Black-hole instabilities

and instabilities in supersonic flows

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Plan

- I. Black hole instabilities: a brief review.
- II. Black holes in BEC.
  - Phonon mode equation in BEC: **exact eq.**
  - **Phonon spectra** in **supersonic** flows with
    - **one** sonic BH **or** WH horizon,
    - **a pair** of BH **and** WH horizons.
  - Impact of the **second** horizon on **observables**.
  - **Classical** vs **Quantum** description of **dyn. instabilities**, link with **Quasi Normal Modes**.
  - **General conditions** to have **dyn. instabilities**.

Black hole instabilities. 1. Pre-history

- The **stability** of the Schwarzschild Black Hole

$$ds^2 = -(1 - \frac{r_S}{r})\, dt^2 + \frac{dr^2}{(1 - \frac{r_S}{r})} + r^2(d\theta^2 + sin^2\theta d\phi^2),$$

with $r_S = 2GM/c^2$, was a subject of **controversy** $\rightarrow$ 50's.

- **stability** demonstrated by Wheeler (and others), i.e., The **spectrum** of metric perturbations contains **no complex frequency asympt. bound modes**

- its (astro)-physical relevance recognized.

Black hole instabilities. 2. Super-radiance

A **rotating Black Hole** (Kerr) is subject to a **weak instability**:

- **Classical** waves display a **super-radiance**:

$$\phi^{\text{in}}_{\omega,l,m} \to R_{\omega,l,m}\, \phi^{\text{out}}_{\omega,l,m} + T_{\omega,l,m}\, \phi^{\text{absorbed}}_{\omega,l,m},$$

  with

$$|R_{\omega,l,m}|^2 > 1.$$

  Energy is **extracted** from the hole.
  This is a **stimulated** process.

- At the **Quantum** level, **super-radiance implies** a **steady spontaneous** pair creation process, i.e. a **"vacuum instability"**.

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Black hole instabilities. 3. Black hole Bomb

- When introducing a **reflecting boundary condition**, the **super-radiant instability** induces a **dynamical instability** a **Black Hole Bomb**, Press '70, Kang '97, Cardoso et al '04.
- A **non-zero mass** can induce a **reflection**, Damour et al '76; this is presently used to constrain the mass of 'axions'.
- As in a resonant cavity, the **spectrum** now contains a **discrete set** of modes with **complex** frequencies.

Black hole instabilities. 4. Hawking radiation

- In 1974, Hawking showed that a Schwarzschild Black Hole **spontaneously** emits thermal radiation.
- Even though it is **micro-canonically** **stable**, it is **canonically** **unstable**.
- Indeed, the partition function possesses an **unstable** **bound mode** (Gross-Perry-Yaffe '82).
- **N.B.** The **same** **bound mode** is responsible for the **dynamical instability** of 5 dimensional **Black String** (Gregory-Laflamme '93).

Black hole instabilities, 5. Black Hole Laser

- discovered by Corley & Jacobson in 1999,
- arises in the presence of **two** horizons (charged BH) **and** with **superluminal** **dispersion**,
- the 'trapped' region acts as a **cavity**,
- induces an **exponential growth** of **Hawking radiation**, and constitutes a **dynamical instability**.

- **Naturally** arises in **Acoustic Black Holes**: **supersonic flows** in Bose Einstein condensates,
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II. Black hole lasers (in BEC)

- studied in terms of **time-dep. wave-packets**, both by Corley & Jacobson in '99, and Leonhardt & Philbin in '08.
- instead, in what follows,
  a **spectral analysis** of **stationary modes**.
- see also
  Garay et al. PRL 85 and PRA 63 (2000/1), BH/WH flows in BEC
  Barcelo et al. PRD 74 (2006), Dynam. stab. analysis
  and Jain et al. PRA 76 (2007). Quantum De Laval nozzle

Bose Einstein Condensates

- Set of atoms is described by $\hat{\Psi}(t, \mathbf{x})$ obeying

$$[\hat{\Psi}(t, \mathbf{x}), \hat{\Psi}^{\dagger}(t, \mathbf{x}')] = \delta^3(\mathbf{x} - \mathbf{x}'),$$

  and by a Hamiltonian

$$\hat{H} = \int \mathrm{d}^3x \left\{ \frac{\hbar^2}{2m} \nabla_{\mathbf{x}} \hat{\Psi}^{\dagger} \nabla_{\mathbf{x}} \hat{\Psi} + V(x) \hat{\Psi}^{\dagger} \hat{\Psi} + \frac{g(x)}{2} \hat{\Psi}^{\dagger} \hat{\Psi}^{\dagger} \hat{\Psi} \hat{\Psi} \right\}.$$

- at low temperature, $\hat{\Psi}$ is expanded as

$$\begin{aligned} \hat{\Psi}(t, \mathbf{x}) &= \Psi_0(t, \mathbf{x}) + \hat{\psi}(t, \mathbf{x}) \\ &= \Psi_0(t, \mathbf{x}) \left(1 + \hat{\phi}(t, \mathbf{x})\right), \end{aligned} \qquad (1)$$

  $\Psi_0(t, \mathbf{x})$ describes the **condensed atoms**,
  $\hat{\phi}(t, \mathbf{x})$ describes **relative perturbations**.

