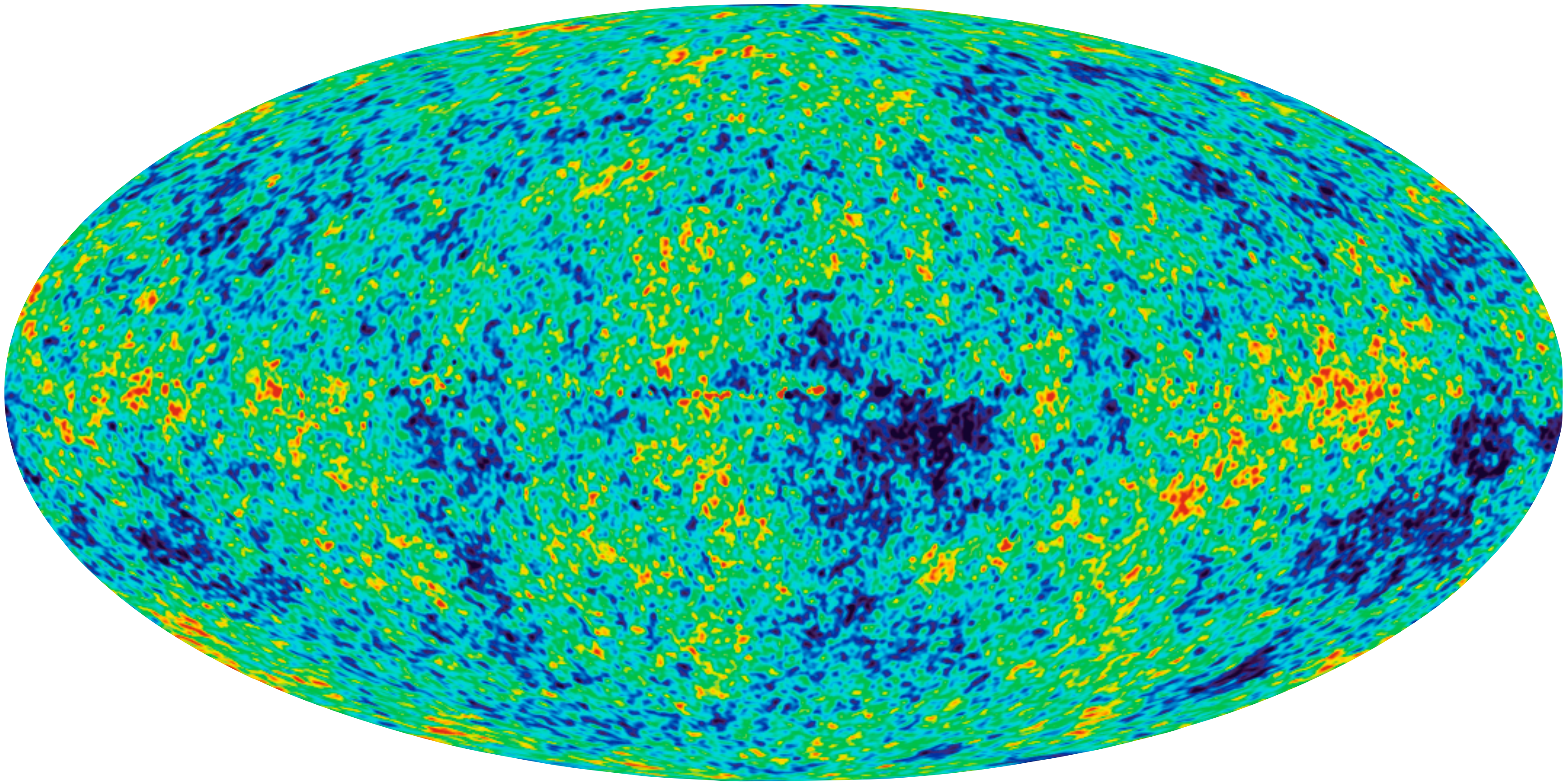
CMB bispectrum on large angular scales

Filippo Vernizzi - IPhT, CEA Saclay

To appear (hopefully) tomorrow!
with Lotfi Boubekeur, Paolo Creminelli, Guido D'Amico and Jorge Noreña
(see also JCAP 0808: 028, 2008)

IAP, Paris, 4 Juin 2009

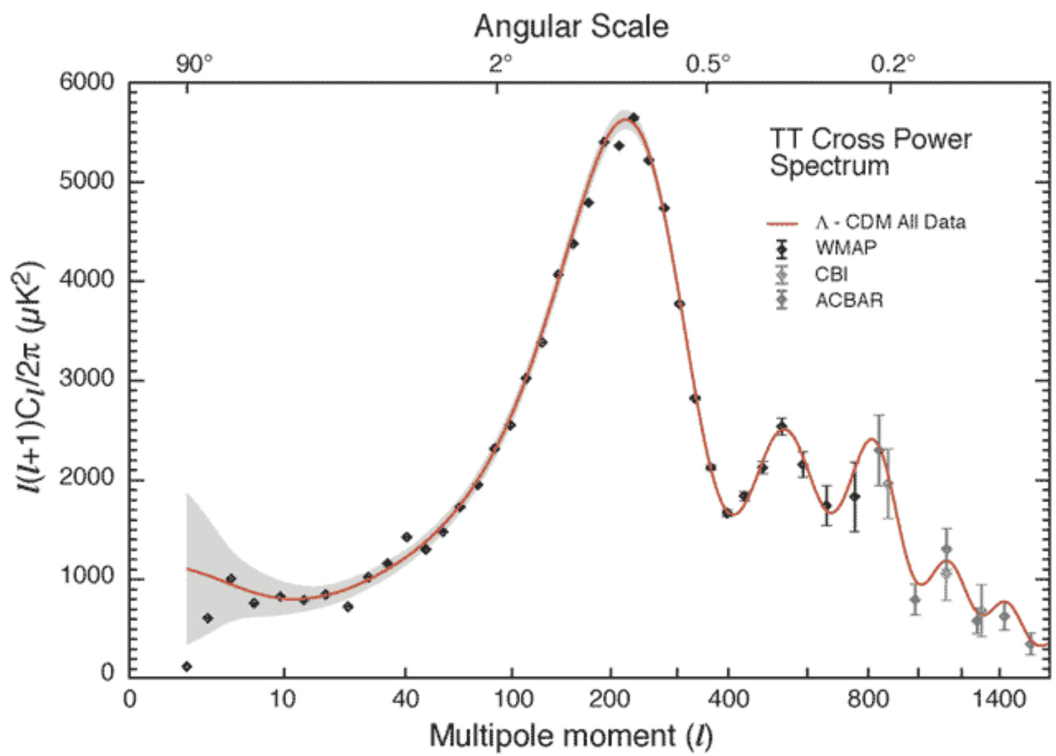
WMAP map of the sky



CMB spectrum

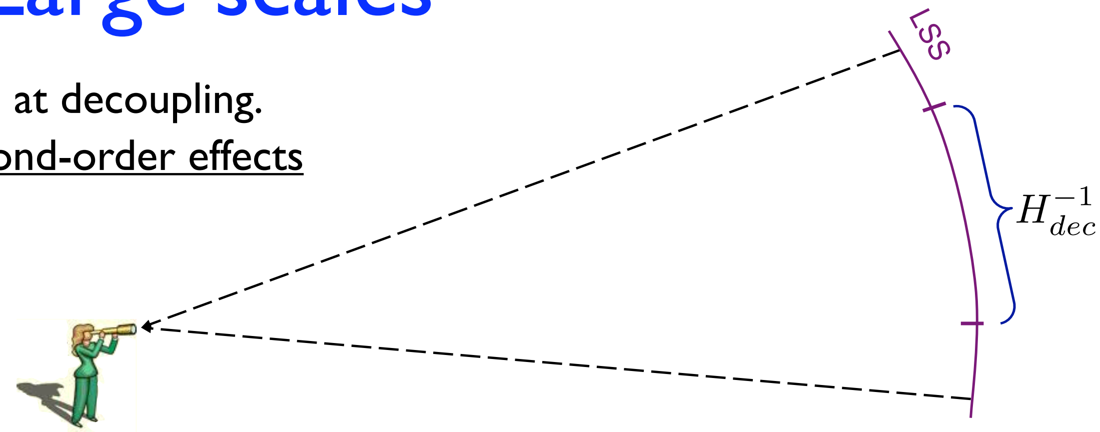
- If data are Gaussian, they are fully characterized by the power spectrum (FT of 2-pf):

$$\langle \frac{\Delta T}{T}(\hat{n}) \frac{\Delta T}{T}(\hat{n}') \rangle \qquad \Rightarrow \qquad C_l \equiv \sum_m \langle a_{lm} a_{l'm'} \rangle \delta_{ll'} \delta_{mm'}$$

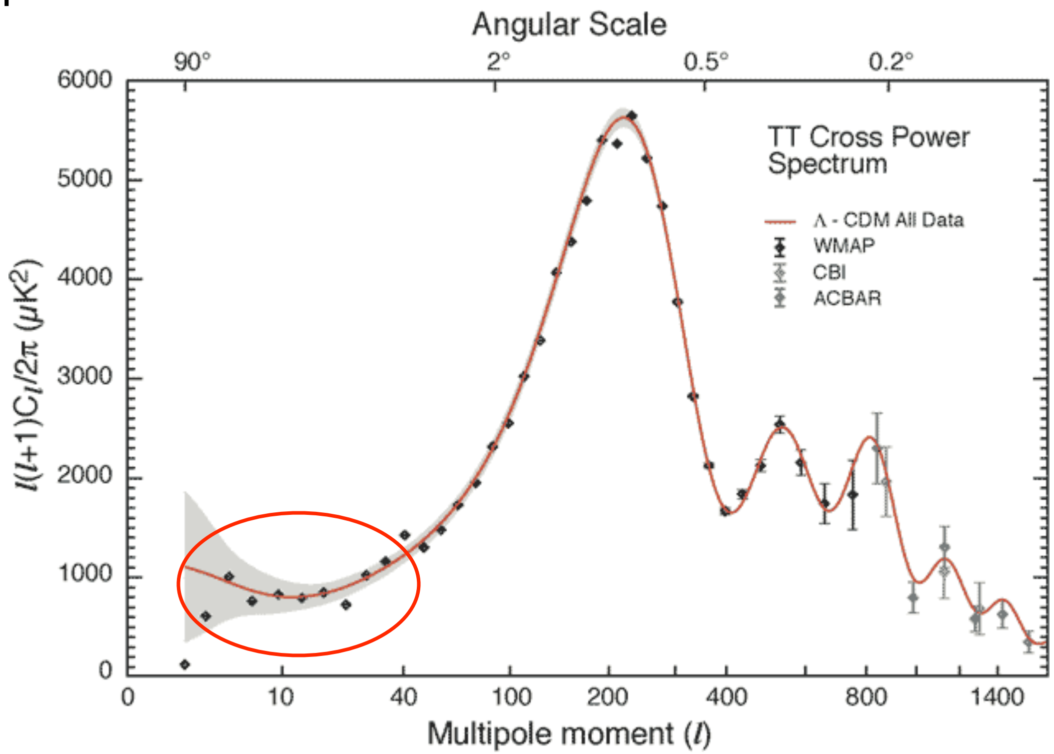


Large scales

• Scales larger than Hubble radius at decoupling.
Dominant effect: <u>gravitational second-order effects</u>



• Low multipoles:

