LIVING IN ROUGH (SPACE)TIMES

PHENOMENOLOGY OF COHERENT STATE NON-COMMUTATIVITY

Massimiliano Rinaldi

Departement de Physique Theorique
Universite de Geneve

April 19, 2010

Outline

What happens at the Planck length?

What happens at the Planck length?

- The Planck length is a combination of fundamental constants

$$L_p = \sqrt{\frac{\hbar G}{c^3}} \simeq 1.616\ 10^{-35}\ \mathrm{m}$$

but what's so special about it?

- To probe short distances we need high energies. To see below the Planck length we need a particle with Compton length λ_C such that

$$\lambda_C = \frac{\hbar}{Mc} \leq L_p \ , \qquad \rightarrow \qquad M \geq \frac{\hbar}{L_p c} \simeq 10^{19}\mathrm{GeV}$$

- According to General Relativity

$$R_S = \frac{2GM}{c^2} = 2L_p$$

- By probing the Planck length we create a black hole larger than it!

When Gravity Faces Quantum Mechanics...

Fundamental theories

- String Theory
- Loop Quantum Gravity
- Causal Dynamical Triangulations
- Deformed Lorentz Groups
- Path Integral Duality
- Star-Product Non-Commutativity
- Coherent States Non-Commutativity
- Hořava-Lifschitz

Phenomenology

- Modified Gravity
- Minimal Lengths
- Modified Dispersion Relations
  - Einstein-Aether theory
  - Analogue Models of Gravity

Modified Dispersion Relations

Modified Dispersion Relations

- Preferred frame encoded by a unit and dynamical timelike vector field u^μ (Jacobson and Mattingly, 2004)

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R + K^{ab}{}_{mn}\nabla_a u^m \nabla_b u^n + \lambda(g_{ab}u^a u^b - 1)\right]$$

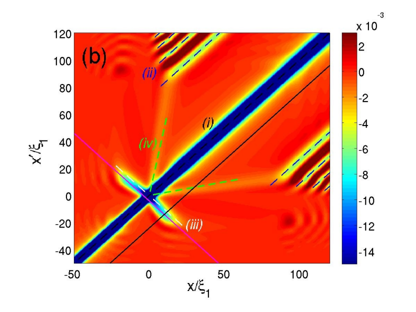
$$K^{ab}{}_{mn} = b_1 g^{ab} g_{mn} + b_2 \delta^a{}_m \delta^b{}_n + b_3 \delta^a{}_n \delta^b{}_m + b_4 u^a u^b g_{mn}$$

- General covariance is preserved. The unit constraint avoid negative-energy solutions.
- Cosmological and black hole solutions; $b_{1\ldots 4}$ are constrained by PPN analysis.
- When matter is coupled to u^μ we have modified dispersion relations

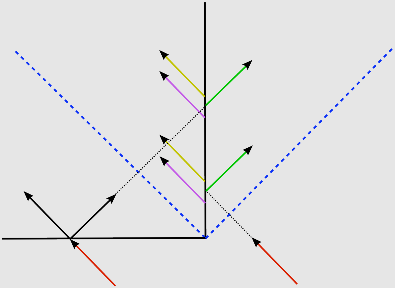
$$\left(\Box + m^2 + \sum_n \alpha_{2n}\nabla^{2n}\right)\phi = 0, \quad \omega^2 = m^2 + k^2 + \sum_n \alpha_{2n}|\vec{k}|^{2n}$$

- Unruh 1981: phonons propagate in superfluids as photons on a curved geometry. Sub-supersonic configuration forms an acoustic black hole.
- In Bose-Einstein condensates, the "healing length" sets a scale for Lorentz symmetry violation $\omega^2 = m^2 + |\vec{k}|^2 + \frac{|\vec{k}|^4}{k_0^2}$

MDR in the Lab



I. Carusotto et. al, New J. Phys. 10:103001, 2008



R. Balbinot, S. Fabbri, C. Mayoral, M. Rinaldi, in progress

- Hawking radiation is robust in black holes (Unruh, Jacobson et. al.) and in analogue models

- Unruh effect is robust M. Rinaldi, Phys. Rev. **D 77** 124029 (2008).

- Transplanckian problem in cosmology still open (Starobinski vs Brandenberger).

