Loss of linear momentum from nonspinning inspiralling compact binaries in quasi-circular orbits and associated recoil

Chandra Kant Mishra
Raman Research Institute & Indian Institute of Science

With
K. G. Arun (*CMI*) & Bala R. Iyer (*RRI*)

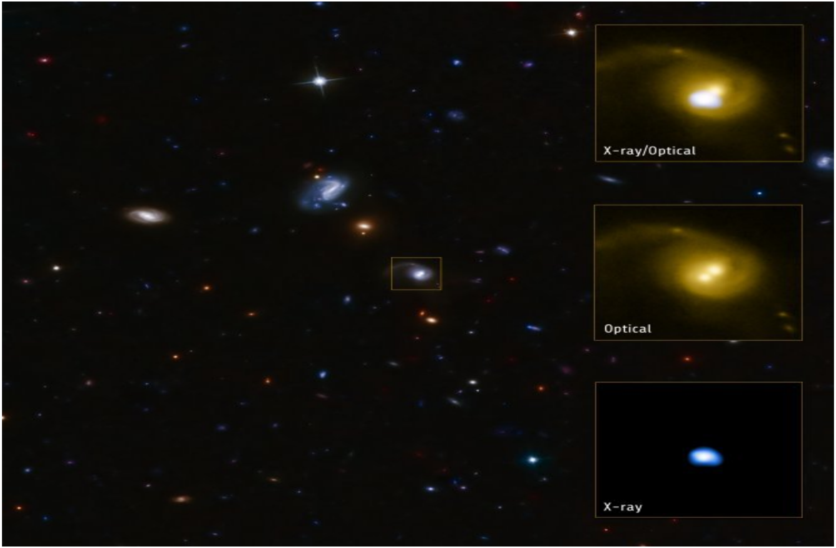
Institut d'astrophysique de Paris

June 25, 2012

Part I

Motivation and Statement of the Problem

CID-42



Understanding the GW Recoil

- If the gravitational field of an qqobject changes, these changes propagate through space in the form of ripples of space-time to far away regions of space. These propagating disturbances of space-time are called gravitational waves (GWs).
- Binary systems composed of compact objects such as neutron stars and/or black holes are considered to be the most important sources of GWs.
- Gravitational waves from inspiralling compact binaries carry both energy and momentum of the source.
- In case of a symmetric binary, the net loss of linear momentum from the system is zero as the contributions to the linear momentum radiated from the two black holes cancel.
- However, if the binary is asymmetric (e.g., is composed of objects of unequal masses), such cancellation does not occur and hence there is a net loss of linear momentum from the system.

Understanding the GW Recoil

- **If the gravitational field of an qqobject changes, these changes propagate through space in the form of ripples of space-time to far away regions of space. These propagating disturbances of space-time are called gravitational waves (GWs).**
- Binary systems composed of compact objects such as neutron stars and/or black holes are considered to be the most important sources of GWs.
- Gravitational waves from inspiralling compact binaries carry both energy and momentum of the source.
- In case of a symmetric binary, the net loss of linear momentum from the system is zero as the contributions to the linear momentum radiated from the two black holes cancel.
- However, if the binary is asymmetric (e.g. is composed of objects of unequal masses), such cancellation does not occur and hence there is a net loss of linear momentum from the system.

Understanding the GW Recoil

- If the gravitational field of an qqobject changes, these changes propagate through space in the form of ripples of space-time to far away regions of space. These propagating disturbances of space-time are called gravitational waves (GWs).
- **Binary systems composed of compact objects such as neutron stars and/or black holes are considered to be the most important sources of GWs.**
- Gravitational waves from inspiralling compact binaries carry both energy and momentum of the source.
- In case of a symmetric binary, the net loss of linear momentum from the system is zero as the contributions to the linear momentum radiated from the two black holes cancel.
- However, if the binary is asymmetric (e.g. is composed of objects of unequal masses), such cancellation does not occur and hence there is a net loss of linear momentum from the system.

Understanding the GW Recoil

- If the gravitational field of an qqobject changes, these changes propagate through space in the form of ripples of space-time to far away regions of space. These propagating disturbances of space-time are called gravitational waves (GWs).
- Binary systems composed of compact objects such as neutron stars and/or black holes are considered to be the most important sources of GWs.
- **Gravitational waves from inspiralling compact binaries carry both energy and momentum of the source.**
- In case of a symmetric binary, the net loss of linear momentum from the system is zero as the contributions to the linear momentum radiated from the two black holes cancel.
- However, if the binary is asymmetric (e.g. is composed of objects of unequal masses), such cancellation does not occur and hence there is a net loss of linear momentum from the system.

Understanding the GW Recoil

- If the gravitational field of an qqobject changes, these changes propagate through space in the form of ripples of space-time to far away regions of space. These propagating disturbances of space-time are called gravitational waves (GWs).
- Binary systems composed of compact objects such as neutron stars and/or black holes are considered to be the most important sources of GWs.
- Gravitational waves from inspiralling compact binaries carry both energy and momentum of the source.
- **In case of a symmetric binary, the net loss of linear momentum from the system is zero as the contributions to the linear momentum radiated from the two black holes cancel.**
- However, if the binary is asymmetric (e.g. is composed of objects of unequal masses), such cancellation does not occur and hence there is a net loss of linear momentum from the system.

Understanding the GW Recoil

- If the gravitational field of an qqobject changes, these changes propagate through space in the form of ripples of space-time to far away regions of space. These propagating disturbances of space-time are called gravitational waves (GWs).
- Binary systems composed of compact objects such as neutron stars and/or black holes are considered to be the most important sources of GWs.
- Gravitational waves from inspiralling compact binaries carry both energy and momentum of the source.
- In case of a symmetric binary, the net loss of linear momentum from the system is zero as the contributions to the linear momentum radiated from the two black holes cancel.
- **However, if the binary is asymmetric (e.g. is composed of objects of unequal masses), such cancellation does not occur and hence there is a net loss of linear momentum from the system.**

- **As a consequence, the center of mass of the system receives a linear momentum in the opposite direction.**
  ▸ Recoil-Cartoon
- Stages of binary evolution ▸ Binary Evolution
- The recoil accumulates until the black holes merge and settle down to a quiescent state and cease to emit linear momentum.
- At this juncture, the remnant of the binary merger, having acquired a non-zero speed, moves in a random direction in the orbital plane.

- As a consequence, the center of mass of the system receives a linear momentum in the opposite direction. ▸ Recoil-Cartoon
- Stages of binary evolution ▸ Binary Evolution
- The recoil accumulates until the black holes merge and settle down to a quiescent state and cease to emit linear momentum.
- At this juncture, the remnant of the binary merger, having acquired a non-zero speed, moves in a random direction in the orbital plane.

- As a consequence, the center of mass of the system receives a linear momentum in the opposite direction. ▸ Recoil-Cartoon
- Stages of binary evolution ▸ Binary Evolution
- **The recoil accumulates until the black holes merge and settle down to a quiescent state and cease to emit linear momentum.**
- At this juncture, the remnant of the binary merger, having acquired a non-zero speed, moves in a random direction in the orbital plane.

- As a consequence, the center of mass of the system receives a linear momentum in the opposite direction. ▸ Recoil-Cartoon
- Stages of binary evolution ▸ Binary Evolution
- The recoil accumulates until the black holes merge and settle down to a quiescent state and cease to emit linear momentum.
- **At this juncture, the remnant of the binary merger, having acquired a non-zero speed, moves in a random direction in the orbital plane.**

Momentum Ejection and subsequent recoil of a binary system

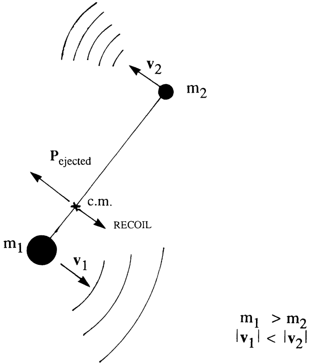
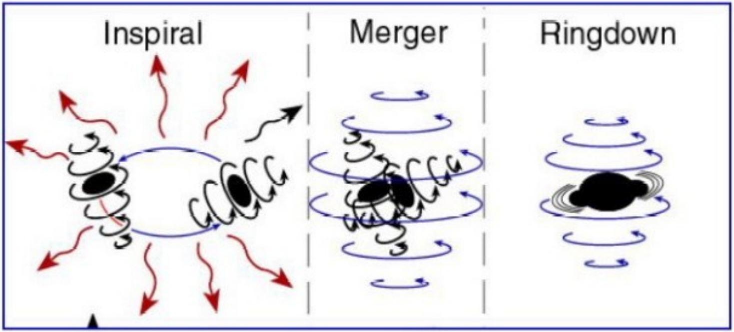


Figure: Figure from Wiseman 1992.

Stages of compact binary evolution



Astrophysics and the GW recoil

- **The phenomenon of gravitational wave recoil is of substantial importance in astrophysics especially if one wants to study the models which suggest the formation and growth of super massive black holes at the center of galaxies through successive mergers from other black holes.**
- If recoil velocity acquired by the remnant exceeds its escape velocity from the host, the host will be unable to retain the remnant and models that grow black holes through mergers from other black holes will not be favored.
- An accurate estimate for the recoil velocities associated with compact binary mergers can lead to address the issues such as
  - Presence of SMBHs at the centers of giant elliptical and spiral galaxies.
  - Absence of massive black holes at centers of dwarf galaxies or globular clusters.
  - Population of compact binary systems in globular clusters.

