Modified gravity and the cosmological constant problem

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Question: Why we should not modify GR

- Theoretical consistency: In 4 dimensions, consider $\mathcal{L} = \mathcal{L}(\mathcal{M}, g, \nabla g, \nabla\nabla g)$. Then Lovelock's theorem in $D = 4$ states that GR with cosmological constant is the unique metric theory emerging from,

$$S_{(4)} = \int_{\mathcal{M}} d^4x \sqrt{-g^{(4)}} \left[R - 2\Lambda\right]$$

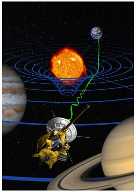
giving,

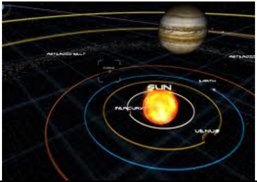
  - Equations of motion of 2^{nd}-order
  - given by a symmetric two-tensor, $G_{\mu\nu} + \Lambda g_{\mu\nu}$
  - and admitting Bianchi identities.

Under these hypotheses GR is the unique massless-tensorial 4 dimensional theory of gravity

Experimental and observational data

- Experimental consistency:
  -Excellent agreement with solar system tests
  -Strong gravity tests on binary pulsars
  -Laboratory tests of Newton's law (tests on extra dimensions)

Time delay of light  Planetary tajectories

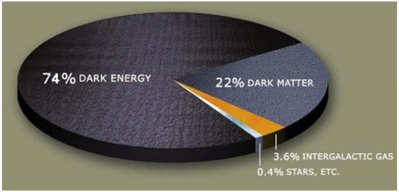


Q: What is the matter content of the Universe today?

Assuming homogeinity-isotropy and GR

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

cosmological and astrophysical observations dictate the matter content of the Universe today:



A: -Only a 4% of matter has been discovered in the laboratory. We hope to see more at LHC. But even then...

If we assume only ordinary sources of matter (DM included) there is disagreement between local, astrophysical and cosmological data.

Universe is accelerating $\longrightarrow$ Enter the cosmological constant

Easiest way out: Assume a tiny cosmological constant $\rho_\Lambda = \frac{\Lambda}{8\pi G} = (10^{-3} eV)^4$, ie modify Einstein's equation by,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- Cosmological constant introduces $\sqrt{\Lambda}$ and generates a cosmological horizon
- $\sqrt{\Lambda}$ is as tiny as the inverse size of the Universe today, $r_0 = H_0^{-1}$
- Note that $\frac{\textbf{Solar system scales}}{\textbf{Cosmological Scales}} \sim \frac{\textbf{10 A.U.}}{H_0^{-1}} = 10^{-14}$
- But things get worse...
- Theoretically, the size of the Universe would not even include the moon!

Cosmological constant problem

Cosmological constant problem, [S Weinberg Rev. Mod. Phys. 1989]

Cosmological constant behaves as vacuum energy which according to the strong equivalence principle gravitates,

- Vacuum energy fluctuations are at the UV cutoff of the QFT $\Lambda_{vac}/8\pi G \sim m_{Pl}^4 \ldots$
- Vacuum potential energy from spontaneous symmetry breaking $\Lambda_{EW} \sim (200 GeV)^4$
- Bare gravitational cosmological constant Λ_{bare}

$\Lambda_{obs} \sim \Lambda_{vac} + \Lambda_{pot} + \Lambda_{bare}$

Enormous Fine-tuning inbetween theoretical and observational value

- Why such a discrepancy between theory and observation? Weinberg no-go theorem **big CC**
- Why is Λ_{obs} so small and not exactly zero? small cc
- Why do we observe it now ?

Gravitational theories

- Extra dimensions : Extension of GR to Lovelock theory with modified yet second order field equations. Braneworlds: In general relevant as UV modifications, problematic in the IR (ghosts, strong coupling problems etc).
- 4-dimensional modification of GR: Scalar-tensor galileon or f(R), Einstein-Aether, Hořava gravity: Tension with local and strong gravity tests, some theoretical problems/questions with Lorentz breaking and flowing back to GR in the IR.
- Massive gravity: Is there a Higgs mechanism for gravity? Not as yet a robust covariant theory, only perturbative windows available, often dressed with stability problems. Some recent progress

We will consider the simplest of cases, namely scalar tensor theory. Generically these theories can result as IR endpoints of more complex theories of gravity or from higher dimensional theories by KK reduction.