Phenomenological Implications of MDR

Are MDR realistic? October 2009, LAT collaboration

doi:10.1038/nature08574 nature

LETTERS

A limit on the variation of the speed of light arising from quantum gravity effects

A list of authors and their affiliations appears at the end of the paper

A cornerstone of Einstein's special relativity is Lorentz invariance—the postulate that all observers measure exactly the same speed of light in vacuum, independent of photon-energy. While special relativity assumes that there is no fundamental length-scale associated with such invariance, there is a fundamental scale (the Planck scale, $l_{\mathrm{Planck}} \approx 1.62 \times 10^{-33}$ cm or $E_{\mathrm{Planck}} = M_{\mathrm{Planck}}c^2 \approx 1.22 \times 10^{19}$ GeV), at which quantum effects are expected to strongly affect the nature of space–time. There is great interest in the (not yet validated) idea that Lorentz invariance might break near the Planck scale. A key test of such violation of Lorentz invariance is a possible variation of photon speed with energy[1–7]. Even a tiny variation in photon speed, when accumulated over cosmological light-travel times, may be revealed by observing sharp features in γ-ray burst (GRB) light-curves[2]. Here we report the detection of emission up to ~31 GeV from the distant and short GRB 090510. We find no evidence for the violation of Lorentz invariance, and place a lower limit of $1.2E_{\mathrm{Planck}}$ on the scale of a linear energy dependence (or an inverse wavelength dependence), subject to reasonable assumptions about the emission (equivalently we have an upper limit of $l_{\mathrm{Planck}}/1.2$ on the length scale of the effect). Our results disfavour quantum-gravity theories[3,6,7] in which the quantum nature of space–time on a very small scale linearly alters the speed of light.

scale (when E_{ph} becomes comparable to $E_{\mathrm{Planck}} = M_{\mathrm{Planck}}c^2$). For $E_{\mathrm{ph}} \ll E_{\mathrm{Planck}}$, the leading term in a Taylor series expansion of the classical dispersion relation is $|v_{\mathrm{ph}}/c - 1| \approx (E_{\mathrm{ph}}/M_{\mathrm{QG},n}c^2)^n$, where $M_{\mathrm{QG},n}$ is the quantum gravity mass for order n and $n = 1$ or 2 is usually assumed. The linear case ($n = 1$) gives a difference $\Delta t = \pm(\Delta E/M_{\mathrm{QG},1}c^2)D/c$ in the arrival time of photons emitted together at a distance D from us, and differing by $\Delta E = E_{\mathrm{high}} - E_{\mathrm{low}}$. At cosmological distances this simple expression is somewhat modified (see Supplementary Information section 4).

Because of their short duration (typically with short substructure consisting of pulses or narrow spikes) and cosmological distances, GRBs are well-suited for constraining LIV[2,11,12]. Individual spikes in long[13] (of duration >2 s) GRB light-curves (10–1,000 keV) usually show[14] intrinsic lags: the peak of a spike occurs earlier at higher photon-energies. However, there are either no lags or very short lags of either sign for short GRBs[15]. Thus far, intrinsic lags have been seen only on timescales of up to the width of individual spikes in a light curve, which for GRB 090510 are $\sim 10^{-2}$ s. Intrinsic lags have not yet been measured at high energies; if they are also present there, it is reasonable to assume that their behaviour is similar to that at low-energies (at least approximately).

When allowing for LIV-induced time-delays, the measured arrival

Coherent State Non-Commutativity

An alternative model of NC - Smailagic and Spallucci, 2003

- 2-dimensional space: coordinate operators such that $[\hat{x}_1, \hat{x}_2] = iL^2$
- Define

$$\hat{A} = \frac{1}{\sqrt{2}L}(\hat{x}_1 + i\hat{x}_2), \quad \hat{A}^\dagger = \frac{1}{\sqrt{2}L}(\hat{x}_1 - i\hat{x}_2), \quad [\hat{A}, \hat{A}^\dagger] = 1$$

- Coherent states are defined by $\hat{A}|\alpha\rangle = \alpha|\alpha\rangle$
- Define the ordinary commuting coordinates as the expectation values: $\langle\alpha|\hat{x}_1|\alpha\rangle = \sqrt{2}L\,\mathrm{Re}(\alpha) \equiv y_1 \;\;, \quad \langle\alpha|\hat{x}_2|\alpha\rangle = \sqrt{2}L\,\mathrm{Im}(\alpha) \equiv y_2$
- The vector $\vec{y} = (y_1, y_2)$ describes the mean position of the particle.

