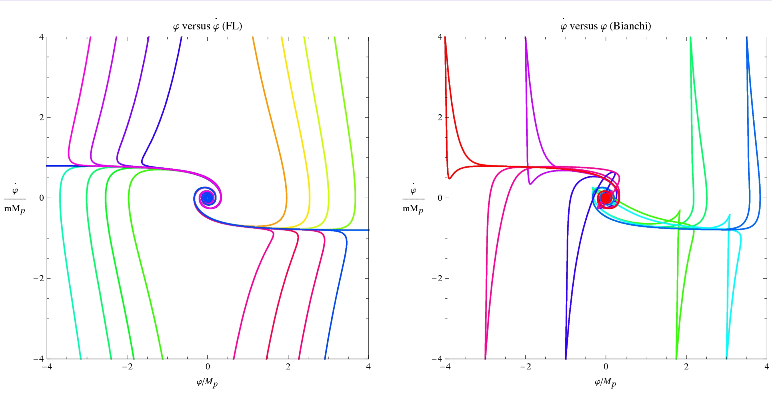
Chaotic inflation: $V = \frac{1}{2}m^2\varphi^2$

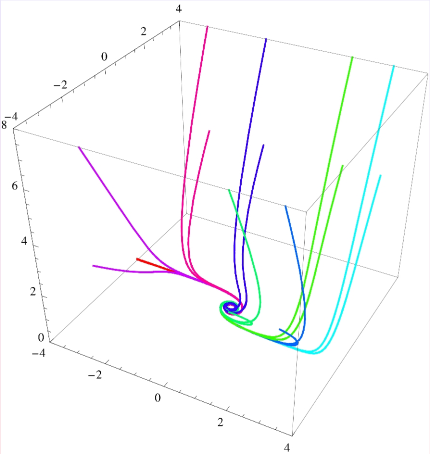


Bianchi I vs FL

- Scale factor
- Scalar field
- Dynamics is only slightly changed: lasts slightly longer

Reaching the attractor





Attraction towards the FL attractor

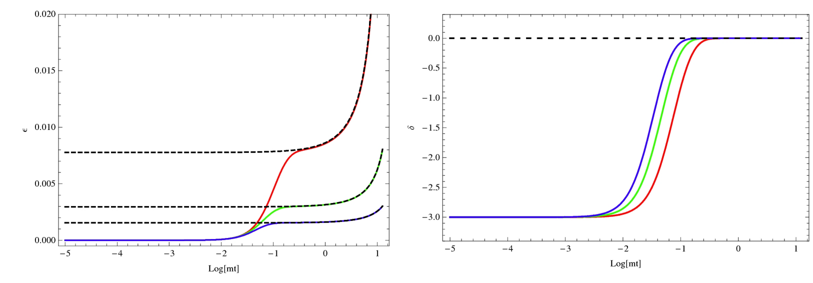
On the attractor

Slow-roll parameters

- $\epsilon \equiv 3\frac{\varphi'^2}{\varphi'^2+2S^2V}$
- $\delta \equiv 1 - \frac{\varphi''}{\mathcal{H}\varphi'} = -\frac{\ddot{\varphi}}{H\dot{\varphi}}$

Dynamics of ϵ and δ

- $\epsilon' = 2\mathcal{H}\epsilon(\epsilon - \delta)$
- $\delta' = \mathcal{H}(3-\delta)\left[\frac{\epsilon^2-3\delta}{3-\epsilon} - x^2(3-\epsilon)\right]$



Dynamics on the attractor near the singularity

- $\epsilon \longrightarrow 0 \qquad \delta \longrightarrow -3\,,$
- $\delta' = \mathcal{O}(\epsilon, \delta), \qquad \epsilon' = \mathcal{O}(\epsilon, \delta)$
- $H \simeq \frac{1}{3t}$
- $x \simeq 1$

Scalar Vector Tensor (SVT) decomposition

- $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g^{(S)}_{\mu\nu} + \delta g^{(V)}_{\mu\nu} + \delta g^{(T)}_{\mu\nu}$
- $\varphi = \bar{\varphi} + \delta\varphi$

Perturbation variables

- $\mathrm{d}s^2 = S^2\left[-(1+2A)\,\mathrm{d}\eta^2 + 2B_i\mathrm{d}x^i\mathrm{d}\eta + \left(\gamma_{ij} + h_{ij}\right)\mathrm{d}x^i\mathrm{d}x^j\right]$
- $B_i = \partial_i B + \bar{B}_i$
- $h_{ij} = 2C\left(\gamma_{ij} + \frac{\sigma_{ij}}{\mathcal{H}}\right) + 2\partial_i\partial_j E + 2\partial_{(i}E_{j)} + 2E_{ij}$