Weinberg no-go theorem

- Assume an effective, conserved and covariant 4 dimensional theory
  Consider gravity action including all contributions of cosmological constant in the scalar potential term,

  $$S[\pi_1, ..., \pi_N, g_{\mu\nu}] = \int d^4x\sqrt{-g}R + L(\pi_1, ..., \pi_N, g_{\mu\nu}, \partial^m) + Matter$$

  If $g_{\mu\nu} = \eta_{\mu\nu}, \pi^i = constant$. Then $\Lambda = 0$
- It is impossible to find trivial solutions to Einstein's field equations without fine tuning the cosmological constant to zero.

Let us consider a *self* – *tuning*, or, non-trivial scalar field...

Self-Tuning: general idea

Question: What if we break Poincaré invariance at the level of the scalar field?
Keep $g_{\mu\nu} = \eta_{\mu\nu}$ locally but allow for $\phi \neq \mathit{constant}$.
Can we have a portion of flat spacetime whatever the value of the cosmological constant and without fine-tuning any of the parameters of the theory?
Toy model theory of self-adjusting scalar field.

- Solving this problem classically means that vacuum energy does not gravitate and we break SEP not EEP.
- Beyond leading order $O(\Lambda^4)$, radiative corrections $O(\Lambda^6/M_{Pl^2})$ may spoil self-tuning.

We need:

1. A cosmological background
2. A sufficiently general theory to work with

A general scalar tensor theory

- Consider ϕ and $g_{\mu\nu}$ as gravitational DoF.
- Consider $\mathcal{L} = \mathcal{L}(g_{\mu\nu}, g_{\mu\nu,i_1}, ..., g_{\mu\nu,i_1...i_p}, \phi, \phi_{,i_1}, ..., \phi_{,i_1...i_q})$ with $p, q \geq 2$ but finite
- $\mathcal{L}$ has higher than second derivatives

What is the most general scalar-tensor theory giving second order field equations?

Similar to Lovelock's theorem but for the presence of higher derivatives in $\mathcal{L}$.

Here second order field equations in principle protect vacua from ghost instabilities.

Horndeski's theory

rather complex...
loads of derivatives,
Kronecker δ's
K-essence terms
free functions of scalar and kinetic term
but just what we need

and it gets simpler as we go on...

The Horndeski action [Horndeski 1974, Int. J. Theor. Phys.], [Deffayet et al.]

$$
\begin{aligned}
\mathcal{L} &= \kappa_1(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla^\mu\nabla_i\phi R_{jk}{}^{\nu\sigma} - \frac{4}{3}\kappa_{1,\rho}(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla^\mu\nabla_i\phi\nabla^\nu\nabla_j\phi\nabla^\sigma\nabla_k\phi \\
&+\kappa_3(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla_i\phi\nabla^\mu\phi R_{jk}{}^{\nu\sigma} - 4\kappa_{3,\rho}(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla_i\phi\nabla^\mu\phi\nabla^\nu\nabla_j\phi\nabla^\sigma\nabla_k\phi \\
&+F(\phi,\rho)\delta^{ij}_{\mu\nu}R_{ij}{}^{\mu\nu} - 4F(\phi,\rho)_{,\rho}\delta^{ij}_{\mu\nu}\nabla_i\phi\nabla^\mu\phi\nabla^\nu\nabla_j\phi \\
&-3[2F(\phi,\rho)_{,\phi} + \rho\kappa_8(\phi,\rho)]\nabla_\mu\nabla^\mu\phi + 2\kappa_8(\phi,\rho)\delta^{ij}_{\mu\nu}\nabla_i\phi\nabla^\mu\phi\nabla^\nu\nabla_j\phi \\
&+\kappa_9(\phi,\rho), \\
\rho &= \nabla_\mu\phi\nabla^\mu\phi,
\end{aligned}
$$

where $\kappa_i(\phi,\rho)$, $i = 1,3,8,9$ are 4 arbitrary functions of the scalar field ϕ and its kinetic term denoted as ρ and

$$F_{,\rho} = \kappa_{1,\phi} - \kappa_3 - 2\rho\kappa_{3,\rho}$$

$$\delta^{i_1\cdots i_h}_{j_1\cdots j_h} = h!\delta^{i_1}_{[j_1}\ldots\delta^{i_h}_{j_h]}$$

Field equations are second order in metric $g_{\mu\nu}$ and ϕ and theory is unique.
Most general galileon theory