A 1D stationary condensate is described by

$$\Psi_0(t,x) = e^{-i\mu t/\hbar} \times \sqrt{\rho_0(x)}\, e^{i\theta_0(x)},$$

ρ_0 is the **mean density** and $v = \frac{\hbar}{m}\partial_x\theta_0$ the **mean velocity**.

ρ_0, v are determined by V and g through
the **Gross Pitaevskii** eq.

$$\mu = \frac{1}{2}mv^2 - \frac{\hbar^2}{2m}\frac{\partial_x^2\sqrt{\rho_0}}{\rho_0} + V(x) + g(x)\,\rho_0,$$

which also gives

$$\partial_x(v\rho_0) = 0.$$

BdG equation for relative density fluctuations

- In a BEC, density fluctuations obey the **BdG equation**. For relative fluctuations, this eq. is

$$i\hbar(\partial_t + v\partial_x)\,\hat{\phi} = \left[T_v + mc^2\right]\hat{\phi} + mc^2\hat{\phi}^\dagger, \tag{2}$$

$$c^2(x) \equiv \frac{g(x)\rho_0(x)}{m},$$

is the x-dep. **speed of sound** and T_v a kinetic term

$$T_v \equiv -\frac{\hbar^2}{2m}\,v\partial_x \frac{1}{v}\partial_x .$$

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Covariantizing the BdG equation ?

Phonons

- only see the macrosc. **mean** fields $c(x), \mathbf{v}(x), \rho_0(x)$,
- are **insensitive** to microsc. qtts $g(x)$, $V(x)$ and Q.pot.

Hence one can

- forget about the (fundamental) theory of the condensate, when **computing the phonon spectrum**.
- consider the phonon field from a **4D point of view** by **covariantizing** the BdG eq. introducing **4D tensors**
  - the (Unruh) acoustic metric $g_{\mu\nu}(t, x)$
  - the (Jacobson) unit time-like vector field $u^\mu(t, x)$
  - extra scalars ...

Not just an **analogy**, but an **equivalent point of view**.

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Computing phonon spectra. 1.

- basically **equivalent** to that of a **hermitian scalar field**.
- to handle the complex character of $\hat{\phi}$, it is useful (Leonhardt et al. '03) to introduce the **doublet**

$$\hat{W} \equiv \begin{pmatrix} \hat{\phi} \\ \hat{\phi}^{\dagger} \end{pmatrix},$$

invariant under a **pseudo-Hermitian conjugation** **(**$pH.c.$**)**

$$\hat{W} = \bar{\hat{W}} \equiv \sigma_1 \hat{W}^{\dagger}.$$

- The mode decomposition of $\hat{W}$ is

$$\hat{W} = \sum_n (W_n \, \hat{a}_n + \bar{W}_n \, \hat{a}_n^{\dagger}) = \sum_n (W_n \, \hat{a}_n + pH.c.), \quad (3)$$

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Computing spectra. 2. The inner product

- The **conserved inner product**

$$\langle W_1 | W_2 \rangle \equiv \int \mathrm{d}x \, \rho_0(x) \, W_1^*(t,x) \, \sigma_3 \, W_2(t,x), \qquad (4)$$

is **not** **positive definite** (c.f. the Klein-Gordon product).

- **As usual**, mode orthogonality

$$\langle W_n | W_m \rangle = -\langle \bar{W}_n | \bar{W}_m \rangle = \delta_{nm},$$

implies canonical commutators

$$[\hat{a}_n, \hat{a}_m^\dagger] = \delta_{nm},$$

where

$$\hat{a}_n = \langle W_n | \hat{W} \rangle.$$

Computing spectra.
3. The notion of Asympt. Bound Modes

For stationary backgrounds with **infinite spatial extension** the **solutions** of

$$H\, W_\lambda(x) = \lambda\, W_\lambda(x), \tag{5}$$

which **belong to the spectrum must be Asymptotically Bound**: bound for $x \to \pm\infty$.

- N.B.1. Hence, in certain **non-homogeneous** backgrounds, **the freq.** λ can be **complex**.
- N.B.2. **Quasi Normal Modes** are **not ABM**, hence are **not** in the spectrum.

