Bondi-Sachs Formulation of General Relativity (GR) and the Vertices of the Null Cones

Thomas Mädler
Observatoire de Paris/Meudon,
Max Planck Institut for Astrophysics

Sept 10, 2012 - IAP seminar

Astrophysical Motivation

Astrophysical Motivation

Astrophysical Motivation

Supernova 1987A was a core-collapse supernova, which ...

- ☑ ... is the explosion of a star undergoing a gravitational collapse
- ☑ ... involves time varying multi-poles of the stellar mass-energy distributions

... generate gravitational waves ....

Gravitational Waves

-Motivation of Bondi's Idea-

metric = Minkowski + perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

choose coordinates y^α

equation of motions

$$\left(\delta^{ij}\frac{\partial^2}{\partial y^i \partial y^j} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)h_{\mu\nu} = 2\kappa T_{\mu\nu}$$

Gravitational Waves -Motivation of Bondi's Idea-

metric = Minkowski + perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

choose coordinates y^α

equation of motions $$\left(\delta^{ij}\frac{\partial^2}{\partial y^i \partial y^j} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)h_{\mu\nu} = 2\kappa T_{\mu\nu}$$

PROBLEM of THIS DESCRIPTION

linearised metric theory is not gauge invariant

other coordinates also yield : metric = Minkowski + perturbation

Eddington (1922) : Gravitational waves travel at the speed of thought

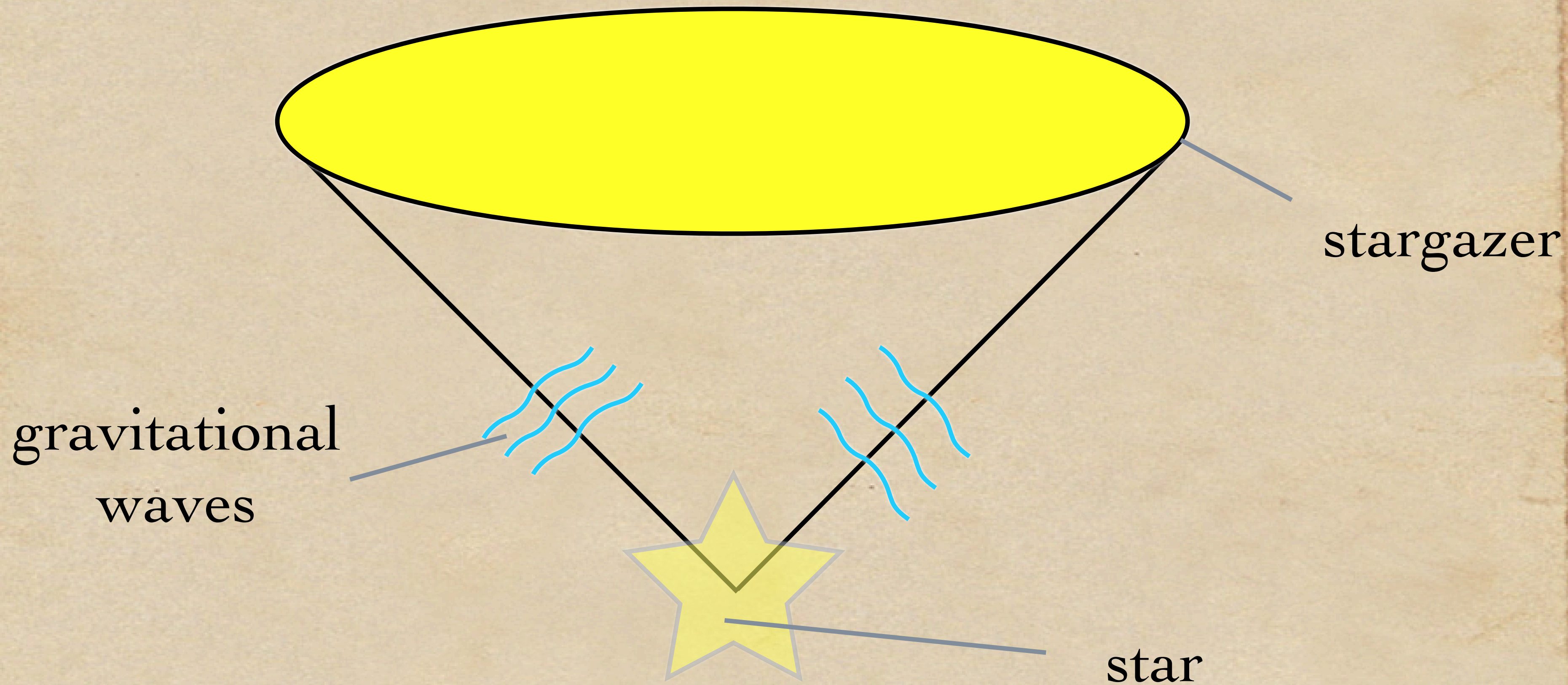
Gravitational Waves
-Motivation of Bondi's Idea-

Pirani showed in 1957...

- gravitational radiation is characterised by the Riemann tensor
- fundamental speed is the speed of light

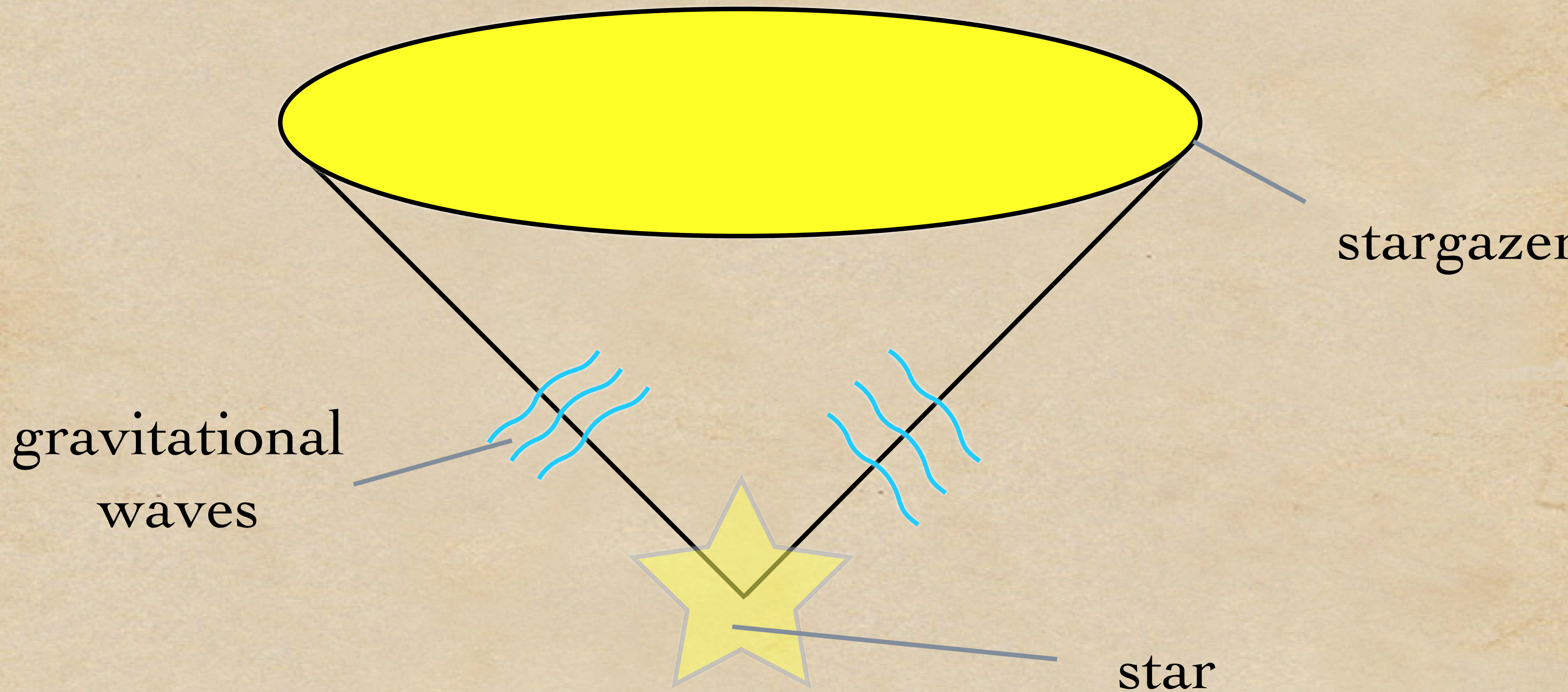
Light-cone approach of GR

The Bondi-Sachs Formulation



Light-cone approach of GR

The Bondi-Sachs Formulation



What are the properties of the Bondi-Sachs Formulation of GR and which role plays the vertex?