Horndeski cosmology

Consider cosmological background:

- Assume, $ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$, $\phi = \phi(t)$
- Insert into action or use EoM,
- Cosmological Langrangian $\mathcal{L}(a, \dot{a}, \phi, \dot{\phi}) = a^3 \sum_{n=0}^{3}(X_n - Y_n \frac{\kappa}{a^2})H^n$
- $$\begin{aligned} X_0 &= -\tilde{Q}_{7,\phi}\dot{\phi} + \kappa_9, \; X_1 = -12F_{,\phi}\dot{\phi} + 3(Q_7\dot{\phi} - \tilde{Q}_7) + 6\kappa_8\dot{\phi}^3 \\ X_2 &= 12F_{,\rho}\rho - 12F, \; X_3 = 8\kappa_{1,\rho}\dot{\phi}^3 \\ Y_0 &= \tilde{Q}_{1,\phi}\dot{\phi} + 12\kappa_3\dot{\phi}^2 - 12F, \; Y_1 = \tilde{Q}_1 - Q_1\dot{\phi}, \; Y_2 = Y_3 = 0 \end{aligned}$$
- With, $-12\kappa_1 = Q_1 := \frac{\partial \tilde{Q}_1}{\partial \dot{\phi}}$, and $6F_{,\phi} - 3\dot{\phi}^2\kappa_8 = Q_7 := \frac{\partial \tilde{Q}_7}{\partial \dot{\phi}}$
- and $H = \frac{\dot{a}}{a}$. $\mathcal{L}$ polynomial of third order in H.

Cosmological field equations

1. Cosmological Langrangian density

$$\mathcal{L}(a, \dot{a}, \phi, \dot{\phi}) = a^3 \sum_{n=0}^{3} (X_n - Y_n \frac{\kappa}{a^2}) H^n$$

2. Modified Friedmann eq (with some matter source).

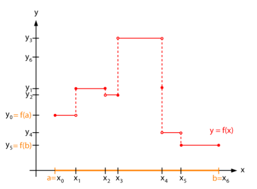
$$\mathcal{H}(a, \dot{a}, \phi, \dot{\phi}) = \frac{1}{a^3}(\frac{\partial \mathcal{L}}{\partial \dot{a}}\dot{a} + \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\dot{\phi} - \mathcal{L}) = -\rho_m$$

3. Scalar eq.

$$\mathcal{E}(a, \dot{a}, \ddot{a}, \phi, \dot{\phi}, \ddot{\phi}) = -\frac{d}{dt}(\partial \mathcal{L}/\partial \dot{\phi}) + \partial \mathcal{L}/\partial \phi = 0$$
$$= \ddot{\phi} f(\phi, \dot{\phi}, a, \dot{a}) + g(\phi, \dot{\phi}, a, \dot{a}, \ddot{a}) = 0$$

Linear in $\ddot{\phi}$ and $\ddot{a}$.
Also have 2nd Friedmann equation or usual energy conservation.

Main Assumptions

- Vacuum energy does not gravitate.
- Assume that $\rho_m = \rho_\Lambda$, a piecewise discontinuous step function of time t. Discontinuous points, $t = t_\star$, are phase transitions which are point like and arbitrary in time.
- 

  $x = time$, and $y = \rho_\Lambda$.
- Assume that spacetime is flat or a flat portion for all t
- $H^2 + \frac{\kappa}{a^2} = 0$, with $\kappa = 0$, or $\kappa = -1$ Milne spacetime $(a(t) = t)$
- ϕ not constant but in principle a function of time t!

The self tuning filter

Mathematical regularity imposed by a distributional source

1. We are going to set $H^2 + \frac{\kappa}{a^2}{=}0$, with $\rho(\Lambda)$ piecewise discontinuous. Then
2. $$\mathcal{H}(a,\phi,\dot\phi) = \frac{1}{a^3}\left(\frac{\partial\mathcal{L}}{\partial\dot a}\dot a + \frac{\partial\mathcal{L}}{\partial\dot\phi}\dot\phi - \mathcal{L}\right) = -\rho_\Lambda$$
   $a(t)$, $\dot a$ and $\phi(t)$ are continuous whereas $\dot\phi$ is discontinuous at $t = t_\star$.
   $\mathcal{H}$ has to depend on $\dot\phi$
3. Scalar eq. on shell is
   $$\mathcal{E}(a,\phi,\dot\phi,\ddot\phi) = \ddot\phi f(\phi,\dot\phi,a) + g(\phi,\dot\phi,a) = 0$$
   $\ddot\phi$ has a $\delta(t - t_\star)$ singularity at $t = t_\star$ Hence
   $$f(\phi,\dot\phi,a) = 0, \qquad g(\phi,\dot\phi,a) = 0$$
   Since $t = t_\star$ is arbitrary we finally get $\ddot\phi_\Lambda f(a) + g(a) = 0$
4. Hence on shell, $\mathcal{E}$ has no dependance on ϕ. ϕ fixed by Friedmann eq.
5. In the presence of matter cosmology must be non trivial. Hence $\mathcal{E}$ must depend on $\ddot a$