Mode decomposition

$$f\left(x^j,\eta\right) = \int \frac{\mathrm{d}^3 k_i}{(2\pi)^{3/2}} \hat{f}\left(k_i,\eta\right) \mathrm{e}^{\mathrm{i}k_i x^i}$$

- $k_i' = 0$ but $k^{i'} = (\gamma^{ij}k_j)' = -2\sigma^{ij}k_j \neq 0$
- $k^2 = k_i k^i$ is time dependent $k'/k = \sigma^{ij}k_i k_j$

Polarisation basis

Base $\{e^1, e^2\}$ orthonormal to $\hat{k}^i = k^i/k$

- $e_i^a k_j \gamma^{ij} = 0$
- $e_i^a e_j^b \gamma^{ij} = \delta^{ab}$

Projector orthogonal to k^i is $P_{ij} \equiv e_i^1 e_j^1 + e_i^2 e_j^2$.

SVT Decomposition in Fourier space

Vectors

$V_i = \sum_a V_a e_i^a$

Tensors

$T_{ij} = \sum_{\lambda=+,\times} T_\lambda \epsilon_{ij}^\lambda$

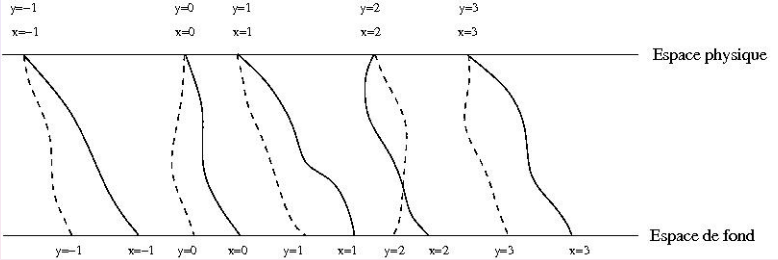
$\varepsilon_{ij}^\lambda = \frac{e_i^1 e_j^1 - e_i^2 e_j^2}{\sqrt{2}} \delta_+^\lambda + \frac{e_i^1 e_j^2 + e_i^2 e_j^1}{\sqrt{2}} \delta_\times^\lambda$

$\varepsilon_{ij}^\lambda \gamma^{ij} = 0$ (traceless) $\varepsilon_{ij}^\lambda k^i = 0(\textit{transverse})$

Decomposition of the shear

$$\sigma_{ij} = \frac{3}{2}\left(\hat{k}_i \hat{k}_j - \frac{1}{3}\gamma_{ij}\right)\sigma_{\parallel} + 2\sum_{a=1,2} \sigma_{\mathrm{V}a}\, \hat{k}_{(i} e_{j)}^a + \sum_{\lambda=+,\times} \sigma_{\mathrm{T}\lambda}\, \varepsilon_{ij}^\lambda$$

Gauge transformation



A vector ξ^μ on the background space-time generates the gauge transformation.

At first order any tensor transforms as

$$\delta \mathbf{T}_{\mu\nu} \rightarrow \delta \mathbf{T}_{\mu\nu} + \mathcal{L}_\xi \bar{\mathbf{T}}_{\mu\nu}, \qquad \xi^\mu = (T, \partial^i L + L^i)$$

Scalars

- $A \rightarrow A + T' + \mathcal{H}T$
- $B \rightarrow B - T + \frac{(k^2 L)'}{k^2}$
- $C \rightarrow C + \mathcal{H}T$
- $E \rightarrow E + L$

Vectors

- $\bar{B}_i \rightarrow \bar{B}_i + \gamma_{ij}(L^j)' - 2\mathrm{i}k^j \sigma_{lj} P^l{}_i L$
- $E_i \rightarrow E_i + L_i$

Tensors

$E_{ij} \rightarrow E_{ij}$

Gauge invariant variables

Scalars

- $\Phi \equiv A + \frac{1}{S}\left\{S\left[B - \frac{\left(k^2 E\right)'}{k^2}\right]\right\}'$
- $\Psi \equiv -C - \mathcal{H}\left[B - \frac{\left(k^2 E\right)'}{k^2}\right]$
- $v = a\left(\delta\varphi - \frac{C}{\mathcal{H}}\bar{\varphi}'\right)$