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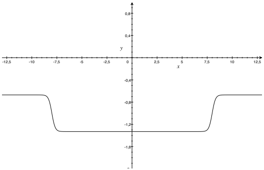
The background stationary profiles

Flows with **one** or **two** **sonic horizons**, $c = |v|$:
That is, $v(x) < 0$ and, for **one** horizon:

$$c(x) + v(x) = c_{\mathrm{H}} D \tanh\left(\frac{\kappa_{\mathrm{B}} x}{c_{\mathrm{H}} D}\right),$$

where $\kappa_{\mathrm{B}} = \partial_x (c + v)|_{\mathrm{hor.}}$, Carter's **decay rate** $\sim$ surf. gravity,
and for **two** horizons:

$$c(x) + v(x) = c_{\mathrm{H}} D \tanh\left(\frac{\kappa_{\mathrm{W}}(x + L)}{c_{\mathrm{H}} D}\right) \tanh\left(\frac{\kappa_{\mathrm{B}}(x - L)}{c_{\mathrm{H}} D}\right),$$



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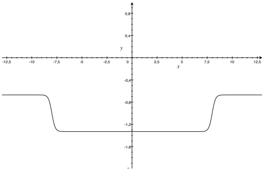

Spectrum of W_n for one B/W sonic horizon

The **complete set of modes** is (Macher-RP 2009)

- a **continuous** set of **real frequency** modes which contains
- for $\omega > \omega_{\max}$, **two positive** norm modes, as in flat space, W_ω^u, W_ω^v, which resp. describe right/left moving phonons,
- for $0 < \omega < \omega_{\max}$, **three** modes: 2 positive norm W_ω^u, W_ω^v + 1 **negative** norm mode $\bar{W}_{-\omega}^u$.
- The **threashold** freq. $\omega_{\max}$ scales $1/\text{healing length} = mc/\hbar$, but **also** depends on $D = (v_{\text{asympt.}} + c_{\text{asympt.}})/c_H$.

- **Lessons:**
  - There are **no complex** freq. **ABM**,
  - **Same spectrum** for **White Holes** and **Black Holes**, because **invariant** under $v \to -v$.
  - Hence **White Hole** flows are **dyn. stable**, as **BH** ones.

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The scattering of *in*-modes

- For $\omega > \omega_{\rm max}$, there is an **elastic** scattering:

$$W^{u,\,in}_\omega = T_\omega W^{u,\,out}_\omega + R_\omega W^{v,\,out}_\omega, \quad \text{with,}$$
$$|T_\omega|^2 + |R_\omega|^2 = 1.$$

- For $0 < \omega < \omega_{\rm max}$, there is a $3{\times}3$ matrix, e.g.

$$W^{u,\,in}_\omega = \alpha_\omega W^{u,\,out}_\omega + R_\omega W^{v,\,out}_\omega + \beta_\omega \bar{W}^{u,\,out}_{-\omega}, \qquad (6)$$

with

$$|\alpha_\omega|^2 + |R_\omega|^2 - |\beta_\omega|^2 = 1.$$

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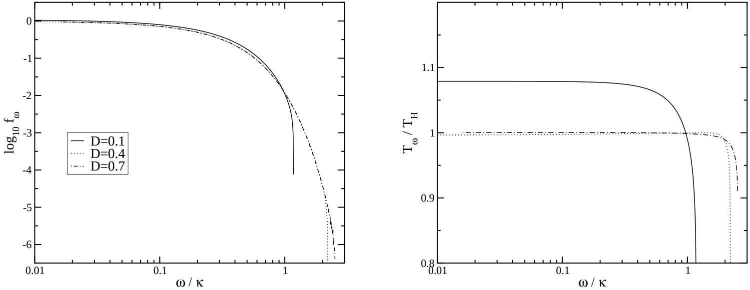
The (numerical) properties of this radiation

For $\omega_{\rm max} \geq 3\kappa$, the energy spectrum $f_\omega = \omega\,|\beta_\omega|^2$ is (JM-RP '09)

- Planckian (up to $\omega_{\rm max}$) and
- with a temperature $= \kappa/2\pi = T_{\rm Hawking}$, $\quad (f_\omega = \omega/(e^{\omega/T_\omega} - 1)$,

"exactly" as **predicted** by the **gravitational analogy**.



N.B. The above spectra are obtained from the BdG eq. **only**.

Spectrum of W_n for two sonic horizons

The (complete) set of modes contains (AC+RP 2010)

- a **continuous** spectrum of **real** freq. modes W^u_ω, W^v_ω with $0 < \omega < \infty$, with **positive norm only**, and of dim. **2**.
- a **discrete** set of **pairs** of **complex** freq. modes (V_a, Z_a) with cc freq. (λ_a, λ_a^*), where $a = 1, 2, ...N < \infty$.