Out-look

- Simplifying Assumptions
- The Bondi-Sachs Metric
- The Field equations
- Mass-loss and Gravitational Waves
- Regularity Conditions @ the Vertex
- Summary

Simplifying Assumptions

Simplifying Assumptions

Axial Symmetry

Simplifying Assumptions

Axial Symmetry

+

Vacuum Space-Times

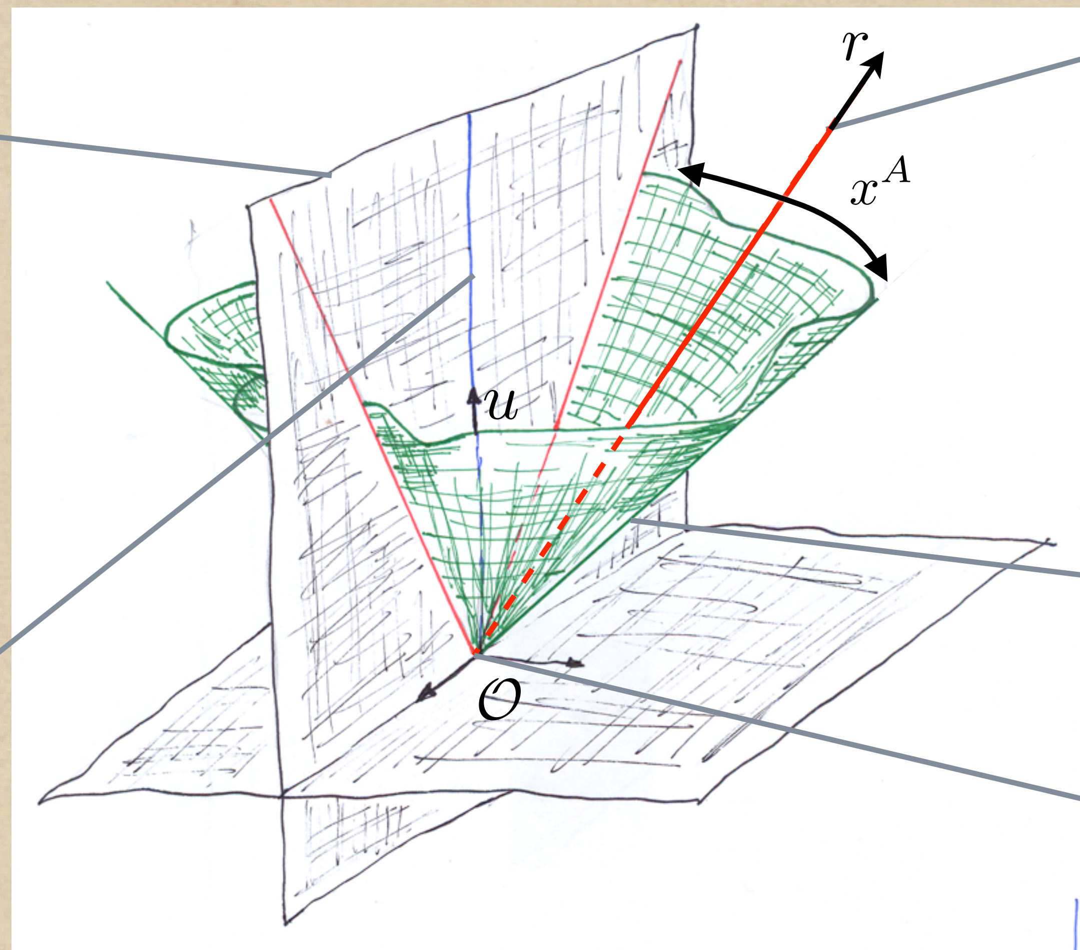
The Origin of the Coordinate System in Axial Symmetry

- Axis of symmetry $\mathcal{A}$ in a four-dimensional space-time is a totally-geodesic, time-like 2-surface
- $\mathcal{A}$ contains time-like geodesic curves given by the symmetry
- choose in $\mathcal{A}$ a time-like geodesic (world-line of an observer) which traces the vertices of null cones

VERTEX = ORIGIN

Coordinates @ a Light-Cone

axis of symmetry



out-going light ray

$u = const$

$x^A = const$

r varies

out-going light cone

$u = const$

axial observer

vertex

The Bondi-Sachs Metric

$x^0 := u$ is null coordinate $g^{\alpha\beta}\nabla_\alpha u \nabla_\beta u = 0$

$k^\alpha := g^{\alpha\beta}\nabla_\beta u$ null vector of the out-going null rays

$x^A := (x^2, x^3)$ are constant along null rays $k^\alpha \nabla_\alpha x^A = 0$

$x^1 := r$ is a luminosity distance

$$\frac{\det(g_{AB})}{r^4} = f(x^A)$$

The Bondi-Sachs Metric

$x^0 := u$ is null coordinate $g^{\alpha\beta}\nabla_\alpha u \nabla_\beta u = 0$ → $g^{00} = g_{11} = 0$

$k^\alpha := g^{\alpha\beta}\nabla_\beta u$ null vector of the out-going null rays

$x^A := (x^2, x^3)$ are constant along null rays $k^\alpha \nabla_\alpha x^A = 0$

$x^1 := r$ is a luminosity distance

$$\frac{\det(g_{AB})}{r^4} = f(x^A)$$

The Bondi-Sachs Metric

$x^0 := u$ is null coordinate $g^{\alpha\beta}\nabla_\alpha u \nabla_\beta u = 0$ → $g^{00} = g_{11} = 0$

$k^\alpha := g^{\alpha\beta}\nabla_\beta u$ null vector of the out-going null rays

$x^A := (x^2, x^3)$ are constant along null rays $k^\alpha \nabla_\alpha x^A = 0$ → $g^{0A} = g_{1A} = 0$

$x^1 := r$ is a luminosity distance

$$\frac{\det(g_{AB})}{r^4} = f(x^A)$$

The Bondi-Sachs Metric

$x^0 := u$ is null coordinate $g^{\alpha\beta}\nabla_\alpha u \nabla_\beta u = 0$ → $g^{00} = g_{11} = 0$

$k^\alpha := g^{\alpha\beta}\nabla_\beta u$ null vector of the out-going null rays

$x^A := (x^2, x^3)$ are constant along null rays $k^\alpha \nabla_\alpha x^A = 0$ → $g^{0A} = g_{1A} = 0$

$x^1 := r$ is a luminosity distance → $g^{01} \neq -1$, $g_{AB} = r^2 h_{AB}$

$$\frac{\det(g_{AB})}{r^4} = f(x^A)$$

The Bondi-Sachs Metric

$x^0 := u$ is null coordinate $g^{\alpha\beta}\nabla_\alpha u \nabla_\beta u = 0$ → $g^{00} = g_{11} = 0$

$k^\alpha := g^{\alpha\beta}\nabla_\beta u$ null vector of the out-going null rays

$x^A := (x^2, x^3)$ are constant along null rays $k^\alpha \nabla_\alpha x^A = 0$ → $g^{0A} = g_{1A} = 0$

$x^1 := r$ is a luminosity distance → $g^{01} \neq -1$, $g_{AB} = r^2 h_{AB}$

$$\frac{\det(g_{AB})}{r^4} = f(x^A)$$

$$ds^2 = -e^{2\Phi+4\beta}du^2 - 2e^{2\beta}dudr + r^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du)$$

$$h_{AB}dx^A dx^B = e^{2\gamma}\cosh(2\delta)d\theta^2 + 2\sin\theta\sinh(2\delta)d\theta d\phi + e^{-2\gamma}\sin^2\theta\cosh(2\delta)d\phi^2$$

Bondi [1960], Sachs [1962], van der Burg [1966]

Einstein Equations

6 main equations

4 hyper-surface equations

$$\mathcal{R}_{rr} = 0$$

$$\mathcal{R}_{rA} = 0$$

$$g^{AB}\mathcal{R}_{AB} = 0$$

2 evolution equations

$$\mathcal{R}_{AB} - \frac{1}{2} g_{AB} \left(g^{CD}\mathcal{R}_{CD} \right) = 0$$

3 conservation equations

$$\mathcal{R}_{uu} = 0$$

$$\mathcal{R}_{uA} = 0$$

1 trivial equation

$$\mathcal{R}_{ur} = 0$$

Tamburino et al. [1966]

Structure of the Field Equations

$$\nabla_\alpha \left(R^\alpha{}_\beta - \frac{1}{2}\delta^\alpha{}_\beta R^\mu{}_\mu \right) = 0$$

imply....

Bondi-Sachs Lemma: *If the main equations hold on one null cone and the optical expansion rate of the null rays does not vanish, i.e.* $\beta \neq \infty$, *on this cone, then the trivial equation is fulfilled algebraically and the supplementary equations hold everywhere on this null cone provided they are fulfilled at one radius r=R>0.*