- Momenta $\vec{p} = (p_1, p_2)$ are commuting and the new "plane-wave function" of a free point particle on the NC plane is $[p_\pm = (p_1 \pm ip_2)/2]$

$$e^{i\vec{p}\cdot\vec{x}} \to \langle\alpha|e^{ip_1\hat{x}_1 + ip_2\hat{x}_2}|\alpha\rangle = \langle\alpha|e^{ip_+\hat{A}^\dagger}e^{ip_-\hat{A}}e^{-\theta p_+ p_-}|\alpha\rangle = e^{-\frac{L^2}{4}(p_1^2+p_2^2)+i\vec{p}\cdot\vec{y}}$$

- Note the relative sign between p_1^2 and p_2^2: it is independent of the metric signature.
- The Fourier transform is modified:

$$F(y) = (2\pi)^{-2}\int d^2p\,\tilde{F}(p)\,e^{-\frac{L^2}{4}(p_1^2+p_2^2)+i\vec{p}\cdot\vec{y}}$$

NC Quantum Field Theory - M. Rinaldi - ArXiv:1003.2408

- Scalar field of mass m satisfies the usual Klein-Gordon equation in Minkowski space with (mean) coordinates (t,x) $(\Box + m^2)\phi(t,x) = 0$ but the mode normalization is modified according to

$$u_p(t,x) = \frac{e^{-L^2(\omega^2+p^2)}}{\sqrt{4\pi\omega}}\, e^{-i\omega t + i\vec{p}\cdot\vec{x}}, \quad \omega^2 = m^2 + p^2$$

- The Klein-Gordon product reflects the non-orthogonality of the coherent states:

$$(u_p, u_p') = e^{-2L^2(\omega^2+p^2)}\delta(p-p')$$

- A scalar field can be represented as the usual mode sum:

$$\phi(t,x) = \int \frac{d\vec{p}}{\sqrt{4\pi\omega}} \left[\hat{a}_p u_p(t,x) + \hat{a}_p^\dagger u_p^*(t,x)\right] \ , \ [\hat{a}_p, \hat{a}_{p'}^\dagger] = 4\pi\omega\delta(p-p')$$

- The equal-time commutator reads

$$[\phi(t,x), \dot{\phi}(t,x')] = \frac{i}{4\sqrt{\pi}L}\, e^{-2L^2m^2 - \frac{(x-x')^2}{16L^2}} .$$

In the limit $L \to 0$ we recover the strandard $i\delta(x-x')$.

NC Quantum Field Theory - M. Rinaldi - ArXiv:1003.2408

- The Wightman functions are

$$G^+(x^\mu, x'^\mu) \equiv \langle 0|\phi(x^\mu)\phi(x'^\mu)|0\rangle = \int \frac{d\vec{p}}{4\pi\omega}\, e^{-2L^2(\omega^2+p^2)-ip_\mu(x^\mu-x'^\mu)}$$

- The Feynman propagator reads

$$G_F = -i \int \frac{d\vec{p}}{4\pi\omega}\, e^{-2L^2(\omega^2+p^2)} \left[\theta(t-t')\, e^{-ip_\mu(x^\mu-x'^\mu)} + \theta(t'-t)\, e^{ip_\mu(x^\mu-x'^\mu)}\right]$$

- This propagator satisfies the equation

$$(\Box + m^2)G_F(x^\mu, x'^\mu) = -\frac{i}{8\pi L^2}\, e^{-\frac{(\Delta t^2+\Delta x^2)}{8L^2}}$$