Applying self-tuning filter to cosmological Horndesky

- Using the form of Horndeski cosmological equations:
  -linearity of second order terms in a and ϕ
  -polynomial form of $\mathcal{H}$
- We obtain

$$
\begin{aligned}
\kappa_1 &= \frac{1}{8} V_{ringo}{}'(\phi)\left(1+\frac{1}{2}\ln|\rho|\right) + \frac{1}{4}V_{paul}(\phi)\rho - \frac{1}{12}B(\phi) \\
\kappa_3 &= \frac{1}{16} V_{ringo}{}''(\phi)\ln|\rho| + \frac{1}{12}V'_{paul}(\phi)\rho - \frac{1}{12}B'(\phi) + p(\phi) - \frac{1}{2}V_{john}(\phi)(1-\ln|\rho|) \\
\kappa_8 &= 2p'(\phi) + V'_{john}(\phi)\ln|\rho| - \lambda(\phi) \\
\kappa_9 &= c_0 + \frac{1}{2}V''_{george}(\phi)\rho + \lambda'(\phi)\rho^2 \\
F &= -\frac{1}{12}V_{george}(\phi) - p(\phi)\rho - \frac{1}{2}V_{john}(\phi)\rho\ln|\rho|
\end{aligned}
$$

- All ρ dependance integrated out.
- Free functions V_{fab4}, c_0 cosmological constant , B, p, λ B, p, λ total derivatives

Relevant and irelevant terms

- Remember the Horndeski action

$$\begin{aligned}
\mathcal{L} &= \kappa_1(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla^\mu\nabla_i\phi R_{jk}{}^{\nu\sigma} - \frac{4}{3}\kappa_{1,\rho}(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla^\mu\nabla_i\phi\nabla^\nu\nabla_j\phi\nabla^\sigma\nabla_k\phi \\
&+\kappa_3(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla_i\phi\nabla^\mu\phi R_{jk}{}^{\nu\sigma} - 4\kappa_{3,\rho}(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla_i\phi\nabla^\mu\phi\nabla^\nu\nabla_j\phi\nabla^\sigma\nabla_k\phi \\
&+F(\phi,\rho)\delta^{ij}_{\mu\nu}R_{ij}{}^{\mu\nu} - 4F(\phi,\rho)_{,\rho}\delta^{ij}_{\mu\nu}\nabla_i\phi\nabla^\mu\phi\nabla^\nu\nabla_j\phi \\
&-3[2F(\phi,\rho)_{,\phi} + \rho\kappa_8(\phi\,\rho)]\nabla_\mu\nabla^\mu\phi + 2\kappa_8\delta^{ij}_{\mu\nu}\nabla_i\phi\nabla^\mu\phi\nabla^\nu\nabla_j\phi \\
&+\kappa_9(\phi,\rho)
\end{aligned}$$

- The self-tuning filter gave,

$$\begin{aligned}
\kappa_1 &= \frac{1}{8}V'_{ringo}(\phi)\left(1+\frac{1}{2}\ln|\rho|\right) + \frac{1}{4}V_{paul}(\phi)\rho \\
\kappa_3 &= \frac{1}{16}V''_{ringo}(\phi)\ln|\rho| + \frac{1}{12}V'_{paul}(\phi)\rho - \frac{1}{2}V_{john}(\phi)(1-\ln|\rho|) \\
\kappa_8 &= V'_{john}(\phi)\ln|\rho| \\
\kappa_9 &= \frac{1}{2}V''_{george}(\phi)\rho \\
F &= -\frac{1}{12}V_{george}(\phi) - \frac{1}{2}V_{john}(\phi)\rho\ln|\rho|
\end{aligned}$$

- Are these terms recognisable geometric quantities?