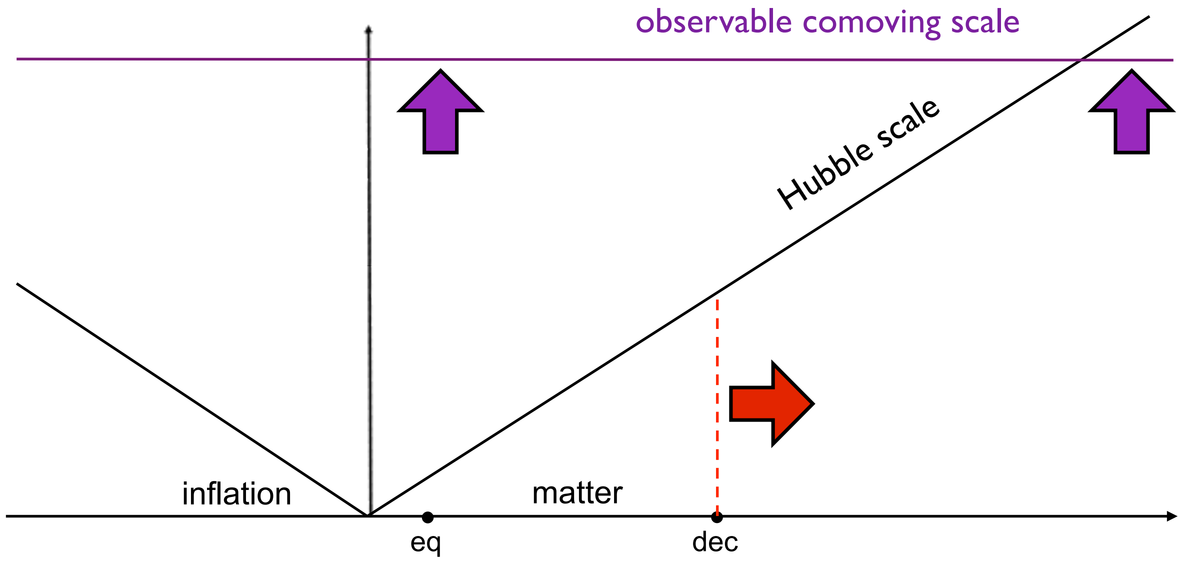


Sachs-Wolfe effect

$$\frac{\Delta T}{T}(\hat{n}) = \frac{1}{3}\Phi(\vec{x}_{\rm dec})$$ [Sachs-Wolfe '67]

- Adiabatic I.C.
- Matter dominance only
- Large angles: gravity only, no sub-Hubble plasma physics

Physically consistent limit: the calculation is exact if recombination takes place much after equality, there is no Lambda and the universe is so old that observed scales are infinitely larger than Hubble radius at recombination



Sachs-Wolfe calculation

$$\frac{\Delta T}{T}(\hat{n}) = \frac{1}{3}\Phi(\vec{x}_{\rm dec})$$ [Sachs-Wolfe '67]

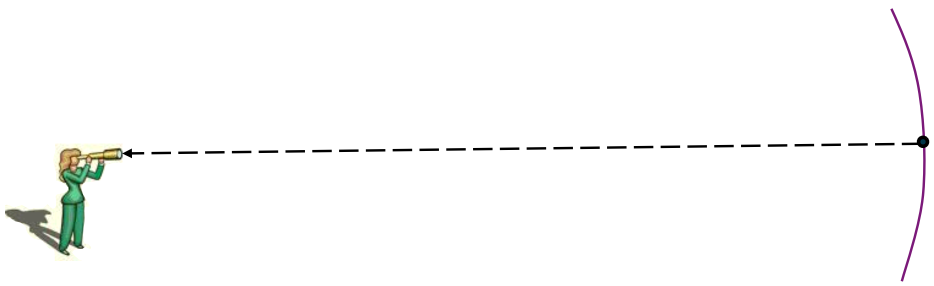
Because of Liouville's theorem: $T_o(\hat{n}) = \frac{\omega_o}{\omega_e} T_e(\vec{x}_e)$

Perturbed metric in conformal time: $ds^2 = a^2(\tau)\left[-(1+2\Phi(\vec{x}))d\tau^2 + (1-2\Phi(\vec{x}))d\vec{x}^2\right]$

cf linear Poisson eq: $\nabla^2\Phi = 4\,\pi G\rho(t)a^2(t)\delta(t,\vec{x})$

- Photon redshift: $\frac{\omega_o}{\omega_e} = \frac{a_e}{a_o}\sqrt{\frac{g_{00}(\vec{x}_e)}{g_{00}(\vec{x}_o)}} = \frac{a_e}{a_o}(1+\Phi_e-\Phi_o)$

- Intrinsic anisotropy (adiabatic I.C.): $T_e \propto \tilde{t}_e^{-2/3} = [t(1+\Phi)]_e^{-2/3} \propto \frac{1}{a_e}\left(1-\frac{2}{3}\Phi_e\right)$



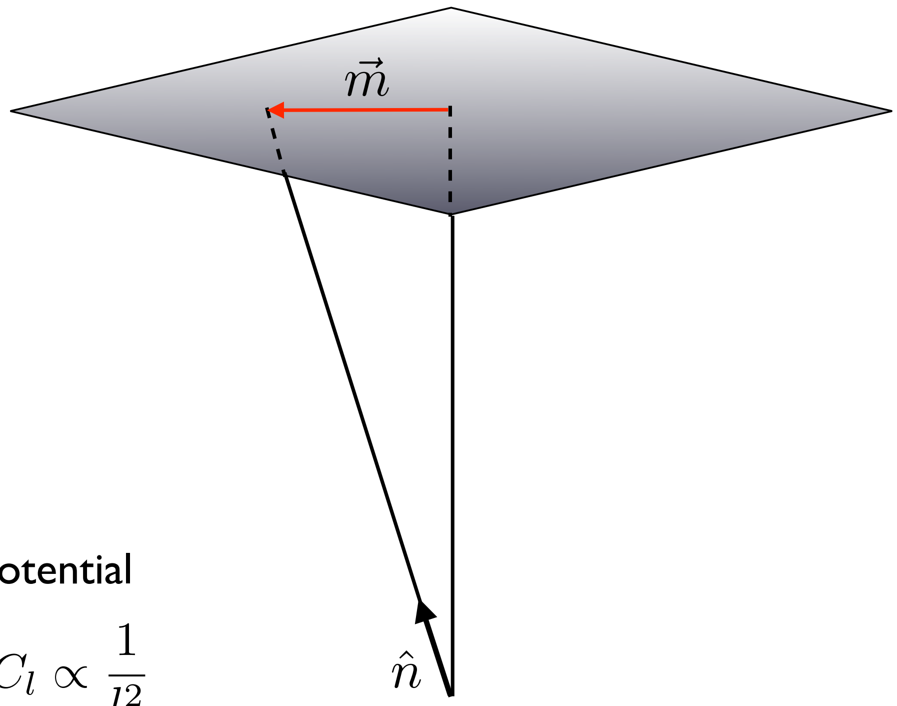
$\vec{x}_e = \hat{n}(\tau_o - \tau_e)$

Power spectrum

- Sachs-Wolfe effect: $\frac{\Delta T}{T}(\hat{n}) = \frac{1}{3}\Phi(\vec{x}_{\rm dec})$

- Flat-sky approximation: not very good on large angular scales; however simpler and more transparent expressions

$$a_{\vec{l}} = \int d^2\vec{m}\,\frac{\delta T}{T}(\hat{n})\,e^{-i\vec{l}\cdot\vec{m}}$$



- Power spectrum in flat sky:

$$\langle a_{\vec{l}}\, a_{\vec{l}'}\rangle = (2\pi)^2\delta(\vec{l}+\vec{l}')C_l$$

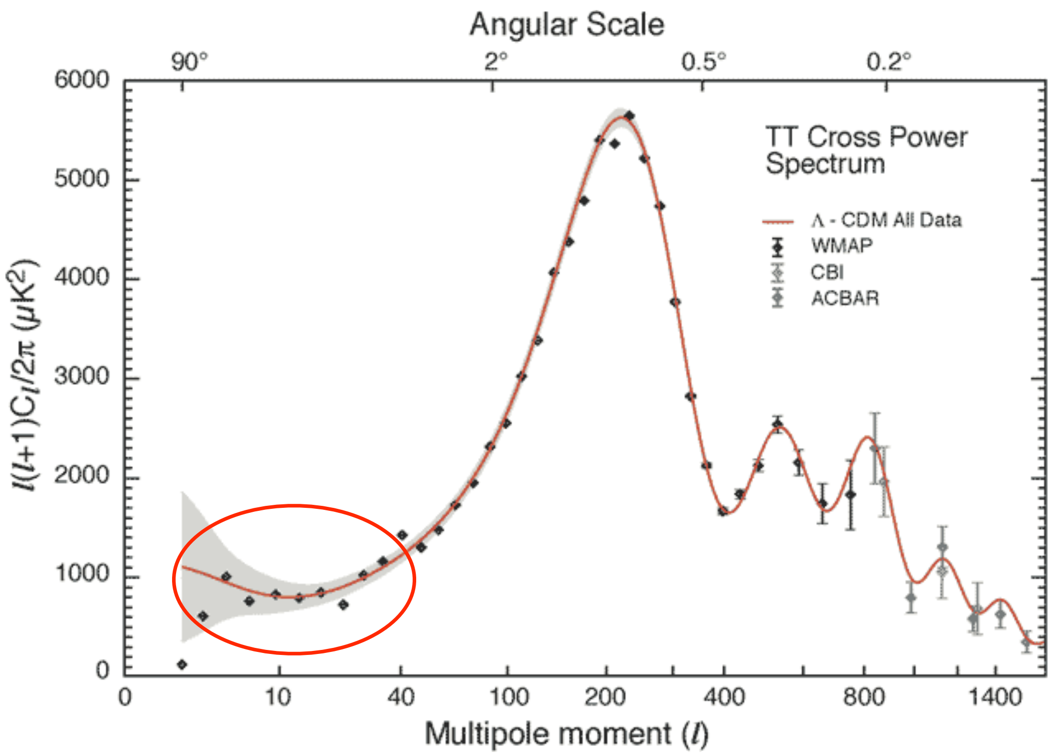
- Use the power spectrum of gravitational potential

$$\langle \Phi_{\vec{k}}\Phi_{\vec{k}'}\rangle \propto \delta(\vec{k}+\vec{k}')\frac{1}{k^3} \qquad \Rightarrow \qquad C_l \propto \frac{1}{l^2}$$

Power spectrum

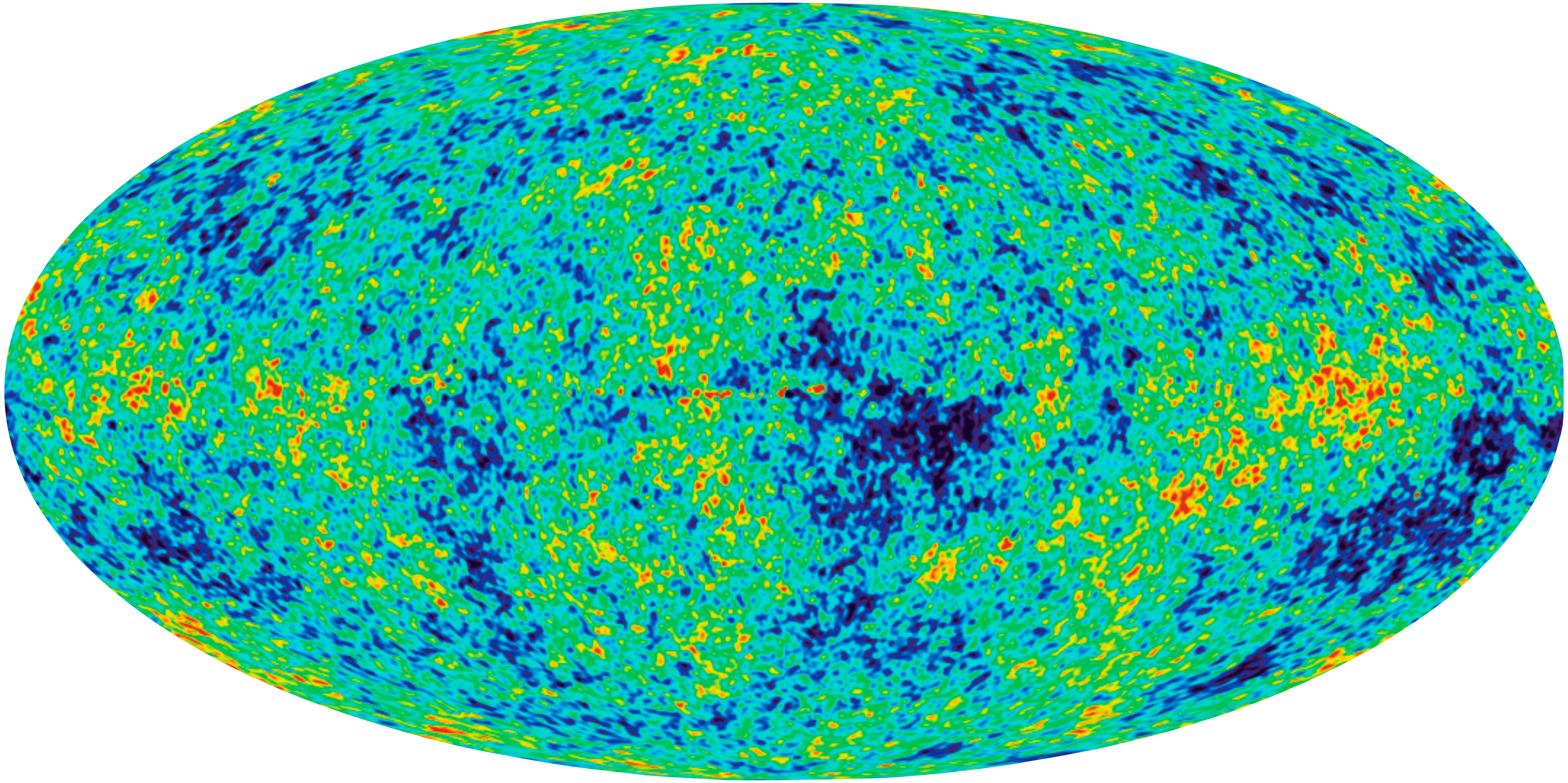
- If data are Gaussian, they are fully characterized by the power spectrum (FT of 2-pf):

$$\langle \frac{\Delta T}{T}(\hat{n}) \frac{\Delta T}{T}(\hat{n}') \rangle \qquad \Rightarrow \qquad C_l \propto \frac{1}{l^2}$$