- The Feynman propagator can also be written as

$$G_F(x^\mu, x'^\mu) = i \int \frac{d^2p}{(2\pi)^2}\, \frac{e^{-2L^2(\omega^2+p^2)-ip_\mu(x^\mu-x'^\mu)}}{\omega^2-p^2-m^2}\ ,$$

  from which we can easily read off the momentum space propagator

$$\tilde{G}_F(\omega, p) = \frac{e^{-2L^2(\omega^2+p^2)}}{\omega^2-p^2-m^2}$$

NC Quantum Field Theory - M. Rinaldi - ArXiv:1003.2408

- In (mean) coordinate space we find:

$$G(t,x;t',x')_{m=0} = -\frac{1-e^{-\frac{(t-t')^2+(x-x')^2}{2L^2}}}{4\pi^2[(t-t')^2-(x-x')^2]}$$

$$G(t,x;t',x')_{m\to 0} = -\frac{1}{8\pi^2 L^2} + m^2 e^{\frac{m^2L^2}{2}}\,\mathrm{Ei}\left(\frac{m^2L^2}{2}\right) + \dots$$

In the coincident limit $(x,t)\to(x',t')$ the propagator is UV finite.

- The Hamiltonian operator becomes

$$\hat{H} = \frac{1}{2}\int d^2x\left[\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right] = \frac{1}{2}\int d\vec{p}\, e^{-2L^2(p^2+\omega^2)}\omega\left(\hat{a}_p\hat{a}_p^\dagger + \hat{a}_p^\dagger\hat{a}_p\right)$$

- Normal ordering is no longer necessary as

$$\langle 0|\hat{H}|0\rangle_{m\neq 0} = e^{-2L^2m^2}\int_0^\infty dp\, e^{-4L^2p^2}\sqrt{p^2+m^2} < \infty$$

$$\langle 0|\hat{H}|0\rangle_{m=0} = \frac{1}{8L^2}$$

NC Quantum Field Theory - M. Rinaldi - ArXiv:1003.2408

- In curved space no global killing vectors and positive and negative frequency mixing so $\phi(x) = \sum_i (\,\hat{a}_i u_i + \text{h.c.}) = \sum_j (\,\hat{b}_j v_j + \text{h.c.})$
- The relations between u and v mode sets are non-trivial $\quad v_j = \sum_i (\alpha_{ij} u_i + \beta_{ij} u_i^*)$.

  α_{ij} , β_{ij} are the Bogolubov coefficients. When $\beta_{ij} = (v_j, u_i^*) \neq 0$ we have particle creation. The vacuum state with respect to u is seen as a populated state by v.

- In NC the situation *apparently* does not change. We write the damped modes as

$$U_i = g_u u_i \;, \quad V_i = g_v v_i \;, \quad g_{u,v} \sim e^{-L^2(\omega_{u,v}^2 + p^2)}$$

  The β_{ij} coefficient is unchanged as $\beta_{jl} = -\frac{1}{g_u g_v}(V_j, U_l^*) \equiv (v_j, u_l^*)$. Thus $\langle N_i \rangle = \sum_j |\beta_{ij}|^2$ is unchanged.

- However, the energy density $\hat{H}_i = \frac{1}{2} \int d\vec{p}\, e^{-2L^2(p^2+\omega^2)} \omega \left(\frac{1}{2} + \hat{N}_i \right)$ is damped. High frequency modes do not contribute to the energy density!
- Solution to the trans-Planckian problem?

NC Quantum Field Theory in Higher Dimensions

- We can extend the above construction to higher dimensions.
- Assume that $2n$ coordinates do not commute $[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}$, where $\Theta^{\mu\nu}$ is an anti-symmetric, constant Lorentz tensor.
- Use Lorentz transformation to write, in D-dimensions

$$\Theta_{\mu\nu} = \mathrm{diag}(\Theta_1, \Theta_2, \ldots, \Theta_{D/2}) \ , \quad \Theta_i = \theta_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- We define $\frac{D}{2}$ planes, on each of which we repeat the construction above.
- The momentum space propagator is

$$G(\vec{p}_1, \ldots, \vec{p}_{D/2}) = \frac{1}{\left(\vec{p}_1^{\,2} + \vec{p}_2^{\,2} + \ldots + \vec{p}_{D/2}^{\,2} + m^2\right)} \exp\left(-\frac{1}{2}\sum_{j=1}^{D/2} \theta_j \vec{p}_j^{\,2}\right)$$

- It can be shown that if $\theta_i = \theta$ for all i the propagator is covariant (Smailagic and Spallucci, 2004).