George is easy

- Start with $\mathcal{L}_{George}$
- Set everybody else to zero
- $$\kappa_9 = \frac{1}{2} V''_{george} \rho, \qquad F = -\frac{1}{12} V_{george}$$
- $$\mathcal{L}_{george} \quad = \quad -\frac{1}{6} V_{george}(\phi) R + \frac{1}{2} \nabla_\mu \left[V'_{george} \partial^\mu \phi \right] . \cong -\frac{1}{6} V_{george}(\phi) R$$
- Einstein-Hilbert non-minimally coupled with a free scalar field

EoM help for Ringo and John

- Switch on only V_{Ringo} in EoM. We find,
- $$K_1 = \frac{1}{16} V'_{ringo}, \qquad K_3 = \frac{1}{16} V''_{ringo}$$
- The equation of motion reads,

$$\begin{aligned}
\mathcal{E}^{ik}_{ringo} &= \sqrt{-g} K_1(\phi,\rho) \delta^{aijk}_{\lambda\mu\nu\sigma} g^{\lambda b} \nabla^\mu \nabla_i \phi R_{jk}{}^{\nu\sigma} + K_3(\phi,\rho) \delta^{aijk}_{\lambda\mu\nu\sigma} g^{\lambda b} \nabla_i \phi \nabla^\mu \phi R_{jk}{}^{\nu\sigma} \\
&= \sqrt{-g} (*R*)^{ijkl} \left(4K_1 \nabla_l \nabla_j \phi + 4K_3 \nabla_l \phi \nabla_j \phi \right) \\
&= \frac{1}{4} \sqrt{-g} (*R*)^{ijkl} \nabla_l \nabla_j V_{ringo}(\phi)
\end{aligned}$$

- While at the same time we have,

$$\delta \left[\int_{\mathcal{M}} d^4x \sqrt{-g}\, V(\phi) \hat{\mathcal{G}} \right]$$
$$= \int_{\mathcal{M}} d^4x \sqrt{-g}\, \delta g^{ij} \left[2\phi H_{ij} + 4(*R*)_{ikjl} \nabla^l \nabla^k V(\phi) \right] + \delta\phi [\partial_\phi V(\phi) \hat{\mathcal{G}}]$$

- Hence $\mathcal{L}_{Ringo} = V_{Ringo}(\phi) \hat{\mathcal{G}}$
- Similarly $\mathcal{L}_{John} = V_{john} G_{ij} \nabla^i \phi \nabla^j \phi$.

The double dual tensor

- In 4 dimensions we can define a dual of the curvature tensor by dualising each pair of indices much like the Faraday tensor in EM

$$*F^{ab} = \frac{1}{2}\varepsilon^{abcd}\ F_{cd}$$

- Double Dual $(*R*)$

$$(*R*)_{\mu\nu\sigma\lambda} = -\frac{1}{4}\varepsilon_{\mu\nu}{}^{ij}\ R_{ijkl}\ \varepsilon_{\sigma\lambda}{}^{kl} = \tfrac{1}{4}\delta^{ijkl}_{\mu\nu\sigma\lambda}\ R_{ijkl}$$

As appearing in the Horndeski action

1. Same index properties as R-tensor
2. Divergence free:
$$\nabla_i (*R*)_{jkl}{}^{i} = 0$$
3. Simple trace is Einstein
$$(*R*)^{ik}{}_{jk} = -G^i_j,$$

Paul

- Last term is not recognisable. However, numerous Padilla tricks bring it to the form,
-

$$\begin{aligned}\mathcal{L}_{paul} &= \sqrt{-g}V_{Paul}(\phi)\left[R^{\mu\nu\alpha\beta}\nabla_\mu\phi\nabla_\alpha\phi\nabla_\nu\nabla_\beta\phi+\right.\\ &\quad +G^{\mu\nu}(\nabla_\mu\phi\nabla_\alpha\phi-g_{\mu\alpha}(\nabla\phi)^2)\nabla^\alpha\nabla_\nu\phi\\ &\quad +R^{\mu\nu}(\nabla_\mu\nabla_\alpha\phi-g_{\mu\alpha}\Box\phi)\nabla^\alpha\phi\nabla_\nu\phi]\end{aligned}$$

- ???However,

$$(*R*)^{\mu\nu\alpha\beta}=R^{\mu\nu\alpha\beta}+2R^{\nu[\alpha}g^{\beta]\mu}-2R^{\mu[\alpha}g^{\beta]\nu}+Rg^{\mu[\alpha}g^{\beta]\nu}\ ,$$