Beyond Gaussianity

~10^6 pixels



$$\langle \frac{\Delta T}{T}(\hat{n}) \frac{\Delta T}{T}(\hat{n}') \rangle$$ 1000 numbers << 10^6

$$\langle \frac{\Delta T}{T}(\hat{n}_1) \frac{\Delta T}{T}(\hat{n}_2) \frac{\Delta T}{T}(\hat{n}_3) \rangle$$ 3-p statistic

Bispectrum

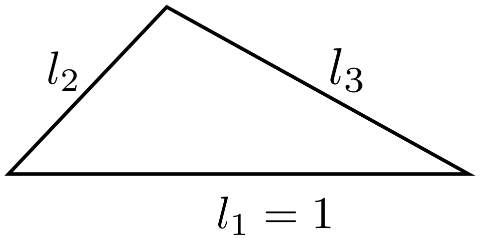
- Deviation from Gaussianity are imprinted in the bispectrum (FT of 3-pf)

$$\langle \frac{\Delta T}{T}(\hat{n}_1)\frac{\Delta T}{T}(\hat{n}_2)\frac{\Delta T}{T}(\hat{n}_3)\rangle \quad \Rightarrow \quad \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3}\rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{l_1 l_2 l_3}$$

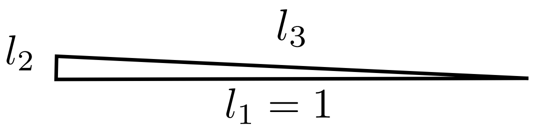
- Flat sky: $\langle a_{\vec{l}_1} a_{\vec{l}_2} a_{\vec{l}_3}\rangle = (2\pi)^3 \delta(\vec{l}_1 + \vec{l}_2 + \vec{l}_3) B(\vec{l}_1, \vec{l}_2, \vec{l}_3)$

- The bispectrum is a function of 6 parameters: -2 from translational, -1 from rotational, -1 from scale invariance = 2 independent parameters

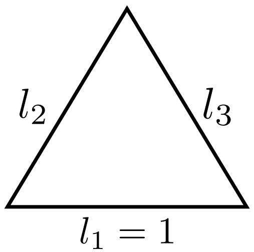
For instance: $r_2 \equiv l_2/l_1; \quad r_3 \equiv l_3/l_1; \quad 1 - r_3 \leq r_2 \leq r_3$ [Babich, Creminelli, Zaldarriaga '04]



- Squeezed: $l_2 \to 0$



- Equilateral: $l_1 = l_2 = l_3$



Primordial non-Gaussianities

- Simple, single field slow-roll inflation predicts very small non-Gaussianities [Maldacena '02]
- Other models predict larger non-Gaussianities:

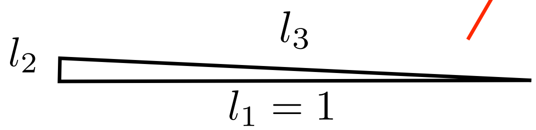
  ✓ Local shape non-Gaussianity, generated on super-Hubble scales (curvaton, modulated reheating, new ekpyrosis...):

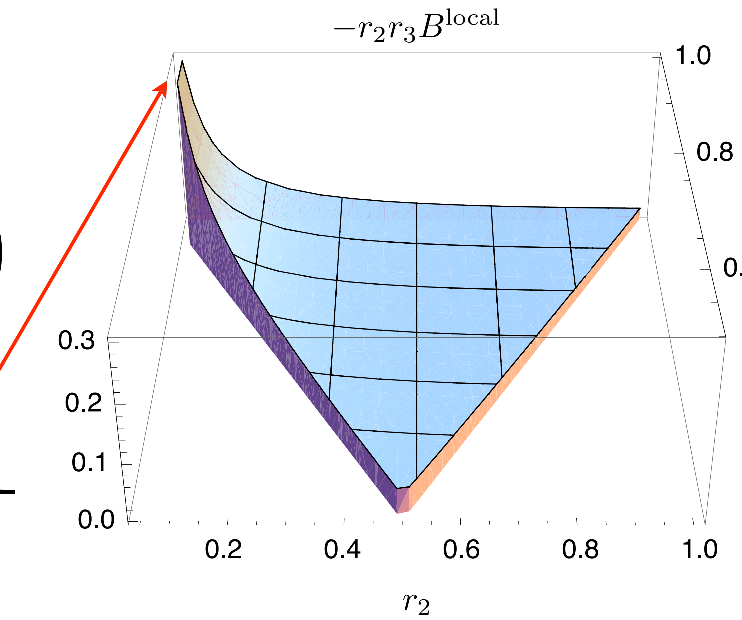
$$\Phi(\vec{x}) = \phi_g(\vec{x}) - f_{\rm NL}^{\rm local}\phi_g^2(\vec{x})$$

$$\langle \Phi_{\vec{k}_1}\Phi_{\vec{k}_2}\Phi_{\vec{k}_3}\rangle \propto \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{1}{k_1^3k_2^3} + \text{ perms} \quad \Rightarrow$$ Maximized in squeezed limit

- Bispectrum: $$B \propto f_{\rm NL}^{\rm local}\left(\frac{1}{l_1^2l_2^2}+\frac{1}{l_1^2l_3^2}+\frac{1}{l_2^2l_3^2}\right)$$
$$\propto f_{\rm NL}^{\rm local}\frac{1}{l_1^4}\left(\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_2^2r_3^2}\right)$$

- Squeezed: $l_2 \to 0$





Primordial non-Gaussianities

✓ <u>Equilateral shape</u> non-Gaussianity, generated at Hubble-crossing (DBI, ghost inflation...):

[Babich, Creminelli, Zaldarriaga '04]

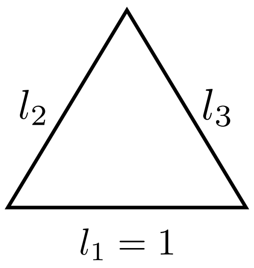
$$\langle \Phi_{\vec{k}_1} \Phi_{\vec{k}_2} \Phi_{\vec{k}_3} \rangle \propto \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \left(-\frac{1}{2k_1^3 k_2^3} - \frac{1}{3k_1^2 k_2^2 k_3^2} - \frac{1}{k_1 k_2^2 k_3^3} + \text{ perms} \right)$$

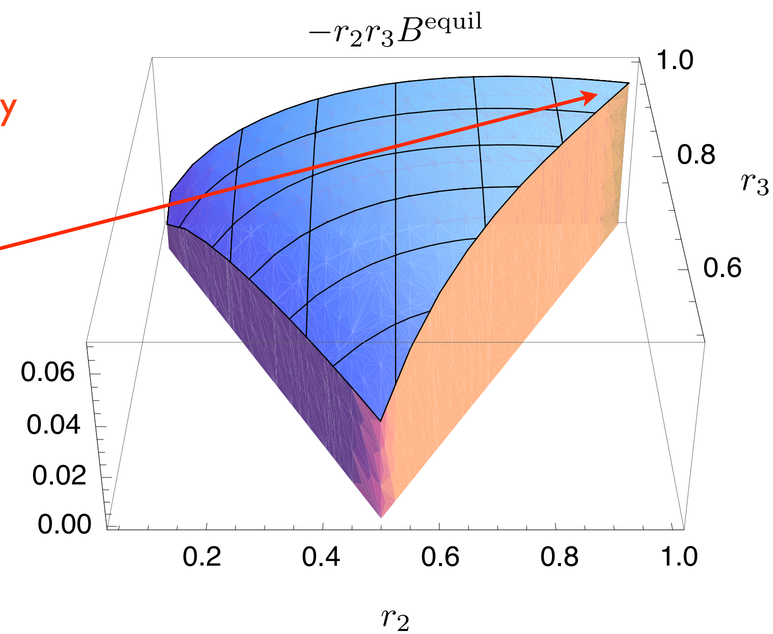
Divergences removed by cancellation in the squeezed limit

- Bispectrum: $B \propto f_{\rm NL}^{\rm equil} \cdot \frac{1}{l_1^4} \cdot F(r_2, r_3)$

obtained numerically

- Equilateral: $l_1 = l_2 = l_3$





Shape matters

- Signal/noise is (2d): $(S/N)^2 = \frac{1}{\pi} \int \frac{d^2 l_2 d^2 l_3}{(2\pi)^2} \frac{B(\vec{l}_1, \vec{l}_2, \vec{l}_3)^2}{6 C_{l_1} C_{l_2} C_{l_3}}$ [Hu '00]

$$(S/N)^2 \propto \int dr_2 dr_3 \left[\frac{r_2^{3/2} r_3^{3/2}}{(2r_2^2 + 2r_3^2 + 2r_2^2 r_3^2 - 1 - r_2^4 - r_3^4)^{1/4}} B(1, r_2, r_3) \right]^2 \longrightarrow \boxed{r_2\, r_3\, B(1, r_2, r_3)}$$

- We can define a scalar product between shapes:

$$B_1 \cdot B_2 \equiv \frac{1}{\pi} \int \frac{d^2 l_2 d^2 l_3}{(2\pi)^2} \frac{B_1(\vec{l}_1, \vec{l}_2, \vec{l}_3)\, B_2(\vec{l}_1, \vec{l}_2, \vec{l}_3)}{6 C_{l_1} C_{l_2} C_{l_3}}$$

$$\cos(B_1, B_2) \equiv \frac{B_1 \cdot B_2}{\sqrt{B_1 \cdot B_1}\sqrt{B_2 \cdot B_2}}$$

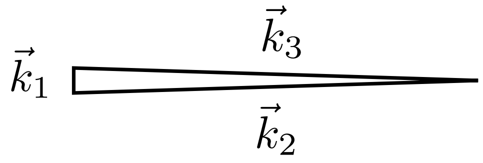
- Very important to constrain non-Gaussianities!

Ex: $\cos(B_{\rm local} \cdot B_{\rm equil}) = 0.30$

Constraints on non-Gaussianities

- Current constraints on “local” and “equilateral” non-Gaussianity from WMAP data:

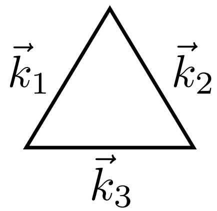
✓ Local type: $\Phi(\vec{x}) = \phi_g(\vec{x}) - f_{\rm NL}^{\rm local}\phi_g^2(\vec{x})$



$$-4 < f_{\rm NL}^{\rm local} < 80 \quad (95\%\ {\rm CL})$$

[Smith, Senatore, Zaldarriaga ’09]

✓ Equilateral type:



$$-151 < f_{\rm NL}^{\rm equil} < 253 \quad (95\%\ {\rm CL})$$

[Komatsu et al. ’08]

- Future constrains with perfect CMB experiment (including polarization):

$$|f_{\rm NL}^{\rm local}| < 1.8 \quad (95\%\ {\rm CL})$$

We expect a plethora of second order effects ~$(10^{-5})^2$ even in absence of primordial non-Gaussianities: experiments are getting close to them!