Astrophysics and the GW recoil

- The phenomenon of gravitational wave recoil is of substantial importance in astrophysics especially if one wants to study the models which suggest the formation and growth of super massive black holes at the center of galaxies through successive mergers from other black holes.
- **If recoil velocity acquired by the remnant exceeds its escape velocity from the host, the host will be unable to retain the remnant and models that grow black holes through mergers from other black holes will not be favored.**
- An accurate estimate for the recoil velocities associated with compact binary mergers can lead to address the issues such as
  - Presence of SMBHs at the centers of giant elliptical and spiral galaxies.
  - Absence of massive black holes at centers of dwarf galaxies or globular clusters.
  - Population of compact binary systems in globular clusters.

Astrophysics and the GW recoil

- The phenomenon of gravitational wave recoil is of substantial importance in astrophysics especially if one wants to study the models which suggest the formation and growth of super massive black holes at the center of galaxies through successive mergers from other black holes.
- If recoil velocity acquired by the remnant exceeds its escape velocity from the host, the host will be unable to retain the remnant and models that grow black holes through mergers from other black holes will not be favored.
- **An accurate estimate for the recoil velocities associated with compact binary mergers can lead to address the issues such as**
  - **Presence of SMBHs at the centers of giant elliptical and spiral galaxies.**
  - **Absence of massive black holes at centers of dwarf galaxies or globular clusters.**
  - **Population of compact binary systems in globular**

Astrophysics and the GW recoil

- The phenomenon of gravitational wave recoil is of substantial importance in astrophysics especially if one wants to study the models which suggest the formation and growth of super massive black holes at the center of galaxies through successive mergers from other black holes.
- If recoil velocity acquired by the remnant exceeds its escape velocity from the host, the host will be unable to retain the remnant and models that grow black holes through mergers from other black holes will not be favored.
- An accurate estimate for the recoil velocities associated with compact binary mergers can lead to address the issues such as
  - **Presence of SMBHs at the centers of giant elliptical and spiral galaxies.**
  - Absence of massive black holes at centers of dwarf galaxies or globular clusters.
  - Population of compact binary systems in globular clusters.

Astrophysics and the GW recoil

- The phenomenon of gravitational wave recoil is of substantial importance in astrophysics especially if one wants to study the models which suggest the formation and growth of super massive black holes at the center of galaxies through successive mergers from other black holes.
- If recoil velocity acquired by the remnant exceeds its escape velocity from the host, the host will be unable to retain the remnant and models that grow black holes through mergers from other black holes will not be favored.
- An accurate estimate for the recoil velocities associated with compact binary mergers can lead to address the issues such as
  - Presence of SMBHs at the centers of giant elliptical and spiral galaxies.
  - **Absence of massive black holes at centers of dwarf galaxies or globular clusters.**
  - Population of compact binary systems in globular clusters.

Astrophysics and the GW recoil

- The phenomenon of gravitational wave recoil is of substantial importance in astrophysics especially if one wants to study the models which suggest the formation and growth of super massive black holes at the center of galaxies through successive mergers from other black holes.
- If recoil velocity acquired by the remnant exceeds its escape velocity from the host, the host will be unable to retain the remnant and models that grow black holes through mergers from other black holes will not be favored.
- An accurate estimate for the recoil velocities associated with compact binary mergers can lead to address the issues such as
  - Presence of SMBHs at the centers of giant elliptical and spiral galaxies.
  - Absence of massive black holes at centers of dwarf galaxies or globular clusters.
  - **Population of compact binary systems in globular clusters.**

Analytical estimates for GW recoil using post-Newtonian approximations

- Peres 1962: **First formal theory for GW recoil.**
- Fitchett 1983: Leading order (Newtonian) effect for two point particles in Keplerian orbits.
- Wiseman 1992: Corrections next to the leading order added.
- BQW 2005: A more accurate post-Newtonian analysis accounting for contributions from both inspiral and plunge phase for compact binary systems in circular orbits.
- TBW 2010: Ringdown contribution was added, results with 2PN accuracy. ▸ NR & AR
- Present work
  - Recoil effects at 2.5PN order using Multipolar post-Minkowskian approach.
  - Nonspinning ICB with unequal mass components
  - Contributions from inspiral and plunge phase

Analytical estimates for GW recoil using post-Newtonian approximations

- Peres 1962: First formal theory for GW recoil.
- Fitchett 1983: **Leading order (Newtonian) effect for two point particles in Keplerian orbits.**
- Wiseman 1992: Corrections next to the leading order added.
- BQW 2005: A more accurate post-Newtonian analysis accounting for contributions from both inspiral and plunge phase for compact binary systems in circular orbits.
- TBW 2010: Ringdown contribution was added, results with 2PN accuracy. ▸ NR & AR
- Present work
  - Recoil effects at 2.5PN order using Multipolar post-Minkowskian approach.
  - Nonspinning ICB with unequal mass components.
  - Contributions from inspiral and plunge phase.

Analytical estimates for GW recoil using post-Newtonian approximations

- Peres 1962: First formal theory for GW recoil.
- Fitchett 1983: Leading order (Newtonian) effect for two point particles in Keplerian orbits.
- Wiseman 1992: **Corrections next to the leading order added.**
- BQW 2005: A more accurate post-Newtonian analysis accounting for contributions from both inspiral and plunge phase for compact binary systems in circular orbits.
- TBW 2010: Ringdown contribution was added, results with 2PN accuracy. NR & AR
- Present work
  - Recoil effects at 2.5PN order using Multipolar post-Minkowskian approach.
  - Nonspinning ICB with unequal mass components.
  - Contributions from inspiral and plunge phase.

Analytical estimates for GW recoil using post-Newtonian approximations

- Peres 1962: First formal theory for GW recoil.
- Fitchett 1983: Leading order (Newtonian) effect for two point particles in Keplerian orbits.
- Wiseman 1992: Corrections next to the leading order added.
- BQW 2005: **A more accurate post-Newtonian analysis accounting for contributions from both inspiral and plunge phase for compact binary systems in circular orbits.**
- TBW 2010: Ringdown contribution was added, results with 2PN accuracy. NR & AR
- Present work
  - Recoil effects at 2.5PN order using Multipolar post-Minkowskian approach.
  - Nonspinning ICB with unequal mass components
  - Contributions from inspiral and plunge phase

Analytical estimates for GW recoil using post-Newtonian approximations

- Peres 1962: First formal theory for GW recoil.
- Fitchett 1983: Leading order (Newtonian) effect for two point particles in Keplerian orbits.
- Wiseman 1992: Corrections next to the leading order added.
- BQW 2005: A more accurate post-Newtonian analysis accounting for contributions from both inspiral and plunge phase for compact binary systems in circular orbits.
- TBW 2010: **Ringdown contribution was added, results with 2PN accuracy.** NR & AR
- Present work
  - Recoil effects at 2.5PN order using Multipolar post-Minkowskian approach.
  - Nonspinning ICB with unequal mass components.
  - Contributions from inspiral and plunge phase.

Analytical estimates for GW recoil using post-Newtonian approximations

- Peres 1962: First formal theory for GW recoil.
- Fitchett 1983: Leading order (Newtonian) effect for two point particles in Keplerian orbits.
- Wiseman 1992: Corrections next to the leading order added.
- BQW 2005: A more accurate post-Newtonian analysis accounting for contributions from both inspiral and plunge phase for compact binary systems in circular orbits.
- TBW 2010: Ringdown contribution was added, results with 2PN accuracy. NR & AR
- Present work
  - Recoil effects at 2.5PN order using Multipolar post-Minkowskian approach.
  - Nonspinning ICB with unequal mass components.
  - Contributions from inspiral and plunge phase.

Analytical estimates for GW recoil using post-Newtonian approximations

- Peres 1962: First formal theory for GW recoil.
- Fitchett 1983: Leading order (Newtonian) effect for two point particles in Keplerian orbits.
- Wiseman 1992: Corrections next to the leading order added.
- BQW 2005: A more accurate post-Newtonian analysis accounting for contributions from both inspiral and plunge phase for compact binary systems in circular orbits.
- TBW 2010: Ringdown contribution was added, results with 2PN accuracy. NR & AR
- Present work
  - **Recoil effects at 2.5PN order using Multipolar post-Minkowskian approach.**
  - Nonspinning ICB with unequal mass components.
  - Contributions from inspiral and plunge phase.

Analytical estimates for GW recoil using post-Newtonian approximations

- Peres 1962: First formal theory for GW recoil.
- Fitchett 1983: Leading order (Newtonian) effect for two point particles in Keplerian orbits.
- Wiseman 1992: Corrections next to the leading order added.
- BQW 2005: A more accurate post-Newtonian analysis accounting for contributions from both inspiral and plunge phase for compact binary systems in circular orbits.
- TBW 2010: Ringdown contribution was added, results with 2PN accuracy. NR & AR
- Present work
  - Recoil effects at 2.5PN order using Multipolar post-Minkowskian approach.
  - **Nonspinning ICB with unequal mass components.**
  - Contributions from inspiral and plunge phase.

