

On the Poynting-Robertson effect in General Relativity

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PR-effect: an overview

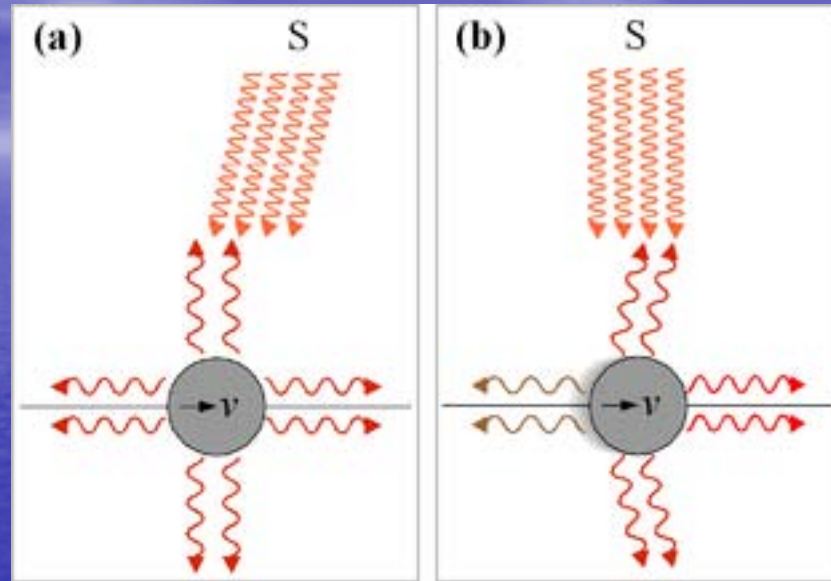
- Consider a fixed background spacetime with superimposed a radiation field (either a test field or the source of the gravitational background field).
- Let a test particle be moving in a spacetime region.
- The particle interacts with the radiation field (actually, it absorbs and re-emits radiation).
- The interaction is modeled by a drag or friction force. (related to Thomson scattering cross section).
- The particle is no longer moving on a geodesic, due to this force (PR effect).
- Interesting features arise on its motion (gravitational force balance effects).

Poynting J H 1903 *Phil. Trans. Roy. Soc.* **203** 525
Robertson H P 1937 *MNRAS* **97** 423

Plan of the talk

- Introduction to PR effect
- Formalization in stationary and axisymmetric spacetimes
- Applications to Schwarzschild and Kerr
- Generalizations:
 - ➔ PR drag force vs MPD spin, em forces etc-
 - ➔ Non-stationary spacetimes (Vaidya)
 - ➔ Matter fields and not radiation fields
 - ➔ Cosmology

Typical example (wiki)



Radiation from the Sun (S) and thermal radiation from a particle (grain of dust) seen

(a) from an observer moving with the particle;

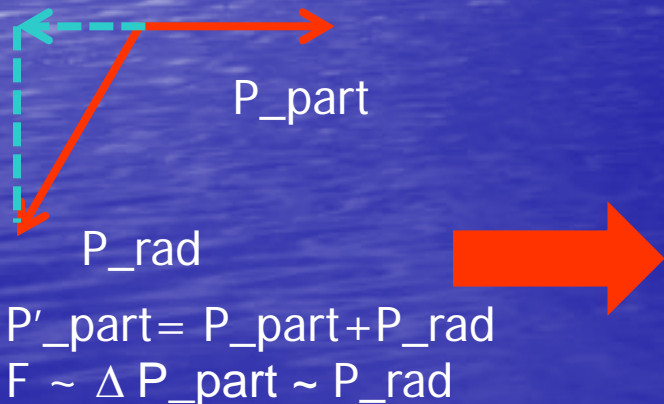
(b) from an observer at rest with respect to the Sun.

From the perspective of the grain of dust circling the Sun (panel (a) of the figure):

the Sun's radiation appears to be coming from a slightly forward direction (aberration of light).

Therefore:

the absorption of this radiation leads to a force with a component against the direction of movement.



(The angle of aberration is extremely small, since the radiation is moving at the speed of light while the dust grain is moving many orders of magnitude slower than that.)

From the perspective of the Solar System as a whole (panel (b) of the figure):

the dust grain absorbs sunlight entirely in a radial direction, thus the grain's angular momentum remains unchanged.

However, in absorbing photons, the dust acquires mass via mass-energy equivalence.

In order to conserve angular momentum (which is proportional to mass), the dust grain must drop into a lower orbit.

Re-emitting

The re-emission of photons, which is isotropic in the frame of the grain (a), does not affect the dust particle's orbital motion.

However, in the frame of the Solar System (b), the emission is beamed anisotropically, and hence the photons carry away angular momentum from the dust grain (i.e. again there exist a force which is opposite to the motion).

The **Poynting–Robertson drag** can be understood as an effective force opposite the direction of the dust grain's orbital motion, leading to a drop in the grain's angular momentum.

Explicit examples in stationary axisymmetric spacetimes

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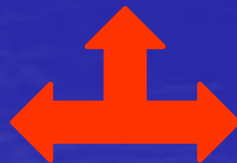
doi:10.1088/0264-9381/26/5/055009

The general relativistic Poynting–Robertson effect

Donato Bini^{1,2}, Robert T Jantzen^{2,3} and Luigi Stella⁴

Stationary, axisymmetric and reflection symmetric spacetimes

Schwarzschild



Kerr

The background geometry

→ **Coords: BL or BL-like** $\{t, r, \theta, \phi\}$ (adapted to the Killing symmetries)
 ∂_t (timelike) and ∂_ϕ (spacelike, with closed coordinate lines) Killing

→ **Metric** $ds^2 = g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2$

→ **Fiducial observers (ZAMOs)** $n = N^{-1}(\partial_t - N^\phi \partial_\phi)$

(lapse, shift functions) $N = (-g^{tt})^{-1/2}$ $N^\phi = g_{t\phi}/g_{\phi\phi}$

ZAMOs OAF $e_{\hat{t}} = n, \quad e_{\hat{r}} = \frac{1}{\sqrt{g_{rr}}} \partial_r, \quad e_{\hat{\theta}} = \frac{1}{\sqrt{g_{\theta\theta}}} \partial_\theta, \quad e_{\hat{\phi}} = \frac{1}{\sqrt{g_{\phi\phi}}} \partial_\phi$

ZAMOs kinematical quantities
(vorticity-free obs)

$$a(n) = a(n)^{\hat{r}} e_{\hat{r}} + a(n)^{\hat{\theta}} e_{\hat{\theta}} = \partial_{\hat{r}}(\ln N) e_{\hat{r}} + \partial_{\hat{\theta}}(\ln N) e_{\hat{\theta}},$$

$$\theta_{\hat{\phi}}(n) = \theta_{\hat{\phi}}(n)^{\hat{r}} e_{\hat{r}} + \theta_{\hat{\phi}}(n)^{\hat{\theta}} e_{\hat{\theta}} = -\frac{\sqrt{g_{\phi\phi}}}{2N} [\partial_{\hat{r}} N^\phi e_{\hat{r}} + \partial_{\hat{\theta}} N^\phi e_{\hat{\theta}}]$$

$$k_{(\text{lie})}(n) = k_{(\text{lie})}(n)^{\hat{r}} e_{\hat{r}} + k_{(\text{lie})}(n)^{\hat{\theta}} e_{\hat{\theta}} = -[\partial_{\hat{r}}(\ln \sqrt{g_{\phi\phi}}) e_{\hat{r}} + \partial_{\hat{\theta}}(\ln \sqrt{g_{\phi\phi}}) e_{\hat{\theta}}]$$

Super-posed photon test field

$$T^{\alpha\beta} = \Phi^2 k^\alpha k^\beta, \quad k^\alpha k_\alpha = 0,$$

K =null geo, known
 Φ = obtained by integrating
 conservation eqs of T

k is assumed to be tangent to an affinely parametrized outgoing null geodesic in the equatorial plane, i.e., $k^\alpha \nabla_\alpha k^\beta = 0$ with $k^\theta = 0$. We will only consider photons in the equatorial plane which are in outward radial motion with respect to the ZAMOs, namely with 4-momentum

$$k = E(n)[n + \hat{v}(k, n)], \quad \hat{v}(k, n) = e_{\hat{r}}, \quad (2.9)$$

$$E(n) = E/N \quad E = -k_t \quad L = k_\phi = 0$$

$$T^{\alpha\beta}_{;\beta} = 0$$



$$\Phi = \frac{\Phi_0}{[g_{\theta\theta}g_{\phi\phi}]^{1/4}}$$

Photons with $L=0$:
 Special situation,
 to be generalized
 later

Test-particle

A test particle of mass m moves on the equatorial plane $\theta = \pi/2$

$$U = \gamma(U, n)[n + v(U, n)], \quad v(U, n) \equiv v^{\hat{r}} e_{\hat{r}} + v^{\hat{\phi}} e_{\hat{\phi}} = v \sin \alpha e_{\hat{r}} + v \cos \alpha e_{\hat{\phi}}$$

accelerated by the radiation field

$$\mathcal{F}_{(\text{rad})}(U)^\alpha = -\sigma P(U)^\alpha{}_\beta T^\beta{}_\mu U^\mu$$



$$ma(U) = \mathcal{F}_{(\text{rad})}(U)$$



Robertson

Notation:

$$A = \tilde{\sigma} \Phi_0^2 E^2 \quad \tilde{\sigma} = \sigma/m$$

$A > 0$: constant introduced by Robertson to combine all the constants associated with the intensity of the radiation at the source, the conserved photon energy and its cross section per unit mass for absorption at the particle

Motion equations

$$\frac{dv}{d\tau} = -\frac{\sin \alpha}{\gamma} [a(n)^{\hat{r}} + 2v \cos \alpha \theta(n)^{\hat{r}}_{\hat{\phi}}] + \frac{A}{N^2 \sqrt{g_{\theta\theta} g_{\phi\phi}}} (1 - v \sin \alpha) (\sin \alpha - v),$$

$$\frac{d\alpha}{d\tau} = -\frac{\gamma \cos \alpha}{v} [a(n)^{\hat{r}} + 2v \cos \alpha \theta(n)^{\hat{r}}_{\hat{\phi}} + v^2 k_{(\text{lie})}(n)^{\hat{r}}] + \frac{A}{N^2 \sqrt{g_{\theta\theta} g_{\phi\phi}}} \frac{(1 - v \sin \alpha) \cos \alpha}{v},$$

$$\frac{dr}{d\tau} = \frac{\gamma v \sin \alpha}{\sqrt{g_{rr}}},$$

Mostly **numerically integrated**.