CMB bispectrum from 2nd order perts

- Complete calculation is extremely challenging: 2nd order Boltzmann equations on all scales
- On sub-Hubble scales at recombination people focussed on particular effects:

✓ Dark matter non-linearities on short scales [Bernardeau, Pitrou, Uzan '08; Bartolo, Riotto '08]

$$\frac{\delta T}{T} = \frac{\delta T_{rec}}{T_{rec}} + \Phi \qquad \Rightarrow \qquad f_{\rm NL}^{\rm equil} \sim 10$$

✓ Perturbed recombination [Senatore, Tassev, Zaldarriaga '09; Khatri, Wandelt '09]

$$\frac{\delta n_e}{n_e} \approx \frac{\dot{n}_e}{n_e}\delta t \approx 5\frac{\delta n_b}{n_b} \qquad \Rightarrow \qquad f_{\rm NL}^{\rm local} \sim 4$$

- On super-Hubble scales we are reduced to a 2nd order GR problem:

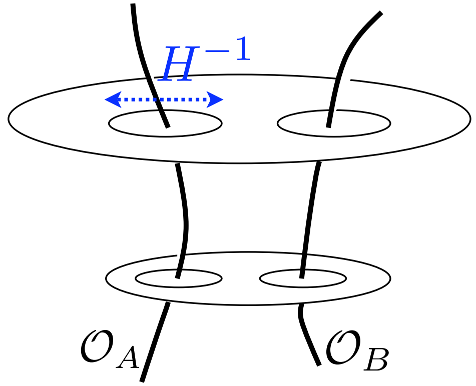
[Pyne, Carroll 00; Mollerach, Matarrese '97; Bartolo, Matarrese, Riotto '04]

$$\frac{\delta T}{T} = \frac{1}{3}\phi + F_{\rm NL}(\phi \star \phi) \qquad \Rightarrow \qquad \langle \frac{\delta T}{T}\frac{\delta T}{T}\frac{\delta T}{T}\rangle = \frac{1}{9}F_{\rm NL}\langle \phi\, \phi\, (\phi \star \phi)\rangle + \text{perms}$$

Separate universe

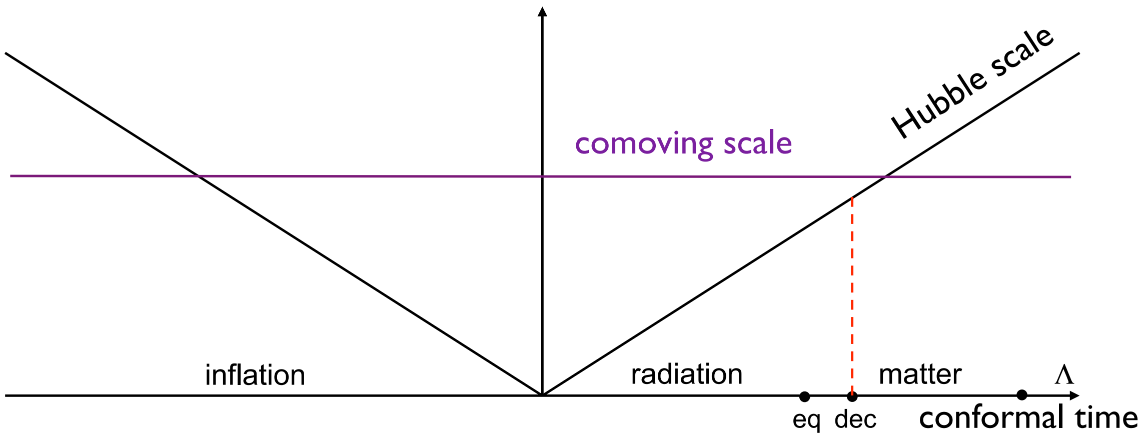
- On large scales the metric reads:

$$ds^2 = -dt^2 + a^2(t)e^{2\zeta_0(\vec{x})}d\vec{x}^2 \,, \qquad k \ll aH$$



- If only one clock (adiabatic perturbations): ζ is conserved on super-Hubble scales

- Primordial non-Gaussianity are encoded in ζ



- We assume there is no primordial non-Gaussianity (e.g., single field inflation)

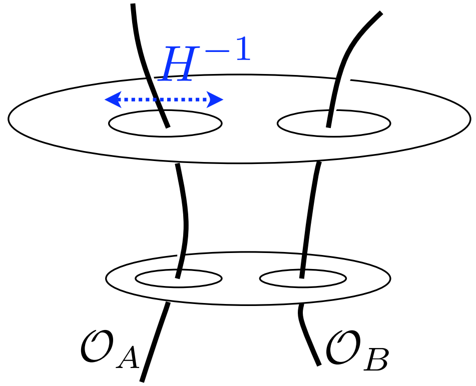
$$\langle \zeta_0(\vec{x}_1)\zeta_0(\vec{x}_2)\zeta_0(\vec{x}_3)\rangle = 0$$

Any guess? Squeezed limit

- Squeezed limit (separation of scales): effect of a very long wavelength mode on the 2-pf:

[Maldacena, '02]

$$\langle a_{\vec{l}_L} a_{\vec{l}_S} a_{-\vec{l}_S} \rangle \propto \langle \zeta_{l_L} C_{l_S} \rangle$$



- Consistency relation: if a long mode is out of the horizon today it should not affect physical observables

[Creminelli, Zaldarriaga, '04]

$$\langle a_{\vec{l}_L} a_{\vec{l}_S} a_{-\vec{l}_S} \rangle \to 0 \qquad \Rightarrow \qquad f_{\rm NL}^{\rm local} = 0 \quad ✓$$

- Second-order calculation keeping only scalar perts. [Bartolo, Matarrese, Riotto, '04]

$$\frac{\Delta T}{T} = \frac{1}{3}\Phi_{\rm dec} + \frac{1}{18}\Phi_{\rm dec}^2 \qquad \Rightarrow \qquad f_{\rm NL}^{\rm local} = -\frac{1}{6} \quad ✓$$

Which of these two results is correct?

Second-order metric in MD

$$ds^2 = a^2(\tau)\left\{-(1+2\Phi)d\tau^2 + 2\omega_i dx^i d\tau + [(1-2\Psi)\delta_{ij} + \gamma_{ij}]dx^i dx^j\right\}$$

$$\omega_{i,i} = 0 \text{ and } \gamma_{ii} = 0 = \gamma_{ij,i}$$

• Second-order metric is time-dependent: non-linear coupling of the dark matter (sub-Hubble Newtonian regime) and generation of vector (non-vortical) and tensor modes

$$\begin{aligned}
\Phi =& \phi + \left[\phi^2 + \partial^{-2}(\partial_j\phi)^2 - 3\partial^{-4}\partial_i\partial_j(\partial_i\phi\partial_j\phi)\right] \\
&+ \frac{2}{21a^2H^2}\partial^{-2}\left[2(\partial_i\partial_j\phi)^2 + 5(\partial^2\phi)^2 + 7\partial_i\phi\partial_i\partial^2\phi\right] , \\
\Psi =& \phi - \left[\phi^2 + \frac{2}{3}\partial^{-2}(\partial_i\phi)^2 - 2\partial^{-4}\partial_i\partial_j(\partial_i\phi\partial_j\phi)\right] \\
&+ \frac{2}{21a^2H^2}\partial^{-2}\left[2(\partial_i\partial_j\phi)^2 + 5(\partial^2\phi)^2 + 7\partial_i\phi\partial_i\partial^2\phi\right] , \\
\omega_i =& -\frac{8}{3aH}\partial^{-2}\left[\partial^2\phi\partial_i\phi - \partial^{-2}\partial_i\partial_j(\partial^2\phi\partial_j\phi)\right] , \\
\gamma_{ij} =& -20\left(\frac{1}{3} - \frac{j_1(k\tau)}{k\tau}\right)\partial^{-2}P^{\rm TT}_{ij\,kl}\left(\partial_k\phi\partial_l\phi\right) .
\end{aligned}$$

$$\frac{1}{a^2H^2} \propto a$$

in matter dominance

$$\nabla^2\Phi(t,\vec{x}) = 4\pi G\rho(t)\; a^2(t)\;\delta(t,\vec{x})$$

[Bartolo, Matarrese, Riotto '06; Boubekeur, Creminelli, Norena, FV '08]

• Gaussian initial condition: $k \ll aH \quad \Rightarrow \quad \phi = -\frac{3}{5}\zeta_0$ Gaussian variable

Sachs-Wolfe at second order

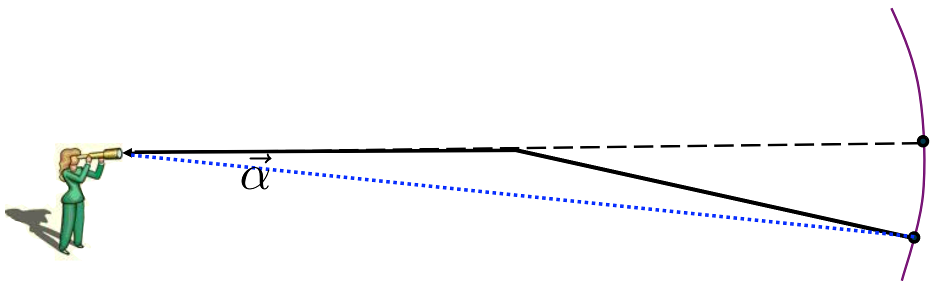
$$ds^2 = a^2(\tau)\left\{-(1+2\Phi)d\tau^2 + 2\omega_i dx^i d\tau + [(1-2\Psi)\delta_{ij}+\gamma_{ij}]dx^i dx^j\right\}$$

$$T(\hat{n}) = \frac{\omega_o}{\omega_e} T_e(\vec{x}_e)$$

- Photon redshift: $$\frac{\omega_o}{\omega_e} = \frac{a_e}{a_o}\sqrt{\frac{1+2\Phi_e}{1+2\Phi_o}}\left[1 + \int_{\tau_e}^{\tau_o} d\tau \left(\Phi' + \Psi' + \omega_i' \hat{n}^i - \frac{1}{2}\gamma_{ij}' \hat{n}^i \hat{n}^j\right)\right]$$
- Intrinsic anisotropy (adiabatic I.C.): $T_e \propto \tilde{t}_e^{-2/3} = [t(1+2\Phi)^{1/2}]_e^{-2/3} \propto \frac{1}{a_e}(1+2\Phi_e)^{-1/3}$
- Lensing: $T_e(\vec{x}_e) = T_e(\hat{n}(\tau_o - \tau_e)) - \vec{\alpha}\cdot\vec{\nabla}_{\hat{n}} T_e$

$$\vec{\alpha} = -2\int_{\tau_e}^{\tau_o} d\tau \frac{\tau - \tau_e}{\tau_o - \tau_e}\vec{\nabla}_\perp \phi$$

lensing deflection angle



$$\vec{x}_e = (\hat{n} - \vec{\alpha})(\tau_o - \tau_e)$$

Sachs-Wolfe at second order

$$\mathrm{d}s^2 = a^2(\tau)\left\{-(1+2\Phi)\mathrm{d}\tau^2 + 2\omega_i \mathrm{d}x^i \mathrm{d}\tau + [(1-2\Psi)\delta_{ij} + \gamma_{ij}]\mathrm{d}x^i \mathrm{d}x^j\right\}$$

$$T(\hat{n}) = \frac{\omega_o}{\omega_e} T_e(\vec{x}_e)$$

- Photon redshift: $$\frac{\omega_o}{\omega_e} = \frac{a_e}{a_o}\sqrt{\frac{1+2\Phi_e}{1+2\Phi_o}}\left[1 + \int_{\tau_e}^{\tau_o} \mathrm{d}\tau \left(\Phi' + \Psi' + \omega_i' \hat{n}^i - \frac{1}{2}\gamma_{ij}' \hat{n}^i \hat{n}^j\right)\right]$$
- Intrinsic anisotropy (adiabatic I.C.): $$T_e \propto \tilde{t}_e^{-2/3} = [t(1+2\Phi)^{1/2}]_e^{-2/3} \propto \frac{1}{a_e}(1+2\Phi_e)^{-1/3}$$
- Lensing: $$T_e(\vec{x}_e) = T_e(\hat{n}(\tau_o - \tau_e)) - \vec{\alpha}\cdot\vec{\nabla}_{\hat{n}} T_e \qquad \vec{\alpha} = -2\int_{\tau_e}^{\tau_o} \mathrm{d}\tau \frac{\tau - \tau_e}{\tau_o - \tau_e}\vec{\nabla}_\perp \phi$$

lensing deflection angle

$$\frac{\delta T}{T}(\hat{n}) = \left[\frac{1}{3}\phi + \frac{1}{18}\phi^2 + \frac{1}{3}\partial^{-2}\left((\partial_i\phi)^2 - 3\partial^{-2}\partial_i\partial_j(\partial_i\phi\partial_j\phi)\right)\right]_e + \int_{\tau_e}^{\tau_o} \mathrm{d}\tau \left(\Phi' + \Psi' + \omega_i' \hat{n}^i - \frac{1}{2}\gamma_{ij}' \hat{n}^i \hat{n}^j\right) + \frac{1}{3}\vec{\alpha}\cdot\vec{\nabla}_{\hat{n}}\phi_e\,,$$