N.B. **Negative norm** modes $\bar{W}_{-\omega}$ are **no longer** in the spectrum; hence there is **no** Bogoliubov transformation in the present case.

The field operator thus reads

$$\hat{W} = \int_0^\infty d\omega \sum_{\alpha=u,v} \left[e^{-i\omega t} W^\alpha_\omega(x)\, \hat{a}^\alpha_\omega + pH.c. \right] + \sum_a \left[e^{-i\lambda_a t} V_a(x)\, \hat{b}_a + e^{-i\lambda_a^* t} Z_a(x)\, \hat{c}_a + pH.c. \right]. \quad (7)$$

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Norms and commutators

- The **real** freq., the modes W_ω^α and operators $\hat{a}_\omega^\alpha$ obey

$$\langle W_\omega^\alpha | W_{\omega'}^{\alpha'} \rangle = \delta(\omega - \omega')\delta_{\alpha\alpha'} = -\langle \bar{W}_\omega^\alpha | \bar{W}_{\omega'}^{\alpha'} \rangle$$

and

$$[\hat{a}_\omega^\alpha, \hat{a}_{\omega'}^{\alpha'\dagger}] = \delta(\omega - \omega')\delta_{\alpha\alpha'}.$$

- Instead for **complex frequency** λ_a, one has

$$\langle V_a | V_{a'} \rangle = 0 = \langle Z_a | Z_{a'} \rangle, \quad \langle V_a | Z_{a'} \rangle = i\delta_{aa'}, \tag{8}$$

and

$$[\hat{b}_a, \hat{b}_{a'}^\dagger] = 0, \quad [\hat{b}_a, \hat{c}_{a'}^\dagger] = i\delta_{aa'}. \tag{9}$$

The two-mode sectors with complex freq. λ_a

Each pair $(\hat{b}_a, \hat{c}_a)$ **always** describes
one complex, rotating, unstable oscillator:

- Its (Hermitian) Hamiltonian is

$$\hat{H}_a = -i\lambda_a \hat{c}_a^\dagger \hat{b}_a + H.c. \tag{10}$$

- Writing

$$\lambda_a = \omega_a + i\Gamma_a,$$

  with ω_a, Γ_a real > 0,
  $\Re\lambda_a = \omega_a$ fixes the angular velocity,
  $\Im\lambda_a = \Gamma_a$ fixes the **growth rate**.

Computing the spectrum of ABM

The method:

- **A.** use WKB waves to
  - 1. **decompose** the exact modes,
  - 2. obtain **algebraic relations** (valid **beyond WKB**) between the $\mathbb{R}$ freq. W_ω and the $\mathbb{C}$ freq. V_a, Z_a
- **B.** a numerical analysis to validate the predictions.

N.B. The W_ω are **deeply connected** to the V_a, Z_a because

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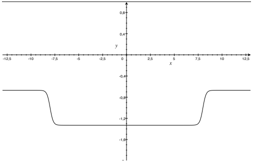
The scattering of real freq. u-mode



- On the **left** of the White hor. $W_\omega^{u,\,in} \to W_\omega^u$, the WKB sol.
- **Between** the two horizons, **for** $\omega < \omega_{\rm max}$,

$$W_\omega^{u,\,in} = \mathcal{A}_\omega \, W_\omega^u + \mathcal{B}_\omega^{(1)} \, \bar{W}_{-\omega}^{(1)} + \mathcal{B}_\omega^{(2)} \, \bar{W}_{-\omega}^{(2)}, \qquad (11)$$

- On the **right** of the Black horizon, $W_\omega^{u,\,in} \to e^{i\theta_\omega} \, W_\omega^u$.
- N.B.1. **Negative norm/freq** WKB modes $\bar{W}_{-\omega}^{(i)}$ in (11). Hence "*anomalous* scattering" ($\sim$ Bogoliubov transf.).
- N.B.2. Modes **fully described** by $\mathcal{A}_\omega, \mathcal{B}_\omega^{(1)}, \mathcal{B}_\omega^{(2)}$ and θ_ω.

Computing $\mathcal{A}_\omega, \mathcal{B}^{(1)}_\omega, \mathcal{B}^{(2)}_\omega$ and θ_ω

- algebraically achieved by introd. a 2-vector $(W^u_\omega, \bar{W}_{-\omega})$, on which acts a 2×2 S**-matrix** (Leonhardt 2008)
- this S-matrix can be decomposed as

$$S = U_4 \, U_3 \, U_2 \, U_1 .$$

where

- U_1 describes the **scattering** on the **WH** horizon.
- U_2 the **propagation from** the WH to the BH
- U_3 the **scattering** on the **BH** horizon.
- U_4 the **escape** to the right of W^u_ω and the **return** of $\bar{W}^{(2)}_{-\omega}$ to the WH horizon.