- Therefore

$$\mathcal{L}_{paul}=\sqrt{-g}V_{paul}(\phi)(*R*)^{\mu\nu\alpha\beta}\nabla_\mu\phi\nabla_\alpha\phi\nabla_\nu\nabla_\beta\phi$$

- Also a higher KK Lovelock density [K V Akoleyen]

Fab 4

Putting it all together
from Horndeski s general action,

$$\begin{aligned}\mathcal{L} = \; & \kappa_1(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla^\mu\nabla_i\phi R_{jk}{}^{\nu\sigma} - \frac{4}{3}\kappa_{1,\rho}(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla^\mu\nabla_i\phi\nabla^\nu\nabla_j\phi\nabla^\sigma\nabla_k\phi \\ & +\kappa_3(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla_i\phi\nabla^\mu\phi R_{jk}{}^{\nu\sigma} - 4\kappa_{3,\rho}(\phi,\rho)\delta^{ijk}_{\mu\nu\sigma}\nabla_i\phi\nabla^\mu\phi\nabla^\nu\nabla_j\phi\nabla^\sigma\nabla_k\phi \\ & +F(\phi,\rho)\delta^{ij}_{\mu\nu}R_{ij}{}^{\mu\nu} - 4F(\phi,\rho)_{,\rho}\delta^{ij}_{\mu\nu}\nabla_i\phi\nabla^\mu\phi\nabla^\nu\nabla_j\phi \\ & -3[2F(\phi,\rho)_{,\phi} + \rho\kappa_8(\phi\,\rho)]\nabla_\mu\nabla^\mu\phi + 2\kappa_8\delta^{ij}_{\mu\nu}\nabla_i\phi\nabla^\mu\phi\nabla^\nu\nabla_j\phi \\ & +\kappa_9(\phi,\rho)\end{aligned}$$

Self-tuning filter

$$\begin{aligned}\mathcal{L}_{john} &= \sqrt{-g}V_{john}(\phi)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \\ \mathcal{L}_{paul} &= \sqrt{-g}V_{paul}(\phi)(*R*)^{\mu\nu\alpha\beta}\nabla_\mu\phi\nabla_\alpha\phi\nabla_\nu\nabla_\beta\phi \\ \mathcal{L}_{george} &= \sqrt{-g}V_{george}(\phi)R \\ \mathcal{L}_{ringo} &= \sqrt{-g}V_{ringo}(\phi)\hat{G}\end{aligned}$$

Cosmology equations and self tuning

- Friedmann equation reads $\mathcal{H} = -\rho_\Lambda$
-

$$\mathcal{H}_{john} = 3V_{john}(\phi)\dot{\phi}^2\left(H^2+\frac{\kappa}{a^2}\right)+6V_{john}(\phi)\dot{\phi}^2H^2$$

$$\mathcal{H}_{paul} = -9V_{paul}(\phi)\dot{\phi}^3H\left(H^2+\frac{\kappa}{a^2}\right)-6V_{paul}(\phi)\dot{\phi}^3H^3$$

$$\mathcal{H}_{george} = -6V_{george}(\phi)\left[\left(H^2+\frac{\kappa}{a^2}\right)+H\dot{\phi}\frac{V'_{george}}{V_{george}}\right]$$

$$\mathcal{H}_{ringo} = -24V'_{ringo}(\phi)\dot{\phi}H\left(H^2+\frac{\kappa}{a^2}\right)$$

- First find self tuning vacuum setting $H^2+\frac{\kappa}{a^2}=0$
- Algebraic equation with respect to $\dot{\phi}$. Hence ϕ is a function of time t with discontinuous first derivatives at $t = t_*$
- Ringo cannot self-tune without a little help from his friends.