Vectors

$\Phi_i \equiv \bar{B}_i - \gamma_{ij}\left(E^j\right)' + 2\mathrm{i}k^j\sigma_{lj}P^l{}_i E$

Tensors

E_{ij} or $\mu_{ij} = aE_{ij}$

Degrees of freedom

10 (metric) + 1 (φ) - 4 (Gauge fixing) - 4 (Constraints)= 3

- Scalars: $4 + 1 - 2 - 2 = 1$
- Vectors: $4 + 0 - 2 - 2 = 0$
- Tensors: $2 + 0 - 0 - 0 = 2$

Canonical degrees of freedom

- v
- μ_+
- $\mu_\times$

They generalize the Mukhanov-Sasaki variables.

ADM Formalism

We start form the Hilbert-Einstein action and split it in space and time with the ADM formalism:

$$S = \frac{1}{2}\int dt d^3x\sqrt{-g}\left[NR^{(3)} + N\left(K_{ij}K^{ij} - K^2\right) - N\left(g^{ij}\partial_i\varphi\partial_j\varphi + 2V(\varphi)\right) + N^{-1}\left(\dot{\varphi} - N^i\partial_i\varphi\right)^2\right]$$

where

$$K_{ij} \equiv \frac{N^{-1}}{2}\left(\dot{g}_{ij} - 2\nabla_{(i}N_{j)}\right), \quad K = K_i^i$$

Second order action

Einstein-Hilbert Action at second order

$$S_2 = \frac{1}{2}\int d\eta d^3k \left(|U'|^2 + U^T\Omega^2 U^*\right)$$

$$U \equiv (v, \mu_+, \mu_\times)$$

$$\Omega = \begin{pmatrix} -\omega_v^2 & \aleph_+ & \aleph_\times \\ \aleph_+ & -\omega_+^2 & \beth \\ \aleph_\times & \beth & -\omega_\times^2 \end{pmatrix}$$

ω^2

- $\omega_v^2(\eta, k_i) \equiv k^2 - \left[\frac{S''}{S} - S^2 V_{,\varphi\varphi} + \frac{1}{S^2}\left(\frac{2S^2\kappa\varphi'^2}{2\mathcal{H}-\sigma_{\parallel}}\right)'\right]$
- $\omega_+^2(\eta, k_i) \equiv k^2 - \left[\frac{S''}{S} + 2\sigma_{\mathrm{T}\times}^2 + \frac{1}{S^2}\left(S^2\sigma_{\parallel}\right)' + \frac{1}{S^2}\left(\frac{2S^2\sigma_{\mathrm{T}+}^2}{2\mathcal{H}-\sigma_{\parallel}}\right)'\right]$
- $\omega_\times^2(\eta, k_i) \equiv k^2 - \left[\frac{S''}{S} + 2\sigma_{\mathrm{T}+}^2 + \frac{1}{S^2}\left(S^2\sigma_{\parallel}\right)' + \frac{1}{S^2}\left(\frac{2S^2\sigma_{\mathrm{T}\times}^2}{2\mathcal{H}-\sigma_{\parallel}}\right)'\right]$

Coupling terms

- $\aleph_{+}(\eta, k_i) \equiv \frac{1}{S^2}\sqrt{\kappa}\left(\frac{2S^2\varphi'\sigma_{\mathrm{T}+}}{2\mathcal{H}-\sigma_{\|}}\right)'$
- $\aleph_{\times}(\eta, k_i) \equiv \frac{1}{S^2}\sqrt{\kappa}\left(\frac{2S^2\varphi'\sigma_{\mathrm{T}\times}}{2\mathcal{H}-\sigma_{\|}}\right)'$
- $\daleth(\eta, k_i) \equiv \frac{1}{S^2}\left(\frac{2^2\sigma_{\mathrm{T}\times}\sigma_{\mathrm{T}+}}{2\mathcal{H}-\sigma_{\|}}\right)' - 2\sigma_{\mathrm{T}\times}\sigma_{\mathrm{T}+}$

Equations

- $v'' + \omega_v^2(k_i, \eta) v = \aleph_+(k_i, \eta)\mu_+ + \aleph_\times(k_i, \eta)\mu_\times$
- $\mu_+'' + \omega_+^2(k_i, \eta)\mu_+ = \aleph_+(k_i, \eta) v + \daleth(k_i, \eta)\mu_\times$
- $\mu_\times'' + \omega_\times^2(k_i, \eta)\mu_\times = \aleph_\times(k_i, \eta) v + \daleth(k_i, \eta)\mu_+$