Special solutions: $\alpha = \pm\pi/2$
(radial inward/outward motion wrt ZAMOs)

A=0: particle in geodesic motion

$$U = \gamma(U, n)[n + v(U, n)], \quad v(U, n) \equiv v^{\hat{r}} e_{\hat{r}} + v^{\hat{\phi}} e_{\hat{\phi}} = v \sin \alpha e_{\hat{r}} + v \cos \alpha e_{\hat{\phi}}$$

Schwarzschild spacetime

$$g_{tt} = -N^2, \quad g_{t\phi} = 0, \quad g_{rr} = 1/N^2, \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2 \sin^2 \theta$$

The radially outgoing (geodesic) photons on the equatorial plane have 4-momentum

$$k = E \left[\left(1 - \frac{2M}{r}\right)^{-1} \partial_t + \partial_r \right] = E(n)[n + e_{\hat{r}}].$$

$$\begin{aligned} \frac{dv}{d\tau} &= -\frac{N \sin \alpha}{r\gamma} v_K^2 + \frac{A}{r^2 N^2} (1 - v \sin \alpha)(\sin \alpha - v), \\ \frac{d\alpha}{d\tau} &= \frac{N\gamma \cos \alpha}{rv} (v^2 - v_K^2) + \frac{A}{r^2 N^2 v} (1 - v \sin \alpha) \cos \alpha, \\ \frac{dr}{d\tau} &= \gamma v N \sin \alpha, \end{aligned}$$

$$v_K = \sqrt{\frac{M}{r - 2M}}$$

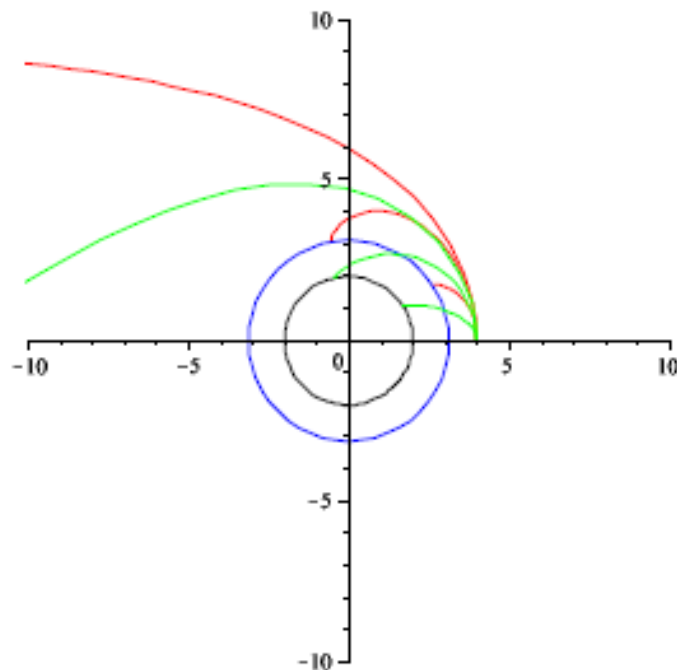


Equilibrium solution

$v=0$, indep of α , $r=\text{const.}$

$$\frac{A}{M} = \left(1 - \frac{2M}{r}\right)^{1/2} \rightarrow r = r_{(\text{crit})} \equiv \frac{2M}{1 - A^2/M^2}$$

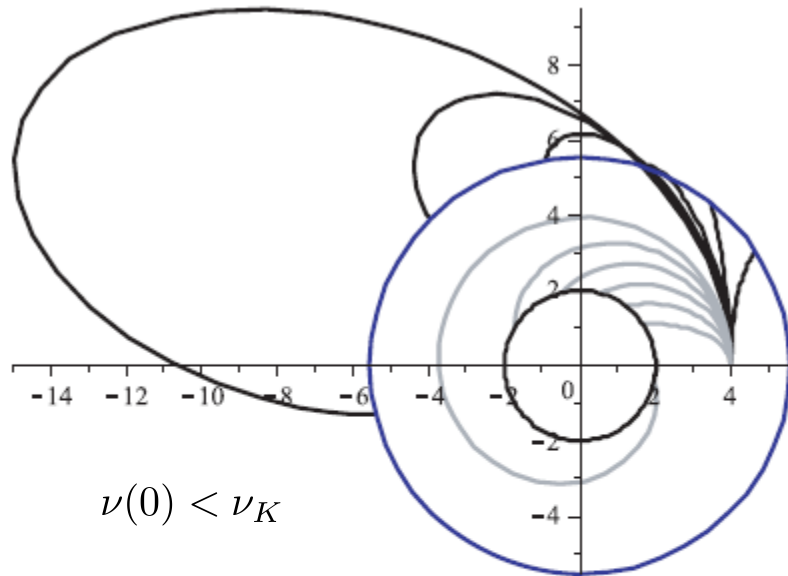
Schwarzschild spacetime



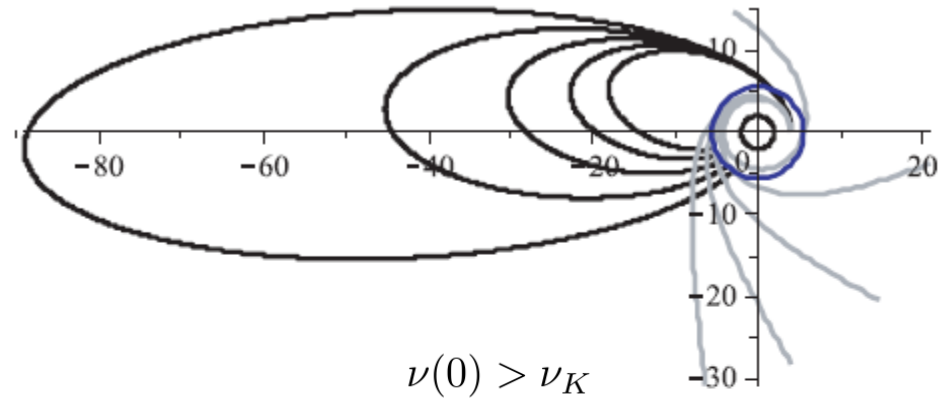
Geos in green

. The orbit of the particle in the Schwarzschild spacetime with $M = 1$, $A/\tilde{M} = 0.6$, $r_{\text{crit}} = 3.125M$. The inner circle is the horizon $r = 2M$, while the outer circle is at the critical radius which is inside the initial data position. Initial conditions have $(r(0), \phi(0), \alpha(0)) = (4M, 0, 0)$ and $\nu(0) = 0.2, 0.5, 0.8$. The corresponding geodesics $A/M = 0$ are in gray.

Schwarzschild spacetime



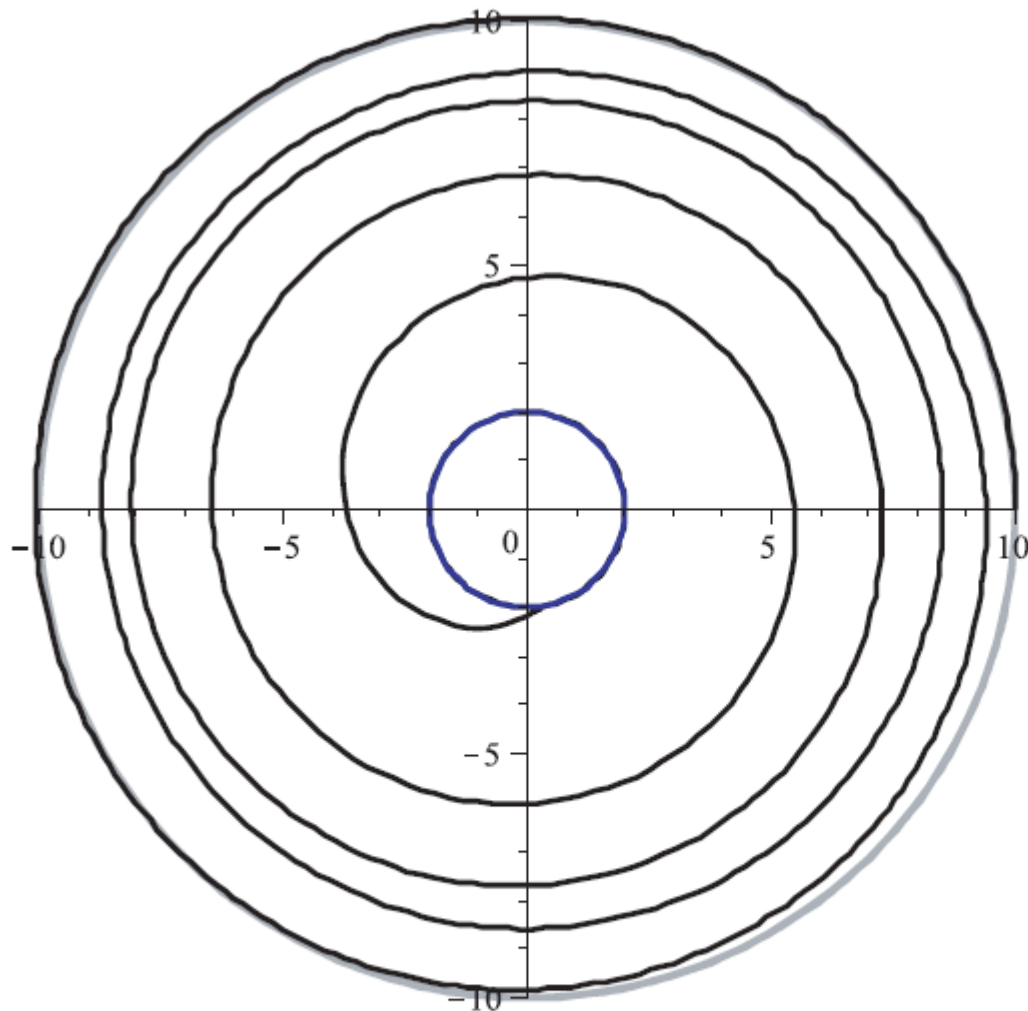
$$\nu(0) < \nu_K$$



$$\nu(0) > \nu_K$$

The orbit of the particle in the Schwarzschild spacetime with $M = 1$, $A/M = 0.8$, $r_{\text{crit}} = 5.5M$, $\nu_K = 0.7071$. The inner circle is the horizon $r = 2M$, while the outer circle is at the critical radius which is outside the initial data position. Initial conditions have $(r(0), \phi(0), \alpha(0)) = (4M, 0, 0)$ and for the left figure $\nu(0) = 0.2, 0.3, \dots, 0.7 < \nu_K$ while for the right figure $0.71, 0.72, \dots, 0.75 > \nu_K$. The corresponding geodesics $A/M = 0$ are in gray, and those in the left figure come to rest at the horizon in the Schwarzschild coordinates.

Schwarzschild spacetime



The orbit of the particle in the Schwarzschild spacetime with $M = 1$, $A/M = 0.01$, $r_{\text{crit}} \approx 2M$ and initial conditions $(r(0), \phi(0), \alpha(0)) = (10M, 0, 0)$ with the circular geodesic speed $v(0) = v_K = 0.3536$. The circular geodesic (gray) is shown with the in-spiraling orbit (black) with the same initial conditions.