Analytical estimates for GW recoil using post-Newtonian approximations

- Peres 1962: First formal theory for GW recoil.
- Fitchett 1983: Leading order (Newtonian) effect for two point particles in Keplerian orbits.
- Wiseman 1992: Corrections next to the leading order added.
- BQW 2005: A more accurate post-Newtonian analysis accounting for contributions from both inspiral and plunge phase for compact binary systems in circular orbits.
- TBW 2010: Ringdown contribution was added, results with 2PN accuracy. NR & AR
- Present work
  - Recoil effects at 2.5PN order using Multipolar post-Minkowskian approach.
  - Nonspinning ICB with unequal mass components.
  - **Contributions from inspiral and plunge phase.**

Comparison of Numerical and analytical estimates for GW recoil

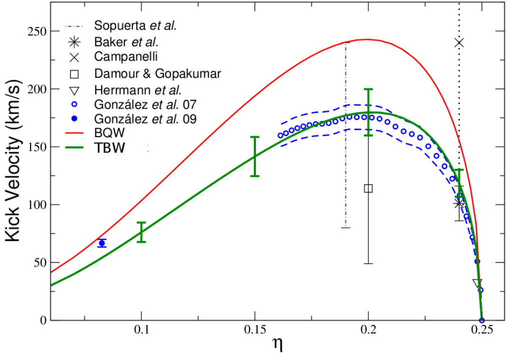


Figure: Figure from TBW 2010

Part II
Computing the Recoil Effect

Structure of the post-Newtonian formula for linear momentum flux

- Generalized multipole expansion for linear momentum loss in the far-zone of the source in terms of symmetric trace-free radiative multipole moment is given as

$$\mathcal{F}_P^i(U) = \frac{G}{c^7}\left\{\left[\frac{2}{63}\,U_{ijk}^{(1)}\,U_{jk}^{(1)} + \frac{16}{45}\,\varepsilon_{ijk}\,U_{ja}^{(1)}\,V_{ka}^{(1)}\right] + \frac{1}{c^2}\left[\frac{1}{1134}\,U_{ijkl}^{(1)}\,U_{jkl}^{(1)} + \frac{4}{63}\,V_{ijk}^{(1)}\,V_{jk}^{(1)} + \frac{1}{126}\,\varepsilon_{ijk}\,U_{jab}^{(1)}\,V_{kab}^{(1)}\right] + \frac{1}{c^4}\left[\frac{1}{59400}\,U_{ijklm}^{(1)}\,U_{jklm}^{(1)} + \frac{2}{945}\,V_{ijkl}^{(1)}\,V_{jkl}^{(1)} + \frac{2}{14175}\,\varepsilon_{ijk}\,U_{jabc}^{(1)}\,V_{kabc}^{(1)}\right] + \mathcal{O}(6)\right\}$$

Structure of the post-Newtonian formula for linear momentum flux

- **Generalized multipole expansion for linear momentum loss in the far-zone of the source in terms of symmetric trace-free radiative multipole moment is given as**

$$
\begin{aligned}
\mathcal{F}_P^i(U) \;=\; & \frac{G}{c^7}\left\{\left[\frac{2}{63}\,U^{(1)}_{ijk}\,U^{(1)}_{jk}+\frac{16}{45}\,\varepsilon_{ijk}\,U^{(1)}_{ja}\,V^{(1)}_{ka}\right]\right.\\
& +\frac{1}{c^2}\left[\frac{1}{1134}\,U^{(1)}_{ijkl}\,U^{(1)}_{jkl}+\frac{4}{63}\,V^{(1)}_{ijk}\,V^{(1)}_{jk}\right.\\
& \left.+\frac{1}{126}\,\varepsilon_{ijk}\,U^{(1)}_{jab}\,V^{(1)}_{kab}\right]+\frac{1}{c^4}\left[\frac{1}{59400}\,U^{(1)}_{ijklm}\,U^{(1)}_{jklm}+\right.\\
& \left.\left.\frac{2}{945}\,V^{(1)}_{ijkl}\,V^{(1)}_{jkl}+\frac{2}{14175}\,\varepsilon_{ijk}\,U^{(1)}_{jabc}\,V^{(1)}_{kabc}\right]+\mathcal{O}(6)\right\}
\end{aligned}
$$

- **In the MPM formalism, radiative moments (U_L, V_L) are connected to canonical moments (M_L, S_L) by the relation**

Memory term

Tail term

$$U_{ij}(U) = M_{ij}^{(2)}(U) + \frac{2GM}{c^3}\int_0^\infty d\tau \left[\ln\left(\frac{\tau}{2\tau_0}\right) + \frac{11}{12}\right] M_{ij}^{(4)}(U-\tau)$$
$$+ \frac{G}{c^5}\left\{-\frac{2}{7}\int_0^\infty d\tau M_{a\langle i}^{(3)}(U-\tau)M_{j\rangle a}^{(3)}(U-\tau) + \frac{1}{7}M_{a\langle i}^{(5)}M_{j\rangle a}\right.$$
$$\left. -\frac{5}{7}M_{a\langle i}^{(4)}M_{j\rangle a}^{(1)} - \frac{2}{7}M_{a\langle i}^{(3)}M_{j\rangle a}^{(2)} + \frac{1}{3}\varepsilon_{ab\langle i}M_{j\rangle a}^{(4)}S_b\right\}$$
$$+ \mathcal{O}\left(\frac{1}{c^6}\right)$$

Instantaneous term

- In the MPM formalism, radiative moments (U_L, V_L) are connected to canonical moments (M_L, S_L) by the relation

Memory term Tail term

$$U_{ij}(U) = M_{ij}^{(2)}(U) + \frac{2GM}{c^3}\int_0^\infty d\tau \left[\ln\left(\frac{\tau}{2\tau_0}\right) + \frac{11}{12}\right] M_{ij}^{(4)}(U-\tau)$$
$$+ \frac{G}{c^5}\left\{-\frac{2}{7}\int_0^\infty d\tau M_{a\langle i}^{(3)}(U-\tau)M_{j\rangle a}^{(3)}(U-\tau) + \frac{1}{7}M_{a\langle i}^{(5)}M_{j\rangle a}\right.$$
$$\left. - \frac{5}{7}M_{a\langle i}^{(4)}M_{j\rangle a}^{(1)} - \frac{2}{7}M_{a\langle i}^{(3)}M_{j\rangle a}^{(2)} + \frac{1}{3}\varepsilon_{ab\langle i}M_{j\rangle a}^{(4)}S_b\right\}$$
$$+ \mathcal{O}\left(\frac{1}{c^6}\right)$$

Instantaneous term

- In the MPM formalism, radiative moments (U_L, V_L) are connected to canonical moments (M_L, S_L) by the relation

Memory term

Tail term

$$U_{ij}(U) = M^{(2)}_{ij}(U) + \frac{2GM}{c^3}\int_0^\infty d\tau \left[\ln\left(\frac{\tau}{2\tau_0}\right) + \frac{11}{12}\right] M^{(4)}_{ij}(U-\tau)$$
$$+ \frac{G}{c^5}\left\{-\frac{2}{7}\int_0^\infty d\tau M^{(3)}_{a\langle i}(U-\tau)M^{(3)}_{j\rangle a}(U-\tau) + \frac{1}{7}M^{(5)}_{a\langle i}M_{j\rangle a}\right.$$
$$\left. - \frac{5}{7}M^{(4)}_{a\langle i}M^{(1)}_{j\rangle a} - \frac{2}{7}M^{(3)}_{a\langle i}M^{(2)}_{j\rangle a} + \frac{1}{3}\varepsilon_{ab\langle i}M^{(4)}_{j\rangle a}S_b\right\}$$
$$+ \mathcal{O}\left(\frac{1}{c^6}\right)$$

Instantaneous term

- In the MPM formalism, radiative moments (U_L, V_L) are connected to canonical moments (M_L, S_L) by the relation

Memory term

Tail term

$$U_{ij}(U) = M_{ij}^{(2)}(U) + \frac{2GM}{c^3}\int_0^{\infty} d\tau \left[\ln\left(\frac{\tau}{2\tau_0}\right) + \frac{11}{12}\right] M_{ij}^{(4)}(U-\tau)$$
$$+ \frac{G}{c^5}\left\{-\frac{2}{7}\int_0^{\infty} d\tau M_{a\langle i}^{(3)}(U-\tau)M_{j\rangle a}^{(3)}(U-\tau) + \frac{1}{7}M_{a\langle i}^{(5)}M_{j\rangle a}\right.$$
$$\left. - \frac{5}{7}M_{a\langle i}^{(4)}M_{j\rangle a}^{(1)} - \frac{2}{7}M_{a\langle i}^{(3)}M_{j\rangle a}^{(2)} + \frac{1}{3}\varepsilon_{ab\langle i}M_{j\rangle a}^{(4)}S_b\right\}$$
$$+ \mathcal{O}\left(\frac{1}{c^6}\right)$$

Instantaneous term

- **Canonical moments and source type moments ($I_L, J_L, W_L, X_L, Y_L, Z_L$) are connected by**

$$M_{ij} = I_{ij} + \frac{4G}{c^5}\left[W^{(2)} I_{ij} - W^{(1)} I_{ij}^{(1)}\right] + \mathcal{O}\left(\frac{1}{c^7}\right)$$

Computation of the flux

- $\mathcal{F}_P^i = \left(\mathcal{F}_P^i\right)_{\rm inst} + \left(\mathcal{F}_P^i\right)_{\rm hered}$
- $I_L ... Z_L$ and $I_L^{(n)} ... Z_L^{(n)}$ give $\left(\mathcal{F}_P^i\right)_{\rm inst}$.
- $I_L ... Z_L$ and $I_L^{(n)} ... Z_L^{(n)}$ give $\left(\mathcal{F}_P^i\right)_{\rm hered}$ after the integrals over time are performed.
- Note that memory integral is a time antiderivative and thus becomes instantaneous in the flux.