Hierarchy of Main Equations

- Hyper-surface equations

$$R_{rr} = 0: \;\; \beta_{,r} = J_{(0)}(\gamma,\, \delta)$$

$$R_{rA} = 0: \;\; U^A_{,rr} = J^A(\beta,\, \gamma,\, \delta)$$

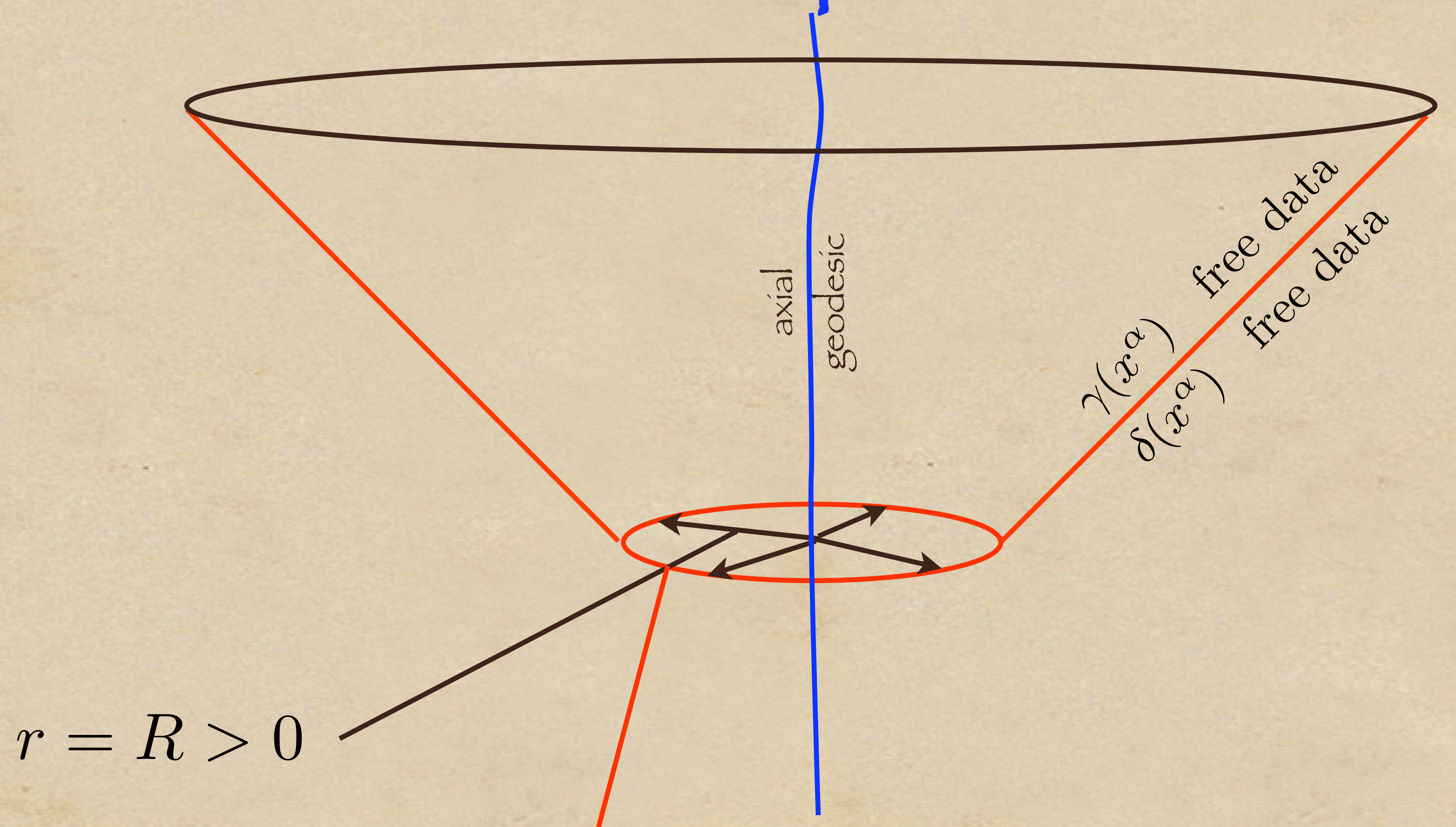
$$g^{AB} R_{AB} = 0: \;\; \Phi_{,r} = J_{(1)}(U^A, \beta,\, \gamma,\, \delta)$$

- evolution equations

$$\gamma_{,ur} = J_{(\gamma)}(\gamma,\, \delta,\, U^A,\, \beta,\, \Phi)$$

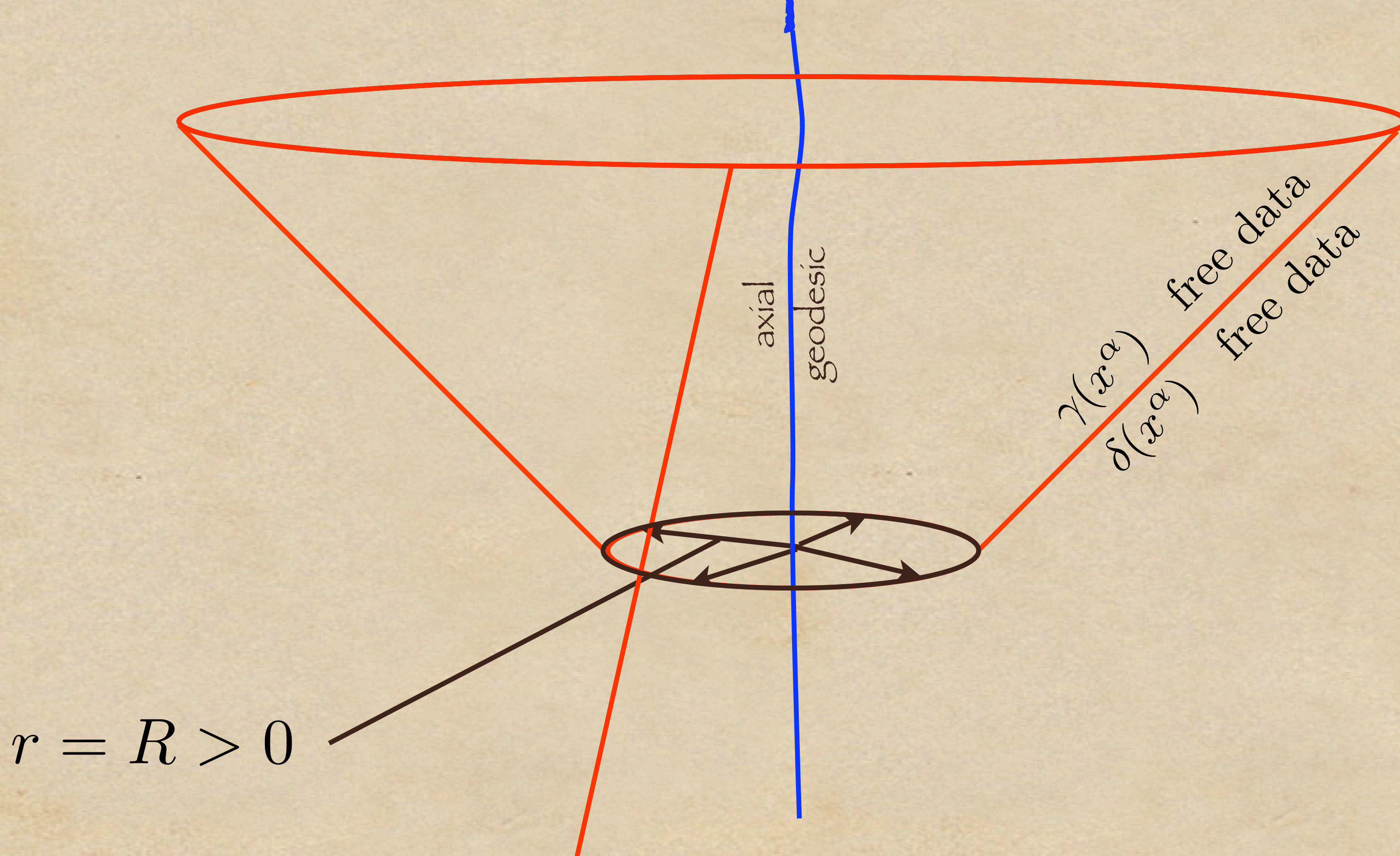
$$\delta_{,ur} = J_{(\delta)}(\gamma,\, \delta,\, U^A,\, \beta,\, \Phi)$$

Initial values to Integrate the Einstein Equations (1)



specify β, U^A, Φ and $U^A_{,r}$ for all values of x^A

Initial values to Integrate the Einstein Equations (2)



specify β, U^A, Φ and $U^A_{,r}$ for all values of x^A

Condition for Out-Going Gravitational Waves

... is the Sommerfeld radiation condition:

$$\lim_{r\to\infty} r\gamma\Big|_{\substack{x^A = const \\ u = const}} = const \quad , \qquad \lim_{r\to\infty} r\delta\Big|_{\substack{x^A = const \\ u = const}} = const$$

... ansatz for the asymptotic solution

$$\gamma(u,r,x^D) = \frac{c(u,x^D)}{r} + \mathcal{O}(r^{-3}) \quad , \qquad \delta(u,r,x^D) = \frac{d(u,x^D)}{r} + \mathcal{O}(r^{-3})$$

Bondi et. al. [1962], van der Burg [1966]

Asymptotic Solution of the Main equations

... consequence of the integration of the main equations

$$\beta(x^\alpha) = \beta_\infty(u, x^A) + \mathcal{O}(r^{-2})$$
$$U^A(x^\alpha) = U^A_\infty(u, x^A) + \mathcal{O}(r^{-2})$$
$$e^{2\Phi(x^\alpha)} = 1 - \frac{2M(u, x^A)}{r} + \mathcal{O}(r^{-2})$$

Bondi's choise of the functions of integration

$$\beta_\infty(u, x^A) = 0$$
$$U^A_\infty(u, x^A) = 0$$

Asymptotic Solution of the Main equations

... consequence of the integration of the main equations

$$\beta(x^\alpha) = \beta_\infty(u, x^A) + \mathcal{O}(r^{-2})$$
$$U^A(x^\alpha) = U^A_\infty(u, x^A) + \mathcal{O}(r^{-2})$$
$$e^{2\Phi(x^\alpha)} = 1 - \frac{2M(u, x^A)}{r} + \mathcal{O}(r^{-2})$$

Bondi's choise of the functions of integration

$$\beta_\infty(u, x^A) = 0$$
$$U^A_\infty(u, x^A) = 0$$

Bondi chooses an asymptotic Minkowskian observer

The Bondi-Mass & Mass-Loss

- the function of integration $M(u, x^A)$ is the Mass aspect of the isolated system
- the Bondi Mass $m(u)$ of an isolated system is

$$m(u) = \frac{1}{4\pi} \oint M(u, \theta) \sin^2 \theta d\theta d\phi$$

- the time variation of the Bondi mass as measured by the asymptotic observer is

$$\frac{d}{du} m(u) = -\frac{1}{4\pi} \oint \left[c^2_{,u}(u, \theta) + d^2_{,u}(u, \theta) \right] \sin^2 \theta d\theta d\phi$$