Kerr spacetime

In the equatorial plane of the Kerr metric, the metric is

$$g_{tt} = -\left(1 - \frac{2M}{r}\right), \quad g_{t\phi} = -\frac{2aM}{r}, \quad g_{\phi\phi} = \frac{r^3 + a^2r + 2a^2M}{r},$$
$$g_{rr} = \frac{r^2}{\Delta}, \quad g_{\theta\theta} = r^2,$$

$$N = \sqrt{\frac{r\Delta}{r^3 + a^2r + 2a^2M}} \sim 1 - \frac{M}{r}, \quad N^\phi = -\frac{2aM}{r^3 + a^2r + 2a^2M} \sim -\frac{2aM}{r^3}$$

The nonvanishing components of the ZAMO kinematical fields are

$$a(n)^{\hat{r}} = \frac{M[(r^2 + a^2)^2 - 4a^2Mr]}{r^2\sqrt{\Delta}(r^3 + a^2r + 2a^2M)} \sim \frac{M}{r^2},$$
$$\theta_{\hat{\phi}}(n)^{\hat{r}} = -\frac{aM(3r^2 + a^2)}{r^2(r^3 + a^2r + 2a^2M)} \sim -\frac{3aM}{r^3},$$
$$k_{(\text{lie})}(n)^{\hat{r}} = -\frac{\sqrt{\Delta}(r^3 - a^2M)}{r^2(r^3 + a^2r + 2a^2M)} \sim -\frac{1}{r} + \frac{M}{r^2}.$$

Kerr spacetime

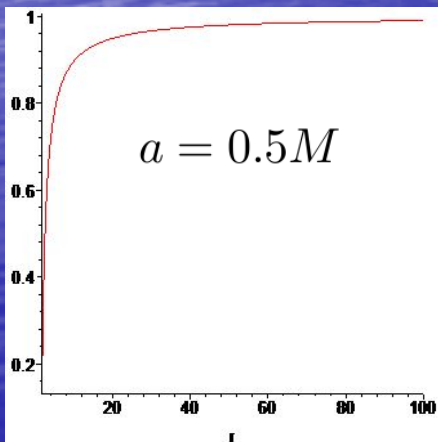
$$\frac{dv}{d\tau} = -\frac{\sin \alpha k_{(\text{lie})}(n)^{\hat{r}}}{\gamma} [v_+ v_- - v \cos \alpha (v_+ + v_-)] + \frac{A(1 - v \sin \alpha)(\sin \alpha - v)}{r \sqrt{g_{\phi\phi}} N^2},$$

$$\frac{d\alpha}{d\tau} = -\frac{\gamma \cos \alpha k_{(\text{lie})}(n)^{\hat{r}}}{v} [v_+ v_- - v \cos \alpha (v_+ + v_-) + v^2] + \frac{A(1 - v \sin \alpha) \cos \alpha}{r \sqrt{g_{\phi\phi}} N^2 v},$$

$$\frac{dr}{d\tau} = \frac{\gamma v \sin \alpha}{\sqrt{g_{rr}}}.$$

Equilibrium solution

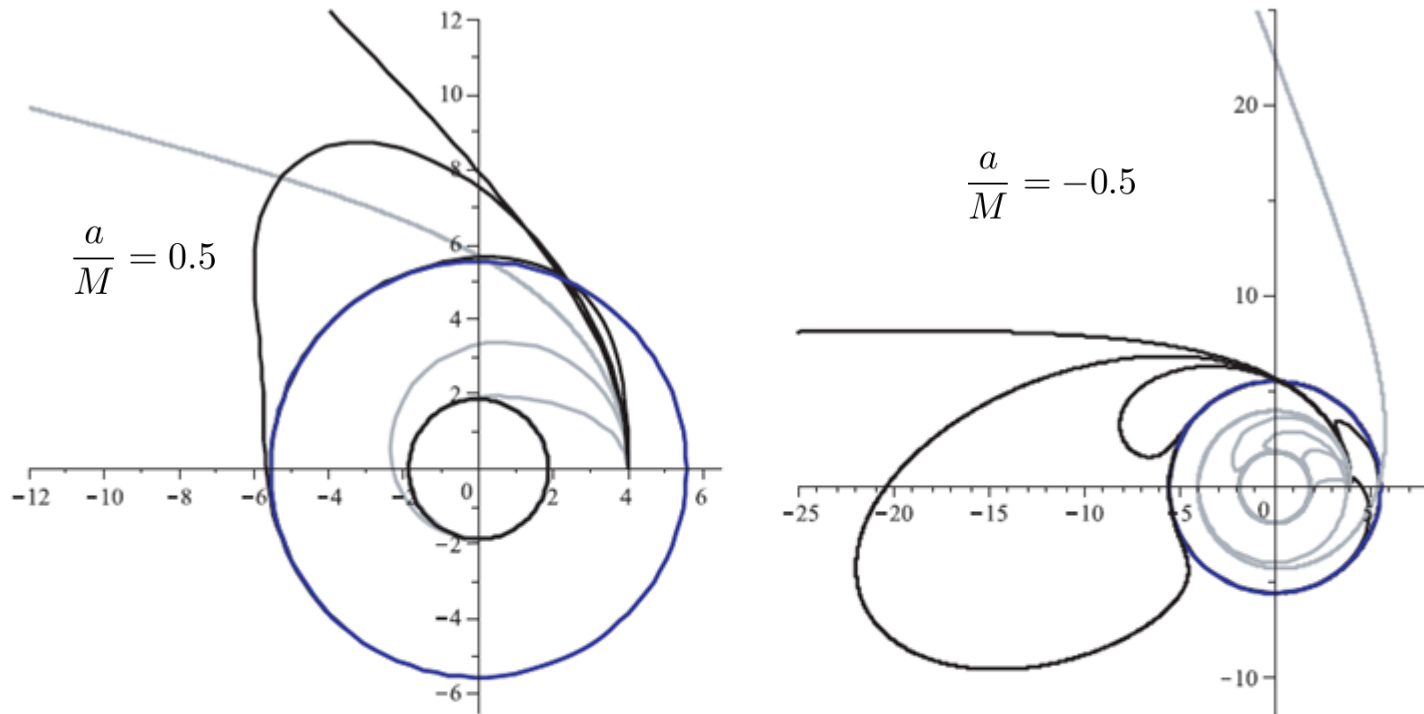
$$\frac{A}{M} = \frac{[(r^2 + a^2)^2 - 4a^2 M r] \sqrt{\Delta}}{r^2 (g_{\phi\phi})^{3/2}}$$



Solve numerically

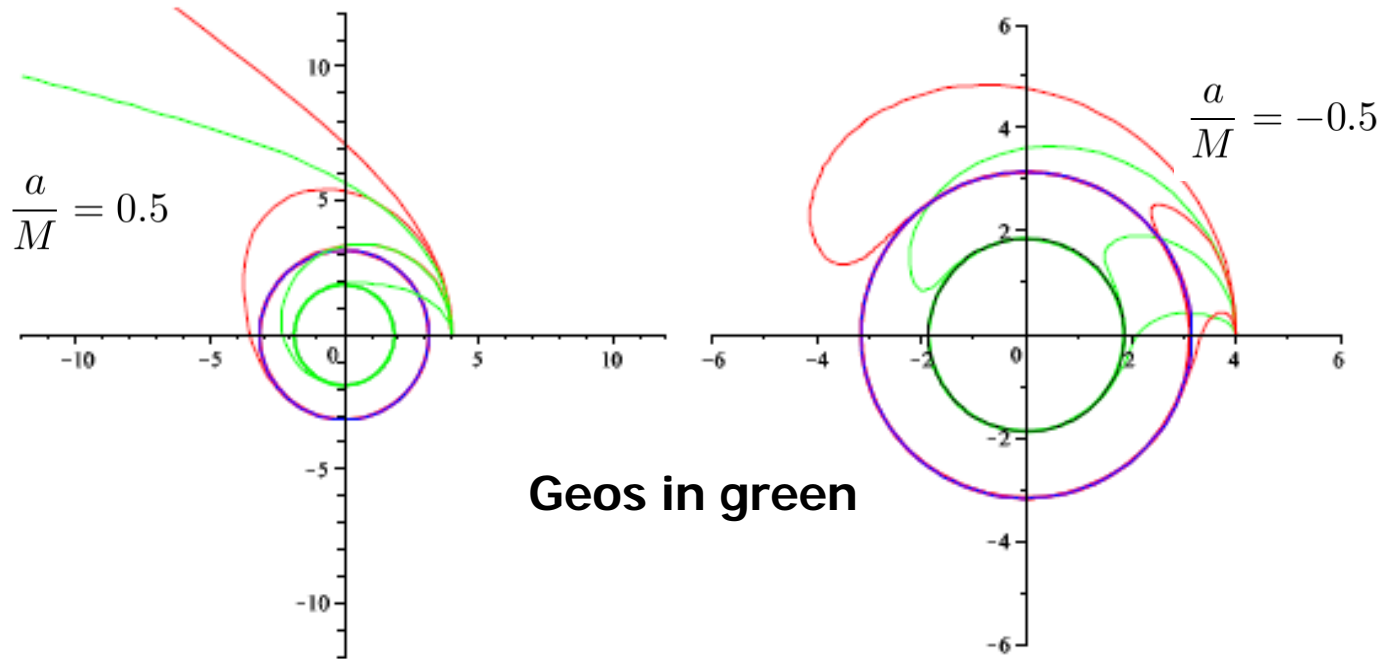
$$r_{(\text{crit})} \approx 2M + 2M \left(\frac{A}{M} \right)^2 - \frac{M}{2} \left(\frac{a}{M} \right)^2$$

Kerr spacetime



The orbit of the particle in the Kerr spacetime with $M = 1$, $a = 0.5$ (left figure), $a = -0.5$ (right figure), $A/M = 0.8$, $r_{\text{crit}} = 5.551M$. The inner circle is the horizon $r = 1.866M$, while the outer circle is at the critical radius which is outside the initial data position. Initial conditions have $(r(0), \phi(0), \alpha(0)) = (4M, 0, 0)$ and $v(0) = 0.2, 0.5, 0.8$ for the left figure, while in the right figure $v(0) = 0.2, 0.5, 0.8, 0.847$ for both the accelerated and geodesic curves and then finally $v(0) = 0.9$ for the accelerated curve and $v(0) = 0.848$ for the geodesic, both of which escape to infinity. The corresponding geodesics $A/M = 0$ are in gray. The bound orbits end up co-rotating with the hole at the horizon (geodesics) or at the critical radius (accelerated).

Kerr spacetime



The orbit of the particle in the Kerr spacetime with $M = 1$, $a = 0.5$ (left figure), $a = -0.5$ (right figure), $A/M = 0.6$, $r_{\text{crit}} = 3.154M$. The inner circle is the horizon $r = 1.866M$, while the outer circle is at the critical radius which is inside the initial data position. Initial conditions have $(r(0), \phi(0), \alpha(0)) = (4M, 0, 0)$ and $\nu(0) = 0.2, 0.5, 0.8$. The corresponding geodesics $A/M = 0$ are in gray.