Main features

- Directional dependence in k^2
- Coupling between degrees of freedom
- Tensor polarization not on the same footing
- Vector perturbations do not vanish (because of constraints)

Standard procedure

Canonical variables are promoted to operators (Heisenberg picture)

- $\hat{v}(\mathbf{x},\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \left[v_k(\eta) e^{i\mathbf{k}.\mathbf{x}} \hat{a}_{\mathbf{k}} + v_k^*(\eta) e^{-i\mathbf{k}.\mathbf{x}} \hat{a}_{\mathbf{k}}^\dagger \right]$
- $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta^{(3)}(\mathbf{k}-\mathbf{k}')$
- $v_k'' + \omega_v^2(k,\eta) v_k = 0$

Bunch-Davies vacuum choice

At small scale we want to recover standard oscillator quantization and thus we ask
$v_k \longrightarrow \frac{1}{\sqrt{2k}} e^{-ik\eta}$
This sets the initial conditions for v_k.

FL case with slow-roll approximation

$\omega_v^2 = k^2 - \frac{2+6\epsilon-3\delta}{\eta^2}, \qquad \omega_+^2 = \omega_\times^2 = k^2 - \frac{2+3\epsilon}{\eta^2}$

We have an exact solution for v_k.

WKB approximation

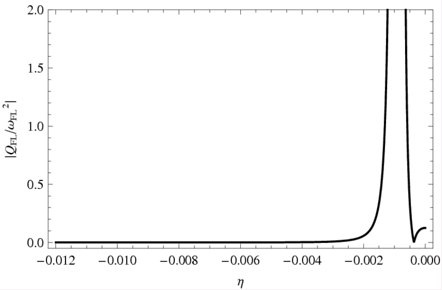
If no exact solution v_k is known. We can build an approximate solution

- $v_k^{\rm WKB}(\eta) = \frac{1}{\sqrt{2\omega_v}} e^{\pm i \int \omega_v {\rm d}\eta}$
- $v_k^{\rm WKB''} + \left(\omega_v^2 - Q_{\rm WKB}\right) v_k^{\rm WKB} = 0$
- $Q_{\rm WKB} = \frac{3}{4}\left(\frac{\omega_v'}{\omega_v}\right)^2 - \frac{1}{2}\frac{\omega_v''}{\omega_v}$

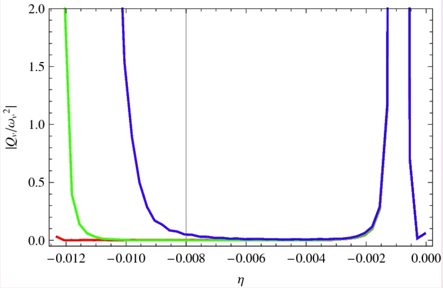
Valid if $\left|Q_{\rm WKB}/\omega_v^2\right| \ll 1$. Then we have a well defined vacuum.

WKB condition asymptotically

FL



Bianchi I

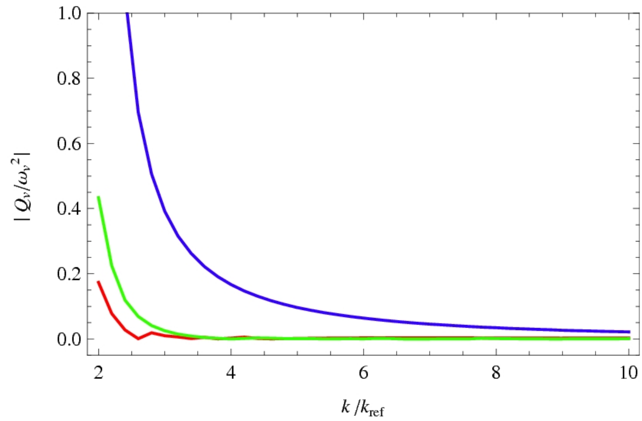


Procedure:

Set the initial condition when $\left|Q_{\rm WKB}/\omega_v^2\right| \ll 1$ is minimum.