The Bondi-Mass & Mass-Loss

- the function of integration $M(u, x^A)$ is the Mass aspect of the isolated system
- the Bondi Mass $m(u)$ of an isolated system is

$$m(u) = \frac{1}{4\pi} \oint M(u, \theta) \sin^2 \theta d\theta d\phi$$

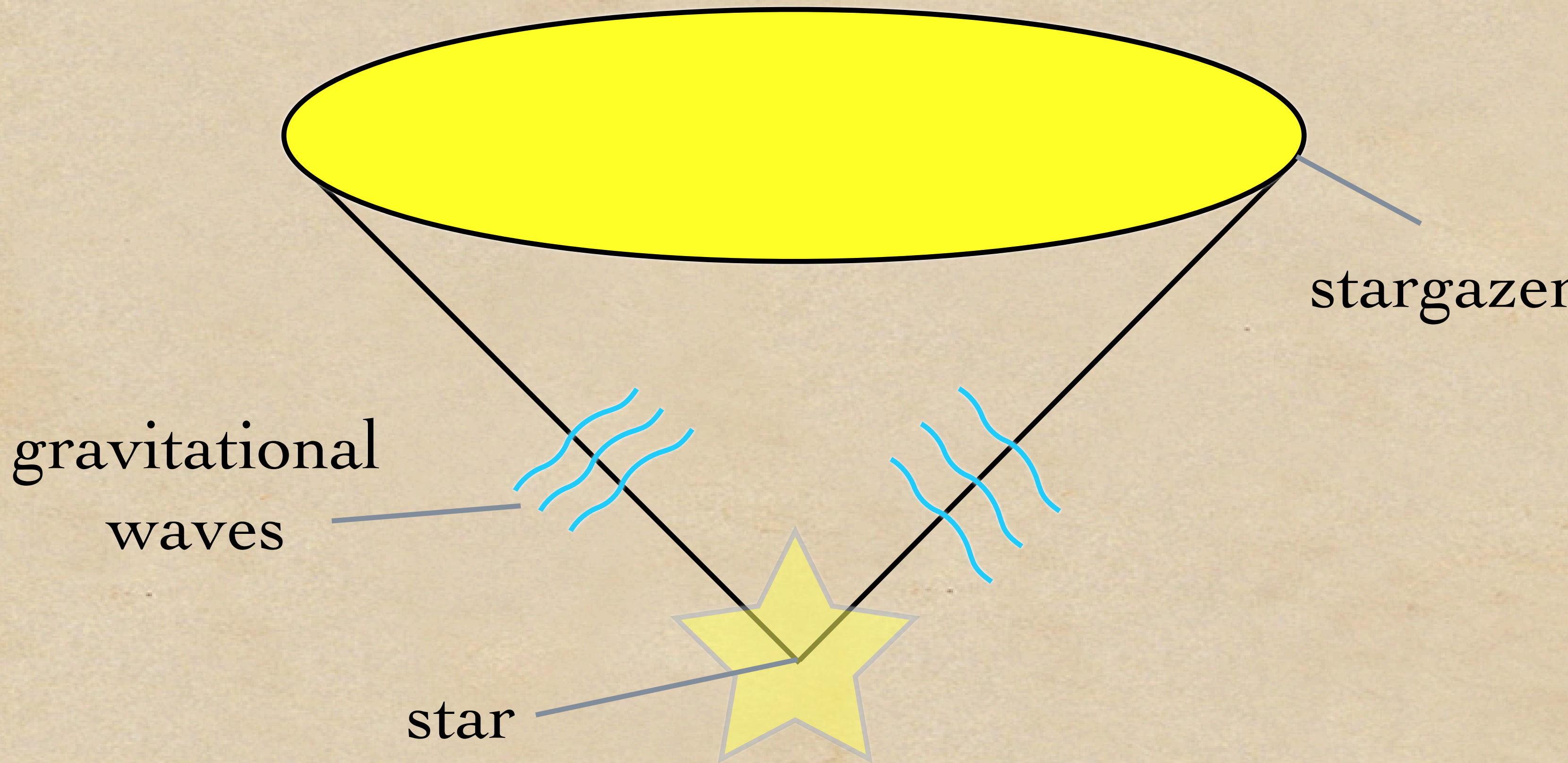
- the time variation of the Bondi mass as measured by the asymptotic observer is

$$\frac{d}{du} m(u) = -\frac{1}{4\pi} \oint \left[c^2_{,u}(u, \theta) + d^2_{,u}(u, \theta) \right] \sin^2 \theta d\theta d\phi$$

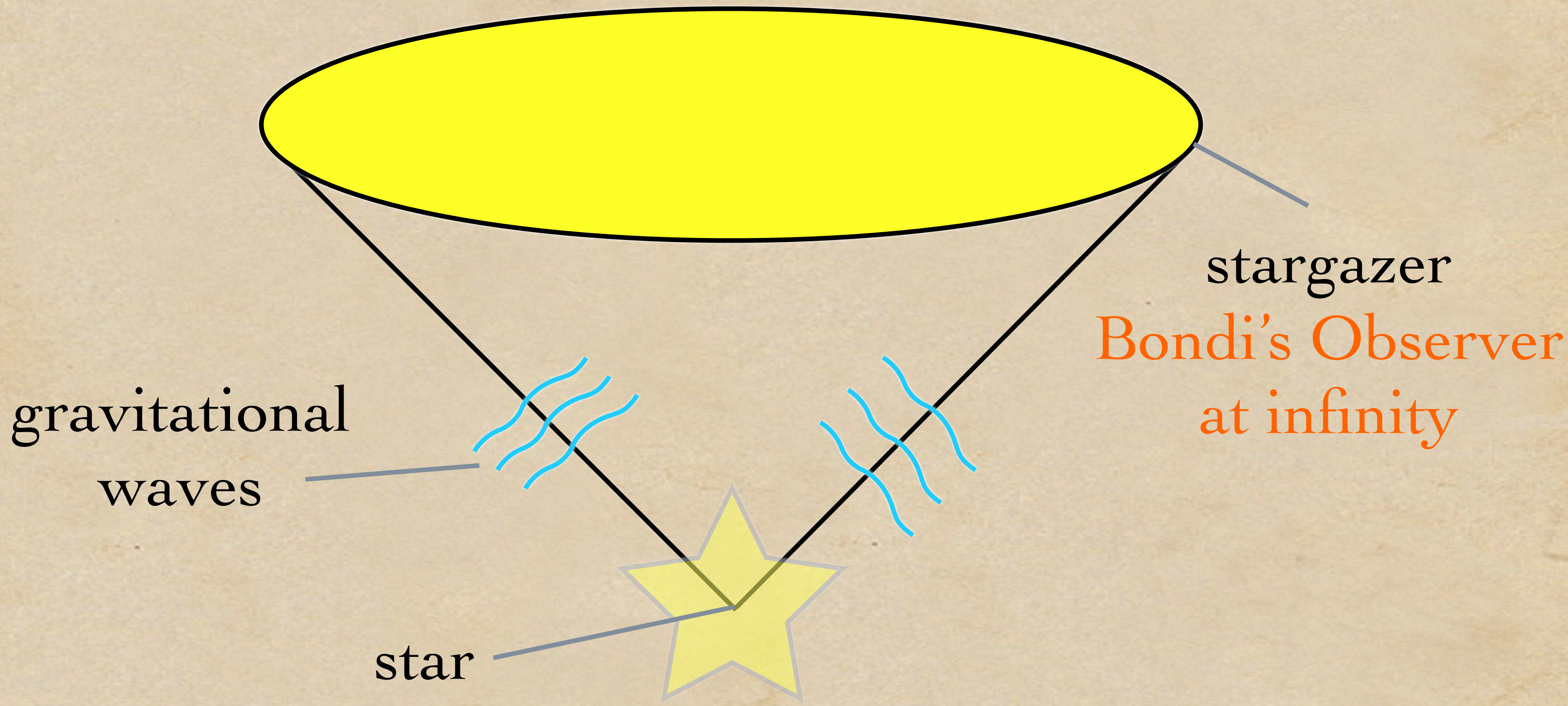
-an isolated system can only lose mass -