Computing $\mathcal{A}_\omega, \mathcal{B}_\omega^{(1)}, \mathcal{B}_\omega^{(2)}$ and θ_ω

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  - U_1 describes the **scattering** on the **WH** horizon.
  - U_2 the **propagation from** the WH to the BH
  - U_3 the **scattering** on the **BH** horizon.
  - U_4 the **escape** to the right of W_ω^u
    and the **return** of $\bar{W}_{-\omega}^{(2)}$ to the WH horizon.

The four U matrices, (Leonhardt et al.)

Explicitly,

$$U_1 = S_{WH} = \begin{pmatrix} \alpha_\omega & \alpha_\omega z_\omega \\ \tilde{\alpha}_\omega z_\omega^* & \tilde{\alpha}_\omega \end{pmatrix}, \quad U_2 = \begin{pmatrix} e^{iS_\omega^u} & 0 \\ 0 & e^{-iS_{-\omega}^{(1)}} \end{pmatrix},$$

$$U_3 = S_{BH} = \begin{pmatrix} \gamma_\omega & \gamma_\omega w_\omega \\ \tilde{\gamma}_\omega w_\omega^* & \tilde{\gamma}_\omega \end{pmatrix}, \quad U_4 = \begin{pmatrix} 1 & 0 \\ 0 & e^{iS_{-\omega}^{(2)}} \end{pmatrix},$$

where

$$S_\omega^u \equiv \int_{-L}^{L} dx\, k_\omega^u(x), \quad S_{-\omega}^{(i)} \equiv \int_{-L_\omega}^{R_\omega} dx \left[-k_\omega^{(i)}(x) \right], \quad i = 1, 2,$$

are H-Jacobi actions, and L_ω and R_ω are the two turning points.
By unitarity, one has $|\alpha_\omega|^2 = |\tilde{\alpha}_\omega|^2$, $|\alpha_\omega|^2 = 1/(1 - |z_\omega|^2)$.

The single-valued real freq. mode

The mode $W_\omega^{u,\,in}(x)$ **must be single-valued.**

Hence the **trapped** piece $\mathcal{B}_\omega^{(2)}$ of

$W_\omega^{u,\,in} = \mathcal{A}_\omega W_\omega^u + \mathcal{B}_\omega^{(1)} \bar{W}_{-\omega}^{(1)} + \mathcal{B}_\omega^{(2)} \bar{W}_{-\omega}^{(2)}$ must obey

$$\begin{pmatrix} e^{i\theta_\omega} \\ \mathcal{B}_\omega^{(2)} \end{pmatrix} = S \begin{pmatrix} 1 \\ \mathcal{B}_\omega^{(2)} \end{pmatrix}, \qquad (12)$$

which implies

$$\mathcal{B}_\omega^{(2)} = \frac{S_{21}(\omega)}{1 - S_{22}(\omega)}. \qquad (13)$$

The first key equation. (Valid beyond WKB.)

The complex frequency ABModes

When $Im\,\lambda = \Gamma > 0$, $\to Im\, k_\lambda^u > 0$, hence **growth** for $x \to -\infty$.
So any single-valued **ABMode** must satisfy

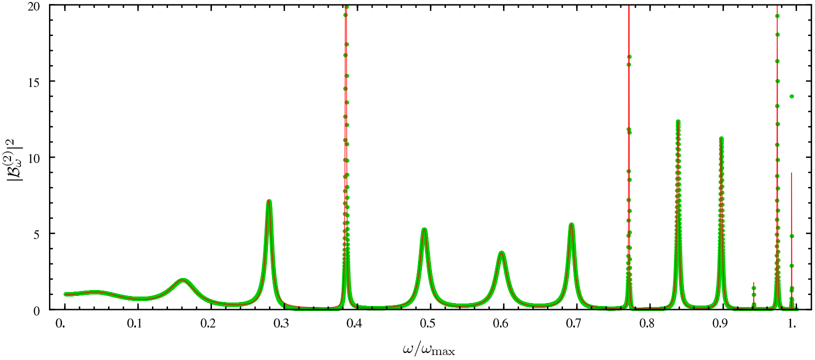
$$\begin{pmatrix} \beta_a(\lambda) \\ 1 \end{pmatrix} = S(\lambda) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (14)$$

This implies

$$S_{22}(\lambda) = 1, \quad \beta_a = S_{12}(\lambda). \quad (15)$$

Second key result:
The **poles** of $\mathcal{B}_\omega^{(2)} = S_{21}/(1 - S_{22})$ **correspond** to the **complex freq.** λ_a.

$|\mathcal{B}_\omega^{(2)}|^2$, as a function of ω **real**.