Relax $L=0$ condition for photons

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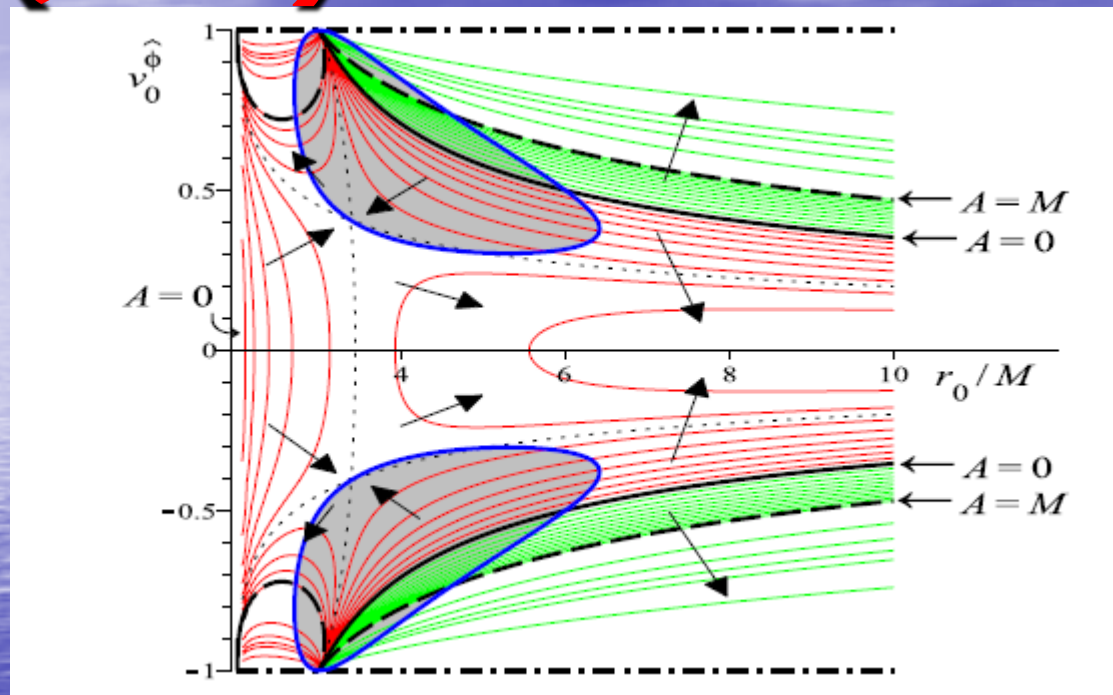
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The general relativistic Poynting–Robertson effect: II. A photon flux with nonzero angular momentum

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Oldřich Semerák⁶ and Luigi Stella⁷

Schwarzschild spacetime ($L \neq 0$)

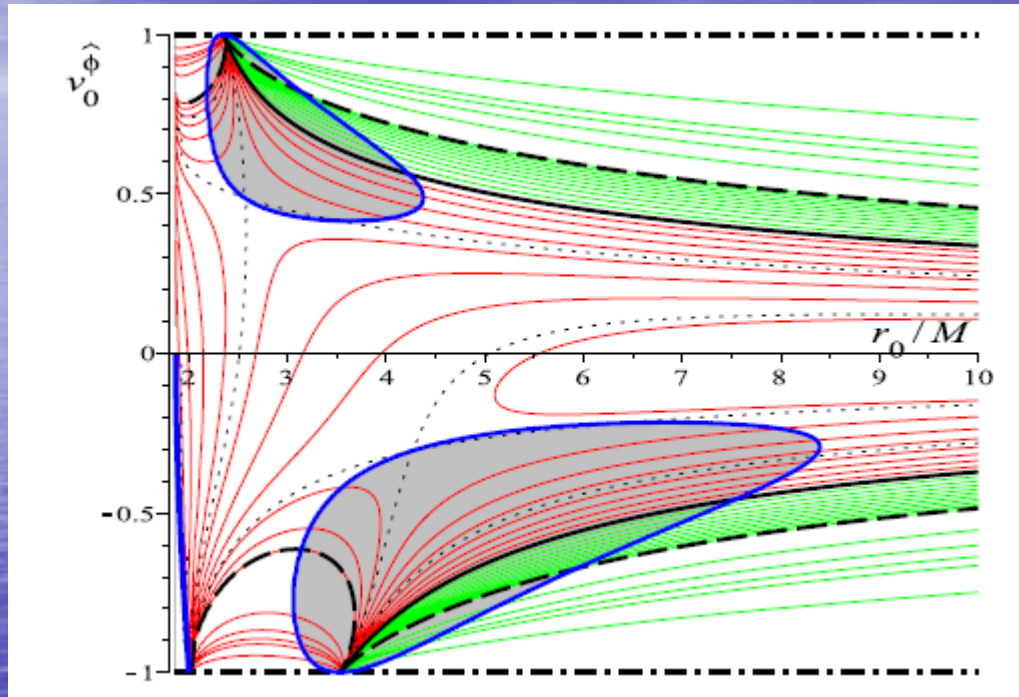


Critical "equilibrium" orbits are circular orbits with speed depending on A/M . There exist stability regions (in gray)

Critical orbits at r -fixed can be more than 1 at the same A/M , if one is close to the horizon

The critical azimuthal velocity v_0^ϕ versus critical radius $r_0/M > 2$ for the Schwarzschild case for selected values of A/M . Physical velocities are confined to the interval $-1 < v_0^\phi < 1$, with $A/M \rightarrow \infty$ corresponding to $|v_0^\phi| = 1$ indicated by the thick dot-dash lines, and the thick solid curves indicating the geodesic velocities corresponding to $A = 0$, enclosing the outgoing photon region, outside of which is the ingoing photon region. The thick closed loop curves enclosing the shaded regions are explained below. For context the thick dashed curves correspond to $A/M = 1$. Arrows indicate the direction of increasing values of A . Note the two pairs of accumulation points of the family of curves near the horizon at unit velocity.

Kerr spacetime ($L \neq 0$)

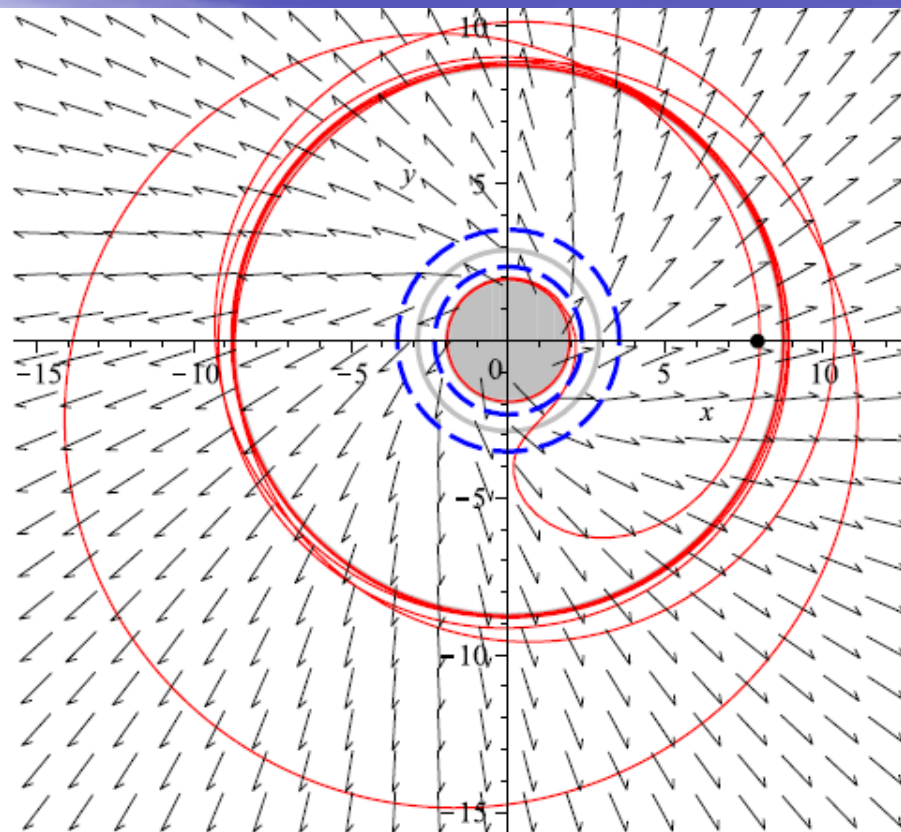


$$a/M = 0.5.$$

Slightly deformed picture
in comparison with the
analogous in
Schwarzschild

The critical azimuthal velocity v_0^ϕ versus critical radius r_0/M for the Kerr case for
equally spaced values of A

Kerr spacetime



The spacetime and photon parameters are $a/M = 0.5$, $A/M = 0.3$ and $b/M = 3$, showing two orbits moving initially from the bullet point on the horizontal axis in the two azimuthal directions just inside the outer critical radius with 1.2 times the critical speed for that counterclockwise critical orbit. The unit velocity direction field $\hat{v}(k, n)$ of the radiation with respect to the ZAMOs is superimposed on the plot, showing the additional counterclockwise rotation of the photon trajectories with respect to the counterclockwise rotating ZAMOs. The dashed circles are the two null circular geodesics orbits. The gray filled circle extends to the horizon. The counterclockwise moving orbit settles down to the outermost critical orbit, while the clockwise moving orbit quickly falls into the innermost critical orbit near the horizon. The gray circle between the null orbits is the unstable critical orbit. The axes show units of r/M .

Radiation field not a test field: Vaidya spacetime

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Effect of radiation flux on test-particle motion in the Vaidya spacetime

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and Oldřich Semerák⁵

Vaidya

$$M = M(t, r), \text{ Free choice}$$

$$ds^2 = - \left(\frac{M_{,t}}{M_{,r}} \right)^2 \frac{dt^2}{1 - 2M/r} + \frac{dr^2}{1 - 2M/r} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$du = \mp \frac{r}{r - 2M} \frac{dM}{M_{,r}}$$

$$N = \sqrt{1 - 2M(u)/r}$$

Upper sign:
out-going radiation

$$ds^2 = -N^2 du^2 \mp 2 du dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$T_{uu} = T^{rr} = \mp \frac{M_{,u}}{4\pi r^2} > 0$$

$$T^{\mu\nu} = \Phi^2 k^\mu k^\nu \quad k^\mu = \pm 2^{-1/2} \delta^\mu_r$$

$$T^{rr} = \frac{\Phi^2}{2} \implies \Phi^2 = \mp \frac{M_{,u}}{2\pi r^2}$$

In the entire article, upper signs correspond to $M_{,u} \leq 0$ (outgoing radiation), while lower signs to $M_{,u} \geq 0$ (ingoing radiation), so that $\pm M_{,u} \leq 0$. The advanced null coordinate is conventionally denoted by v , but we keep u in both cases (distinguishing them by signs).

The most natural test observers suitable for physical interpretation are those at rest in the spatial coordinate grid at $r = \text{const}$, $\theta = \text{const}$ and $\phi = \text{const}$; their 4-velocity field is

$$\hat{u} \equiv e_{\hat{u}} = \frac{1}{N} \partial_u.$$

A convenient spatial orthonormal triad tied to this observer congruence is

$$e_{\hat{r}} = N \left(\partial_r \mp \frac{1}{N^2} \partial_u \right), \quad e_{\hat{\theta}} = \frac{1}{\sqrt{g_{\theta\theta}}} \partial_\theta, \quad e_{\hat{\phi}} = \frac{1}{\sqrt{g_{\phi\phi}}} \partial_\phi.$$

$$U = \gamma(U, \hat{u}) [\hat{u} + v(U, \hat{u})^{\hat{a}} e_{\hat{a}}]$$

$$ma(U)^\alpha = -\sigma P(U)^\alpha{}_\mu T^\mu{}_\nu U^\nu \equiv \mathcal{F}_{\text{rad}}(U)^\alpha$$

For a particle moving in the equatorial plane $\theta = \pi/2$, so that $v(U, \hat{u})^{\hat{\theta}} = 0 = U^{\theta}$, the equations of motion (3.1) become

$$\frac{dv^{\hat{r}}}{d\tau} = -\frac{\gamma}{rN^3}(1 - (v^{\hat{r}})^2) \left[\frac{M}{r}N^2 \mp M_{,u}(1 \mp v^{\hat{r}}) \right] + \frac{\gamma N}{r}(v^{\hat{\phi}})^2 \pm \frac{\tilde{\sigma} T_{uu}}{N^2}(1 \mp v^{\hat{r}})^2,$$

$$\frac{dv^{\hat{\phi}}}{d\tau} = \frac{\gamma}{rN^3}v^{\hat{r}}v^{\hat{\phi}} \left[\frac{M}{r}N^2 \mp M_{,u}(1 \mp v^{\hat{r}}) \right] - \frac{\gamma N}{r}v^{\hat{r}}v^{\hat{\phi}} - \frac{\tilde{\sigma} T_{uu}}{N^2}v^{\hat{\phi}}(1 \mp v^{\hat{r}}),$$

with $\gamma(U, \hat{u}) \equiv \gamma = 1/\sqrt{1 - (v^{\hat{r}})^2 - (v^{\hat{\phi}})^2}$. To complete this system one must add the evolution equations for u , r and ϕ , i.e.