- Canonical moments and source type moments $(I_L, J_L, W_L, X_L, Y_L, Z_L)$ are connected by

$$M_{ij} = I_{ij} + \frac{4G}{c^5}\left[W^{(2)} I_{ij} - W^{(1)} I_{ij}^{(1)}\right] + \mathcal{O}\left(\frac{1}{c^7}\right)$$

Computation of the flux

- $\mathcal{F}_P = \left(\mathcal{F}_P\right)_{\rm inst} + \left(\mathcal{F}_P\right)_{\rm hered}$
- $I_L ... Z_L$ and $I_L^{(n)} ... Z_L^{(n)}$ give $\left(\mathcal{F}_P\right)_{\rm inst}$
- $I_L ... Z_L$ and $I_L^{(n)} ... Z_L^{(n)}$ give $\left(\mathcal{F}_P\right)_{\rm hered}$ after the integrals over time are performed
- Note that memory integral is a time antiderivative and thus becomes instantaneous in the flux

- Canonical moments and source type moments $(I_L, J_L, W_L, X_L, Y_L, Z_L)$ are connected by

$$M_{ij} = I_{ij} + \frac{4G}{c^5}\left[W^{(2)} I_{ij} - W^{(1)} I_{ij}^{(1)}\right] + \mathcal{O}\left(\frac{1}{c^7}\right)$$

Computation of the flux

- $\mathcal{F}_P^i = \left(\mathcal{F}_P^i\right)_{\rm inst} + \left(\mathcal{F}_P^i\right)_{\rm hered}$
- $I_L ... Z_L$ and $I_L^{(n)} ... Z_L^{(n)}$ give $\left(\mathcal{F}_P^i\right)_{\rm inst}$.
- $I_L ... Z_L$ and $I_L^{(n)} ... Z_L^{(n)}$ give $\left(\mathcal{F}_P^i\right)_{\rm hered}$ after the integrals over time are performed.
- Note that memory integral is a time antiderivative and thus becomes instantaneous in the flux.

- Canonical moments and source type moments $(I_L, J_L, W_L, X_L, Y_L, Z_L)$ are connected by

$$M_{ij} = I_{ij} + \frac{4G}{c^5}\left[W^{(2)} I_{ij} - W^{(1)} I_{ij}^{(1)}\right] + \mathcal{O}\left(\frac{1}{c^7}\right)$$

Computation of the flux

- $\mathcal{F}_P^i = \left(\mathcal{F}_P^i\right)_{\rm inst} + \left(\mathcal{F}_P^i\right)_{\rm hered}$
- $I_L ... Z_L$ and $I_L^{(n)} ... Z_L^{(n)}$ give $\left(\mathcal{F}_P^i\right)_{\rm inst}$.
- $I_L ... Z_L$ and $I_L^{(n)} ... Z_L^{(n)}$ give $\left(\mathcal{F}_P^i\right)_{\rm hered}$ after the integrals over time are performed.
- Note that memory integral is a time antiderivative and thus becomes instantaneous in the flux.

- Canonical moments and source type moments $(I_L, J_L, W_L, X_L, Y_L, Z_L)$ are connected by

$$M_{ij} = I_{ij} + \frac{4G}{c^5}\left[W^{(2)} I_{ij} - W^{(1)} I_{ij}^{(1)}\right] + \mathcal{O}\left(\frac{1}{c^7}\right)$$

Computation of the flux

- $\mathcal{F}_P^i = \left(\mathcal{F}_P^i\right)_{\rm inst} + \left(\mathcal{F}_P^i\right)_{\rm hered}$
- $I_L ... Z_L$ and $I_L^{(n)} ... Z_L^{(n)}$ give $\left(\mathcal{F}_P^i\right)_{\rm inst}$.
- $I_L ... Z_L$ and $I_L^{(n)} ... Z_L^{(n)}$ give $\left(\mathcal{F}_P^i\right)_{\rm hered}$ after the integrals over time are performed.
- Note that memory integral is a time antiderivative and thus becomes instantaneous in the flux.

- Canonical moments and source type moments $(I_L, J_L, W_L, X_L, Y_L, Z_L)$ are connected by

$$M_{ij} = I_{ij} + \frac{4G}{c^5}\left[W^{(2)} I_{ij} - W^{(1)} I_{ij}^{(1)}\right] + \mathcal{O}\left(\frac{1}{c^7}\right)$$

Computation of the flux

- $\mathcal{F}_P^i = \left(\mathcal{F}_P^i\right)_{\rm inst} + \left(\mathcal{F}_P^i\right)_{\rm hered}$
- $I_L ... Z_L$ and $I_L^{(n)} ... Z_L^{(n)}$ give $\left(\mathcal{F}_P^i\right)_{\rm inst}$.
- $I_L ... Z_L$ and $I_L^{(n)} ... Z_L^{(n)}$ give $\left(\mathcal{F}_P^i\right)_{\rm hered}$ after the integrals over time are performed.
- Note that memory integral is a time antiderivative and thus becomes instantaneous in the flux.

Time derivatives of multipole moments

- **Computing time derivatives of various multipole moments at some PN order requires the knowledge of equations of motion at the same PN order.**
- At 2.5PN order, orbital motion of the binary can be modeled as a quasi-circular orbit decaying under the leading radiation reaction effect.
- This effect is computed by equating the rate of change of the orbital energy with the total energy flux radiated in the form of gravitational waves. This yields,

$$\dot{r} = -\frac{64}{5}\sqrt{\frac{Gm}{r}}\,\nu\,\gamma^{5/2} + \mathcal{O}(7)$$

$$\dot{\omega} = \frac{96}{5}\,\frac{Gm}{r^3}\,\nu\,\gamma^{5/2} + \mathcal{O}(7)$$

where γ is a PN parameter and is given by $\gamma = Gm/rc^2$.

Time derivatives of multipole moments

- Computing time derivatives of various multipole moments at some PN order requires the knowledge of equations of motion at the same PN order.
- **At 2.5PN order, orbital motion of the binary can be modeled as a quasi-circular orbit decaying under the leading radiation reaction effect.**
- This effect is computed by equating the rate of change of the orbital energy with the total energy flux radiated in the form of gravitational waves. This yields,

$$\dot{r} = -\frac{64}{5}\sqrt{\frac{Gm}{r}}\,\nu\,\gamma^{5/2} + \mathcal{O}(7)$$

$$\dot{\omega} = \frac{96}{5}\,\frac{Gm}{r^3}\,\nu\,\gamma^{5/2} + \mathcal{O}(7)$$

where γ is a PN parameter and is given by $\gamma = Gm/rc^2$.

Time derivatives of multipole moments

- Computing time derivatives of various multipole moments at some PN order requires the knowledge of equations of motion at the same PN order.
- At 2.5PN order, orbital motion of the binary can be modeled as a quasi-circular orbit decaying under the leading radiation reaction effect.
- **This effect is computed by equating the rate of change of the orbital energy with the total energy flux radiated in the form of gravitational waves. This yields,**

$$\dot{r} = -\frac{64}{5}\sqrt{\frac{Gm}{r}}\,\nu\,\gamma^{5/2} + \mathcal{O}(7)$$
$$\dot{\omega} = \frac{96}{5}\,\frac{Gm}{r^3}\,\nu\,\gamma^{5/2} + \mathcal{O}(7)$$

where γ is a PN parameter and is given by $\gamma = Gm/rc^2$.

- **Let us fix the orbital plane of the binary to be the $x - y$ plane. Again if $\mathbf{y_1}$ and $\mathbf{y_2}$ be position vectors of individual objects in the binary system, the relative position $\mathbf{x} = \mathbf{y_1} - \mathbf{y_2}$, the relative velocity $\mathbf{v} = \mathrm{d}\mathbf{x}/\mathrm{d}t$, and relative acceleration $\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t$ will be given as**

$$\mathbf{x} = r\,\hat{\mathbf{n}}$$
$$\mathbf{v} = \dot{r}\,\hat{\mathbf{n}} + r\,\omega\,\hat{\lambda}$$
$$\mathbf{a} = (\ddot{r} - r\,\omega^2)\,\hat{\mathbf{n}} + (r\,\dot{\omega} + 2\dot{r}\,\omega)\,\hat{\lambda}$$

- **Here $\hat{\lambda}$ along with the unit vector $\hat{\mathbf{e}}_{\mathbf{z}}$ (perpendicular to the orbital plane of the binary) and $\hat{\mathbf{n}}$ (along relative position vector), forms a set of orthonormal triad such as $\hat{\lambda} = \hat{\mathbf{e}}_{\mathbf{z}} \times \hat{\mathbf{n}}$.**

- **Substituting for $\dot{r}$ and $\dot{\omega}$ back in expressions for $\mathbf{v}$ and $\mathbf{a}$ we get**

$$\mathbf{v} = r\,\omega\,\hat{\lambda} - \frac{64}{5}\sqrt{\frac{Gm}{r}}\;\nu\,\gamma^{5/2}\,\hat{\mathbf{n}} + \mathcal{O}(6)$$
$$\mathbf{a} = -\omega^2\,\mathbf{x} - \frac{32}{5}\sqrt{\frac{Gm}{r^3}}\;\nu\,\gamma^{5/2}\,\mathbf{v} + \mathcal{O}(6)$$

- The last ingredient we would need is the connection between the orbital frequency to binary's radial separation. Such a relation is known up to 3PN order in harmonic coordinates however for our work we need it we need it to be just 2PN accurate and is given as

$$\omega^2 = \frac{Gm}{r^3}\left\{1 + \gamma\Big(-3+\nu\Big) + \gamma^2\left(6 + \frac{41}{4}\nu + \nu^2\right) + \mathcal{O}(6)\right\}$$

- Substituting for $\dot{r}$ and $\dot{\omega}$ back in expressions for $\mathbf{v}$ and $\mathbf{a}$ we get

$$
\begin{aligned}
\mathbf{v} &= r\,\omega\,\hat{\lambda} - \frac{64}{5}\sqrt{\frac{Gm}{r}}\;\nu\,\gamma^{5/2}\,\hat{\mathbf{n}} + \mathcal{O}(6)\\
\mathbf{a} &= -\omega^2\,\mathbf{x} - \frac{32}{5}\sqrt{\frac{Gm}{r^3}}\;\nu\,\gamma^{5/2}\,\mathbf{v} + \mathcal{O}(6)
\end{aligned}
$$