- Green dots are **numerical values**,
  the continuous red line is a sum of Lorentzians.
- Near a complex frequency λ_a, solution of $S_{22} = 1$,
  $|\mathcal{B}_\omega^{(2)}|^2 \sim C_a / \, |\omega - \omega_a - i\Gamma_a|^2$, i.e. a Lorentzian.
- Above $\omega_{\max}$ no peaks, because no neg. norm WKB mode.

Computing the complex freq. $\lambda_a = \omega_a + i\Gamma_a$.

- The λ_a's, are fixed by the cond. **ABM + single-valued**.
  Both conditions encoded in $S_{22} = 1$.
- When the **leaking-out amplitudes** are small,
  $|z_\omega|, |w_\omega| = |\beta_\omega/\alpha_\omega| \ll 1$,
  the supersonic region acts as a **cavity**:
- To **zeroth order** in z_ω, w_ω, $S_{22} = 1$ fixes
  $\Re\lambda_a = \omega_a$ by a **Bohr-Sommerfeld** condition

$$S^{(1)}_{-\omega} - S^{(2)}_{-\omega} + \pi = \int_{-L}^{L} dx[-k^{(1)}_\omega(x) + k^{(2)}_\omega(x)] + \pi = 2\pi n,$$

where $n = 1, 2, ..., N$.
This explains the **discreteness** of the set.

To **second order** in z_ω, w_ω, $S_{22} = 1$ fixes $Im\,\lambda_a = \Gamma_a$ to be

$$2\Gamma_a T^b_{\omega_a} = |S_{12}(\omega_a)|^2 = |z_{\omega_a} + w_{\omega_a} e^{i\psi_a}|^2 \tag{16}$$

- $T^b_{\omega_a} > 0$ is the **bounce time**, given by

$$T^b_\omega = \frac{\partial}{\partial\omega}\left(S^{(2)}_{-\omega} - S^{(1)}_{-\omega} + \text{"non HJ terms"}\right) \tag{17}$$

- The phase in the cosine is

$$\psi_a = S^u_{\omega_a} + S^{(1)}_{-\omega_a} + \text{ other "non HJ terms"}$$

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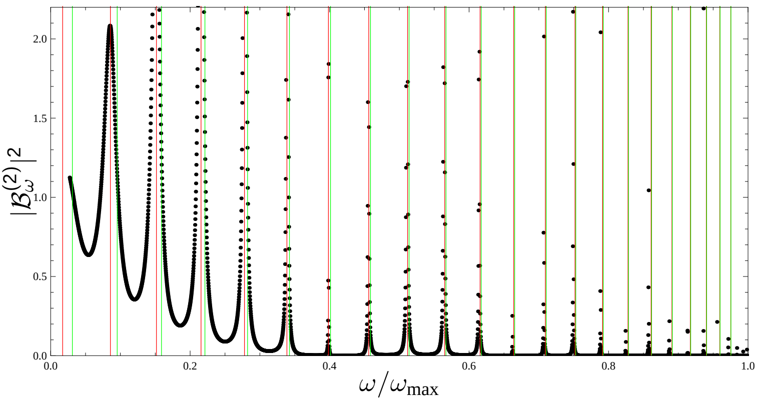
The validity of the 'semi-classical' treatment.

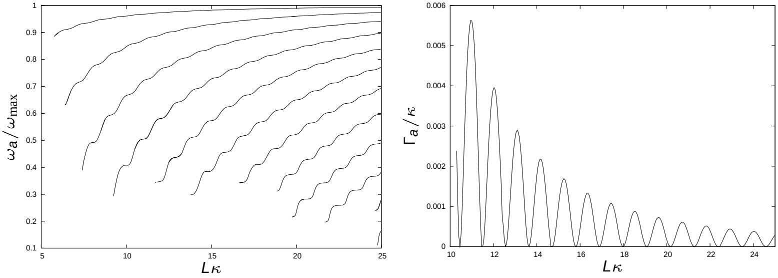


Dots are **numerical values**.

The 22 red lines are the **predictions**.

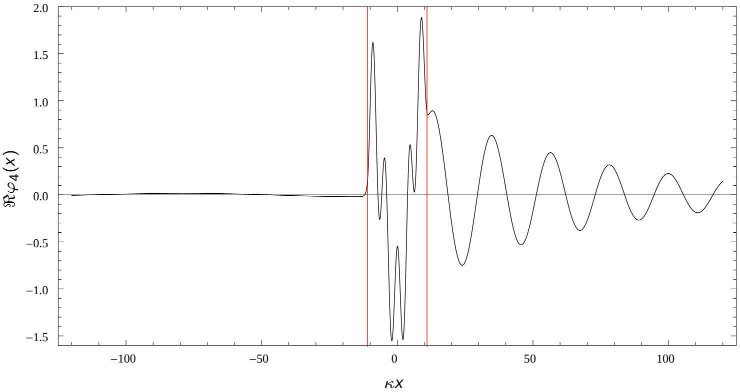
Excellent agreement

The evolution of ω_a and Γ_a in terms of L.