$$\frac{du}{d\tau} = \frac{\gamma}{N}(1 \mp v^{\hat{r}}), \quad \frac{dr}{d\tau} = \gamma N v^{\hat{r}}, \quad \frac{d\phi}{d\tau} = \frac{\gamma v^{\hat{\phi}}}{r}.$$

Equilibrium under special assumptions of $M(u)$

$$M(u) = M_0 \exp(-4\pi ua/\bar{\sigma})$$

$$r = \frac{\bar{\sigma}}{4\pi} \frac{N(a-N)}{a}$$

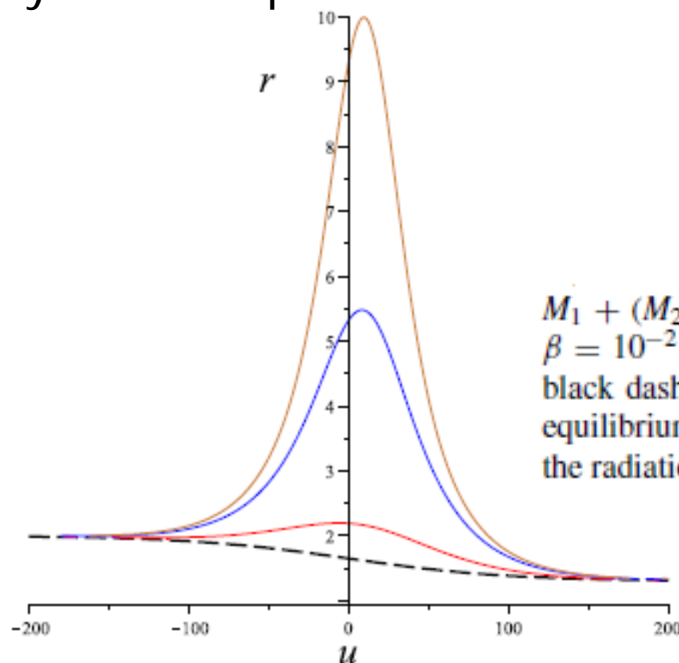
$$M(u) = M_0 - \beta u$$

$$r = N \left(\frac{\bar{\sigma}}{4\pi} - \frac{MN}{\beta} \right)$$

$$M(u) = M_1 + \frac{(M_2 - M_1)}{2} (1 + \tanh \beta u)$$

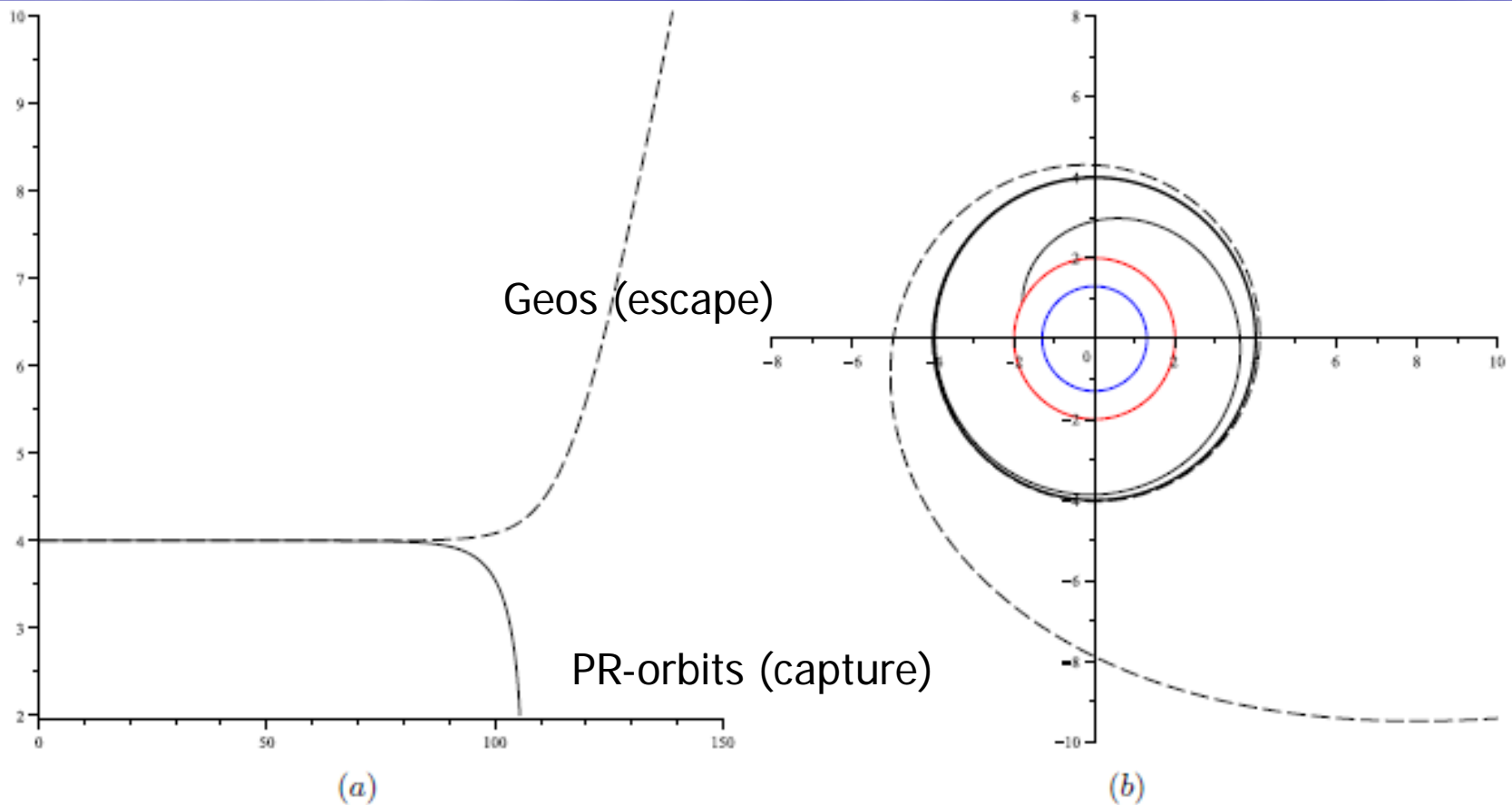
outgoing ($M_1 > M_2$) and ingoing radiation ($M_1 < M_2$) cases

Dynamical equilibrium radius



In this case (tanh-case) one has asymptotically two Schwarzschild spacetimes, of which we already know the properties.

Evolution of the quasi-equilibrium radius r with time u for the mass profile $M(u) = M_1 + (M_2 - M_1)(1 + \tanh \beta u)/2$ with the following parameter choice: $M_1 = 1$, $M_2 = 0.65$, $\beta = 10^{-2}$ and different values of $\bar{\sigma} = [3, 5, 5.5] \times 10^3$, with the axes given in units of M_1 . The black dashed curve corresponds to the apparent horizon. During the transition phase the quasi-equilibrium radius is even more enhanced for increasing values of $\bar{\sigma}$, i.e. when the interaction with the radiation field becomes stronger.

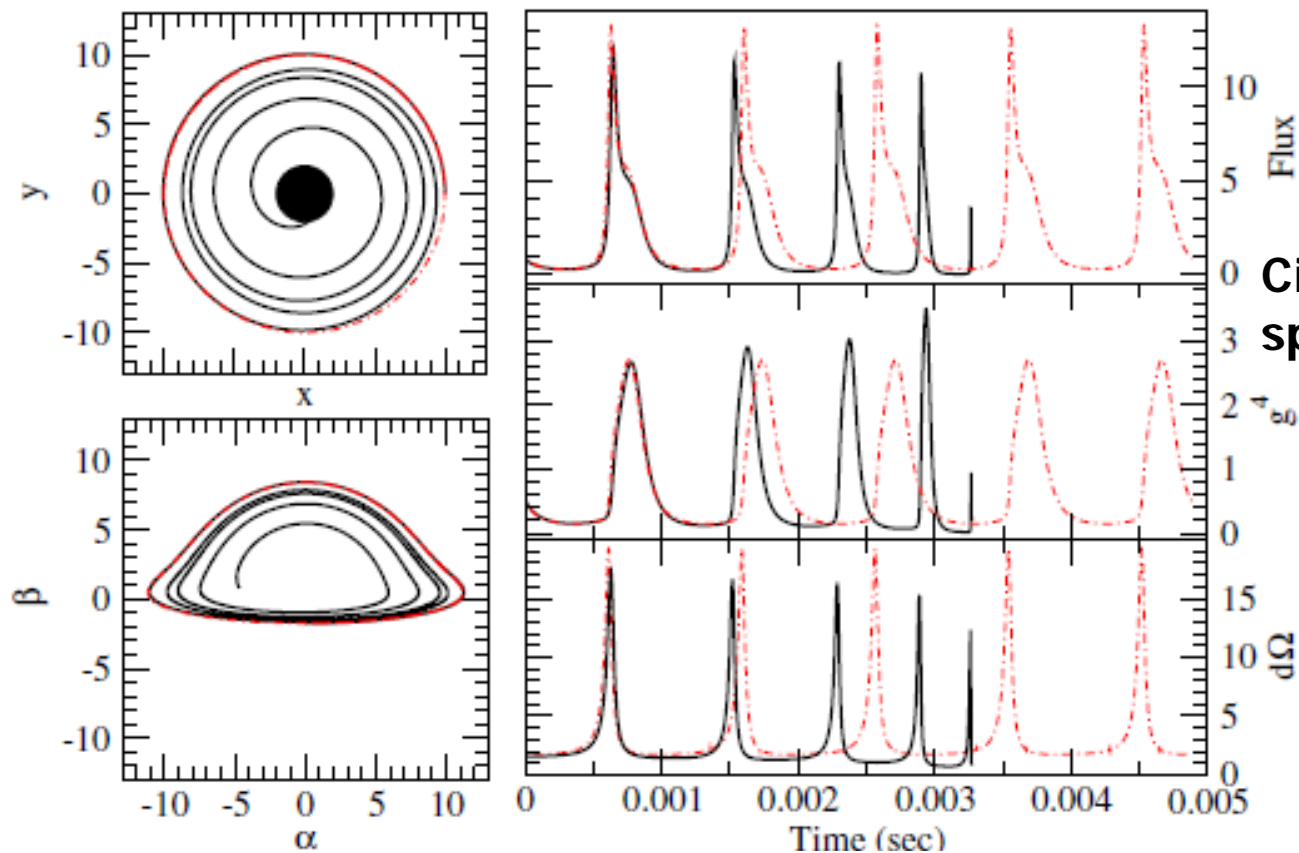


The behavior of $r(\tau)$ is shown in (a) in the case of outgoing radiation with the following parameter choice: $M_1 = 1$, $M_2 = 0.65$, $\beta = 10^{-2}$ and $\tilde{\sigma} = 0$ (geodesic, thick dashed line) and $\tilde{\sigma} = 10^4$ (solid line), with the axes given in units of M_1 . The initial conditions are $u(0) = -1000$, $r(0) = 4$, $\phi(0) = 0$, $v^{\hat{r}}(0) = 0$, $v^{\hat{\phi}}(0) \approx 0.707$, which correspond to a circular geodesic in the past asymptotic Schwarzschild spacetime with mass M_1 . The corresponding orbits are shown in (b). In the geodesic case, the orbit escapes outward after a few loops. In contrast, the accelerated particle spirals toward the apparent horizon, which is reached in a finite proper time interval at $r \approx 2$. The asymptotic inner apparent horizon at $r \approx 1.3$ is also shown.