- **The last ingredient we would need is the connection between the orbital frequency to binary's radial separation. Such a relation is known up to 3PN order in harmonic coordinates however for our work we need it we need it to be just 2PN accurate and is given as**

$$
\omega^2 = \frac{Gm}{r^3}\left\{1 + \gamma\Big(-3+\nu\Big) + \gamma^2\left(6 + \frac{41}{4}\nu + \nu^2\right) + \mathcal{O}(6)\right\}
$$

Instantaneous Contribution

- **Instantaneous part of the LMF at the 2.5PN order reads**

$$\left(\mathcal{F}^i_P\right)_{\rm inst} = -\frac{464}{105}\frac{c^4}{G}\sqrt{1-4\nu}\,\gamma^{11/2}\,\nu^2\left\{\left[1-\left(\frac{1861}{174}+\frac{91}{261}\nu\right)\gamma+\left(\frac{139355}{2871}+\frac{36269}{1044}\nu+\frac{17}{3828}\nu^2\right)\gamma^2\right]\hat{\lambda}_i+\frac{1199}{290}\nu\,\gamma^{5/2}\,\hat{n}_i+\mathcal{O}(6)\right\}$$

- $\hat{n}^i \to \{\cos\phi, \sin\phi, 0\}$ and $\hat{\lambda}^i \to \{-\sin\phi, \cos\phi, 0\}$.

Instantaneous Contribution

- Instantaneous part of the LMF at the 2.5PN order reads

$$\left(\mathcal{F}_P^i\right)_{\rm inst} = -\frac{464}{105}\,\frac{c^4}{G}\,\sqrt{1-4\,\nu}\,\gamma^{11/2}\,\nu^2\left\{\left[1-\left(\frac{1861}{174}\right.\right.\right. \\ \left.+\frac{91}{261}\,\nu\right)\gamma+\left(\frac{139355}{2871}+\frac{36269}{1044}\,\nu\right. \\ \left.\left.+\frac{17}{3828}\,\nu^2\right)\gamma^2\right]\hat{\lambda}_i+\frac{1199}{290}\,\nu\,\gamma^{5/2}\,\hat{n}_i+\mathcal{O}(6)\right\}$$

- $\hat{n}^i \to \{\cos\,\phi, \sin\,\phi, 0\}$ and $\hat{\lambda}^i \to \{-\sin\,\phi, \cos\,\phi, 0\}$.

- Alternatively one can use a new PN parameter, x, instead of γ and is directly connected to orbital frequency through the relation

$$x = \left(\frac{G\, m\, \omega}{c^3}\right)^{2/3}$$

- In terms of the parameter x, $\left(\mathcal{F}_P^i\right)_{\rm inst}$ takes the form

$$\begin{aligned}\left(\mathcal{F}_P^i\right)_{\rm inst} = & -\frac{464}{105}\,\frac{c^4}{G}\,\sqrt{1-4\,\nu}\,x^{11/2}\,\nu^2\left\{\left[1-\left(\frac{452}{87}\right.\right.\right. \\ & \left. +\frac{1139}{522}\,\nu\right)x+\left(-\frac{71345}{22968}+\frac{36761}{2088}\,\nu\right. \\ & \left.\left. +\frac{147101}{68904}\,\nu^2\right)x^2\right]\hat{\lambda}_i+\frac{1199}{290}\,\nu\, x^{5/2}\,\hat{n}_i+\mathcal{O}(6)\right\}\end{aligned}$$

- Alternatively one can use a new PN parameter, x, instead of γ and is directly connected to orbital frequency through the relation

$$x = \left(\frac{G\,m\,\omega}{c^3}\right)^{2/3}$$

- **In terms of the parameter x, $\left(\mathcal{F}_P^i\right)_{\rm inst}$ takes the form**

$$\left(\mathcal{F}_P^i\right)_{\rm inst} = -\frac{464}{105}\frac{c^4}{G}\sqrt{1-4\,\nu}\,x^{11/2}\,\nu^2\left\{\left[1-\left(\frac{452}{87}+\frac{1139}{522}\,\nu\right)x+\left(-\frac{71345}{22968}+\frac{36761}{2088}\,\nu+\frac{147101}{68904}\,\nu^2\right)x^2\right]\hat{\lambda}_i+\frac{1199}{290}\,\nu\,x^{5/2}\,\hat{n}_i+\mathcal{O}(6)\right\}$$

Ingredients for Hereditary computations

- **A hereditary term**

$$\left(\mathcal{F}^i_P\right)^1_{\rm hered} = \frac{4\,G^2\,M}{63\,c^{10}}\,I^{(4)}_{ijk}(U)\int_0^\infty d\tau\left[\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{12}\right] \times I^{(5)}_{jk}(U-\tau)$$

- We can write, $I_L(U) = (\cdots)\,\hat{\mathbf{n}}(U) + (\cdots)\,\hat{\lambda}(U)$ & $I_L(U') = (\cdots)\,\hat{\mathbf{n}}(U') + (\cdots)\,\hat{\lambda}(U')$, with $U' = U - \tau$.
- $\hat{\mathbf{n}}(U)$ and $\hat{\lambda}(U)$

$$\hat{\mathbf{n}}(U) = \cos\phi(U)\,\hat{\mathbf{e}}_{\mathbf{x}} + \sin\phi(U)\,\hat{\mathbf{e}}_{\mathbf{y}}$$

$$\hat{\lambda}(U) = -\sin\phi(U)\,\hat{\mathbf{e}}_{\mathbf{x}} + \cos\phi(U)\,\hat{\mathbf{e}}_{\mathbf{y}}$$

Ingredients for Hereditary computations

- A hereditary term

$$\left(\mathcal{F}_P^i\right)^1_{\mathrm{hered}} = \frac{4\,G^2\,M}{63\,c^{10}}\,I^{(4)}_{ijk}(U)\int_0^\infty d\tau\left[\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{12}\right] \times I^{(5)}_{jk}(U-\tau)$$

- **We can write,** $I_L(U)=(\cdots)\,\hat{\mathbf{n}}(U)+(\cdots)\,\hat{\lambda}(U)$ **&** $I_L(U')=(\cdots)\,\hat{\mathbf{n}}(U')+(\cdots)\,\hat{\lambda}(U')$**, with** $U'=U-\tau$**.**
- $\hat{\mathbf{n}}(U)$ and $\hat{\lambda}(U)$

$$\hat{\mathbf{n}}(U)=\cos\phi(U)\,\hat{\mathbf{e}}_{\mathbf{x}}+\sin\phi(U)\,\hat{\mathbf{e}}_{\mathbf{y}}$$

$$\hat{\lambda}(U)=-\sin\phi(U)\,\hat{\mathbf{e}}_{\mathbf{x}}+\cos\phi(U)\,\hat{\mathbf{e}}_{\mathbf{y}}$$

Ingredients for Hereditary computations

- A hereditary term

$$\left(\mathcal{F}_P^i\right)^1_{\rm hered} = \frac{4\,G^2\,M}{63\,c^{10}}\,I^{(4)}_{ijk}(U)\int_0^\infty d\tau\left[\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{12}\right] \times I^{(5)}_{jk}(U-\tau)$$

- We can write, $I_L(U) = (\cdots)\,\hat{\mathbf{n}}(U) + (\cdots)\,\hat{\lambda}(U)$ & $I_L(U') = (\cdots)\,\hat{\mathbf{n}}(U') + (\cdots)\,\hat{\lambda}(U')$, with $U' = U - \tau$.
- $\hat{\mathbf{n}}(U)$ and $\hat{\lambda}(U)$

$$\hat{\mathbf{n}}(U) = \cos\phi(U)\,\hat{\mathbf{e}}_{\mathbf{x}} + \sin\phi(U)\,\hat{\mathbf{e}}_{\mathbf{y}}$$
$$\hat{\lambda}(U) = -\sin\phi(U)\,\hat{\mathbf{e}}_{\mathbf{x}} + \cos\phi(U)\,\hat{\mathbf{e}}_{\mathbf{y}}$$

- **$\hat{\mathbf{n}}(U')$ and $\hat{\lambda}(U')$ can be written in terms of $\hat{\mathbf{n}}(U)$ and $\hat{\lambda}(U)$ as**

$$\hat{\mathbf{n}}(U') = \cos\left[\phi(U) - \phi(U')\right]\hat{\mathbf{n}}(U) - \sin\left[\phi(U) - \phi(U')\right]\hat{\lambda}(U)$$
$$\hat{\lambda}(U') = \sin\left[\phi(U) - \phi(U')\right]\hat{\mathbf{n}}(U) + \cos\left[\phi(U) - \phi(U')\right]\hat{\lambda}(U)$$

- Using the above and after a few steps of algebra we can write for $I^{(4)}_{ijk}(U)\, I^{(5)}_{(jk)}(U-\tau)$

$$I^{(4)}_{ijk}(U)\, I^{(5)}_{jk}(U') = \frac{16}{5}\,\frac{c^{17}}{G^4\, m^2}\, x^{17/2}\,\sqrt{1-4\nu}\,\nu^2 \Bigg\{ \Big[-203\,\sin(2\,\delta\phi) + x\left(\frac{2657}{2}\,\sin(2\,\delta\phi) - \frac{1341}{2}\,\nu\,\sin(2\,\delta\phi)\right)\Big]\,\hat{n}_i(U) + \Big[202\,\cos(2\,\delta\phi) + x\left(-\frac{9263}{7}\,\cos(2\,\delta\phi) + \frac{4689}{7}\,\nu\,\cos(2\,\delta\phi)\right)\Big]\,\hat{\lambda}_i(U)\Bigg\}$$