Cosmology equations and self tuning

- Scalar equation, $E_\phi = E_{john} + E_{paul} + E_{george} + E_{ringo} = 0$
-

$$E_{john} = 6\frac{d}{dt}\left[a^3 V_{john}(\phi)\dot{\phi}\Delta_2\right] - 3a^3 V'_{john}(\phi)\dot{\phi}^2\Delta_2$$

$$E_{paul} = -9\frac{d}{dt}\left[a^3 V_{paul}(\phi)\dot{\phi}^2 H\Delta_2\right] + 3a^3 V'_{paul}(\phi)\dot{\phi}^3 H\Delta_2$$

$$E_{george} = -6\frac{d}{dt}\left[a^3 V'_{george}(\phi)\Delta_1\right] + 6a^3 V''_{george}(\phi)\dot{\phi}\Delta_1 + 6a^3 V'_{george}(\phi)\Delta_1^2$$

$$E_{ringo} = -24V'_{ringo}(\phi)\frac{d}{dt}\left[a^3\left(\frac{\kappa}{a^2}\Delta_1 + \frac{1}{3}\Delta_3\right)\right]$$

- where

$$\Delta_n = H^n - \left(\frac{\sqrt{-\kappa}}{a}\right)^n$$

- which vanishes on shell as it should
- For non trivial cosmology need

$$\{V_{john}, V_{paul}, V_{george}, V_{ringo}\} \neq \{0, 0, constant, constant\}$$

Conclusions

- Starting from a general scalar tensor theory (Horndeski)
- We have filtered out the theory with self-tuning properties
- Theory has enchanting geometrical properties which we need to understand
- Still have 4 free functions which parametrise the theory. These need to be fixed by cosmology, stability and local constraints.

Many questions unanswered:

1. What is the Fab 4 cosmology? In other words for which of the potentials do we get usual Hot Big Bang cosmology?
2. Usually to escape solar system constraints we take refuge in Veinshtein of chameleon mechanisms...
3. Maybe we can do better by redoing solar system tests from scratch for the self-tuned background in the spirit of [gr-qc/08014339]
4. Black hole solutions of such theories could really help. Also self tuning in different backgrounds.

Sketch of proof

Consider gravity action including all contributions of cosmological constant in the scalar potential term V,

$$S[\pi, g_{\mu\nu}] = \int d^4x\sqrt{-g}R + L(\pi, g_{\mu\nu}, \partial^m, V)$$

Assume $g_{\mu\nu} = \eta_{\mu\nu}, \pi = constant$. Then
On-shell $L_0 = -V_0\sqrt{-g}$ where $L_0 = L(\eta_{\mu\nu}, constant, \Lambda)$
with EoM,
$\frac{\partial L}{\partial g_{\mu\nu}}_{|0} = \frac{\partial L}{\partial \pi}_{|0} = 0$
scalar EoM is related to the trace of gravity equation
Then Lagrangian has remnant symmetry,
$\delta g_{\mu\nu} = \epsilon g_{\mu\nu}$ and $\delta\pi = -\epsilon$
and hence
$L = \sqrt{-\hat{g}}\mathcal{L}(\hat{g}_{\mu\nu}, \partial)$ with $\hat{g}_{\mu\nu} = e^{\pi} g_{\mu\nu}$
All dependance in π has dropped out.
So,on-shell for vacuum we have
$\frac{\partial L}{\partial g_{\mu\nu}}_{|0} = \frac{1}{2} g^{\mu\nu} L_0$
Hence $V_0(\Lambda) = 0$ and thus the cosmological constant is fine tuned

George is easy

- Start with $\mathcal{L}_{George}$
- Set everybody else to zero
- $$\kappa_9 = \frac{1}{2} V''_{george}\rho, \qquad F = -\frac{1}{12} V_{george}$$
- $$\mathcal{L}_{george} = -\frac{1}{6} V_{george}(\phi) R + \frac{1}{2} \nabla_\mu \left[V'_{george} \partial^\mu \phi \right] . \cong -\frac{1}{6} V_{george}(\phi) R$$
- Einstein-Hilbert non-minimally coupled with a free scalar field
- The remaining terms need more work.
- Back to classical GR

The double dual tensor and Lovelock theory

- In 4 dimensions we can define a dual of the curvature tensor by dualising each pair of indices much like the Faraday tensor in EM

$$*F^{ab} = \frac{1}{2}\varepsilon^{abcd}\ F_{cd}$$

- Double Dual $(*R*)$

$$(*R*)_{\mu\nu\sigma\lambda} = -\frac{1}{4}\varepsilon_{\mu\nu}{}^{ij}\ R_{ijkl}\ \varepsilon_{\sigma\lambda}{}^{kl} = \tfrac{1}{4}\delta^{ijkl}_{\mu\nu\sigma\lambda}\ R_{ijkl}$$