The signal from an emitting source moving in a Schwarzschild spacetime under the influence of a radiation field

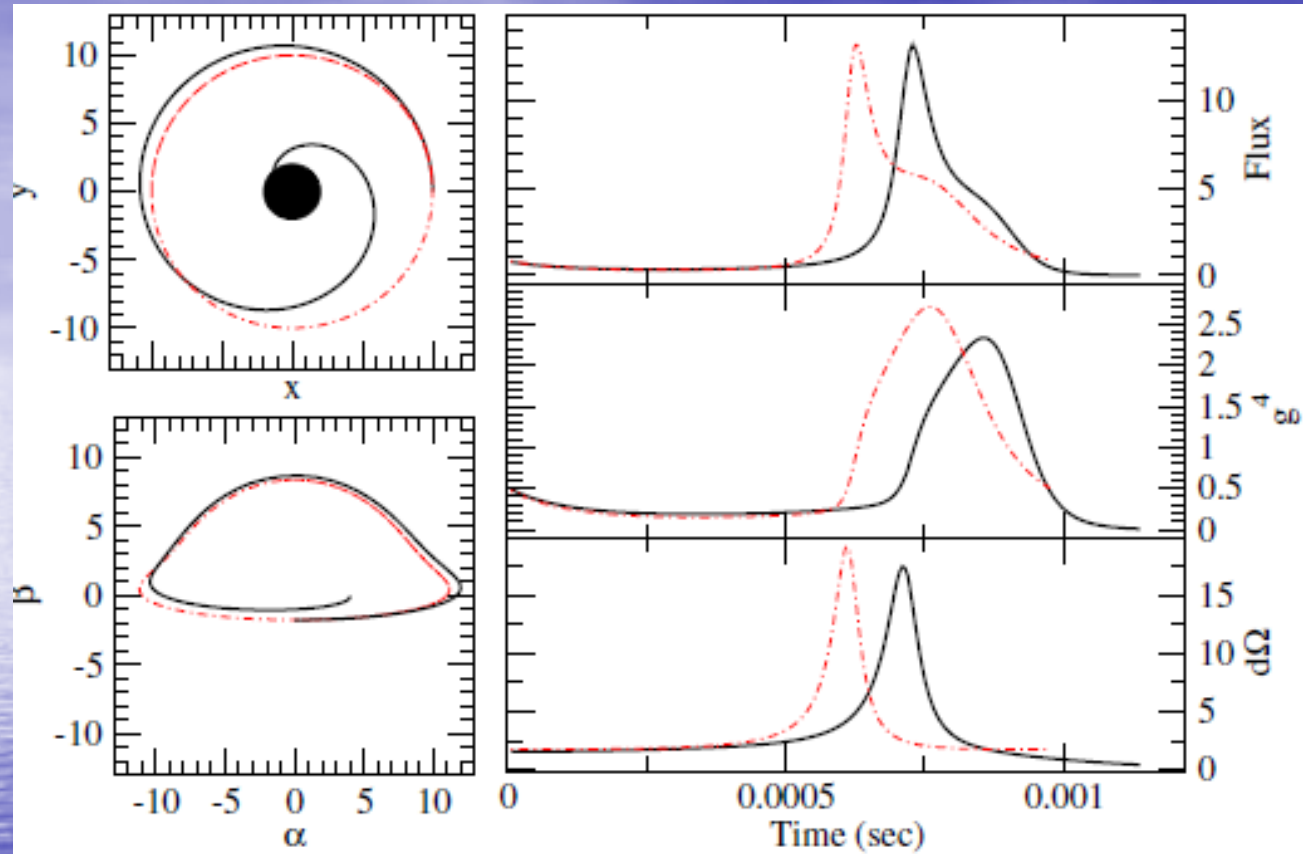
Donato Bini^{1,2}, Maurizio Falanga³, Andrea Geralico^{2,4} and Luigi Stella⁵

The emitting source now undergoes PR effects...i.e. it spirals up to a critical radius



Circular geo emitting spots at $r=10M$ in red

The apparent position (direct image only), light curve, redshift factor and solid angle of the emitting spot are shown for the orbit depicted in the upper left panel in the case of a radially outgoing radiation field. The orbital parameters and initial conditions are $A/M = 0.01$, $r_{\text{em}}(0) = 10M$, $\phi_{\text{em}}(0) = 0$, $v_{\text{em}}(0) = v_K \approx 0.35$ and $\alpha_{\text{em}}(0) = 0$. The distant observer is located at the polar angle $\theta_{\text{obs}} = 80^\circ$. $x = (r_{\text{em}}/M) \cos \phi_{\text{em}}$ and $y = (r_{\text{em}}/M) \sin \phi_{\text{em}}$ are the Cartesian-like coordinates expressed in units of M . The black circle represents the Schwarzschild horizon $r = 2M$. The critical radius approaches the horizon in this case ($r_{\text{crit}} \approx 2.0002M$). The flux is given in arbitrary units as a function of the coordinate time given in seconds, corresponding to the choice of $M = 1.0M_\odot$. The observed time is given by the orbital time plus the light-bending travel time delay. The relative time delay is then evaluated by using the geodesic equations in such a way that the first photon emitted at the starting point defines the reference time which is set to zero. For comparison purposes, the corresponding curves for an emitting spot in circular geodesic orbit at $r_{\text{em}} = 10M$ on the equatorial plane of a Schwarzschild spacetime are shown (dashed curves).



The same as in figure 1 but with $A/M = 0.1$. The critical radius approaches the horizon also in this case ($r_{\text{crit}} \approx 2.02M$).

Spinning bodies immersed in a radiation field: comparing the spin force and the PR one

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Spinning bodies and the Poynting–Robertson effect in the Schwarzschild spacetime

Donato Bini^{1,2,3} and Andrea Geralico^{2,4}

$$\frac{DP^\mu}{d\tau} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} \equiv F^{(\text{spin})\mu}$$

$$S^{\mu\nu}P_\nu = 0$$

$$\frac{DS^{\mu\nu}}{d\tau} = P^\mu U^\nu - P^\nu U^\mu,$$

$$T^{\alpha\beta} = \Phi^2 k^\alpha k^\beta, \quad k^\alpha k_\alpha = 0,$$

$$k = E(n)(n + e_{\hat{r}})$$

$$\Phi = \frac{\Phi_0}{r}$$

$$U = \gamma(U, n)[n + v(U, n)], \quad v(U, n) \equiv v^{\hat{r}} e_{\hat{r}} + v^{\hat{\phi}} e_{\hat{\phi}} = v(\sin \alpha e_{\hat{r}} + \cos \alpha e_{\hat{\phi}})$$

$$F^{(\text{rad})\alpha} = -\sigma \mathcal{P}(U)^\alpha{}_\beta T^\beta{}_\mu U^\mu$$

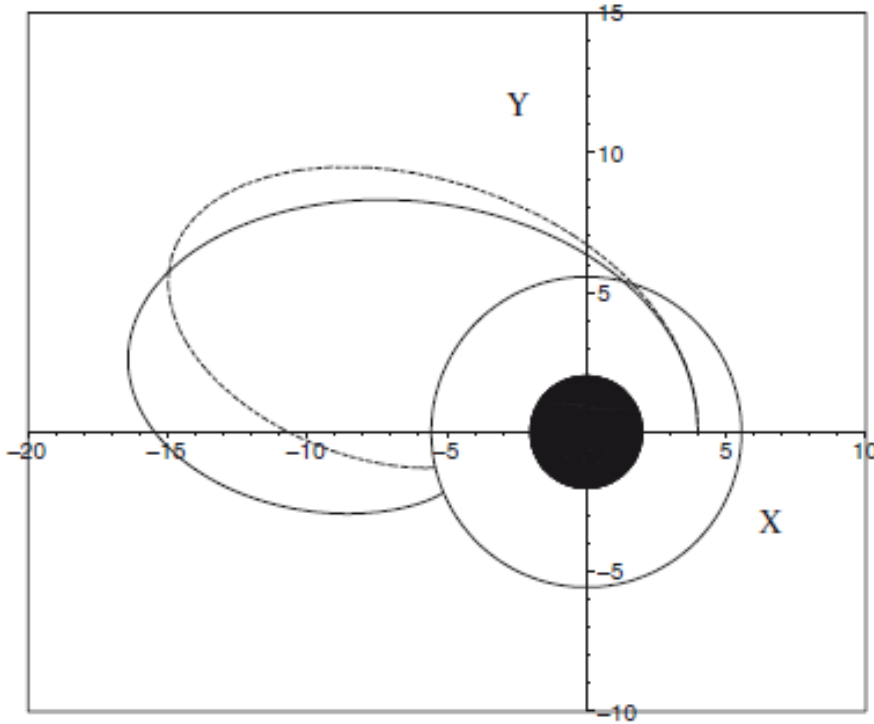
$$\mathcal{P}(U)^\alpha{}_\beta = \delta^\alpha_\beta + U^\alpha U_\beta$$

$$F^{(\text{rad})} = \frac{mA}{N^2 r^2} \gamma^3 (1 - v^{\hat{r}}) [(v^{\hat{r}} - v^2)n + (1 - v^{\hat{r}} - (v^{\hat{\phi}})^2) e_{\hat{r}} - (1 - v^{\hat{r}}) v^{\hat{\phi}} e_{\hat{\phi}}]$$

$$\frac{DP^\mu}{d\tau} = F^{(\text{spin})\mu} + F^{(\text{rad})\mu}$$

$$P = mu$$

$$u = \gamma_u [n + v_u (\sin \alpha_u e_{\hat{r}} + \cos \alpha_u e_{\hat{\phi}})]$$



The orbit of a spinning particle (solid curve) subject to the Poynting–Robertson effect is shown for the choice of the parameters $A/M = 0.8$ and $\hat{s} = 0.5$ ($X = r \cos \phi$ and $Y = r \sin \phi$ are the Cartesian-like coordinates). The starting point is located at $r_0(0) = 4M$ and $\phi_0(0) = 0$ with $v_{u0}(0) = 0.7$, $\alpha_{u0}(0) = 0$, $t_s(0) = 0$, $r_s(0) = 0$ and $\phi_s(0) = 0$, $v_s^r(0) = 0$ and $v_s^{\hat{\phi}}(0) = 0$. The values of the spin parameter have been exaggerated in order to distinguish the difference from the motion of a spinless particle (dashed curve). The inner circle is at the horizon $r = 2M$, while the outer circle is at the critical radius $r_{(\text{crit})} = 5.5M$ which is outside the initial data position.