"Conventional" NC Field Theory

- On $\mathbb{R}^d$, coordinates are operators $[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}$, antysimmetric and constant matrix.
- Based on the $\star$-product:

$$(f \star g)(x) = e^{\frac{i}{2}\Theta^{\mu\nu}\partial^y_\mu \partial^z_\nu} f(y) g(z)\Big|_{y=z=x}$$

- Actions for fields are unchanged $\quad S = \int d^d x \mathcal{L}[\phi]$
- However:

$$\int d^d x\, (\phi \star \phi) = \int d^d x\, \phi^2 \;, \quad \int d^d x\, (\partial\phi \star \partial\phi) = \int d^d x\, (\partial\phi)^2$$

therefore free theories are unchanged. NC is visible only when interactions are present, through a phase factor in the vertex of the Feynman rules. For a ϕ^n theory:

$$V(k_1 \cdots k_n) = \exp\left[-\frac{i}{2} \sum_{i<j} k_{i\mu} \Theta^{\mu\nu} k_{j\nu} \right]$$

- For practical calculations, usually one truncates $\exp\left[\frac{i}{2}\Theta^{\mu\nu}\partial^y_\mu \partial^z_\nu\right]$ loosing non-locality.

Comparison with ⋆-product NC

What are the advantages of the coherent state approach?

- Simplicity:
  - Based on well known QM.
  - No ⋆-product: also free fields feel NC.
  - Minimal modification of QFT.
  - Unitary, no IR/UV mixing, UV-finite.

- The field theory is completely known
  - In theories where $G \sim (\Delta x^2 + \ell_p^2)^{-1}$ (Parker, Padmanabhan) we do not know the KG equation.
  - Dispersion relations are not modified.
  - In four-dimension the theory is covariant.

Unruh Effect - Nicolini and Rinaldi, ArXiv: 0910.2860

- Detector moving in flat spacetime, on accelerated trajectory.
- First order amplitude

$$d\Gamma = i\langle E,\psi| \int_{-\infty}^{+\infty} d\tau L_{\text{int}} |0_{\text{M}}; E_0\rangle, \quad L_{\text{int}} = \gamma\, \mu(\tau)\, \phi[x(\tau)]$$

- Transition probability at leading order $\mu(\tau) \simeq e^{iH_0\tau}\mu(0)\, e^{-iH_0\tau}$:

$$\Gamma \simeq \gamma^2 \sum_E \left|\langle E|\mu(0)|E_0\rangle\right|^2 \mathcal{F}(\Delta E)$$

- Detector's response rate function

$$\mathcal{F}(\Delta E) = \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau'\, e^{-i\Delta\tau\Delta E}\, G^+(x(\tau), x(\tau'))$$

- On a trajectory parameterized by τ, G^+ depends on $\Delta\tau$ $\rightarrow$ response rate

$$\boxed{\dot{\mathcal{F}}(\Delta E) = \int_{-\infty}^{+\infty} d\Delta\tau\, e^{-i\Delta\tau\Delta E}\, G^+(\Delta\tau)}$$

Unruh Effect - Nicolini and Rinaldi, ArXiv: 0910.2860

- Hyperbolic trajectory $z = y = 0$, $x = \sqrt{t^2 + a^{-2}}$, $a =$ acceleration

$$G^+(\Delta x) = -\frac{a^2}{16\pi^2 \sinh^2\left[\frac{a(\tau - \tau' - 2i\epsilon)}{2}\right]}, \quad t = \frac{\sinh(\tau a)}{a}$$

The response rate is

$$\dot{\mathcal{F}}(\Delta E) \sim \frac{1}{e^{2\pi\Delta E/a} - 1}$$

i.e a thermal spectrum with temperature $T = a/(2\pi k_B) \propto$ acceleration

- In Euclidean NC theory

$$G_E(t, x; t', x') = -\frac{1 - e^{-\frac{(t-t')^2 + (x-x')^2}{2L^2}}}{4\pi^2[(t-t')^2 + (x-x')^2]}$$