- **New** bound modes appear as L grows, **with** $\omega = \Gamma = 0$ **?**
- The Γ_a reach their maximal value for $\omega_a/\omega_{\max} \ll 1$.
- Γ_a reach 0 because of (Young) interferences.
  The destruction is imperfect when $z_\omega \neq w_\omega$.
- No bound mode is destroyed as L grows.

A typical growing mode with a high Γ_a ($\Gamma/\omega \sim 1/20$)



- Highest amplitudes in the trapped region.
- Exponential decrease on the Right of the BH horizon.
  The **spatial** damping is proportional to the **rate** $\Gamma_a = Im\lambda_a$.

Physical predictions

- At **late times** w.r.t. the **formation** of the BH-WH, i.e. times $\gg 1/Max\Gamma_a$, the mode with the highest Γ_a dominates **all observables**.
  The **classical** and **quantum** descriptions **coincide**.

- At **earlier times**, **if** the ***in*-state** is (near) vacuum, the **quantum settings must be used**, and **all** complex freq. modes contribute to the **observables**

- At **"early" times**, i.e. $\Delta t < T^{\rm Bounce} = 2\pi/(\omega_a - \omega_{a+1})$ **Hawking radiation** as if the **WH were not present.** the **discreteness** of the λ_a-set is not yet visible, the **resolution in** ω being too small.

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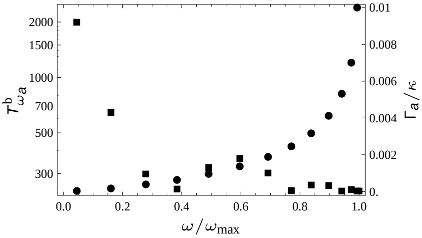
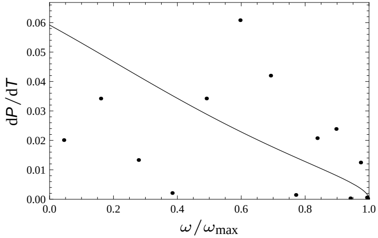
The quantum flux emitted by a BH-WH system, 1

1. A BH-WH system with 13 complex freq. modes.



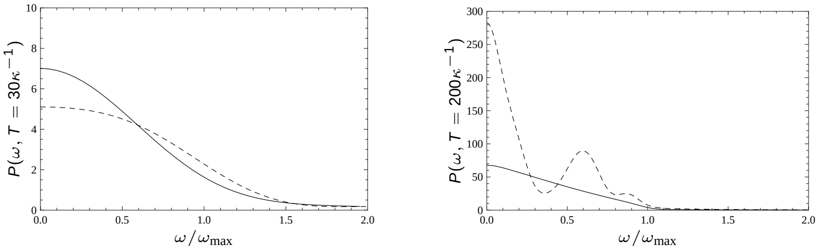


Left: The 13 values of T_a^{Bounce} (dots) and Γ_a (squares)

Right: The **continuous** spectrum obtained **without the WH** vs. the corresponding **discrete** quantity for the **BH-WH** pair.

Very different spectra **in ω-space**.

Fluxes emitted **after a finite lapse of time**
by a single BH (solid line) and the BH-WH pair (dashed).



Left: after $\Delta t = 30/\kappa$, **no** sign yet of discreteness **nor** instab.
the BH-WH pair emits Hawking-like radiation.

Right: after $\Delta t = 200/\kappa$, **discreteness** and **instab.** visible.

The flux emitted by a BH-WH system, 2

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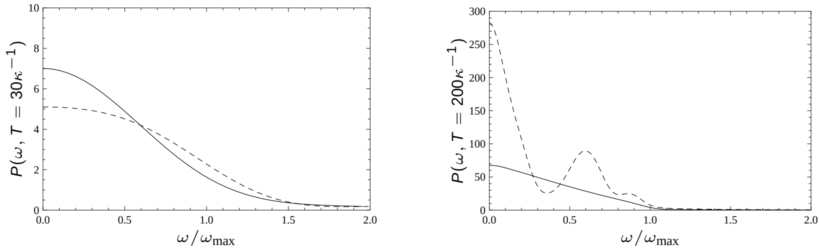

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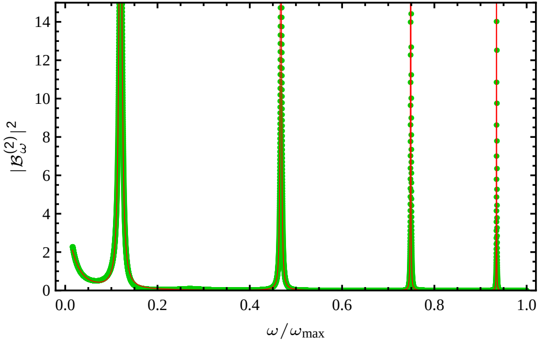
The Technion BH-WH, June 2009, preliminary results



About 4 unstable modes.
Experiment too **short** by a factor of 10 to see the laser effect.
Probably **more** than 4 complex freq. modes.