-via gravitational radiation-

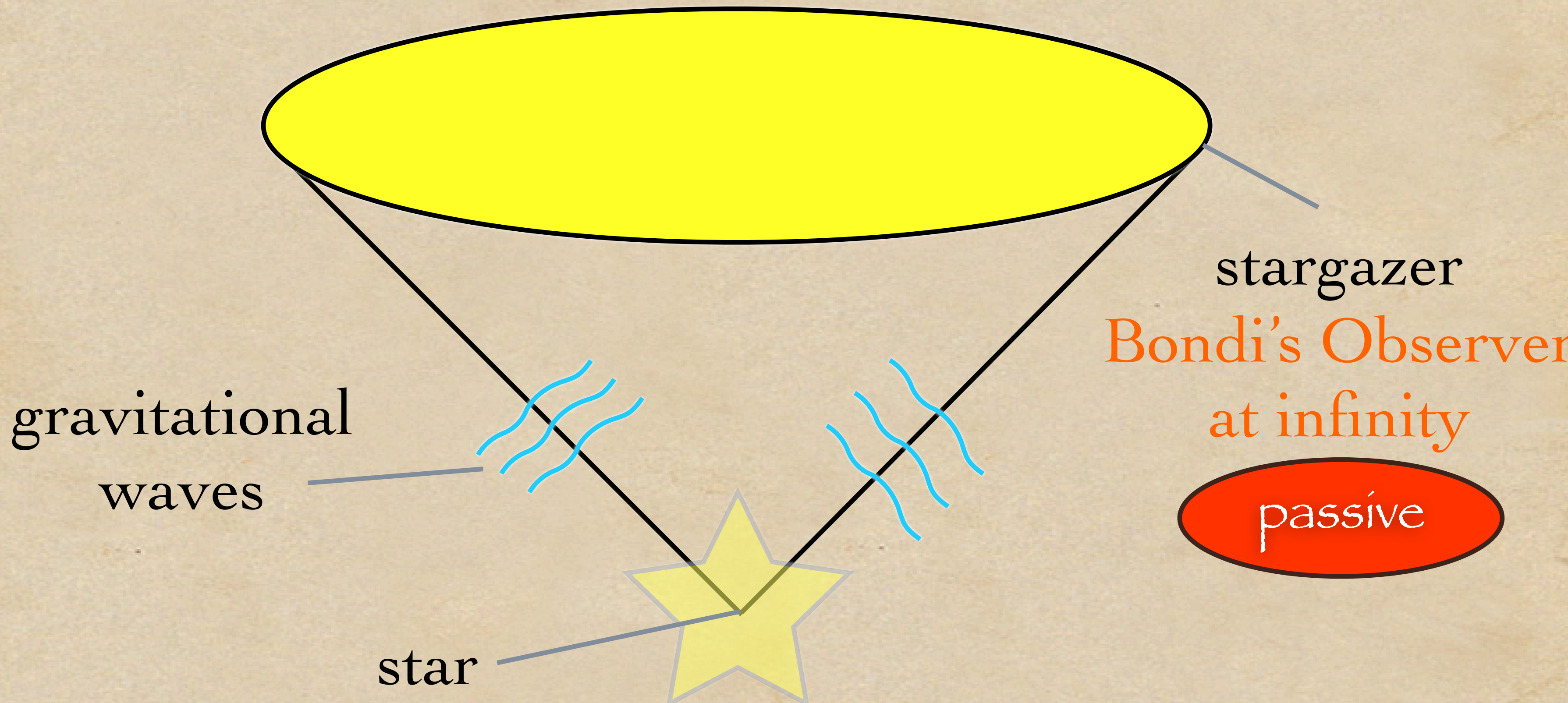
Infinity vs. Origin



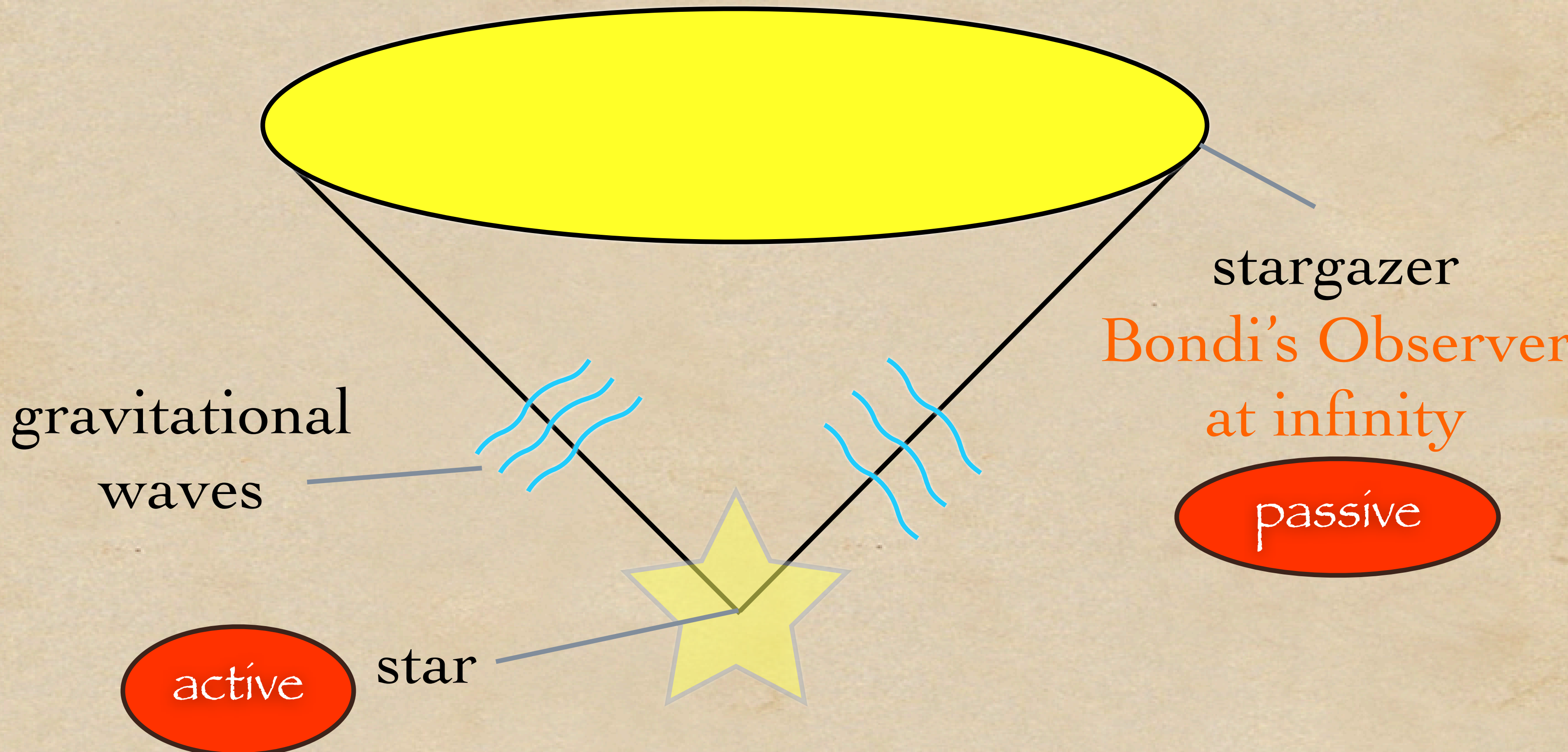
Infinity vs. Origin



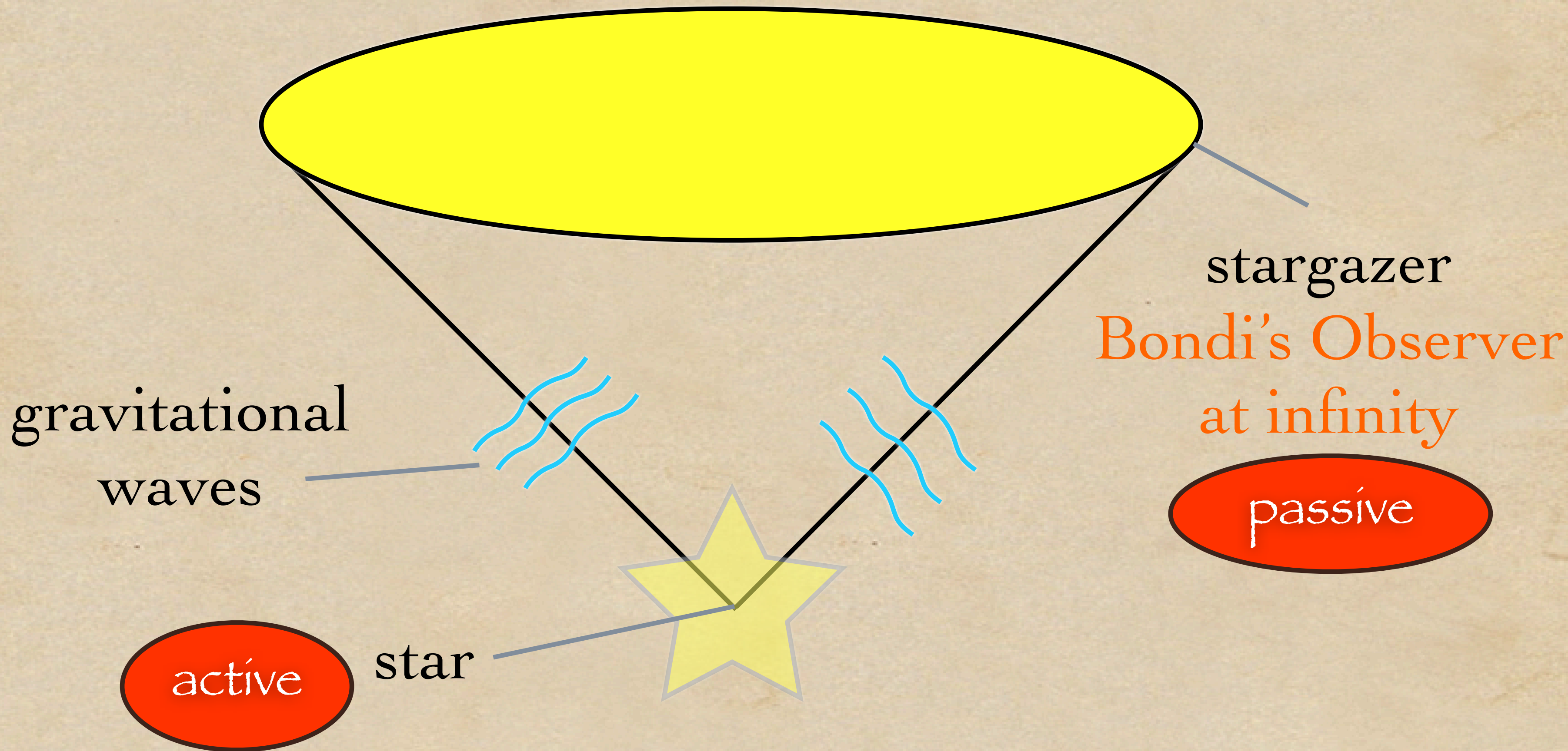
Infinity vs. Origin



Infinity vs. Origin



Infinity vs. Origin



to study a gravitational active source
need inertial observer at the origin

The Vertex - Problems

(1) a null cone is not differentiable at its vertex

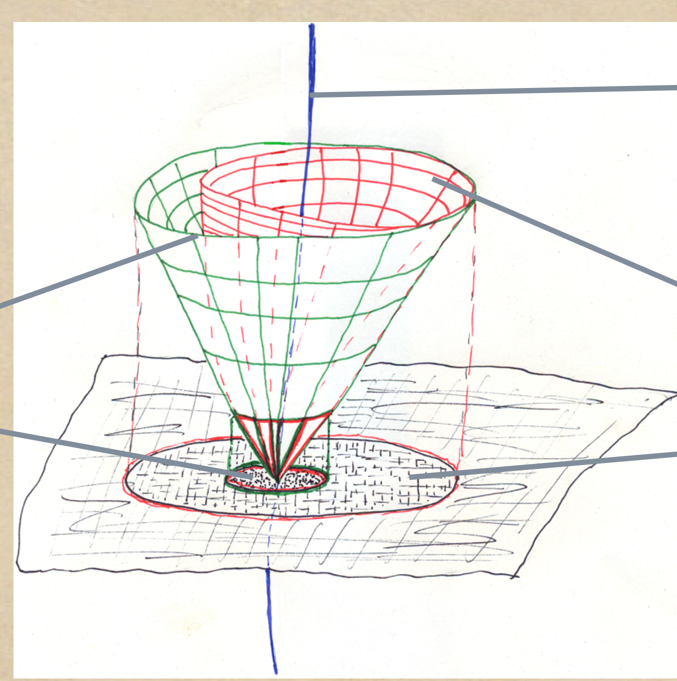
(2) Bondi-Sachs metric is not defined at r=0

(3) require additional regular structure at the vertex with a Taylor expansion in regular coordinates

(4) order of approximation of the metric near the vertex determines how the light rays leave the axial observer

The Vertex - Problems

(4) order of approximation of the metric near the vertex determines how the light rays leave the axial observer



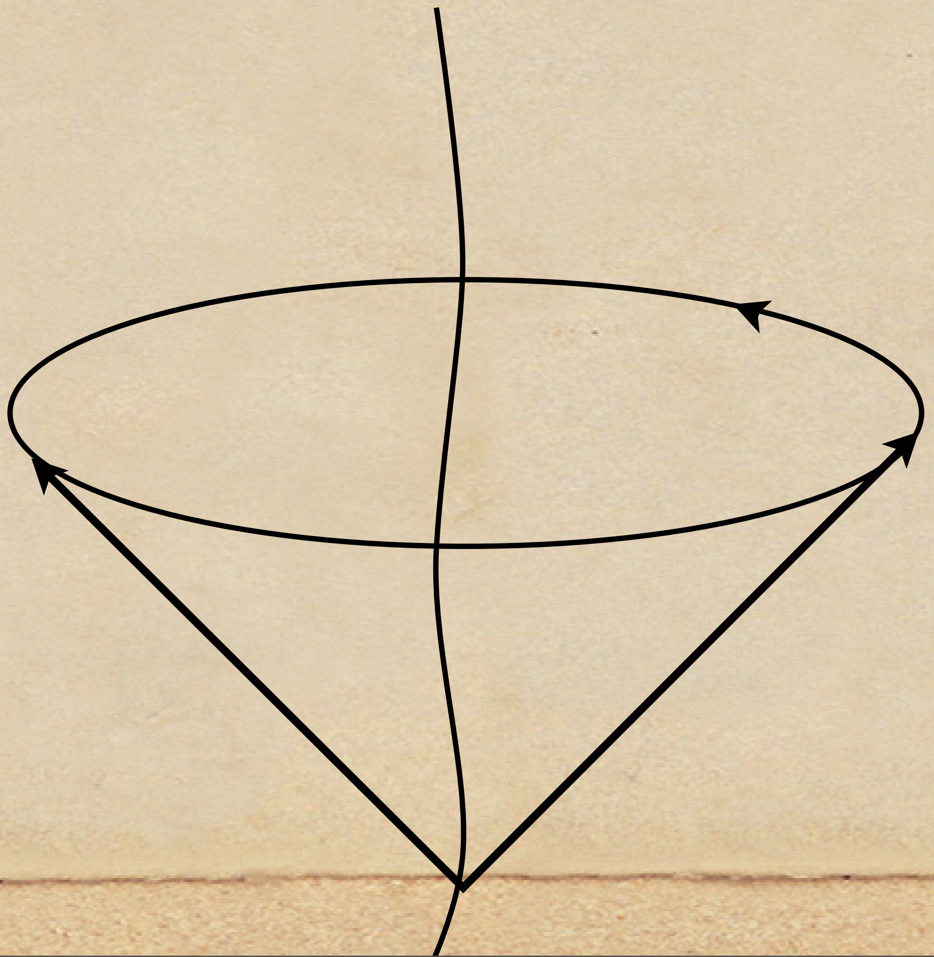
axial observer

spherical emission of the light rays & linear approximation

non-spherical emission of the light rays & quadratic approximation

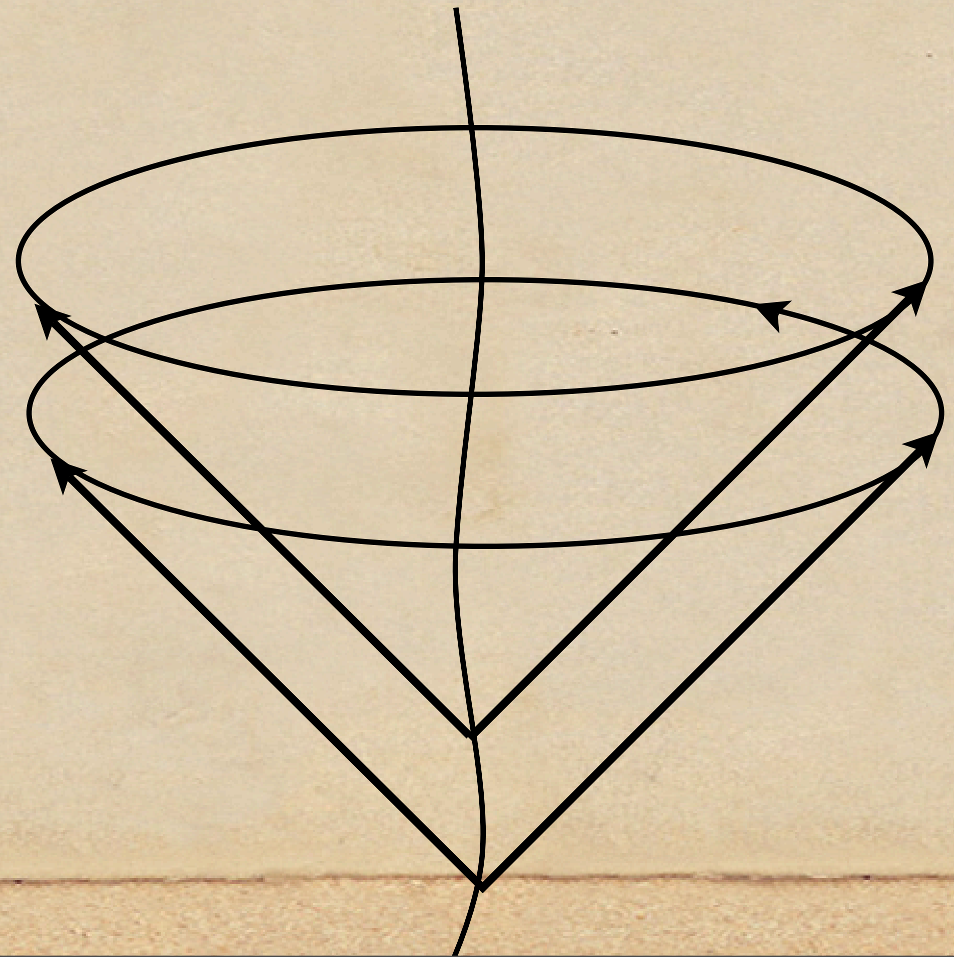
The Vertex - Problems

(5) How does one move the origin of the coordinate system along the time-like curve defining it ?



The Vertex - Problems

(5) How does one move the origin of the coordinate system along the time-like curve defining it ?



Solving the Vertex - Problem

... define a regular coordinate system along the axial geodesic

... define a null cone in the regular coordinate system

... transform the regular metric to a Bondi-Sachs metric

The Metric @ the Vertex

The radial expansion of the Bondi-Sachs metric functions...

...starts at different positive powers

...shows a strict angular behaviour in terms of associated Legendre polynomials in the coefficients

...contains at higher order coefficients time derivatives of the lower order ones

...contains strict numerical factors in the expansion coefficients

Example :
$$\gamma(u,r,\theta) = \Big[\gamma_2(u)P_2^2(\theta)\Big]r^2 + \Big[\gamma_3(u)P_3^2(\theta) + \frac{5}{6}\frac{d\gamma_2}{du}(u)P_2^2(\theta)\Big]r^3 + \mathcal{O}(r^4)$$

The Metric @ the Vertex

The radial expansion of the Bondi-Sachs metric functions...