- $\hat{\mathbf{n}}(U')$ and $\hat{\lambda}(U')$ can be written in terms of $\hat{\mathbf{n}}(U)$ and $\hat{\lambda}(U)$ as

$$\hat{\mathbf{n}}(U') = \cos\left[\phi(U) - \phi(U')\right]\hat{\mathbf{n}}(U) - \sin\left[\phi(U) - \phi(U')\right]\hat{\lambda}(U)$$
$$\hat{\lambda}(U') = \sin\left[\phi(U) - \phi(U')\right]\hat{\mathbf{n}}(U) + \cos\left[\phi(U) - \phi(U')\right]\hat{\lambda}(U)$$

- **Using the above and after a few steps of algebra we can write for $I^{(4)}_{ijk}(U)\, I^{(5)}_{(jk)}(U-\tau)$**

$$I^{(4)}_{ijk}(U)\, I^{(5)}_{jk}(U') = \frac{16}{5}\frac{c^{17}}{G^4\, m^2}\, x^{17/2}\sqrt{1-4\nu}\,\nu^2\Bigg\{\Big[-203\,\sin(2\,\delta\phi) + x\left(\frac{2657}{2}\sin(2\,\delta\phi) - \frac{1341}{2}\nu\,\sin(2\,\delta\phi)\right)\Big]\hat{n}_i(U) + \Big[202\,\cos(2\,\delta\phi) + x\left(-\frac{9263}{7}\cos(2\,\delta\phi) + \frac{4689}{7}\nu\,\cos(2\,\delta\phi)\right)\Big]\hat{\lambda}_i(U)\Bigg\}$$

where we have defined $\delta\phi \equiv \phi(U) - \phi(U-\tau)$.

- **For $\delta\phi$ we can write**

$$
\begin{aligned}
\delta\phi &= \phi(U) - \phi(U-\tau) \\
&= \phi(U) - \left[\phi(U) - \tau\left(\frac{d\phi}{d\tau}\right)_{\tau=U} + \cdots\right] \\
&= \omega\,\tau
\end{aligned}
$$

- With $\delta\phi$ given in terms of τ we can use the following standard integral in order to compute the hereditary contribution

$$
\int_0^\infty \log\left(\frac{\tau}{2\,b}\right) e^{i\,n\,\omega\,\tau}\,d\tau = -\frac{1}{n\,\omega}\left\{\frac{\pi}{2} + i\Big[\ln(2\,b\,n\,\omega) + C\Big]\right\}
$$

- For $\delta\phi$ we can write

$$\begin{aligned}\delta\phi &= \phi(U) - \phi(U-\tau) \\ &= \phi(U) - \left[\phi(U) - \tau \left(\frac{d\phi}{d\tau}\right)_{\tau=U} + \cdots\right] \\ &= \omega\,\tau\end{aligned}$$

- **With $\delta\phi$ given in terms of τ we can use the following standard integral in order to compute the hereditary contribution**

$$\int_0^\infty \log\left(\frac{\tau}{2\,b}\right)\,e^{i\,n\,\omega\,\tau}\,d\tau = -\frac{1}{n\,\omega}\left\{\frac{\pi}{2} + i\Big[\ln(2\,b\,n\,\omega) + C\Big]\right\}$$

Hereditary Contribution

- **Hereditary contributions to the linear momentum flux at 2.5PN order, in terms of the parameter x, takes the following form**

$$
\begin{aligned}
\left(\mathcal{F}_P^i\right)_{\rm hered} \quad = \quad & -\frac{464}{105}\,\frac{c^4}{G}\,\sqrt{1-4\,\nu}\,x^{11/2}\,\nu^2\Bigg\{\Bigg[\frac{309}{58}\,\pi\,\hat{\lambda}_i \\
& +2\,\log\left(\frac{\omega}{\hat{\omega}_0}\right)\hat{n}_i\Bigg]\,x^{3/2}+\Bigg[\Bigg(-\frac{2663}{116}\,\pi \\
& -\frac{2185}{87}\,\pi\,\nu\Bigg)\,\hat{\lambda}_i+\Bigg(-\frac{106187}{50460}+\frac{32835}{841}\log 2 \\
& -\frac{77625}{3364}\log 3-\frac{904}{87}\,\log\left(\frac{\omega}{\hat{\omega}_0}\right)+\Bigg[-\frac{38917}{25230} \\
& -\frac{109740}{841}\log 2+\frac{66645}{841}\log 3-\frac{1400}{261}\,\log\left(\frac{\omega}{\hat{\omega}_0}\right) \\
& \Bigg]\,\nu\Bigg)\,\hat{n}_i\Bigg]\,x^{5/2}+\mathcal{O}(6)\Bigg\}
\end{aligned}
$$

- **Here $\hat{\omega}_0$ appearing in the above provides a scale to the logarithms and is given as**

$$\hat{\omega}_0 = \frac{1}{\tau_0}\exp\left(\frac{5921}{1740} + \frac{48}{29}\log 2 - \frac{405}{116}\log 3 - C\right)$$

- Introducing a new phase phase variable (Blanchet et al. 1996, Arun et al. 2004)

$$\psi = \phi - \frac{2\,G\,M\,\omega}{c^3}\log\left(\frac{\omega}{\hat{\omega}_0}\right)$$

- Here $\hat{\omega}_0$ appearing in the above provides a scale to the logarithms and is given as

$$\hat{\omega}_0 \quad = \quad \frac{1}{\tau_0} \exp\left(\frac{5921}{1740} + \frac{48}{29}\log 2 - \frac{405}{116}\log 3 - C\right)$$

- **Introducing a new phase phase variable (Blanchet et al. 1996, Arun et al. 2004)**

$$\psi = \phi - \frac{2\,G\,M\,\omega}{c^3}\log\left(\frac{\omega}{\hat{\omega}_0}\right)$$

Total LMF

- $$\mathcal{F}_P^i = \left(\mathcal{F}_P^i\right)_{\rm inst} + \left(\mathcal{F}_P^i\right)_{\rm hered}$$

- Final 2.5PN expression for LMF reads

$$\begin{aligned}
\mathcal{F}_P^i = & -\frac{464}{105}\frac{c^4}{G}\sqrt{1-4\nu}\,x^{11/2}\,\nu^2\left\{\left[1-\left(\frac{452}{87}+\frac{1139}{522}\nu\right)x\right.\right. \\
& +\frac{309}{58}\pi\,x^{3/2}+\left(-\frac{71345}{22968}+\frac{36761}{2088}\nu+\frac{147101}{68904}\nu^2\right)x^2 \\
& \left.+\left(-\frac{2663}{116}\pi-\frac{2185}{87}\pi\,\nu\right)x^{5/2}\right]\hat{\lambda}_i+\left[-\frac{106187}{50460}\right. \\
& +\frac{32835}{841}\log 2-\frac{77625}{3364}\log 3+\left(\frac{32698}{12615}-\frac{109740}{841}\log 2\right. \\
& \left.\left.\left.+\frac{66645}{841}\log 3\right)\nu\right]x^{5/2}\,\hat{n}_i+\mathcal{O}(6)\right\}
\end{aligned}$$

Total LMF

- $$\mathcal{F}_P^i = \left(\mathcal{F}_P^i\right)_{\rm inst} + \left(\mathcal{F}_P^i\right)_{\rm hered}$$

- **Final 2.5PN expression for LMF reads**

$$\begin{aligned}
\mathcal{F}_P^i \;=\; & -\frac{464}{105}\,\frac{c^4}{G}\,\sqrt{1-4\,\nu}\,x^{11/2}\,\nu^2 \left\{\left[1-\left(\frac{452}{87}+\frac{1139}{522}\,\nu\right)x\right.\right. \\
& +\frac{309}{58}\,\pi\,x^{3/2}+\left(-\frac{71345}{22968}+\frac{36761}{2088}\,\nu+\frac{147101}{68904}\,\nu^2\right)x^2 \\
& \left.+\left(-\frac{2663}{116}\,\pi-\frac{2185}{87}\,\pi\,\nu\right)x^{5/2}\right]\hat{\lambda}_i+\left[-\frac{106187}{50460}\right. \\
& +\frac{32835}{841}\log 2-\frac{77625}{3364}\log 3+\left(\frac{32698}{12615}-\frac{109740}{841}\log 2\right. \\
& \left.\left.\left.+\frac{66645}{841}\log 3\right)\,\nu\right]x^{5/2}\,\hat{n}_i+\mathcal{O}(6)\right\}
\end{aligned}$$