Unruh Effect - Nicolini and Rinaldi, ArXiv: 0910.2860

Modified response rate for a trajectory $f(\Delta\tau)$ on the Euclidean plane:

$$\dot{\mathcal{F}} = \frac{1}{4\pi^2}\int_{i\infty}^{-i\infty} d\Delta\tau\, e^{-\Delta E\Delta\tau}\left[\frac{1-e^{-\frac{f(\Delta\tau)}{2L^2}}}{f(\Delta\tau)}\right]$$

- If $f(\Delta\tau)$ is smooth enough, there are no poles.
- For a Rindler's trajectory: $f(\Delta\tau) = 4a^{-2}\sin^2(a\Delta\tau/2)$ (periodic on the Euclidean plane) so:

$$\dot{\mathcal{F}} \simeq \frac{1}{16}\left[-\frac{9}{2\sqrt{\theta}} + a^2\sqrt{\theta}\right] e^{-\Delta E^2\theta} + \mathcal{O}(\theta^{1/2}\Delta E^2) + \mathcal{O}(a^2\theta^{3/2}\Delta E^2)$$

- The leading term is negative and does not depend on the acceleration.
- It diverges for $\theta \to 0$: the integral in τ and the limit $\theta = 0$ do not commute!
- The next-to-leading order term depends on a but it is not thermal.
- One needs calibration in order to measure this higher order effect (see Parker et. al.).
- The leading term can be interpreted as a dissipation effect.

Inflationary Universe

Inflationary Universe

- One-loop effective action: massive scalar field

$$\mathcal{L}_{\rm eff} = -\frac{i}{2}\int_{i\theta/2}^{\infty}\frac{ds}{s}\, e^{-im^2 s}\, e^{\frac{\theta}{2}\Box_x} K_{\rm DS}(x,x';s), \qquad L^2 \propto \theta$$

$$\text{Coincident points} \quad \rightarrow \quad K_{\rm DS}(x,x;s) = -\frac{i}{16\pi^2 s^2}\sum_{n=0}^{\infty} a_n(x,x)(is)^n$$

$$a_0(x,x) = 1\,, \quad a_1(x,x) = \frac{R}{6}\,, \quad a_2(x,x) = \frac{R^2}{72} + \cdots$$

- We can integrate: $\mathcal{L}_{\rm eff} = L_0 + L_1\, a_1(x,x) + L_2\, \tilde{a}_2(x,x) + \ldots$

$$L_0 = \frac{m^4}{64\pi^2}\left[\frac{(4+2\theta m^2)\,e^{\frac{\theta}{2}m^2}}{m^4\theta^2} + \mathrm{Ei}\left(1, -\frac{\theta m^2}{2}\right)\right], L_1 = -\frac{m^2}{32\pi^2}\left[\frac{2\,e^{\frac{\theta}{2}m^2}}{m^2\theta} + \mathrm{Ei}\left(1, -\frac{\theta m^2}{2}\right)\right]$$

- If $\mathcal{L} = (16\pi\, G_{\rm bare})^{-1}(R - 2\Lambda_{\rm bare}) + \mathcal{L}_{\rm eff}$ then

$$\boxed{\Lambda_{\rm eff} = \Lambda_{\rm bare}\left(\frac{1 - 8\pi L_1\, G_{\rm eff}}{3}\right) + 8\pi\, G_{\rm eff}\, L_0\,, \quad G_{\rm eff} = \frac{3\, G_{\rm bare}}{3 + 8\pi L_1\, G_{\rm bare}}}$$

- In the massless case $L_0 = (16\pi^2\theta^2)^{-1}$ and $L_1 = (16\pi^2\theta)^{-1}$

Semiclassical approach - M. Rinaldi - ArXiv: 0908.1949

- According to the semiclassical picture: $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle$
- In general $\langle T_{\mu\nu} \rangle$ diverges. With point-splitting

$$\begin{aligned} \langle T_{\mu\nu}(x,x') \rangle_E &= \frac{1}{2} \left(g^{\alpha'}_{\mu} \nabla_{\alpha'} \nabla_\nu + g^{\alpha'}_{\nu} \nabla_\mu \nabla_{\alpha'} \right) G_E(x,x') + \\ &- \frac{1}{2} g_{\mu\nu} \left(g^{\alpha' \beta} \nabla_{\alpha'} \nabla_\beta + m^2 \right) G_E(x,x') \ , \end{aligned}$$