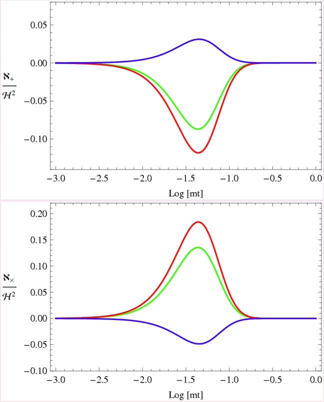
Validity of WKB when setting initial conditions

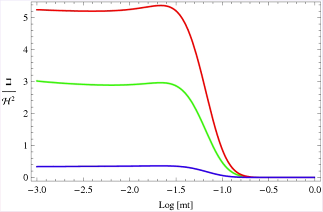
Q_{WKB}/ω_v^2 when CI set



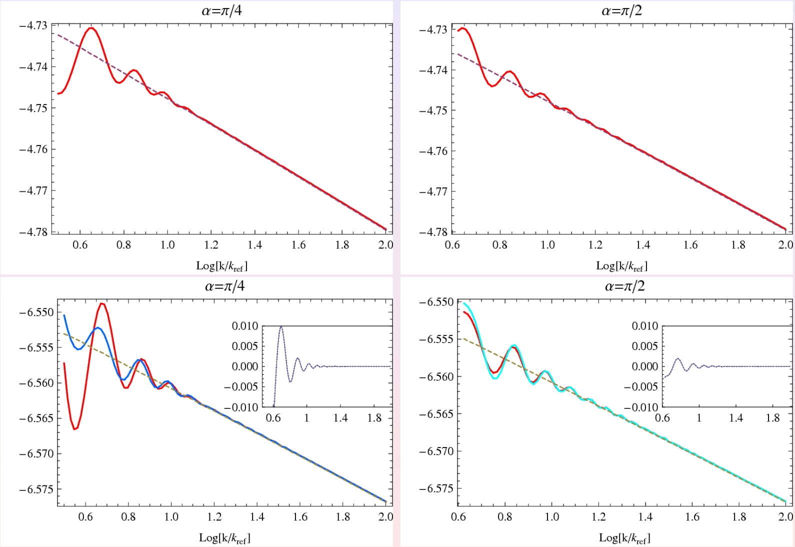
For k too small, this procedure is not valid.

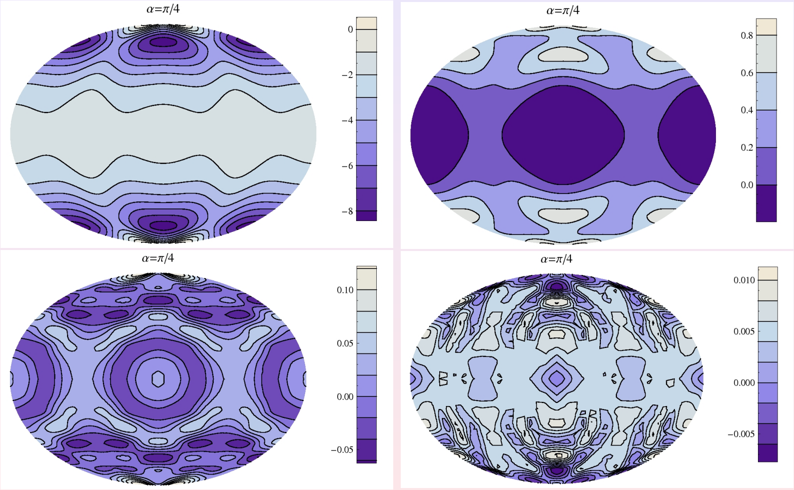
Statistical independence





The coupling terms vanish essentially when the IC are set. Our procedure induces no coupling between the 3 degrees of freedom.


α=π/4
α=π/2
Log[k/k_ref]
α=π/4
α=π/2
Log[k/k_ref]


α=π/4
0
−2
−4
−6
−8
α=π/4
0.8
0.6
0.4
0.2
0.0
α=π/4
0.10
0.05
0.00
−0.05
α=π/4
0.010
0.005
0.000
−0.005

Conclusion

- Perturbation formalism can be generalized. We hope to get it for any Bianchi. We will use the tetrad formalism for that.
- Canonical degrees of freedom identified. Direction dependence and couplings.
- Quantization procedure: no uniform convergence from Bianchi I to FL.
- Bunch Davies vacuum cannot be generalized easily: modes have always been super-Hubble. We cannot handle this in this theoretical frame. String theory?
- However, the spectrum converges towards the nearly scale-invariant isoptropic spectrum. For long inflation, no imprint of early anisotropic era, as expected.

THE END ...

THANK YOU