In the absence of both spin and radiation we assume the geodesic motion of the particle to be circular at $r = r_0$ ($r_0 > 3M$ in order U_K to be timelike), that is

$$U = U_K = \gamma_K (n \pm v_K e_{\hat{\phi}}),$$

where the Keplerian value of speed (v_K) and the associated Lorentz factor (γ_K) and angular velocity (ζ_K) are given by

$$v_K = \sqrt{\frac{M}{r_0 - 2M}}, \quad \gamma_K = \sqrt{\frac{r_0 - 2M}{r_0 - 3M}}, \quad \zeta_K = \sqrt{\frac{M}{r_0^3}}.$$

The parametric equations of U_K are

$$t_K = t_0 + \frac{\gamma_K}{N_0} \tau \equiv t_0 + \Gamma_K \tau,$$

$$r = r_0, \quad \theta = \frac{\pi}{2},$$

$$\phi_K = \phi_0 \pm \frac{\gamma_K v_K}{r_0} \tau \equiv \phi_0 \pm \Omega_K \tau,$$

$$f \equiv \frac{A}{M} \ll 1$$

$$\Gamma_K = \sqrt{\frac{r_0}{r_0 - 3M}}, \quad \Omega_K = \frac{1}{r_0} \sqrt{\frac{M}{r_0 - 3M}}$$

$$t = t_K + f t_f + \hat{s} t_{\hat{s}}, \quad r = r_0 + f r_f + \hat{s} r_{\hat{s}}, \quad \phi = \phi_K + f \phi_f + \hat{s} \phi_{\hat{s}},$$

$$v^{\hat{t}} = f v_f^{\hat{t}} + \hat{s} v_{\hat{s}}^{\hat{t}}, \quad v^{\hat{\phi}} = \pm v_K + f v_f^{\hat{\phi}} + \hat{s} v_{\hat{s}}^{\hat{\phi}},$$

$$v_u = \pm v_K + f v_{uf} + \hat{s} v_{u\hat{s}}, \quad \alpha_u = f \alpha_{uf} + \hat{s} \alpha_{u\hat{s}},$$

$$U = U_K + f U_f + \hat{s} U_{\hat{s}}$$

$$u = U + f u_f + \hat{s} u_{\hat{s}}$$

$$U_f = \left(-v_K \frac{r_f}{r_0} \pm \gamma_K^2 v_f^{\hat{\phi}} \right) \bar{U}_K + \gamma_K v_f^{\hat{r}} e_{\hat{r}},$$

$$U_{\hat{s}} = \left(-v_K \frac{r_{\hat{s}}}{r_0} \pm \gamma_K^2 v_{\hat{s}}^{\hat{\phi}} \right) \bar{U}_K + \gamma_K v_{\hat{s}}^{\hat{r}} e_{\hat{r}}.$$

$$u_f = \gamma_K (v_K^2 v_f^{\hat{\phi}} - v_f^{\hat{r}}) e_{\hat{r}}, \quad u_{\hat{s}} = \gamma_K (v_K^2 v_{\hat{s}}^{\hat{\phi}} - v_{\hat{s}}^{\hat{r}}) e_{\hat{r}}$$

To first order in \hat{s} and f the spin force and radiation force are given by

$$F^{(\text{spin})} = \mp 3mM\hat{s}\gamma_K^2\zeta_K^2 v_K e_{\hat{r}},$$

$$F^{(\text{rad})} = -mf\Omega_K v_K (\gamma_K v_K \bar{U}_K - e_{\hat{r}}),$$

$$\frac{|F^{(\text{spin})}|}{|F^{(\text{rad})}|} = 3 \frac{|\hat{s}|}{f} \left(\frac{M}{r_0} \right)^{3/2} \sqrt{1 - \frac{3M}{r_0}}$$

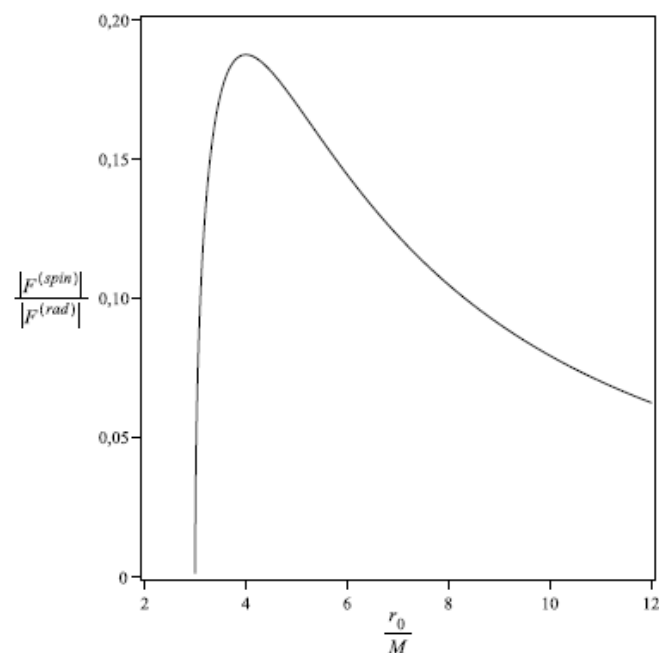
$$\Omega_{\text{ep}} = \sqrt{\frac{M(r_0 - 6M)}{r_0^3(r_0 - 3M)}}$$

$$v_{\hat{s}}^{\hat{r}} = \mp \frac{3M^2}{r_0^2} \frac{\Omega_K}{\Omega_{\text{ep}}} \sin(\Omega_{\text{ep}}\tau),$$

$$v_f^{\hat{r}} = \frac{v_K^2}{r_0 \Omega_{\text{ep}}} \left\{ \sin(\Omega_{\text{ep}}\tau) + 2r_0 \zeta_K \frac{\Omega_K}{\Omega_{\text{ep}}} [\cos(\Omega_{\text{ep}}\tau) - 1] \right\},$$

$$v_{\hat{s}}^{\hat{\phi}} = -\frac{3M\zeta_K^3}{\Omega_{\text{ep}}^2} v_K [\cos(\Omega_{\text{ep}}\tau) - 1],$$

$$v_f^{\hat{\phi}} = \pm \frac{v_K^3}{r_0 \Omega_{\text{ep}}^2} \left\{ \frac{\zeta_K}{r_0 \Omega_{\text{ep}}} [\cos(\Omega_{\text{ep}}\tau) - 1] - \frac{2\zeta_K^2}{\Omega_{\text{ep}}} \sin(\Omega_{\text{ep}}\tau) + \Omega_K^2 \tau \right\}$$



Numbers

Earth-Sun

For instance, for the motion of the Earth about the Sun we find

$$\left\langle \frac{\delta r}{r} \right\rangle \approx f \mp 2 \times 10^{-17} \frac{(s/m)_{\oplus}}{\text{cm}} \approx 3 \times 10^{-15} \mp 4 \times 10^{-15} \approx 10^{-15},$$

since $r_{\oplus} \approx 1.5 \times 10^{13}$ cm, $M = M_{\odot} \approx 1.5 \times 10^5$ cm and the ratio $(s/m)_{\oplus} \approx 200$ cm for the Earth; the friction parameter is related to the ratio between the solar luminosity $\mathcal{L}_{\odot} \approx 3.8 \times 10^{33}$ erg/s and the Eddington luminosity [16, 17] $\mathcal{L}_{\text{Edd}} \approx 1.3 \times 10^{38}$ erg/s, and for the Sun is given by $f \approx 3 \times 10^{-15}$. Therefore, in this case the effect of the radiation field on the orbit is of the same order as that due to spin. Note that the estimate of the contribution due to spin is in agreement with [23].

The effect of the spin may become important when the orbiting extended body is a fast rotating object. To illustrate the order of magnitude of the effect, we may consider the binary pulsar system PSR J0737-3039 as orbiting Sgr A*, the supermassive ($M \simeq 10^6 M_{\odot}$) black hole located at the Galactic Center [24, 25], at a distance of $r \simeq 10^9$ Km. The PSR J0737-3039 system consists of two close neutron stars (their separation is only $d_{AB} \sim 8 \times 10^5$ Km) of comparable masses $m_A \simeq 1.4 M_{\odot}$, $m_B \simeq 1.2 M_{\odot}$), but very different intrinsic spin period (23 ms of pulsar A vs 2.8 s of pulsar B) [26]. Its orbital period is about 2.4 hours, the smallest yet known for such an object. Since the intrinsic rotations are negligible with respect to the orbital period, we can treat the binary system as a single object with reduced mass $\mu_{AB} \simeq 0.7 M_{\odot}$ and intrinsic rotation equal to the orbital period. The spin parameter thus turns out to be equal to $\hat{s} \approx 1.0 \times 10^{-3}$. The luminosity of Sgr A* is about $10^3 \mathcal{L}_{\odot}$, whereas its Eddington luminosity is $\mathcal{L}_{\text{Edd}} \approx 10^{11} \mathcal{L}_{\odot}$, so that $f \approx 10^{-18}$. Therefore, in this case

$$\left\langle \frac{\delta r}{r} \right\rangle \approx 7.6 \times 10^{-19} \mp 1.8 \times 10^{-7} \approx 10^{-7}.$$

PSR J0737-3039

Therefore, in this case the effect of the spin on the orbit dominates with respect to the friction due to the radiation field.

Accelerated orbits in black hole fields: the static case

Donato Bini^{1,2,3}, Fernando de Felice^{4,5} and Andrea Geralico^{2,6}

Charged particles in external magnetic fields

$$ma(U) = f_{(\text{em})}(U)$$

$$\zeta_0 = \frac{qB_0}{m}$$

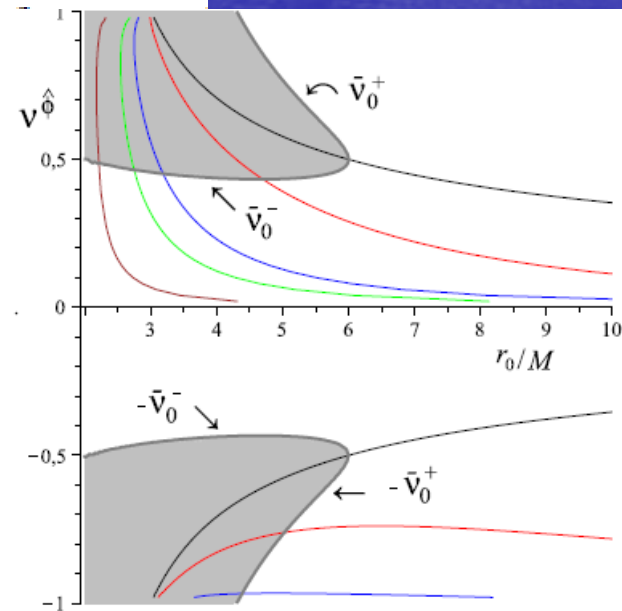
$$f_{(\text{em})}(U) = q\gamma NB_0[-v^{\hat{\phi}} e_{\hat{r}} + v^{\hat{r}} e_{\hat{\phi}}] = m\zeta_0\gamma N[-v^{\hat{\phi}} e_{\hat{r}} + v^{\hat{r}} e_{\hat{\phi}}].$$