- In NC theory:

$$G_E(x,x) = \int_0^\infty ds K(x,x;s) \ , \quad K(x;s) = \frac{e^{-m^2 s}}{16\pi^2 (s+\theta)^2} \left[1 + e^{[s\theta/(s+\theta)]\Box} \sum_{n=1}^{\infty} s^n a_n(x) \right]$$

- by expanding $16\pi^2 G_E(x,x) = \frac{a_0}{\theta} - F_1(\theta m^2) a_1(x) + \cdots$
- also $(\Box_x + m^2) G_E(x,y) = -\, e^{\theta \Box_x} \left[\frac{1}{\sqrt{g}} \delta^{(4)}(x,y) \right]$
- Therefore

$$\langle T_{\mu\nu} \rangle = \nabla_\nu \nabla_\mu G_E(x) + \frac{1}{2} g_{\mu\nu} \lim_{y \to x} e^{\theta \Box_x} \left[\frac{\delta^{(4)}(x,y)}{\sqrt{g}} \right] = \frac{g_{\mu\nu}}{32\pi^2 \theta^2} + \text{curvature corrections}$$

- Maximum and constant energy density $\rho_{NC} \propto \theta^{-2} \propto H^2$: inflationary solution?

Inflationary Universe - Thermodynamics

- NC modifies the bosonic mean occupation number:

$$\langle n_\omega \rangle = \frac{1}{\exp\left(\frac{\hbar\omega}{kT} + \frac{\theta}{2}\omega^2\right) - 1}$$

- Stephan-Boltzman law:

$$\rho = \frac{(kT)^4}{\pi^2 c^3 \hbar^3} \int_0^\infty \frac{x^3\, dx}{e^{x+Ax^2} - 1}\ , \quad A = \frac{\theta}{2}\left(\frac{kT}{\hbar}\right)^2$$

- When T is large

$$\int_0^\infty \frac{x^3\, dx}{e^{x+Ax^2} - 1} \simeq \frac{\pi^2 \hbar^4}{3\theta^2 k^4 T^4} + \mathcal{O}(T^{-5}) \quad \Longrightarrow \rho \simeq \frac{\hbar}{3c^3\theta^2} + \mathcal{O}(T^{-1})$$

- Also, $\omega = \omega(T)$. In the early Universe

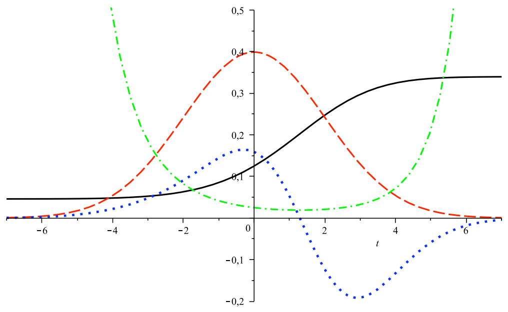
$$H^2 \simeq \frac{\hbar}{9M_p^2 c^3 \theta^2} \quad \Longrightarrow \text{de Sitter phase?} \Longrightarrow \text{Bounce?}$$

Inflationary Universe - Speculations

- NC replaces $\delta(x)$ with Gaussian functions. For black hole, the point mass M becomes $\rho(r) = \frac{M}{(4\pi\theta)^{3/2}}\, e^{-\frac{r^2}{4\theta}}$. This is justified with the "Voros" product
- Can we say the same for the energy density near a singularity? Namely:

$$\rho(t \sim 0) = \frac{\rho_0}{\theta^2}\, e^{-t^2/\theta}$$

If so, the Friedmann equation $H^2 \propto \rho$ gives



Qualitative behaviour of a (solid black line), $\ddot{a}$ (dotted line), H (dashed line), and $(aH)^{-1}$ (dot-dashed line) as functions of time (M. Rinaldi, ArXiv: 0908.1949)

Black Holes

Black Holes

From P. Nicolini, A. Smailagic, E. Spallucci, Phys. Lett. B632 547, 2006:

- Smeared pointlike source $\rho(r) = \frac{M}{(4\pi\theta)^{3/2}}\, e^{-\frac{r^2}{4\theta}}, \quad L^2 \propto \theta$
- Effective stress tensor

$$T^{\mu}{}_{\nu} = -\mathrm{diag}\left(\rho,\, \rho,\, \rho + \frac{r}{2}\frac{\partial\rho}{\partial r},\, \rho + \frac{r}{2}\frac{\partial\rho}{\partial r}\right), \quad T^{\mu\nu}{}_{;\mu} = 0$$

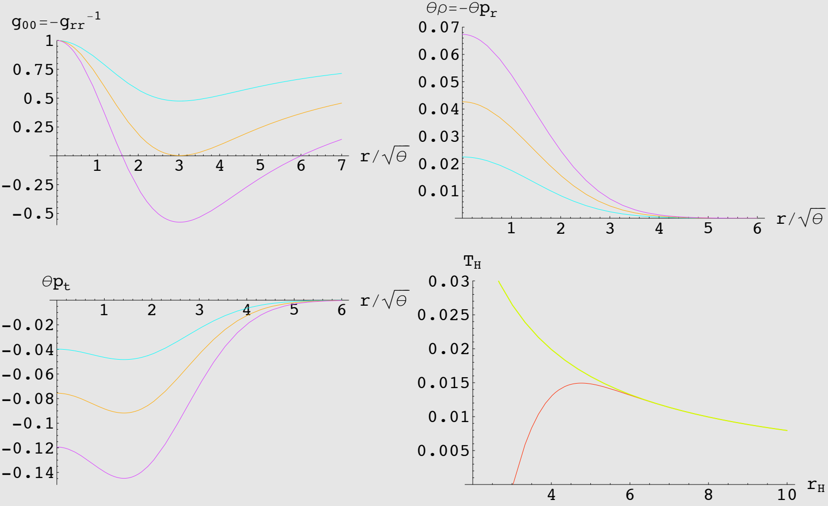
- Metric $ds^2 = -g_{tt}\,dt^2 + g_{rr}\,dr^2 + r^2\,d\Omega^2$ with

$$g_{tt} = -g^{rr} = -\left[1 - \frac{2M}{r}\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right)\right]$$

- There can be zero, one or two horizons. The 2-horizon solution evolves towards the 1-horizon (extremal) configuration.

- Near the origin, geometry is de Sitter. Negative pressure plays the role of a positive cosmological constant.
- Large black holes, $T \sim (4\pi r_H)^{-1}$. For small ones, the temperature reaches a maximum and then drops to zero.
- The Hawking radiation is the same for large black holes but it stops at the extremal configuration. There are stable remnants: dark matter?

Black Holes



Courtesy of P. Nicolini, A. Smailagic, E. Spallucci, Phys. Lett. B632 547, 2006.

Conclusions

We have seen that

- observations seem to rule out short distance modifications of dispersion relations
- we can introduce a minimal length in a covariant way, e.g. NC on coherent states
- this can lead to important effects in QFT, black holes and inflationary cosmology.

What we intend to do is to look at

- cosmological perturbation and possible signatures in CMB (in progress)
- to study the transplanckian problem in the Hawking effect (in progress)
- black hole thermodynamics, early cosmological solutions...

Path Integral Duality

Path Integral Duality (Padmanabhan PRL 78, 1854)

Can we introduce a minimal length in a covariant way?

- Modified euclidean Feynman propagator for $(\hat{H} + m^2)\Phi = 0$

$$G_F(x,x') = \int_0^\infty ds\, e^{-m^2 s}\, e^{-L_p^2/s}\, K(x,x';s), \quad K(x,x';s) = \langle x | e^{-i\hat{H}s} | x' \rangle$$

- Invariant under $ds \to L_p^2/ds$: $\tilde{G}_F(p) \sim \begin{cases} \frac{1}{p^2+m^2} & L_P \ll 1 \\ \frac{\exp(\sqrt{p^2+m^2})}{p^2+m^2} & L_P \gg 1 \end{cases}$
- In coordinate space: $G(x,x') \sim \frac{1}{(x-x')^2+L_p^2}$ $\to$ relativistic propagator.
- Links with string T-duality: same propagator as for the CoM of a bosonic string.

- Small corrections to the Unruh and Casimir effects.
- No visible effects on cosmological spectra.
- In curved space $\to$ deWitt-Schwinger expansion $\to$ Rescaled G_N and Λ