Classical terms: Induced instability

- When sending a **classical wave** $W_{in}(t,x)$, this **induces** the instability.

- N.B. It does it through the overlaps with the **decaying** modes Z_a

$$b_a \equiv \langle Z_a | W_{in} \rangle \tag{18}$$

which fix the amplitude of the **growing** mode V_a :

$$W_{in}(t,x) \to \sum_a \left[e^{-i\lambda_a t} b_a V_a(x) + p.H.c. \right] . \tag{19}$$

Conclusions

- In flows with **one** sonic B/W horizon, the spectrum
  - is **continuous**, and
  - contains **real** freq., of **both signs** for $\omega < \omega_{\max}$.
  - emitted flux is $\sim$ Hawking radiation when $\omega_{\max} > 3\kappa$.
- In flows with **a pair** of BH-WH horizons, one has
  - a **continuous** spectrum of **real** and **positive** freq., and
  - a **discrete** set of pair of **complex** freq., with $Re\,\lambda_a < \omega_{\max}$.
  - At **late time**, the mode with highest Γ_a dominates all obs.
  - At **early time**, BH-WH flux **as that** from the sole BH.
- When $L\kappa$ suff. **small**,
  no complex freq. modes, hence **no** dyn. instability,
  No radiation emitted, even though $\kappa \neq 0$,
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Additional remarks, 1.

- In weak external fields, the discrete set is empty.
- This can be seen from the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int dx \left[(\partial_t \phi)^2 + (c^2 - v^2)(\partial_x \phi)^2 + \frac{1}{\Lambda^2}(\partial_x^2 \phi)^2\right]. \quad (20)$$

- For $v^2 < c^2$, i.e. no horizon, H is positive, and this **suffices** for having no complex freq.

- Another **sufficient** condition for having **no** complex freq., is that the scalar product $(\phi|\psi)$ be **positive definite**, which is the case for **fermions**, but which is **not** the case for **bosons**.

Additional remarks, 2.

- $v^2 > c^2$, is a **necessary** condition for having complex freq.
- However, it is not **sufficient**, as is verified when having only a **single** Black (or White) Hole horizon
- In these cases, there are **negative** real frequencies, but no complex ones.
- These **negative** frequencies are **necessary** to get **Hawking radiation**.

Additional remarks, 3.

There is a ”hierarchy” in the **external** field strength.

- For **weak** fields, **neither** negative **nor** complex freq.
  There is a unique ground state.
  The system is stable (**classically** and **QMcally**).
- For strong fields, one **frequent possibility** is :
  some negative freq. but **no** complex.
  There is no "minimal energy state".
  Weak QM instability, e.g. a **steady** Hawking radiation.
- For strong fields, **under specific conditions,**
  complex eigen-frequencies can be found.
  Both **QM** and **class.** unstable: **dynamical instability**.
- In **many** cases, as in the Black Hole laser,
  the latter is **deeply** related to the former.

General remarks, 4.
Conditions to get a Laser effect

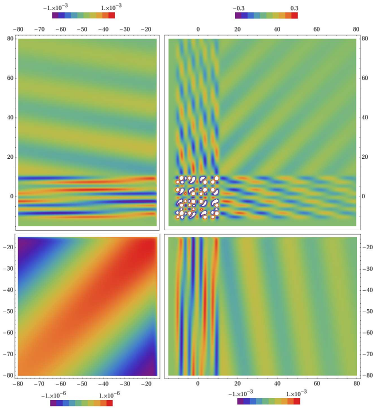
In statio. bgds, the following conditions are **sufficient**

- 1. For some range of ω real, in some region,
  WKB solutions with **both** signs of norm should exist.
  This is a **strong** condition.
- 2. These solutions must **mix** in exact solutions.
  This is a weak condition.
- 3. One of the WKB solution must be **trapped**.
  This is a strong condition.
- 4. The potential should be **deep enough** so that at least one bound mode exist.

NB. When **only** 1 and 2 are met,
one gets a super-radiance, i.e. a **vacuum instability**.

The pattern of density-density fluctuations $\langle \delta\rho\delta\rho \rangle$

In a BEC, the equal time $\langle \delta\rho(x)\, \delta\rho(x')\rangle$ is observable,
as the temperature fluct. $\langle \delta T(x)\, \delta T(x')\rangle$ on the LSS, in cosmology.



Different scales are used, the central square is the trapped region.