Recoil Velocity

- Momentum Balance equation

$$\frac{dP^i}{dt} = -\mathcal{F}_p^i$$

- The net loss of linear momentum

$$\Delta P^i = -\int_{-\infty}^{t} dt' \, \mathcal{F}_p^i$$

- Hence, net recoil velocity

$$V^i = \Delta P^i / m$$

Recoil Velocity

- **Momentum Balance equation**

$$\frac{dP^i}{dt} = -\mathcal{F}_P^i$$

- The net loss of linear momentum

$$\Delta P^i = -\int_{-\infty}^{t} dt'\, \mathcal{F}_P^i$$

- Hence, net recoil velocity

$$V^i = \Delta P^i / m$$

Recoil Velocity

- Momentum Balance equation

$$\frac{dP^i}{dt} = -\mathcal{F}^i_P$$

- **The net loss of linear momentum**

$$\Delta P^i = -\int_{-\infty}^{t} dt'\, \mathcal{F}^i_P$$

- Hence, net recoil velocity

$$V^i = \Delta P^i / m$$

Recoil Velocity

- Momentum Balance equation

$$\frac{dP^i}{dt} = -\mathcal{F}^i_P$$

- The net loss of linear momentum

$$\Delta P^i = -\int_{-\infty}^{t} dt' \, \mathcal{F}^i_P$$

- **Hence, net recoil velocity**

$$V^i = \Delta P^i / m$$

- $$V^i_{\rm recoil} = \frac{464}{105}\, c\, \nu^2 \sqrt{1-4\,\nu}\, x^4 \Bigg\{ \Bigg[1 + \left(-\frac{452}{87} - \frac{1139}{522}\,\nu \right) x + \frac{309}{58}\,\pi\, x^{3/2} + \left(-\frac{71345}{22968} + \frac{36761}{2088}\,\nu + \frac{147101}{68901}\,\nu^2 \right) x^2 + \left(-\frac{2663}{116}\,\pi - \frac{2185}{87}\,\pi\,\nu \right) x^{5/2} \Bigg]\, \hat{n}_i + \Bigg[\frac{106187}{50460} - \frac{32835}{841}\log 2 + \frac{77625}{3364}\log 3 + \left(-\frac{32698}{12615} + \frac{109740}{841}\log 2 - \frac{66645}{841}\log 3 \right) \nu \Bigg] x^{5/2} \hat{\lambda}_i + \mathcal{O}(6) \Bigg\}$$

Contributions from plunge phase and numerical estimates for Recoil velocity

- **Adopting the Effective One Body (EOB) picture, the plunge cab be viewed as that of a test particle moving in the fixed Schwarzshild geometry of mass m.**
- In addition, the effects of the radiation of energy and angular momentum on the plunge orbit have been neglected. Over the small number of orbits constituting the plunge, this seems reasonable.
- The contribution from the plunge phase is estimated using the PN formulae assuming they are valid even beyond ISCO. Since the PN representation is usually not reliable inside ISCO, this should be a source of error and in general this computation is only a crude estimate.
- $$\Delta V_{recoil} = V_{ISCO} + \Delta V_{plunge}$$

Contributions from plunge phase and numerical estimates for Recoil velocity

- Adopting the Effective One Body (EOB) picture, the plunge cab be viewed as that of a test particle moving in the fixed Schwarzshild geometry of mass m.
- **In addition, the effects of the radiation of energy and angular momentum on the plunge orbit have been neglected. Over the small number of orbits constituting the plunge, this seems reasonable.**
- The contribution from the plunge phase is estimated using the PN formulae assuming they are valid even beyond ISCO. Since the PN representation is usually not reliable inside ISCO, this should be a source of error and in general this computation is only a crude estimate.
- $$\Delta V_{recoil} = V_{ISCO} + \Delta V_{plunge}$$

Contributions from plunge phase and numerical estimates for Recoil velocity

- Adopting the Effective One Body (EOB) picture, the plunge cab be viewed as that of a test particle moving in the fixed Schwarzshild geometry of mass m.
- In addition, the effects of the radiation of energy and angular momentum on the plunge orbit have been neglected. Over the small number of orbits constituting the plunge, this seems reasonable.
- **The contribution from the plunge phase is estimated using the PN formulae assuming they are valid even beyond ISCO. Since the PN representation is usually not reliable inside ISCO, this should be a source of error and in general this computation is only a crude estimate.**
- $$\Delta V_{recoil} = V_{ISCO} + \Delta V_{plunge}$$

Contributions from plunge phase and numerical estimates for Recoil velocity

- Adopting the Effective One Body (EOB) picture, the plunge cab be viewed as that of a test particle moving in the fixed Schwarzshild geometry of mass m.
- In addition, the effects of the radiation of energy and angular momentum on the plunge orbit have been neglected. Over the small number of orbits constituting the plunge, this seems reasonable.
- The contribution from the plunge phase is estimated using the PN formulae assuming they are valid even beyond ISCO. Since the PN representation is usually not reliable inside ISCO, this should be a source of error and in general this computation is only a crude estimate.
-
$$\Delta V_{recoil} = V_{ISCO} + \Delta V_{plunge}$$

- **For** V_{ISCO}

$$x \to x_{\rm ISCO}, \;\; \hat{n}^i \to \hat{n}^i_{\rm ISCO}, \;\; \hat{\lambda}^i \to \hat{\lambda}^i_{\rm ISCO}$$

- Let the ISCO be the one defined for a test particle moving around a Schwarzschild black hole with mass equal to the total mass of the binary and the phase $\psi = 0$ at the ISCO. This gives $x \to 1/6, \;\; \hat{n}^i \to \{1,0,0\}\,, \;\; \hat{\lambda}^i \to \{0,1,0\}$
- Geodesic equations in Schwarzschild geometry

$$\frac{dt}{d\tau} = \frac{\tilde{E}}{1 - \frac{2\,G\,m}{c^2\,r_{\rm s}}}$$

$$\frac{d\psi}{d\tau} = \frac{\tilde{L}}{r_{\rm s}^2}$$

$$\left(\frac{dr_{\rm s}}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2\,G\,m}{c^2\,r_{\rm s}}\right)\left(1 + \frac{\tilde{L}^2}{r_{\rm s}^2}\right)$$

- For V_{ISCO}

$$x \to x_{\rm ISCO}, \;\; \hat{n}^i \to \hat{n}^i_{\rm ISCO}, \;\; \hat{\lambda}^i \to \hat{\lambda}^i_{\rm ISCO}$$

- **Let the ISCO be the one defined for a test particle moving around a Schwarzschild black hole with mass equal to the total mass of the binary and the phase** $\psi = 0$ **at the ISCO. This gives** $x \to 1/6, \;\; \hat{n}^i \to \{1,0,0\}\,, \;\; \hat{\lambda}^i \to \{0,1,0\}$
- Geodesic equations in Schwarzschild geometry

$$\frac{dt}{d\tau} = \frac{\tilde{E}}{1 - \frac{2\,G\,m}{c^2\,r_{\rm s}}}$$

$$\frac{d\psi}{d\tau} = \frac{\tilde{L}}{r_{\rm s}^2}$$

$$\left(\frac{dr_{\rm s}}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2\,G\,m}{c^2\,r_{\rm s}}\right)\left(1 + \frac{\tilde{L}^2}{r_{\rm s}^2}\right)$$

- For V_{ISCO}

$$x \to x_{\rm ISCO}, \;\; \hat{n}^i \to \hat{n}^i_{\rm ISCO}, \;\; \hat{\lambda}^i \to \hat{\lambda}^i_{\rm ISCO}$$

- Let the ISCO be the one defined for a test particle moving around a Schwarzschild black hole with mass equal to the total mass of the binary and the phase $\psi = 0$ at the ISCO. This gives $x \to 1/6, \;\; \hat{n}^i \to \{1,0,0\}, \;\; \hat{\lambda}^i \to \{0,1,0\}$
- **Geodesic equations in Schwarzschild geometry**

$$\frac{dt}{d\tau} = \frac{\tilde{E}}{1 - \frac{2\,G\,m}{c^2\,r_{\rm s}}}$$

$$\frac{d\psi}{d\tau} = \frac{\tilde{L}}{r_{\rm s}^2}$$

$$\left(\frac{dr_{\rm s}}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2\,G\,m}{c^2\,r_{\rm s}}\right)\left(1 + \frac{\tilde{L}^2}{r_{\rm s}^2}\right)$$

- **Phase of the plunge orbit**

$$\psi = \int_{y_0}^{y} \left\{ \frac{\bar{L}}{\left[\bar{E}^2 - (1-2\,y)(1+\bar{L}^2\,y^2)\right]^{1/2}} \right\} \mathrm{d}y$$

with $y = (G\,m/r_{\mathrm{s}}\,c^2)$ and ψ is defined to vanish at $y = y_0$ in order to match the phase at the ISCO.

- Accumulated kick during the plunge

$$\Delta V^i_{\mathrm{plunge}} = \frac{1}{m} \int_{t_0}^{t_{\mathrm{Horizon}}} dt\, \frac{dP^i}{dt}$$

- Singular nature of the integral variable t at the horizon leads to the change of integration variable to a new variable $\bar{\omega} = \mathrm{d}\psi/\mathrm{d}\tau$. We can write

$$\Delta V^i_{\mathrm{plunge}} = \frac{1}{m} \int \left(\frac{d\bar{\omega}}{d\bar{\omega}/dt} \right) \frac{dP^i}{dt}$$

- Phase of the plunge orbit

$$\psi = \int_{y_0}^{y} \left\{ \frac{\bar{L}}{\left[\bar{E}^2 - (1 - 2\,y)(1 + \bar{L}^2\,y^2)\right]^{1/2}} \right\} \mathrm{d}y$$

with $y = (G\,m/r_{\rm s}\,c^2)$ and ψ is defined to vanish at $y = y_0$ in order to match the phase at the ISCO.

- **Accumulated kick during the plunge**

$$\Delta V^i_{\rm plunge} = \frac{1}{m} \int_{t_0}^{t_{\rm Horizon}} dt\, \frac{dP^i}{dt}$$

- Singular nature of the integral variable t at the horizon leads to the change of integration variable to a new variable $\bar{\omega} = \mathrm{d}\psi/\mathrm{d}\tau$. We can write

$$\Delta V^i_{\rm plunge} = \frac{1}{m} \int \left(\frac{d\bar{\omega}}{d\bar{\omega}/dt} \right) \frac{dP^i}{dt}$$

- Phase of the plunge orbit

$$\psi = \int_{y_0}^{y} \left\{ \frac{\bar{L}}{\left[\bar{E}^2 - (1 - 2\,y)(1 + \bar{L}^2\,y^2)\right]^{1/2}} \right\} \mathrm{d}y$$

with $y = (G\,m/r_{\rm s}\,c^2)$ and ψ is defined to vanish at $y = y_0$ in order to match the phase at the ISCO.