[see also Pyne, Carroll '96, Mollerach, Matarrese '97]

“Intrinsic” contribution

$$\frac{\delta T}{T}(\hat{n}) = \left[\frac{1}{3}\phi + \frac{1}{18}\phi^2 + \frac{1}{3}\partial^{-2}\left((\partial_i\phi)^2 - 3\partial^{-2}\partial_i\partial_j(\partial_i\phi\partial_j\phi)\right)\right]_e + \int_{\tau_e}^{\tau_o} \mathrm{d}\tau \left(\Phi' + \Psi' + \omega_i'\hat{n}^i - \frac{1}{2}\gamma_{ij}'\hat{n}^i\hat{n}^j\right) + \frac{1}{3}\vec{\alpha}\cdot\vec{\nabla}_{\hat{n}}\phi_e\,,$$

- Local contribution: $f_{\rm NL}^{\rm local} = -\frac{1}{6}$ [Bartolo, Matarrese, Riotto ’04]

- *k*-dependent Kernel: $\left[\frac{1}{3}\frac{\vec{p}_1\cdot\vec{p}_2}{(\vec{p}_1+\vec{p}_2)^2} - \frac{p_1^2p_2^2 + (p_1^2+p_2^2)(\vec{p}_1\cdot\vec{p}_2) + (\vec{p}_1\cdot\vec{p}_2)^2}{(\vec{p}_1+\vec{p}_2)^4}\right]\phi_{\vec{p}_1}\phi_{\vec{p}_2}$

$$B^{\rm intr} = \frac{2A^2}{9(2\pi)^2 l_1^4}\int_{-\infty}^{+\infty}\mathrm{d}y_1\mathrm{d}y_2\left[\frac{1}{(y_1^2+r_1^2)^{3/2}(y_2^2+r_2^2)^{3/2}}\left(\frac{2y_1y_2+r_3^2-r_1^2-r_2^2}{6\left((y_1+y_2)^2+r_3^2\right)} - \frac{4(y_1^2+r_1^2)(y_2^2+r_2^2)+2(y_1^2+r_1^2+y_2^2+r_2^2)(2y_1y_2+r_3^2-r_1^2-r_2^2)+(2y_1y_2+r_3^2-r_1^2-r_2^2)^2}{4\left((y_1+y_2)^2+r_3^2\right)^2}\right) + 2\,\mathrm{cyclic}\right]$$

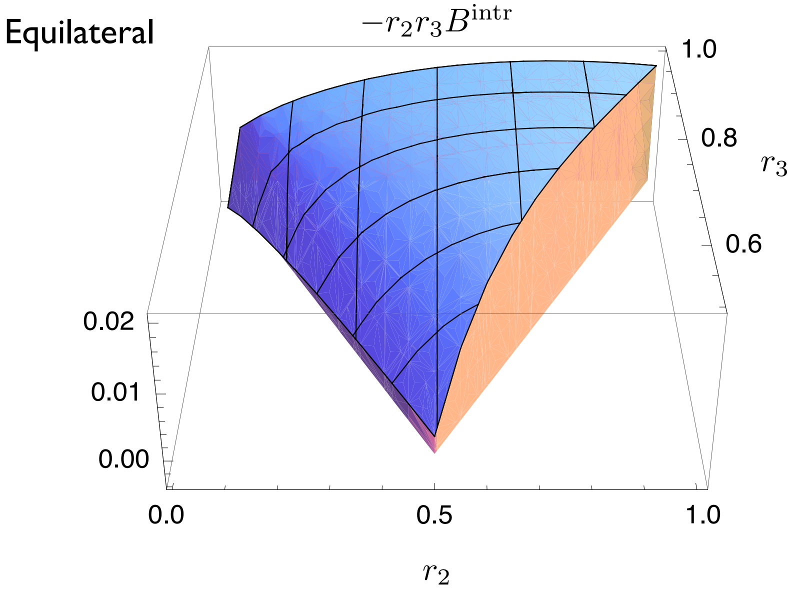
scale invariant

“Intrinsic” contribution

$$\frac{\delta T}{T}(\hat{n}) = \left[\frac{1}{3}\phi + \boxed{\frac{1}{18}\phi^2 + \frac{1}{3}\partial^{-2}\left((\partial_i\phi)^2 - 3\partial^{-2}\partial_i\partial_j(\partial_i\phi\partial_j\phi)\right)}\right]_e + \int_{\tau_e}^{\tau_o} \mathrm{d}\tau \left(\Phi' + \Psi' + \omega_i'\hat{n}^i - \frac{1}{2}\gamma_{ij}'\hat{n}^i\hat{n}^j\right) + \frac{1}{3}\vec{\alpha}\cdot\vec{\nabla}_{\hat{n}}\phi_e\,,$$

- Local contribution: $f_{\rm NL}^{\rm local} = -\frac{1}{6}$ [Bartolo, Matarrese, Riotto ’04]

- *k*-dependent Kernel: $f_{\rm NL}^{\rm equil} \simeq 1.21$



Integrated effects

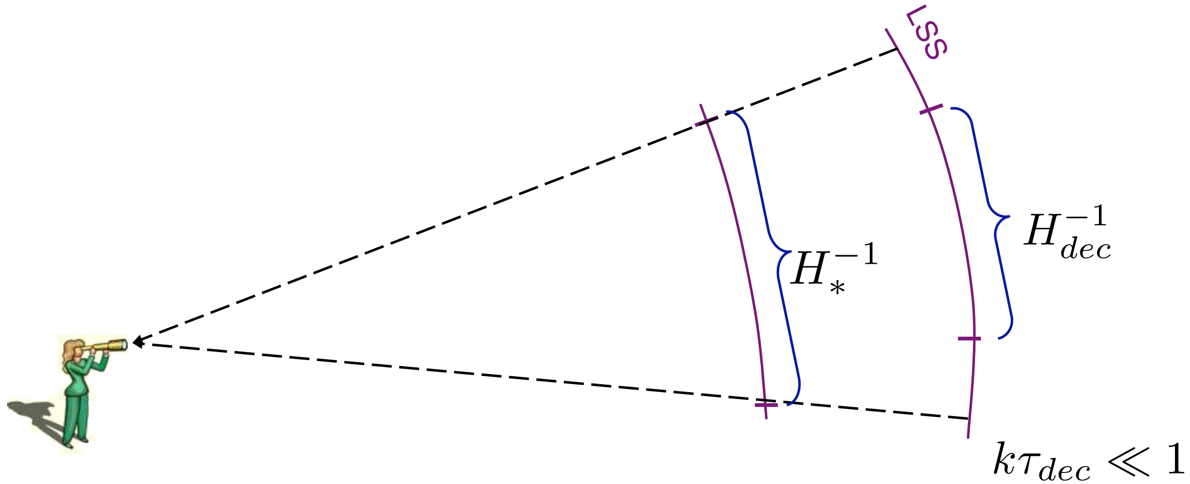
$$\frac{\delta T}{T}(\hat{n}) = \left[\frac{1}{3}\phi + \frac{1}{18}\phi^2 + \frac{1}{3}\partial^{-2}\left((\partial_i\phi)^2 - 3\partial^{-2}\partial_i\partial_j(\partial_i\phi\partial_j\phi)\right)\right]_e$$

$$+ \int_{\tau_e}^{\tau_o} \mathrm{d}\tau \left(\Phi' + \Psi' + \omega_i'\hat{n}^i - \frac{1}{2}\gamma_{ij}'\hat{n}^i\hat{n}^j\right) + \frac{1}{3}\vec{\alpha}\cdot\vec{\nabla}_{\hat{n}}\phi_e ,$$

- We are correlating integrated effects with the last scattering surface: naively they are suppressed by gradients on large scales

Eg, Newtonian second-order evolution: $\Phi(t) = \Psi(t) \propto a(t)$

$$\Phi' + \Psi' = -\tau\frac{4(\vec{p}_1\cdot\vec{p}_2)^2 + 10p_1^2p_2^2 + 7(p_1^2+p_2^2)(\vec{p}_1\cdot\vec{p}_2)}{21(\vec{p}_1+\vec{p}_2)^2}\phi_{\vec{p}_1}\phi_{\vec{p}_2}$$



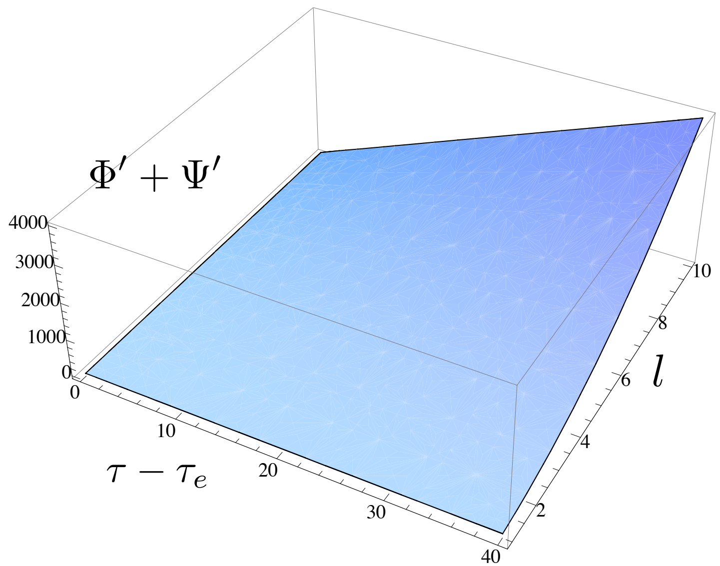
Integrated effects

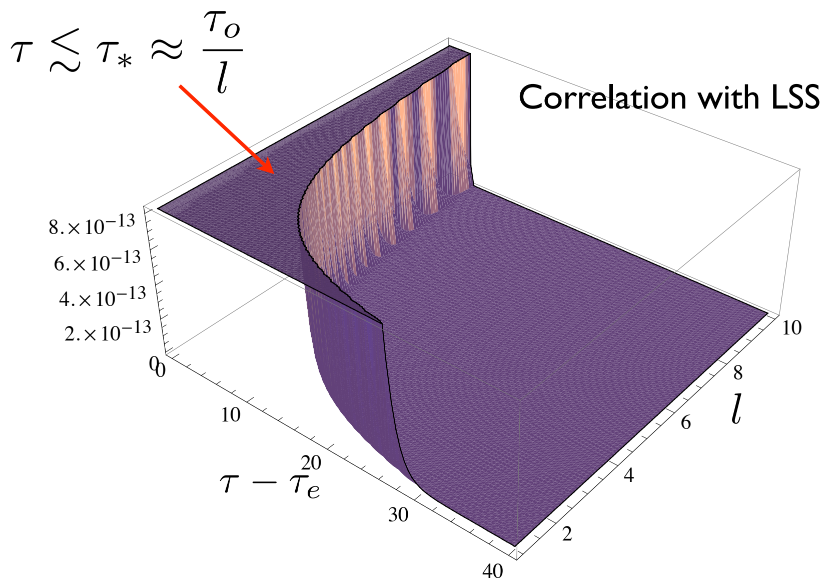
$$\frac{\delta T}{T}(\hat{n}) = \left[\frac{1}{3}\phi + \frac{1}{18}\phi^2 + \frac{1}{3}\partial^{-2}\left((\partial_i\phi)^2 - 3\partial^{-2}\partial_i\partial_j(\partial_i\phi\partial_j\phi)\right)\right]_e$$

$$+ \int_{\tau_e}^{\tau_o} \mathrm{d}\tau \left(\Phi' + \Psi' + \omega_i'\hat{n}^i - \frac{1}{2}\gamma_{ij}'\hat{n}^i\hat{n}^j\right) + \frac{1}{3}\vec{\alpha}\cdot\vec{\nabla}_{\hat{n}}\phi_e,$$

- We are correlating integrated effects with the last scattering surface: naively they are suppressed by gradients on large scales

- But correlation with the last scattering surface does not decay instantaneously!