...starts at different positive powers

...shows a strict angular behaviour in terms of associated Legendre polynomials in the coefficients

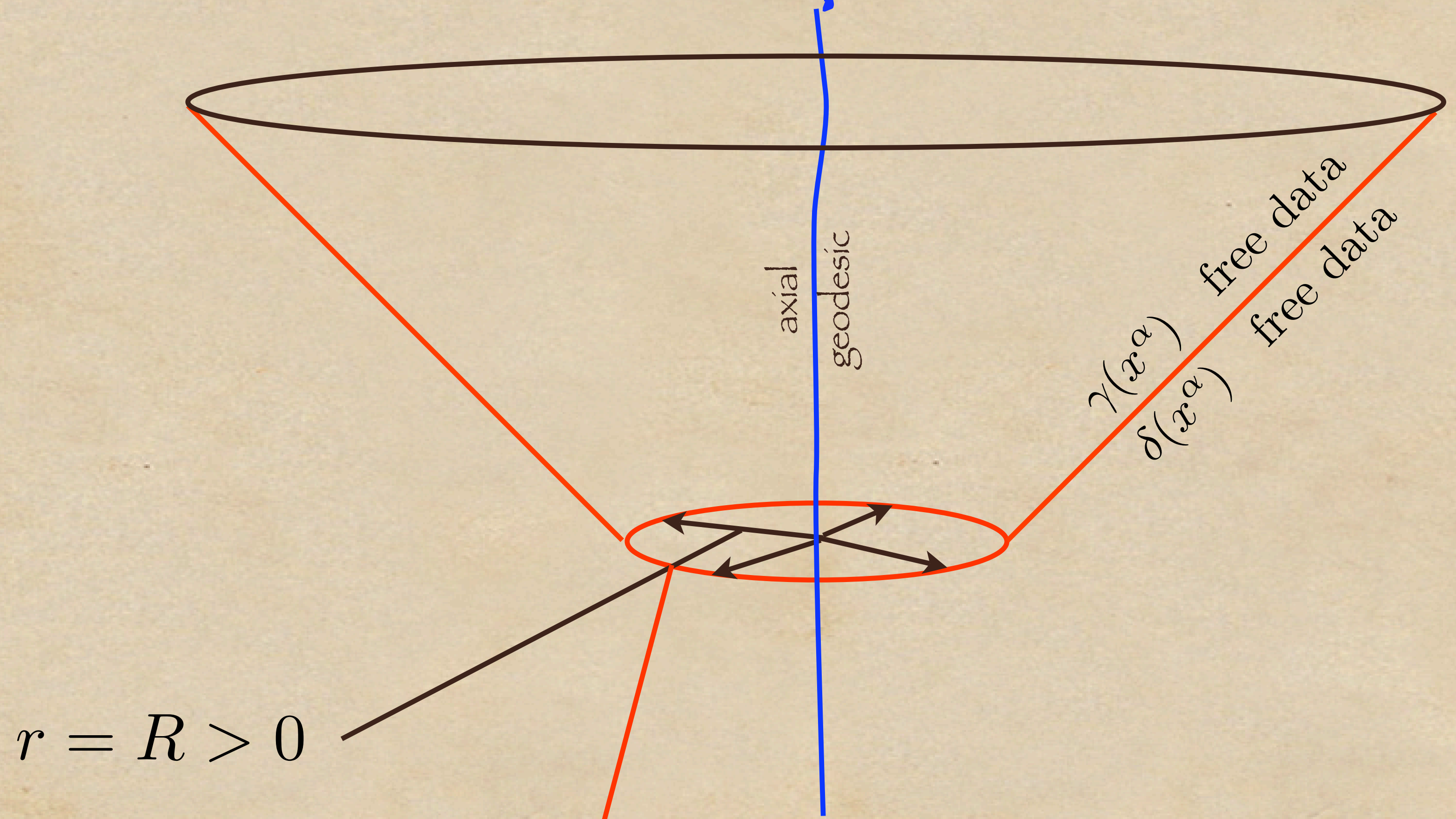
...contains at higher order coefficients time derivatives of the lower order ones

...contains strict numerical factors in the expansion coefficients

Example : $$\gamma(u,r,\theta) = \left[\gamma_2(u)P_2^2(\theta)\right]r^2 + \left[\gamma_3(u)P_3^2(\theta) + \frac{5}{6}\frac{d\gamma_2}{du}(u)P_2^2(\theta)\right]r^3 + \mathcal{O}(r^4)$$

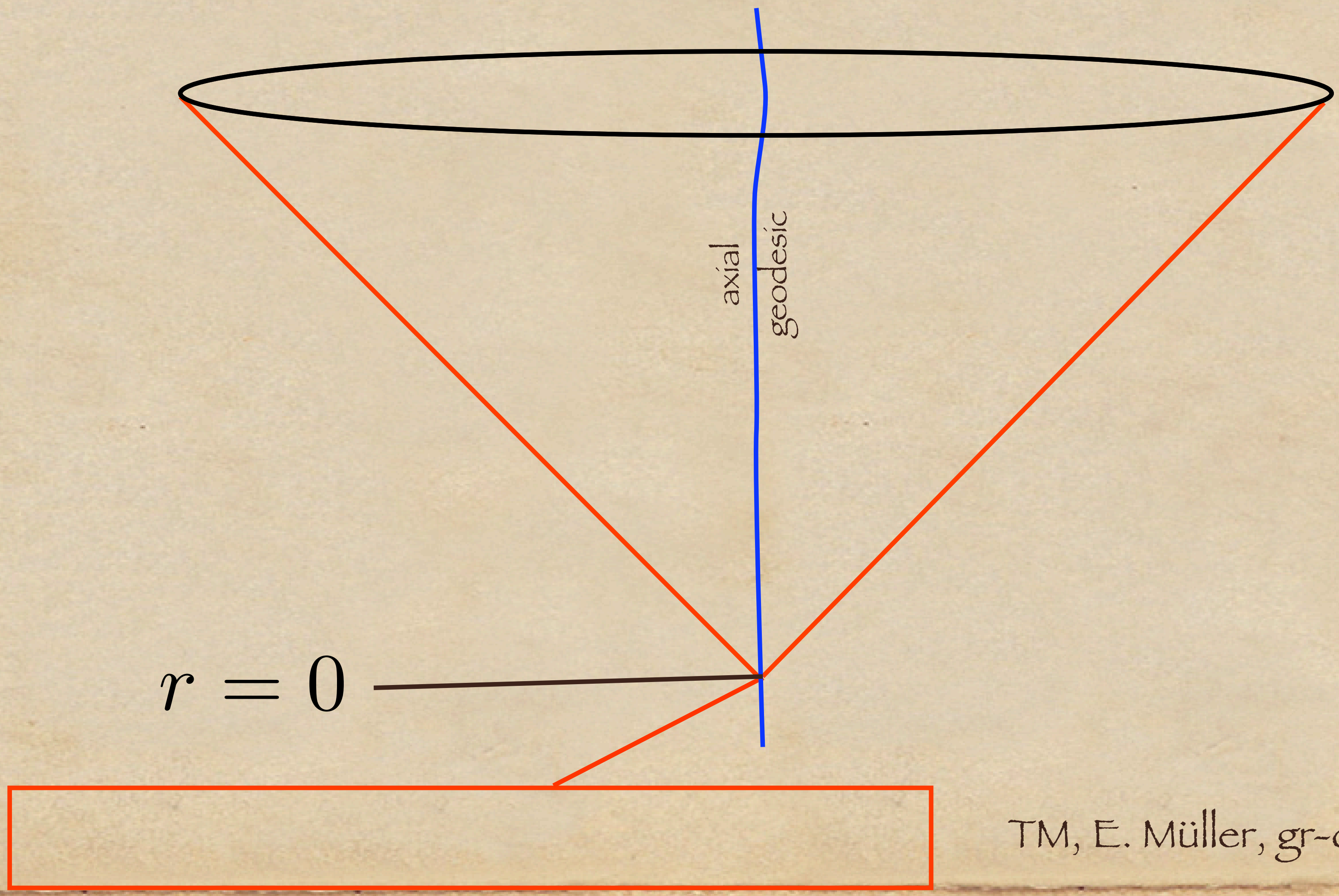
The metric near the vertex is very rigidly fixed

Initial values to Integrate the Einstein Equations (1)