- Accumulated kick during the plunge

$$\Delta V^i_{\rm plunge} = \frac{1}{m} \int_{t_0}^{t_{\rm Horizon}} dt\, \frac{dP^i}{dt}$$

- **Singular nature of the integral variable t at the horizon leads to the change of integration variable to a new variable $\bar{\omega} = \mathrm{d}\psi/\mathrm{d}\tau$. We can write**

$$\Delta V^i_{\rm plunge} = \frac{1}{m} \int \left(\frac{d\bar{\omega}}{d\bar{\omega}/dt} \right) \frac{dP^i}{dt}$$

- **In its final form the integral will look like**

$$\Delta V^i_{\rm plunge} = \frac{G\,\bar{L}}{c^3}\int_{y_0}^{y_{\rm Horizon}} \frac{\left(\frac{1}{x^{3/2}}\,|\frac{dP^i}{dt}|(\cos\,\psi, \sin\,\psi, 0)\right)}{\left[\bar{E}^2-(1-2\,y)(1+\bar{L}^2\,y^2)\right]^{1/2}}\,{\rm d}y$$

where x is connected to y by the relation

$$x = \left[\frac{\bar{L}}{\bar{E}}\,y^2\,(1-2\,y)\right]^{2/3}$$

- With the choice

$$\bar{E}^2 \to \frac{8}{9}\left[1-\frac{9}{4}\frac{1}{c^2}\left(\frac{{\rm d}r_s}{{\rm d}t}\right)^2_{\rm ISCO}\right]^{-1} \; \& \; \bar{L} \to \sqrt{12}$$

we have everything we would need to compute the above integral.

- In its final form the integral will look like

$$\Delta V^i_{\rm plunge} = \frac{G\,\bar{L}}{c^3}\int_{y_0}^{y_{\rm Horizon}} \frac{\left(\frac{1}{x^{3/2}}\,|\frac{dP^i}{dt}|(\cos\,\psi, \sin\,\psi, 0)\right)}{\left[\bar{E}^2-(1-2\,y)(1+\bar{L}^2\,y^2)\right]^{1/2}}\,{\rm d}y$$

where x is connected to y by the relation

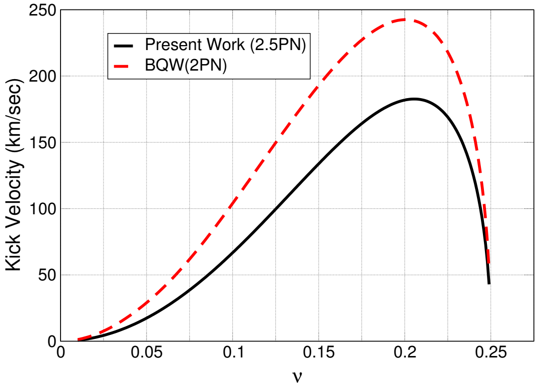
$$x = \left[\frac{\bar{L}}{\bar{E}}\,y^2\,(1-2\,y)\right]^{2/3}$$

- **With the choice**

$$\bar{E}^2 \rightarrow \frac{8}{9}\left[1-\frac{9}{4}\frac{1}{c^2}\left(\frac{{\rm d}r_s}{{\rm d}t}\right)^2_{\rm ISCO}\right]^{-1} \;\&\;\; \bar{L}\rightarrow\sqrt{12}$$

we have everything we would need to compute the above integral.

Result



Present Work (2.5PN)
BQW(2PN)
Kick Velocity (km/sec)
250
200
150
100
50
0
0
0.05
0.1
0.15
0.2
0.25
ν

Comparison of present estimates with other related works and Conclusion

- **Maximum recoil velocity at the end of inspiral and plunge phase is of the order of 4 $\mathrm{km\,s^{-1}}$ and 182 $\mathrm{km\,s^{-1}}$ respectively (compared to BQW estimates of 22 $\mathrm{km\,s^{-1}}$ and 243 $\mathrm{km\,s^{-1}}$ respectively) for binary with $\nu = 0.2$.**
- The 2.5 PN terms not only contribute significantly but also contribute negatively to the recoil velocity estimates.
- In a numerical study (Baker et al. 2006) suggests that the recoil velocity estimates at the end of inspiral phase should be of the order of $\sim 14\,\mathrm{km\,s^{-1}}$ for a binary with $\nu = 0.24$ (agrees well with BQW estimates).
- This is a relatively higher estimate as compared to our estimate of $2.2\,\mathrm{km\,s^{-1}}$ at the ISCO for a system with the same mass ratio.

Comparison of present estimates with other related works and Conclusion

- Maximum recoil velocity at the end of inspiral and plunge phase is of the order of $4\,\mathrm{km\,s^{-1}}$ and $182\,\mathrm{km\,s^{-1}}$ respectively (compared to BQW estimates of $22\,\mathrm{km\,s^{-1}}$ and $243\,\mathrm{km\,s^{-1}}$ respectively) for binary with $\nu = 0.2$.
- **The 2.5 PN terms not only contribute significantly but also contribute negatively to the recoil velocity estimates.**
- In a numerical study (Baker et al. 2006) suggests that the recoil velocity estimates at the end of inspiral phase should be of the order of $\sim 14\,\mathrm{km\,s^{-1}}$ for a binary with $\nu = 0.24$ (agrees well with BQW estimates).
- This is a relatively higher estimate as compared to our estimate of $2.2\,\mathrm{km\,s^{-1}}$ at the ISCO for a system with the same mass ratio.

Comparison of present estimates with other related works and Conclusion

- Maximum recoil velocity at the end of inspiral and plunge phase is of the order of $4\,\mathrm{km\,s^{-1}}$ and $182\,\mathrm{km\,s^{-1}}$ respectively (compared to BQW estimates of $22\,\mathrm{km\,s^{-1}}$ and $243\,\mathrm{km\,s^{-1}}$ respectively) for binary with $\nu = 0.2$.
- The 2.5 PN terms not only contribute significantly but also contribute negatively to the recoil velocity estimates.
- **In a numerical study (Baker et al. 2006) suggests that the recoil velocity estimates at the end of inspiral phase should be of the order of $\sim 14\,\mathrm{km\,s^{-1}}$ for a binary with $\nu = 0.24$ (agrees well with BQW estimates).**
- This is a relatively higher estimate as compared to our estimate of $2.2\,\mathrm{km\,s^{-1}}$ at the ISCO for a system with the same mass ratio.

Comparison of present estimates with other related works and Conclusion

- Maximum recoil velocity at the end of inspiral and plunge phase is of the order of $4\,\mathrm{km\,s^{-1}}$ and $182\,\mathrm{km\,s^{-1}}$ respectively (compared to BQW estimates of $22\,\mathrm{km\,s^{-1}}$ and $243\,\mathrm{km\,s^{-1}}$ respectively) for binary with $\nu = 0.2$.
- The 2.5 PN terms not only contribute significantly but also contribute negatively to the recoil velocity estimates.
- In a numerical study (Baker et al. 2006) suggests that the recoil velocity estimates at the end of inspiral phase should be of the order of $\sim 14\,\mathrm{km\,s^{-1}}$ for a binary with $\nu = 0.24$ (agrees well with BQW estimates).
- **This is a relatively higher estimate as compared to our estimate of $2.2\,\mathrm{km\,s^{-1}}$ at the ISCO for a system with the same mass ratio.**

- **In a situation of such mismatch, we should expect that inclusion of higher order contributions at the 3PN order will contribute to the recoil velocity positively (in contrast to the negative contributions from 2.5PN terms).** BQW05 results
- Including contributions from the final ringdown.
- This will allow one to make more direct comparisons with recoil velocity estimates involving all three phases of binary evolution (such as those due to Numerical Relativity or effective one body approach).
- Paper Reference: Physical Review D, **85** 044021 (2012)/arXiv:1111.2701.

- In a situation of such mismatch, we should expect that inclusion of higher order contributions at the 3PN order will contribute to the recoil velocity positively (in contrast to the negative contributions from 2.5PN terms). ▸ BQW05 results
- **Including contributions from the final ringdown.**
- This will allow one to make more direct comparisons with recoil velocity estimates involving all three phases of binary evolution (such as those due to Numerical Relativity or effective one body approach).
- Paper Reference: Physical Review D, **85** 044021 (2012)/arXiv:1111.2701.

- In a situation of such mismatch, we should expect that inclusion of higher order contributions at the 3PN order will contribute to the recoil velocity positively (in contrast to the negative contributions from 2.5PN terms). BQW05 results
- Including contributions from the final ringdown.
- **This will allow one to make more direct comparisons with recoil velocity estimates involving all three phases of binary evolution (such as those due to Numerical Relativity or effective one body approach).**
- Paper Reference: Physical Review D, **85** 044021 (2012)/arXiv:1111.2701.

- In a situation of such mismatch, we should expect that inclusion of higher order contributions at the 3PN order will contribute to the recoil velocity positively (in contrast to the negative contributions from 2.5PN terms). BQW05 results
- Including contributions from the final ringdown.
- This will allow one to make more direct comparisons with recoil velocity estimates involving all three phases of binary evolution (such as those due to Numerical Relativity or effective one body approach).
- **Paper Reference: Physical Review D, 85 044021 (2012)/arXiv:1111.2701.**

BQW05 results