Integrated effects

$$\frac{\delta T}{T}(\hat{n}) = \left[\frac{1}{3}\phi + \frac{1}{18}\phi^2 + \frac{1}{3}\partial^{-2}\left((\partial_i\phi)^2 - 3\partial^{-2}\partial_i\partial_j(\partial_i\phi\partial_j\phi)\right)\right]_e$$

$$+ \int_{\tau_e}^{\tau_o} \mathrm{d}\tau \left(\Phi' + \Psi' + \omega_i'\hat{n}^i - \frac{1}{2}\gamma_{ij}'\hat{n}^i\hat{n}^j\right) + \frac{1}{3}\vec{\alpha}\cdot\vec{\nabla}_{\hat{n}}\phi_e,$$

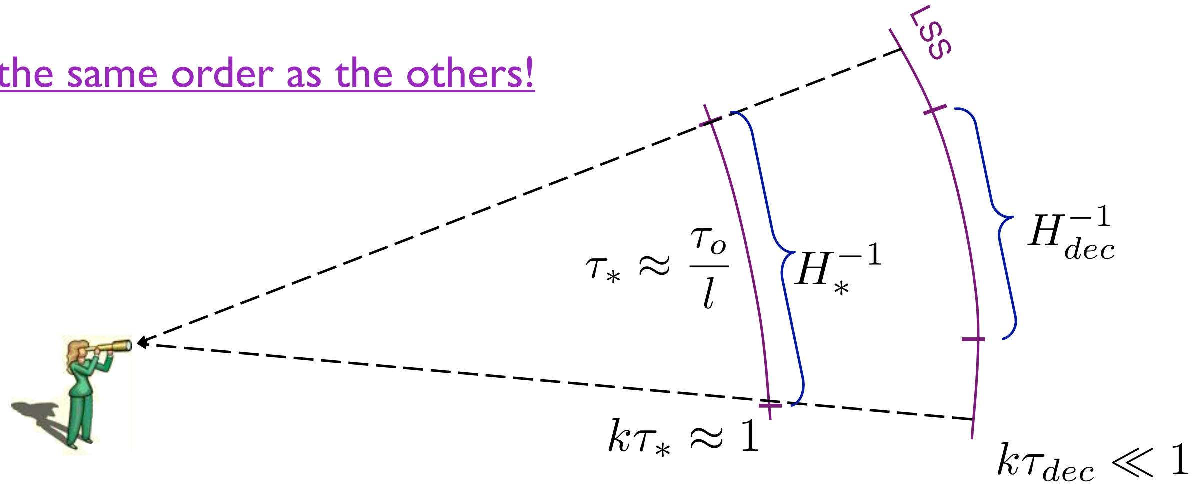
- We are correlating integrated effects with the last scattering surface: naively they are suppressed by gradients on large scales

Eg, Newtonian second-order evolution: $\Phi(t) = \Psi(t) \propto a(t)$

$$\Phi' + \Psi' = -\tau \frac{4(\vec{p}_1\cdot\vec{p}_2)^2 + 10p_1^2p_2^2 + 7(p_1^2+p_2^2)(\vec{p}_1\cdot\vec{p}_2)}{21(\vec{p}_1+\vec{p}_2)^2}\phi_{\vec{p}_1}\phi_{\vec{p}_2}$$

- Integrated terms are of the same order as the others!

$$k\tau_* \approx \frac{l}{\tau_o}\cdot\frac{\tau_o}{l} = 1$$



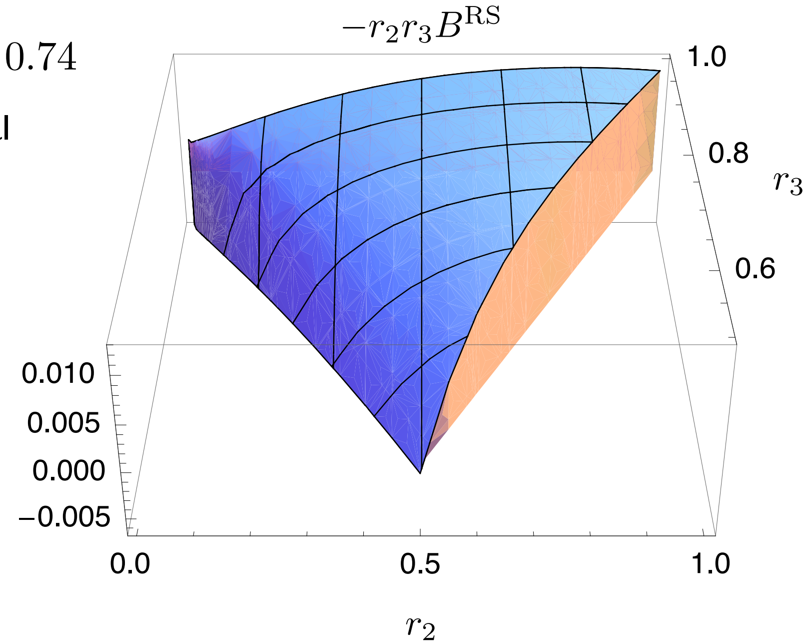
Rees-Sciama effect

$$\frac{\delta T}{T}(\hat{n}) = \left[\frac{1}{3}\phi + \frac{1}{18}\phi^2 + \frac{1}{3}\partial^{-2}\left((\partial_i\phi)^2 - 3\partial^{-2}\partial_i\partial_j(\partial_i\phi\partial_j\phi)\right)\right]_e$$

$$+ \int_{\tau_e}^{\tau_o} \mathrm{d}\tau \left(\Phi' + \Psi' + \omega_i'\hat{n}^i - \frac{1}{2}\gamma_{ij}'\hat{n}^i\hat{n}^j\right) + \frac{1}{3}\vec{\alpha}\cdot\vec{\nabla}_{\hat{n}}\phi_e\,,$$

$f_{\rm NL}^{\rm equil} \simeq 0.74$

Equilateral



Vector contribution

$$\frac{\delta T}{T}(\hat{n}) = \left[\frac{1}{3}\phi + \frac{1}{18}\phi^2 + \frac{1}{3}\partial^{-2}\big((\partial_i\phi)^2 - 3\partial^{-2}\partial_i\partial_j(\partial_i\phi\partial_j\phi)\big)\right]_e + \int_{\tau_e}^{\tau_o} \mathrm{d}\tau\,\left(\Phi' + \Psi' + \omega_i'\hat{n}^i - \frac{1}{2}\gamma_{ij}'\hat{n}^i\hat{n}^j\right) + \frac{1}{3}\vec{\alpha}\cdot\vec{\nabla}_{\hat{n}}\phi_e\,,$$

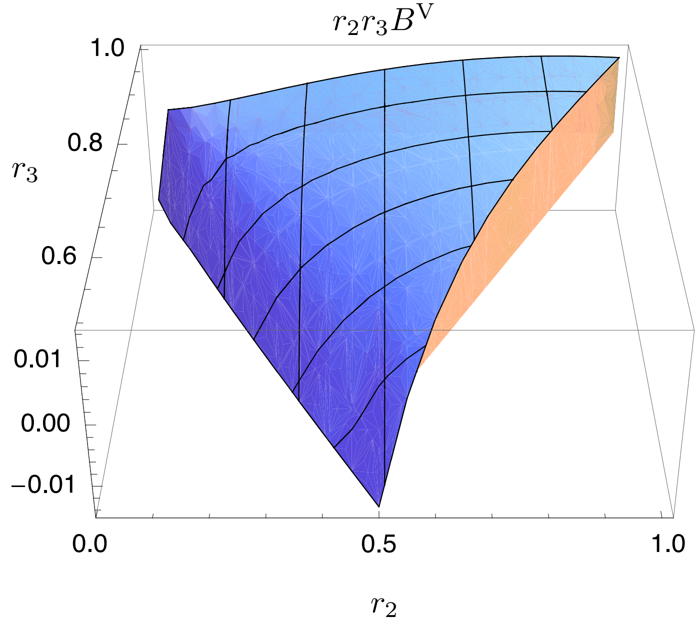
$$\omega_i'\hat{n}^i = -\frac{2i}{3}\left[\frac{p_1^2(\hat{n}\cdot\vec{p}_2) + p_2^2(\hat{n}\cdot\vec{p}_1)}{(\vec{p}_1+\vec{p}_2)^2} - \hat{n}\cdot(\vec{p}_1+\vec{p}_2)\frac{2p_1^2p_2^2 + (p_1^2+p_2^2)(\vec{p}_1\cdot\vec{p}_2)}{(\vec{p}_1+\vec{p}_2)^4}\right]\phi_{\vec{p}_1}\phi_{\vec{p}_2}$$

Total derivative

- No real gauge independent separation between integrated/intrinsic

$f_{\rm NL}^{\rm equil} \simeq -0.84$

Equilateral



Tensor contribution

$$\frac{\delta T}{T}(\hat{n}) = \left[\frac{1}{3}\phi + \frac{1}{18}\phi^2 + \frac{1}{3}\partial^{-2}\left((\partial_i\phi)^2 - 3\partial^{-2}\partial_i\partial_j(\partial_i\phi\partial_j\phi)\right)\right]_e$$

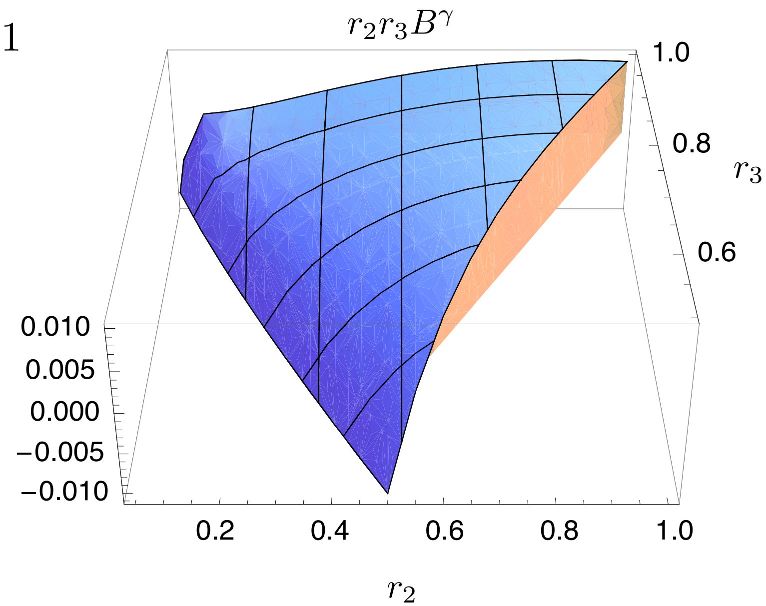
$$+ \int_{\tau_e}^{\tau_o} \mathrm{d}\tau \left(\Phi' + \Psi' + \omega_i'\hat{n}^i - \frac{1}{2}\gamma_{ij}'\hat{n}^i\hat{n}^j\right) + \frac{1}{3}\vec{\alpha}\cdot\vec{\nabla}_{\hat{n}}\phi_e\,,$$

$$\frac{1}{2}\gamma_{ij}'\hat{n}^i\hat{n}^j = j_2(|\vec{p}_1+\vec{p}_2|\tau)\frac{10}{\tau}\left[\frac{(\vec{p}_1\cdot\vec{p}_2)^2 - p_1^2p_2^2}{(\vec{p}_1+\vec{p}_2)^4}\left(1 + \frac{(\hat{n}\cdot(\vec{p}_1+\vec{p}_2))^2}{(\vec{p}_1+\vec{p}_2)^2}\right)\right.$$

$$\left. + \frac{p_1^2(\hat{n}\cdot\vec{p}_2)^2 + p_2^2(\hat{n}\cdot\vec{p}_1)^2 - 2(\vec{p}_1\cdot\vec{p}_2)(\hat{n}\cdot\vec{p}_1)(\hat{n}\cdot\vec{p}_2)}{(\vec{p}_1+\vec{p}_2)^4}\right]\phi_{\vec{p}_1}\phi_{\vec{p}_2}$$