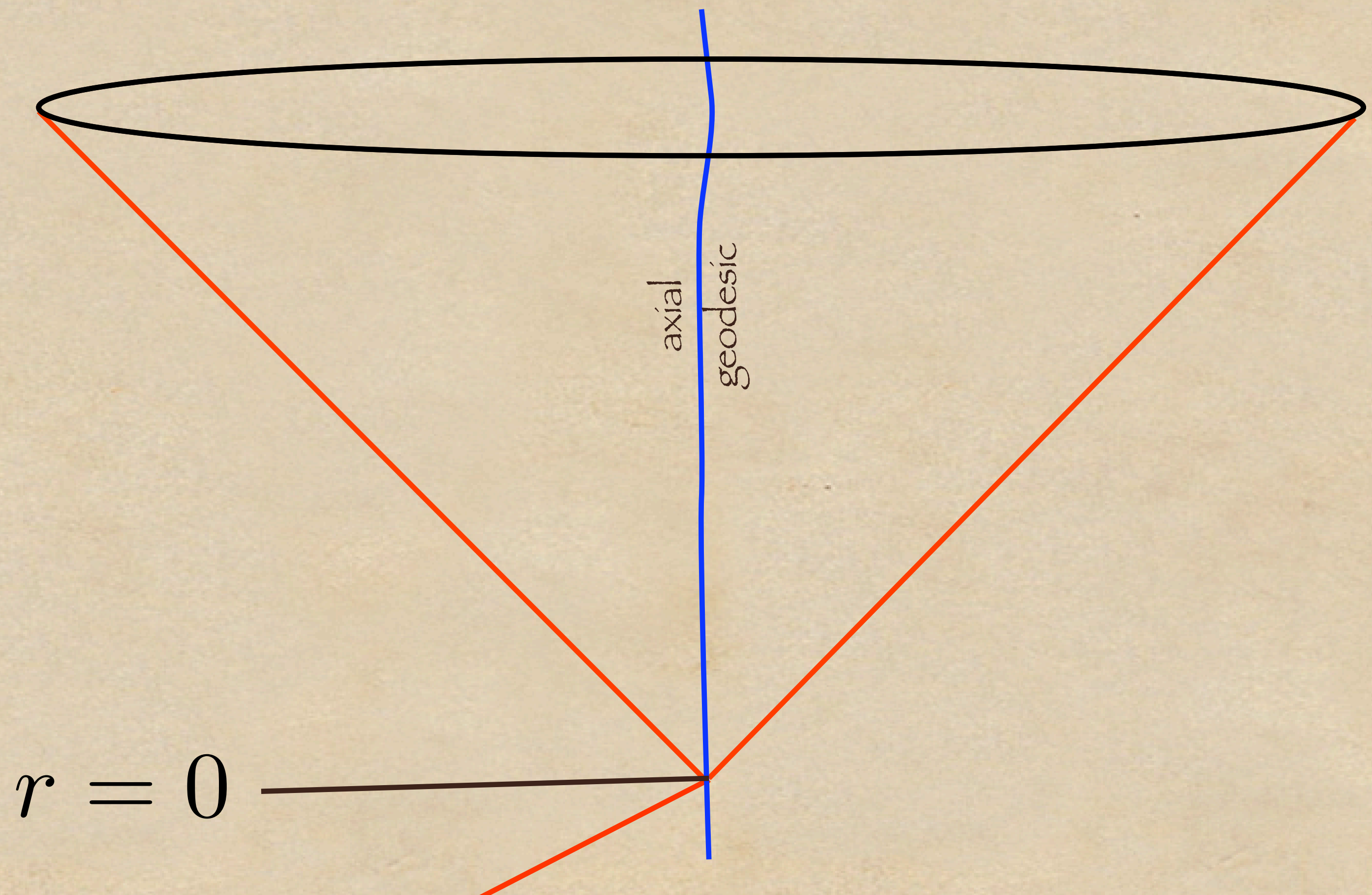
specify β, U^A, Φ and $U^A_{,r}$ for all values of x^A

Implications from the Regularity Conditions at the Vertex for Initial Data on a Light Cone



TM, E. Müller, gr-qc/1211.4980

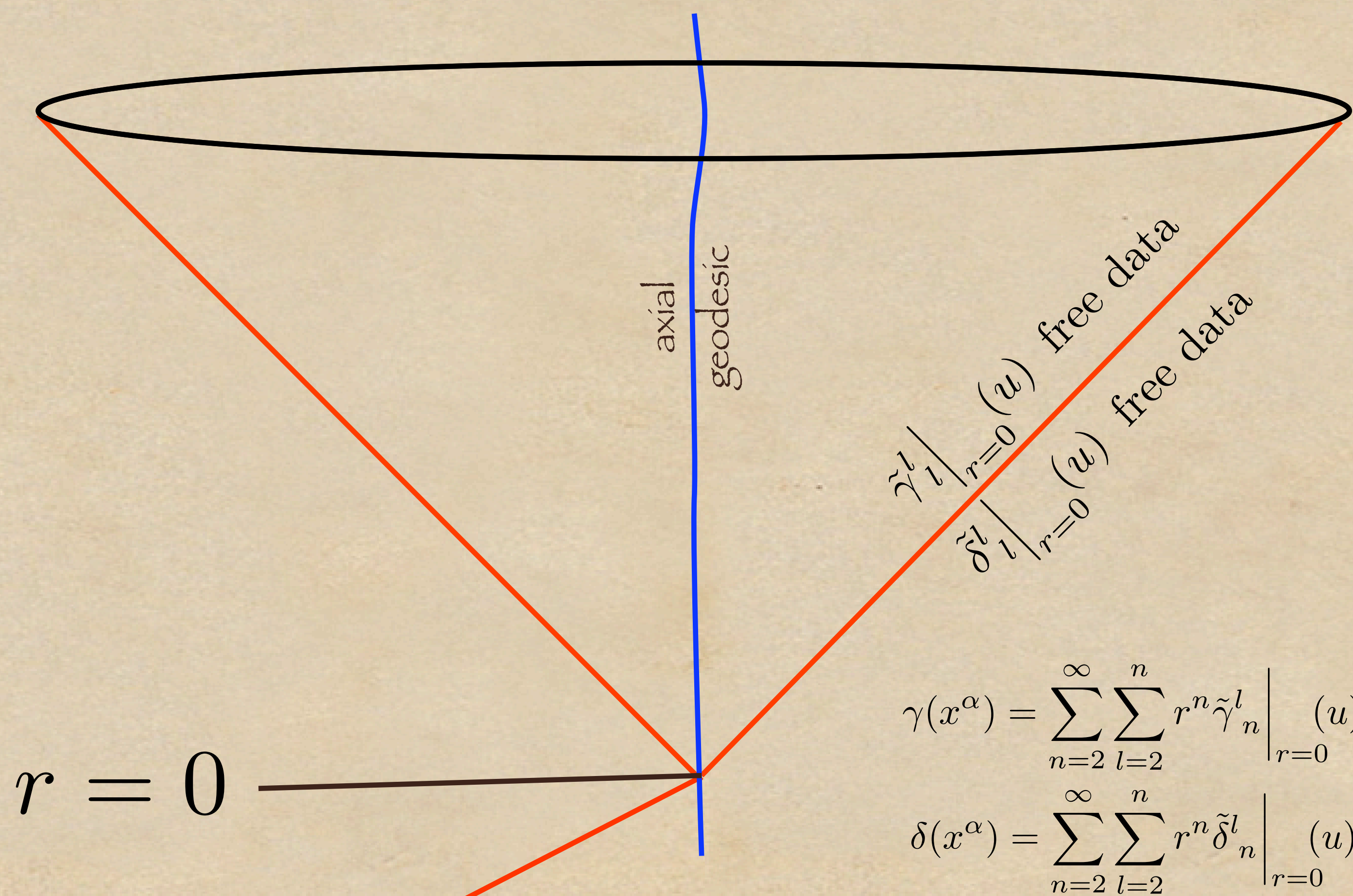
Implications from the Regularity Conditions at the Vertex for Initial Data on a Light Cone



set $\beta = U^A = \Phi = 0$ and determine $U^A_{,r}\Big|_{r=0}(\gamma, \delta)$

TM, E. Müller, gr-qc/1211.4980

Implications from the Regularity Conditions at the Vertex for Initial Data on a Light Cone



$$\gamma(x^\alpha) = \sum_{n=2}^{\infty} \sum_{l=2}^{n} r^n \tilde{\gamma}^l{}_n\Big|_{r=0}(u) P_l^2(\cos\theta)$$

$$\delta(x^\alpha) = \sum_{n=2}^{\infty} \sum_{l=2}^{n} r^n \tilde{\delta}^l{}_n\Big|_{r=0}(u) P_l^2(\cos\theta)$$

set $\beta = U^A = \Phi = 0$ and determine $U^A_{,r}\Big|_{r=0}(\gamma, \delta)$

TM, E. Müller, gr-qc/1211.4980

Regularity at the Vertex and asymptotical flatness

Regularity conditions allow time-dependent initial data that are

TM, gr-qc/1212.3316

Regularity at the Vertex and asymptotical flatness

Regularity conditions allow time-dependent initial data that are

asymptotically flat

TM, gr-qc/1212.3316

Regularity at the Vertex and asymptotical flatness

Regularity conditions allow time-dependent initial data that are

asymptotically flat

not-asymptotically flat

TM, gr-qc/1212.3316

Regularity at the Vertex and asymptotical flatness

Regularity conditions allow time-dependent initial data that are

asymptotically flat

not-asymptotically flat

This can be demonstrated by solutions derived from a quasi-spherical approximation of the Bondi-Sachs metric

TM, gr-qc/1212.3316

Example: the Scalar Wave Equation in Flat-Space Null Coordinates

... the wave equation

$$0 = \Box\psi = \frac{1}{r}\Big[-2(r\psi)_{,ur} + (r\psi)_{,rr} + \frac{1}{r^2}\nabla^A\nabla_B(r\psi)\Big]$$

... a solution that is asymptotical flat

$$\psi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_l\left[\frac{r}{u(u+2r)}\right]^{l+1} Y_{lm}(x^A)$$

... a solution that is not asymptotical flat

$$\psi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} B_l\frac{e^{u+r}I_{l+\frac{1}{2}}(r)}{\sqrt{r}}Y_{lm}(x^A)$$

Summary

- In the Bondi-Sachs formulation it can be shown that an isolated system can only loose mass via gravitational radiation

- vacuum initial data on a light cone are fixed by the regularity conditions at the vertex
  - data are given by free functions along the curve tracing the vertex

- regularity conditions do not restrict whether the initial data are asymptotically flat or not