As appearing in the Horndeski action

1. Same index properties as R-tensor
2. Divergence free:

$$\nabla_i (*R*)_{jkl}{}^{i} = 0$$

3. Simple trace is Einstein

$$(*R*)^{ik}{}_{jk} = -G^i_j,$$

4. Hence

$$\tfrac{1}{4}\delta^{ijk}_{\mu\nu\sigma}\ R_{jk}{}^{\mu\nu} = -G^i_\mu$$

5. $$(*R*)^{\mu\nu\alpha\beta} = R^{\mu\nu\alpha\beta} + 2R^{\nu[\alpha}g^{\beta]\mu} - 2R^{\mu[\alpha}g^{\beta]\nu} + Rg^{\mu[\alpha}g^{\beta]\nu}\ ,$$

6. Finally the 2nd order Lovelock tensor originating from variation of $\hat{\mathcal{G}}$ is:

$$H_{ij} = (*R*)_i{}^{klm}R_{jklm} - \frac{1}{4}g_{ij}\hat{\mathcal{G}}\ .$$

In $D = 4$ $H_{ij} = 0$ hence $(*R*)_i{}^{klm}R_{jklm} = \frac{1}{4}g_{ij}\hat{\mathcal{G}}$

With a little help from my friends

- Switch on only V_{Ringo} in EoM. We find,
- $$K_1 = \frac{1}{16} V'_{ringo}, \qquad K_3 = \frac{1}{16} V''_{ringo}$$
  Note the absence of $\dot{\phi}$; Ringo cannot self-tune without a little help from his friends.
- The equation of motion reads,

$$\begin{aligned}
\mathcal{E}^{ik}_{ringo} &= \sqrt{-g} K_1(\phi, \rho) \delta^{aijk}_{\lambda\mu\nu\sigma} g^{\lambda b} \nabla^\mu \nabla_i \phi R_{jk}{}^{\nu\sigma} + K_3(\phi, \rho) \delta^{aijk}_{\lambda\mu\nu\sigma} g^{\lambda b} \nabla_i \phi \nabla^\mu \phi R_{jk}{}^{\nu\sigma} \\
&= \sqrt{-g} (*R*)^{ijkl} \left(4K_1 \nabla_l \nabla_j \phi + 4K_3 \nabla_l \phi \nabla_j \phi\right) \\
&= \frac{1}{4} \sqrt{-g} (*R*)^{ijkl} \nabla_l \nabla_j V_{ringo}(\phi)
\end{aligned}$$

- While at the same time we have,

$$\delta \left[\int_{\mathcal{M}} d^4x \sqrt{-g}\, V(\phi) \hat{\mathcal{G}} \right]$$
$$= \int_{\mathcal{M}} d^4x \sqrt{-g}\, \delta g^{ij} \left[2\phi H_{ij} + 4(*R*)_{ikjl} \nabla^l \nabla^k V(\phi) \right] + \delta\phi [\partial_\phi V(\phi) \hat{\mathcal{G}}]$$

- Hence $\mathcal{L}_{Ringo} = V_{Ringo}(\phi) \hat{\mathcal{G}}$
- Similarly $\mathcal{L}_{John} = V_{john} G_{ij} \nabla^i \phi \nabla^j \phi$.

Paul

- Last term is not recognisable. However, numerous Padilla tricks bring it to the form,
- $$\begin{aligned}\mathcal{L}_{paul} &= \sqrt{-g}V_{Paul}(\phi)\left[R^{\mu\nu\alpha\beta}\nabla_\mu\phi\nabla_\alpha\phi\nabla_\nu\nabla_\beta\phi+\right. \\ &\quad +G^{\mu\nu}(\nabla_\mu\phi\nabla_\alpha\phi - g_{\mu\alpha}(\nabla\phi)^2)\nabla^\alpha\nabla_\nu\phi \\ &\quad +R^{\mu\nu}(\nabla_\mu\nabla_\alpha\phi - g_{\mu\alpha}\Box\phi)\nabla^\alpha\phi\nabla_\nu\phi]\end{aligned}$$
- ???However remember,
  $$(*R*)^{\mu\nu\alpha\beta} = R^{\mu\nu\alpha\beta} + 2R^{\nu[\alpha}g^{\beta]\mu} - 2R^{\mu[\alpha}g^{\beta]\nu} + Rg^{\mu[\alpha}g^{\beta]\nu} ,$$
- Therefore
  $$\mathcal{L}_{paul} = \sqrt{-g}V_{paul}(\phi)(*R*)^{\mu\nu\alpha\beta}\nabla_\mu\phi\nabla_\alpha\phi\nabla_\nu\nabla_\beta\phi$$