$f_{\rm NL}^{\rm equil} \simeq -0.61$

Equilateral



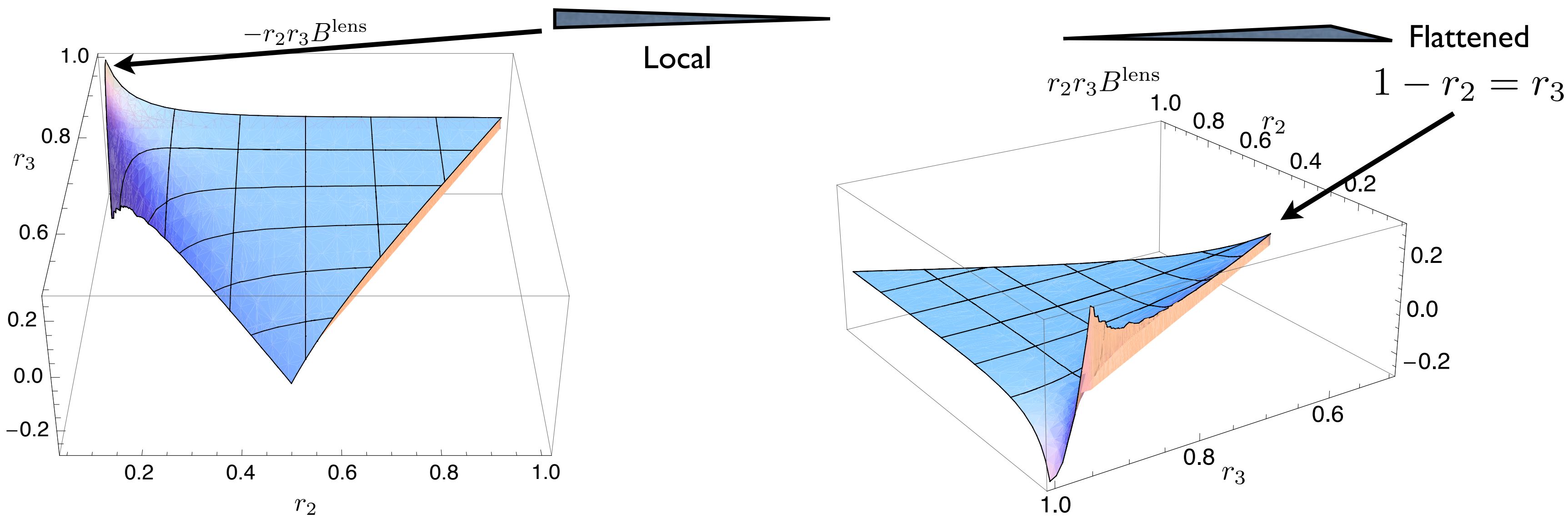
Lensing

$$\frac{\delta T}{T}(\hat{n}) = \left[\frac{1}{3}\phi + \frac{1}{18}\phi^2 + \frac{1}{3}\partial^{-2}\left((\partial_i\phi)^2 - 3\partial^{-2}\partial_i\partial_j(\partial_i\phi\partial_j\phi)\right)\right]_e + \int_{\tau_e}^{\tau_o} \mathrm{d}\tau \left(\Phi' + \Psi' + \omega_i'\hat{n}^i - \frac{1}{2}\gamma'_{ij}\hat{n}^i\hat{n}^j\right) + \frac{1}{3}\vec{\alpha}\cdot\vec{\nabla}_{\hat{n}}\phi_e\,,$$

$$\vec{\alpha} = -2\int_{\tau_e}^{\tau_o} \mathrm{d}\tau \frac{\tau - \tau_e}{\tau_o - \tau_e}\vec{\nabla}_\perp\phi$$

$$f_{\rm NL} \propto \frac{\vec{l}_2\cdot\vec{l}_3}{l_2^2 l_3^2}\left(\frac{1}{l_2^2} + \frac{1}{l_3^2}\right) + 2 \text{ perms}$$

$$f_{\rm NL}^{\rm local} = -\cos(2\theta_{\hat{l}_1\cdot\hat{l}_2}) \qquad l_2 \to 0$$



Total bispectrum

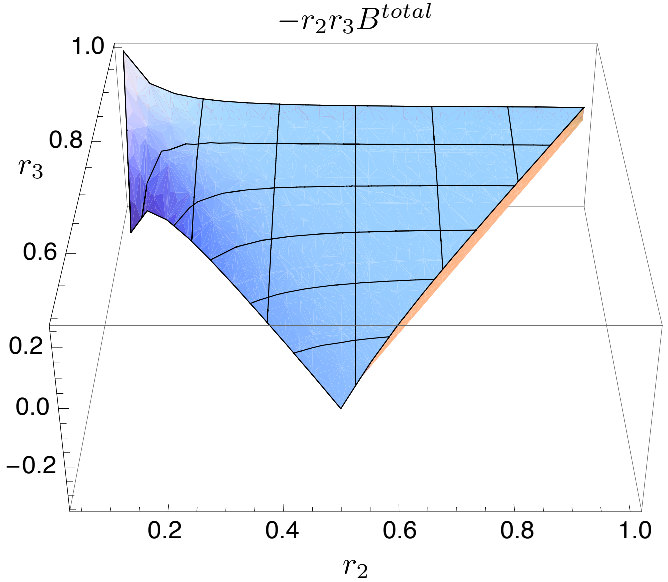
$$\frac{\delta T}{T}(\hat{n}) = \left[\frac{1}{3}\phi + \frac{1}{18}\phi^2 + \frac{1}{3}\partial^{-2}\left((\partial_i\phi)^2 - 3\partial^{-2}\partial_i\partial_j(\partial_i\phi\partial_j\phi)\right)\right]_e$$
$$+ \int_{\tau_e}^{\tau_o} \mathrm{d}\tau \left(\Phi' + \Psi' + \omega_i'\hat{n}^i - \frac{1}{2}\gamma_{ij}'\hat{n}^i\hat{n}^j\right) + \frac{1}{3}\vec{\alpha}\cdot\vec{\nabla}_{\hat{n}}\phi_e\,,$$

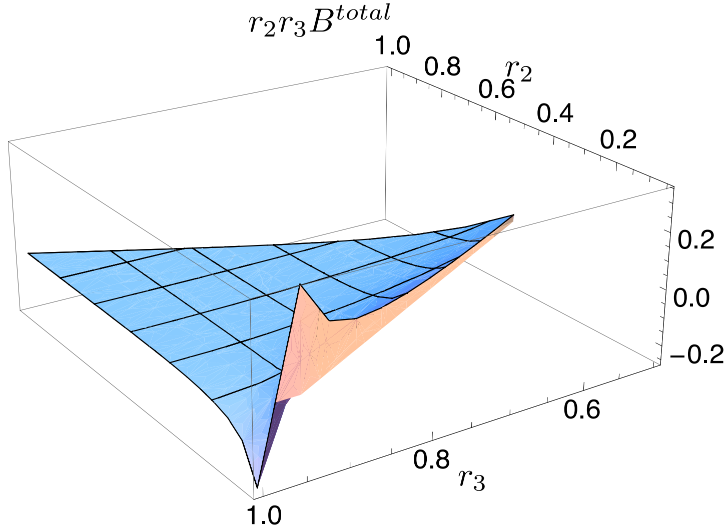
$$f_{\rm NL}^{\rm local} = -\frac{1}{6} - \cos(2\theta)$$

$$f_{\rm NL}^{\rm equil} \simeq 3.13$$

$\cos(B_X \cdot B_Y)$

Shape:	total	local	equil	lens
total	1.00	-0.17	0.41	0.98
local		1.00	0.30	0.03
equil			1.00	0.47
lens				1.00





Squeezed limit, consistency check

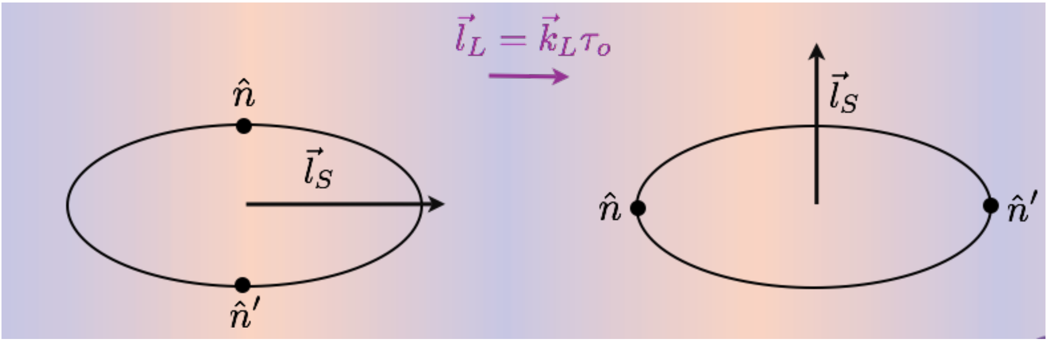
- A long wavelength mode today does not affect physical observables!

$$\frac{\delta T}{T}(\hat{n}) = \left[\frac{1}{3}\phi + \frac{1}{18}\phi^2\right]_e \simeq e^{\phi_e/3} - 1 \quad \Rightarrow \quad \frac{T_o(\hat{n}) - \bar{T}_o}{\bar{T}_o} = \frac{e^{\phi_e/3}}{\langle e^{\phi_e/3}\rangle} - 1$$

$$\Rightarrow \quad -\frac{1}{6} \quad ✓$$

- Lensing consistency relation:

[Similar to consistency relation involving gravity waves, Maldacena '02, Seery, Sloth, FV '08]



$$C_{l_S}^{\text{lensed}} = C_{l_S} + i l_L^j \alpha^i(\vec{l}_L) C_{l_S}\left(\delta_{ij} - 2\frac{l_S^i l_S^j}{l_S^2}\right)$$

$$\langle a_{l_L} C_{l_S}^{\text{lensed}}\rangle \propto i l_L^j \langle a_{l_L} \alpha_{l_L}^i\rangle C_{l_S}\left(\delta_{ij} - 2\frac{l_i l_j}{l^2}\right) \longleftarrow -\cos(2\theta_{\vec{l}_S \cdot \vec{l}_L}) \quad ✓$$

Conclusion

- ✓ Generalization of Sachs-Wolfe effect at 2nd order
- ✓ Local contribution gives $f_{\rm NL}^{\rm local} = -\frac{1}{6} - \cos(2\theta)$; corrects both Creminelli, Zaldarriaga and Bartolo, Matarrese, Riotto; check by physical arguments
- ✓ Equilateral contribution gives $f_{\rm NL}^{\rm equil} \simeq 3.13$: intrinsic and integrated effects physically indistinguishable
- ✓ Signal is very small, even for Planck. May be important to understand the contamination of primordial signal
- ✓ Future:
  - generalization to full-sky and RD + Lambda
  - Squeezed limit with the short modes inside the horizon

Computing the second-order metric

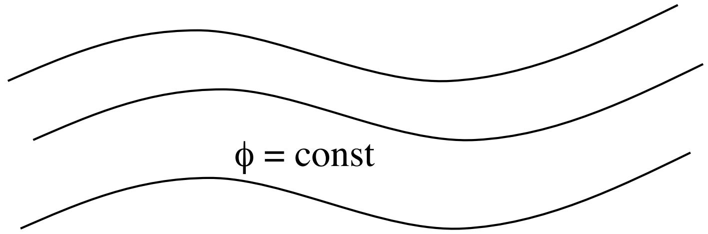
[Boubekeur, Creminelli, Norena, FV '08]

- "Action approach to cosmological perturbation theory." A perfect, irrotational, barotropic fluid has the same symmetries as a scalar field with Lagrangian:

[Taub '54; Shutz '70; for a recent review see Dubovski, Gregoire, Nicolis, Rattazzi '05]

$$\mathcal{L} = P(X) , \qquad X \equiv -\partial_\mu \phi \partial^\mu \phi$$

Warning! ϕ is now a scalar field and not the gravitational potential! Sorry.



- Indeed: $T_{\mu\nu} = 2P'(X)\partial_\mu\phi\partial_\nu\phi + P(X)g_{\mu\nu}$ Perfect fluid

$\rho(X) = 2P'X - P \, , \quad p(X) = P$ Barotropic

$u_\mu = \frac{\partial_\mu \phi}{\sqrt{X}}$ Irrotational

- Around: $\phi = ct$

$$\mathcal{L} = P'(c^2)[\dot{\delta\phi}^2 - (\nabla\delta\phi)^2] + 2P''(c^2)c^2\dot{\delta\phi}^2$$

$$c_s^2 = \left.\frac{P'(X)}{P'(X) + 2XP''(X)}\right|_{X=c^2} = \left.\frac{p'(X)}{\rho'(X)}\right|_{X=c^2} = \frac{dp}{d\rho}$$

w = constant

- Constant equation of state:

Example: relativistic fluid w = 1/3

$$p = w\rho \longrightarrow P = X^{\frac{1+w}{2w}}\,, \quad w \neq 0$$

$$\mathcal{L} = X^2 = (-\partial_\mu\phi\partial^\mu\phi)^2$$

Including gravity: $\partial_\mu[\sqrt{-g}P'(X)\partial^\mu\phi] = 0 \longrightarrow \rho \propto \dot{\phi}^{\frac{1+w}{w}} \propto a^{-3(1+w)}$

- The Lagrangian is like *k*-inflation: we can study metric + fluid perturbations exactly as we do for inflation!

Calculation of the 3-pf in inflation $\longleftrightarrow$ Calculation of 2nd order metric in MD

[Maldacena ’02; Seery, Lidsey ’05; Chen et al. ’06; etc...]

- Dark matter = dust (far from shell crossing, it is a perfect, irrotational and barotropic fluid)

Subtlety: take carefully the limit *w* to 0

Let us start computing

[Maldacena '02; Seery, Lidsey '05; Chen et al. '06; etc...]

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}[R + 2P(X)] \qquad P = X^{\frac{1+w}{2w}} \, , \quad w \neq 0$$

- ADM formalism: solve constraints to get action for scalar + tensors

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

$$S = \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}^3x\sqrt{h}\left[N(R^{(3)} + 2P) + N^{-1}\left(E_{ij}E^{ij} - E^2\right)\right] \qquad E_{ij} \equiv \frac{1}{2}\left(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i\right)$$

- Choose <u>velocity orthogonal gauge</u> (uniform-field):

$$\delta\phi = 0, \qquad h_{ij} = a^2 e^{2\zeta}\hat{h}_{ij} \, , \qquad \hat{h}_{ij} = \delta_{ij} + \gamma_{ij} + \tfrac{1}{2}\gamma_{il}\gamma_{lj} + \dots$$

$$\det \hat{h} = 1 \, , \qquad \gamma_{ii} = 0 \, , \qquad \partial_i\gamma_{ij} = 0 \, ,$$

- Constraint equations:

$$\nabla_i\left[N^{-1}\left(E^i_j - \delta^i_j E\right)\right] = 0 \, ,$$

$$R^{(3)} + 2P - 4XP' - \frac{1}{N^2}\left(E_{ij}E^{ij} - E^2\right) = 0$$

$$N_i \equiv \partial_i\psi + N_{Ti} \qquad \partial_i N_{Ti} = 0 \qquad N = 1 + \delta N$$

Start simple: Ist order

$$\delta N = \frac{\dot\zeta}{H}\,, \qquad N_{T\,i} = 0$$

$$\psi = -\frac{\zeta}{H} + \frac{a^2\epsilon}{w}\partial^{-2}\dot\zeta$$

$\longrightarrow$

$$\epsilon \equiv -\frac{\dot H}{H^2} = \frac{3}{2}(1+w)$$

- 2nd order action:

$$S_2 = \int \mathrm{d}t\,\mathrm{d}^3x\,a^3\frac{\epsilon}{w}\left[\dot\zeta^2 - \frac{w}{a^2}(\partial\zeta)^2\right]$$

ζ is constant on large scales (also for a generic barotropic fluid)

- Solve the action (expand 1st order in $\mathrm{w} \to 0$)

$$\zeta = \zeta_0 + \frac{2w}{5a^2H^2}\partial^2\zeta_0 + \mathcal{O}(w^2) \qquad \delta N = 0\,, \qquad \psi = -\frac{2}{5}\frac{\zeta_0}{H}$$

- ζ_0 sets the initial condition from inflation: <u>everything in terms of the natural variable, no need to mach with inflation!</u>

- We have all the ingredients for the 1st order metric:

$$ds^2 = -dt^2 - \frac{4}{5H}\partial_i\zeta_0\,dtdx^i + a^2(1+2\zeta_0)d\vec{x}^2$$

- Gauge tranformation to Poisson (Newtonian) gauge

$$ds^2 = -(1+2\Phi)dt^2 + a^2(1-2\Psi)d\vec{x}^2 \qquad \Phi = \Psi = -\frac{3}{5}\zeta_0$$

Not so simple (but straigtforward...): 2nd order

- Expanding the action at 3rd order (and after some work...):

[Seery, Lidsey '05;
Chen et al. '06; etc...]

$$S_3 = \int dt\, d^3x\, a^3 \frac{\epsilon}{w}\left[\frac{2}{3}\left(\frac{1}{w}-1\right)\frac{\dot{\zeta}_n^3}{H} + \frac{3}{2}\left(3-\frac{1}{w}\right)\zeta_n\dot{\zeta}_n^2 + \frac{1}{2a^2}(5+w)\zeta_n(\partial_i\zeta_n)^2 - \left(2-\frac{\epsilon}{2}\right)\frac{\epsilon}{w}\,\dot{\zeta}_n\partial_i\zeta_n\partial_i\partial^{-2}\dot{\zeta}_n + \frac{\epsilon^2}{4w}\partial^2\zeta_n(\partial_i\partial^{-2}\dot{\zeta}_n)^2\right].$$

- Field redefinition (new variable):

$$\zeta_n = \zeta - f(\zeta)$$

$$f(\zeta) = \frac{1}{wH}\zeta\dot{\zeta} + \frac{1}{4a^2H^2}\left[-(\partial_i\zeta)^2 + \partial^{-2}\partial_i\partial_j(\partial_i\zeta\partial_j\zeta)\right] + \frac{\epsilon}{2H^2w}\left[\partial_i\zeta\partial_i\partial^{-2}\dot{\zeta} - \partial^{-2}\partial_i\partial_j(\partial_i\zeta\partial_j\partial^{-2}\dot{\zeta})\right].$$

- Solve the action: $\zeta = \zeta_0 - \frac{1}{5a^2H^2}\partial^{-2}\partial_i\partial_j(\partial_i\zeta_0\partial_j\zeta_0)$

- Have we finished? No, we need to solve constraints at 2nd order to obtain the metric:

$$\delta N_2 = \frac{w}{5a^2H^2}\left[(\partial\zeta_0)^2 - 4\zeta_0\partial^2\zeta_0\right] - \frac{2w}{175a^4H^4}\left[3(\partial^2\zeta_0)^2 + 14\partial_i\zeta_0\partial_i\partial^2\zeta_0 + 4(\partial_i\partial_j\zeta_0)^2\right] + \mathcal{O}(w^2)$$

$\longrightarrow$ ψ_2, N_{Ti}

Finally... the metric!

- Do the same for tensor contribution induced by scalars (no tensors at 1st order):

$$\gamma_{ij} = -\frac{4}{5}\left[9\left(\frac{1}{3} - \frac{j_1(k\tau)}{k\tau}\right)\partial^{-2} + \frac{1}{5a^2H^2}\right] P^{\rm TT}_{ij\,kl}\left(\partial_k\zeta_0\partial_l\zeta_0\right)$$

- The final metric in matter dominance:

$$\begin{aligned}
g_{00} &= -1 + \frac{4}{25a^2H^2}(\partial_i\zeta_0)^2 \,, \\
g_{0i} &= -\frac{1}{5H}\partial_i\Big[2\zeta_0 - \partial^{-2}(\partial_j\zeta_0)^2 + 3\partial^{-4}\partial_j\partial_k(\partial_j\zeta_0\partial_k\zeta_0) \\
&\qquad - \frac{4}{5a^2H^2}\partial^{-2}\Big(\frac{3}{7}(\partial^2\zeta_0)^2 + \partial_i\zeta_0\partial_i\partial^2\zeta_0 + \frac{4}{7}(\partial_i\partial_j\zeta_0)^2\Big)\Big] \\
&\quad -\frac{4}{5}\frac{1}{H}\partial^{-2}\left[\partial_i\zeta_0\partial^2\zeta_0 - \partial^{-2}\partial_i\partial_j(\partial_j\zeta_0\partial^2\zeta_0)\right] \,, \\
g_{ij} &= a^2\exp[2\zeta(t)]\delta_{ij} + a^2\gamma_{ij} \,,
\end{aligned}$$

$$\zeta(t) = \zeta_0 - \frac{1}{5a^2H^2}\partial^{-2}\partial_k\partial_l(\partial_k\zeta_0\partial_l\zeta_0) \,,$$

- On large scales: $ds^2 = -dt^2 + a^2(t)e^{2\zeta_0(\vec{x})}d\vec{x}^2 \,, \qquad k \ll aH$
- Initial condition already built in the formalism (we always worked with ζ):

Second-order metric in MD

- After a gauge transformation in the generalized Poisson gauge:

$$ds^2 = a^2(\tau)\left\{-(1+2\Phi)d\tau^2 + 2\omega_i dx^i d\tau + [(1-2\Psi)\delta_{ij} + \gamma_{ij}]dx^i dx^j\right\}$$

$$\omega_{i,i} = 0 \text{ and } \gamma_{ij,i} = 0 = \gamma_{ii}$$

$$\begin{aligned}
\Phi =& \phi + \left[\phi^2 + \partial^{-2}(\partial_j\phi)^2 - 3\partial^{-4}\partial_i\partial_j(\partial_i\phi\partial_j\phi)\right] \\
&+ \frac{2}{21a^2H^2}\partial^{-2}\left[2(\partial_i\partial_j\phi)^2 + 5(\partial^2\phi)^2 + 7\partial_i\phi\partial_i\partial^2\phi\right] , \\
\Psi =& \phi - \left[\phi^2 + \frac{2}{3}\partial^{-2}(\partial_i\phi)^2 - 2\partial^{-4}\partial_i\partial_j(\partial_i\phi\partial_j\phi)\right] \\
&+ \frac{2}{21a^2H^2}\partial^{-2}\left[2(\partial_i\partial_j\phi)^2 + 5(\partial^2\phi)^2 + 7\partial_i\phi\partial_i\partial^2\phi\right] , \\
\omega_i =& -\frac{8}{3aH}\partial^{-2}\left[\partial^2\phi\partial_i\phi - \partial^{-2}\partial_i\partial_j(\partial^2\phi\partial_j\phi)\right] , \\
\gamma_{ij} =& -20\left(\frac{1}{3} - \frac{j_1(k\tau)}{k\tau}\right)\partial^{-2}P^{\mathrm{TT}}_{ij\,kl}\left(\partial_k\phi\partial_l\phi\right) ,
\end{aligned}$$

$$k \ll aH \quad \Rightarrow \quad \phi = -\frac{3}{5}\zeta_0$$

- Matches the metric found by Bartolo, Matarrese and Riotto