$$v_0 = \frac{r_0|\zeta_0|}{\sqrt{1 + r_0^2\zeta_0^2}}$$

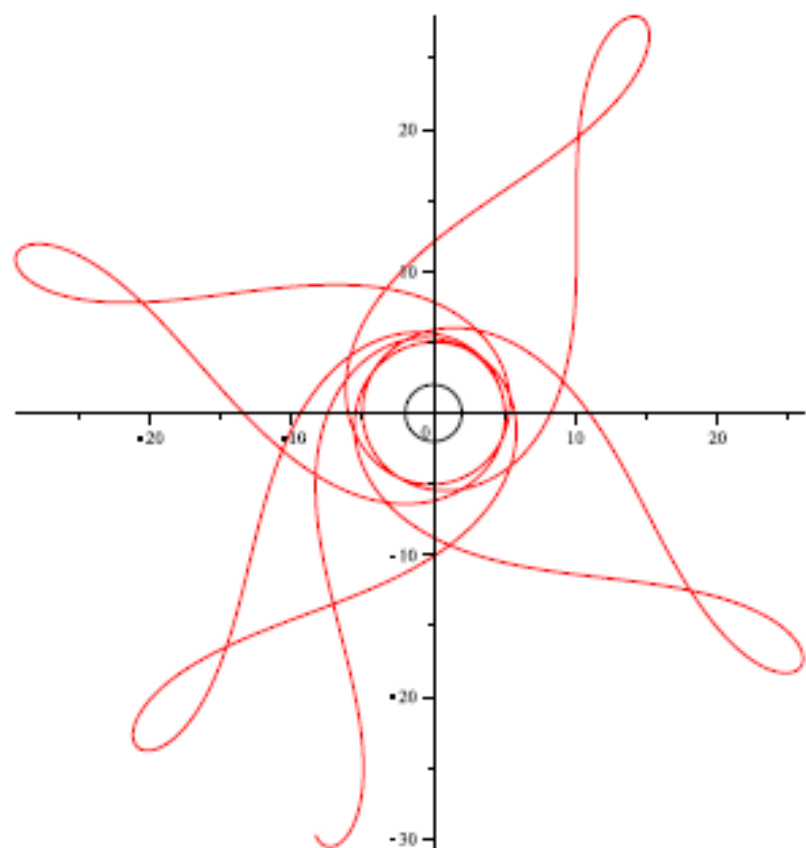
Due to the presence of external fields one may find equilibrium conditions for balance of the gravitational and non-gravitational actions.

→ Compare with PR effect

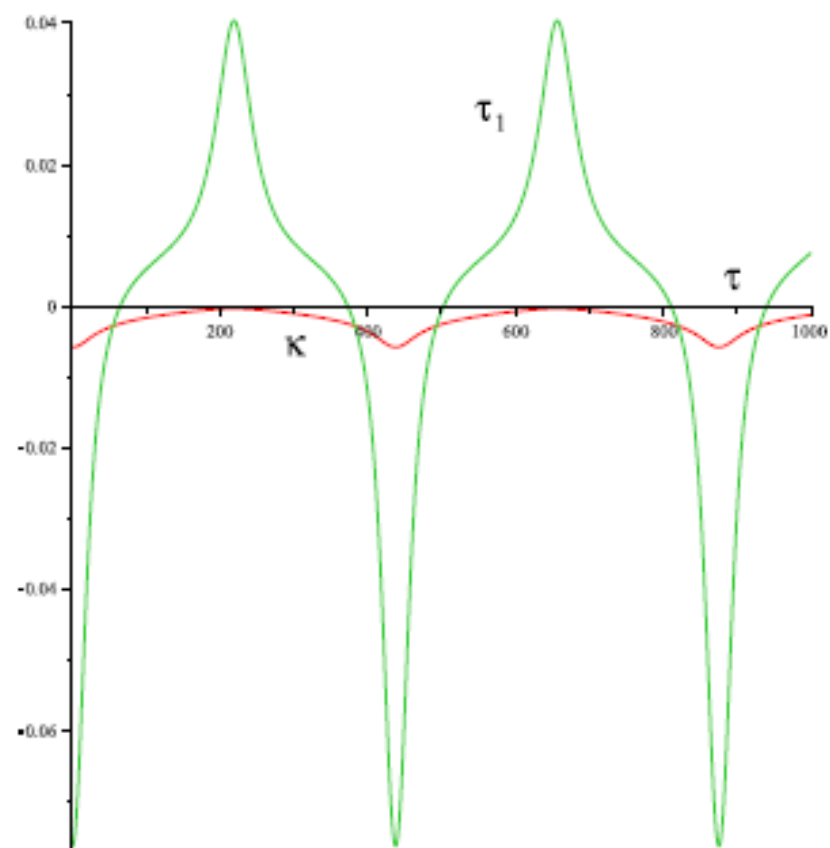
→ Equivalence principle, uncertainty of the measurements



Charged particle in an external magnetic field. The equilibrium azimuthal velocity $v^{\hat{\phi}}$ is plotted as a function of r_0/M for fixed values of $M\zeta_0 = [0 \text{ (black)}, -0.1 \text{ (red)}, -0.5 \text{ (blue)}, -1 \text{ (green)}, -5 \text{ (brown)}]$. The corresponding equilibrium orbits are stable outside the shaded region. For every fixed value of the equilibrium radius r_0/M , there exist in general two values of the azimuthal velocity corresponding to co-rotating and counter-rotating orbits.

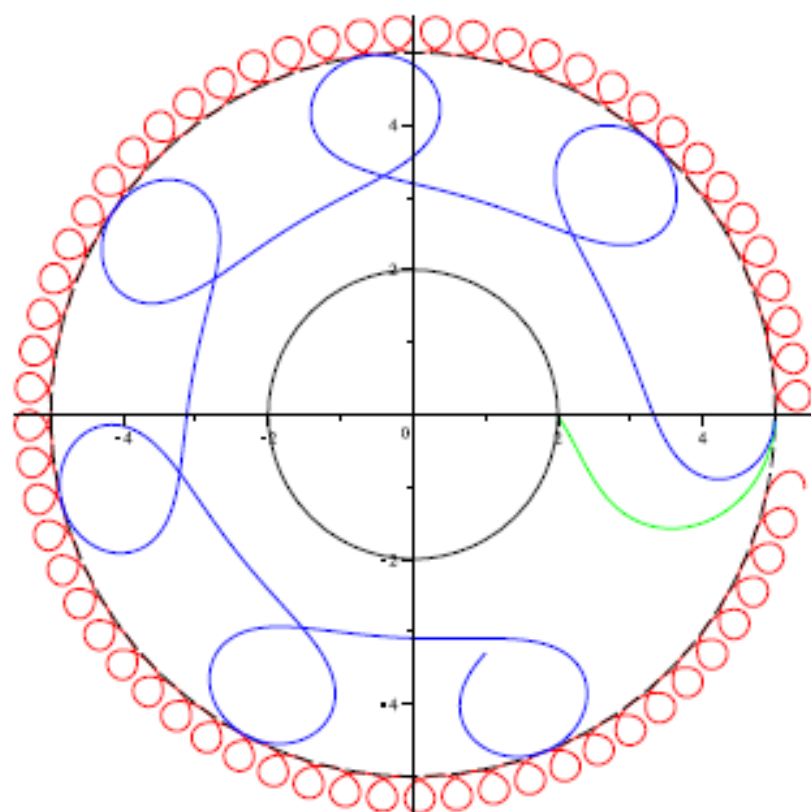


(a)

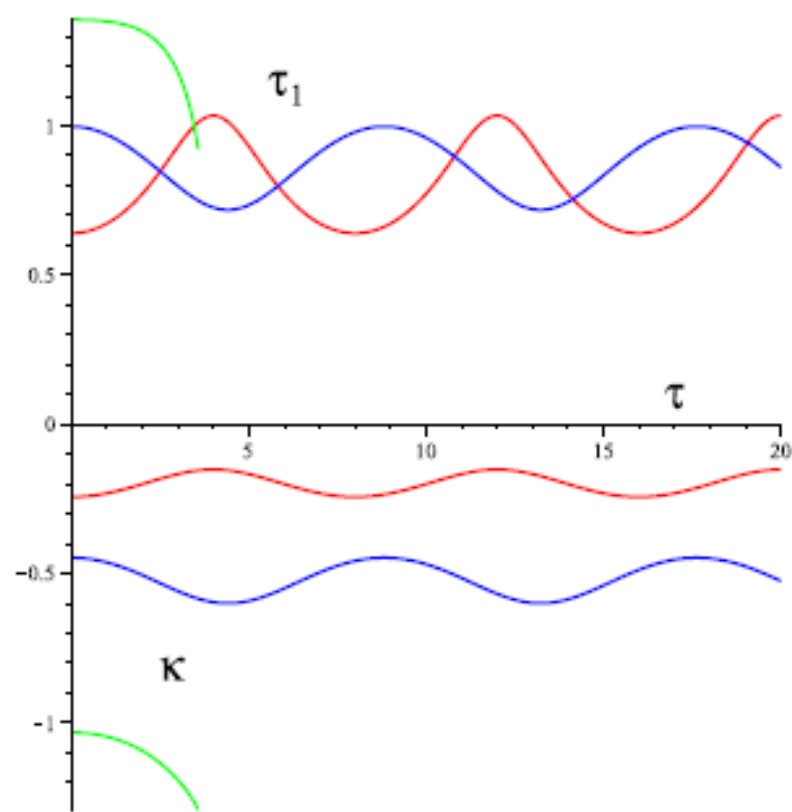


(b)

Charged particle in an external magnetic field. (a) The orbit corresponding to $M\zeta_0 = -0.01$ for the choice of initial conditions $r(0) = 5M$, $\phi(0) = 0$, $v^{\hat{r}}(0) = 0$ and $v^{\hat{\phi}}(0) = 0.6$. The equilibrium values of the azimuthal velocity are $v^{\hat{\phi}}(0) \approx [-0.598, 0.557]$. (b) The corresponding behaviors of the signed magnitude κ and the first torsion τ_1 as functions of the proper time τ .



(a)



(b)

Charged particle in an external magnetic field. (a) The orbit corresponding to $M\zeta_0 = -1$ for the following choices of initial conditions: $r(0) = 5M$, $\phi(0) = 0$, $v^{\hat{r}}(0) = 0$ and $v^{\hat{\phi}}(0) = [-0.8$ (green), -0.5 (blue), 0.066 (black), 0.3 (red)]. For the selected values of $r(0)/M$ and $M\zeta_0$, the equilibrium values of the azimuthal velocity are $v^{\hat{\phi}}(0) \approx [-0.991, 0.066]$. (b) The behaviors of the signed magnitude κ and the first torsion τ_1 as functions of the proper time τ for the same choice of parameters and initial conditions as in (a). The critical values corresponding to equilibrium are $\kappa \approx -0.051$ and $\tau_1 \approx -0.007$.

Particles with a magnetic dipole moment

$$f_{(\text{dip})}(U)^\sigma = \eta^{\alpha\beta\gamma\delta} U_\alpha \mu_\beta \nabla^\sigma F_{\gamma\delta} \quad \mu^\alpha = -(\mu/r)\delta_\theta^\alpha$$

$$f_{(\text{dip})}(U) = \frac{2\mu M\gamma B_0}{r^2} (v^{\hat{r}} n + e_{\hat{r}})$$

Superposed magnetic field and motion of a spinning particle also endowed with a magnetic dipole

$$f_{(\text{ds})}(U) = f_{(\text{dip})}(U) + f_{(\text{spin})}(U) = \left(2\mu B_0 - 3\frac{s\gamma v^{\hat{\phi}}}{r} \right) \frac{M\gamma}{r^2} (v^{\hat{r}} n + e_{\hat{r}})$$

5. Analogies between different kinds of situations

Consider the equilibrium circular orbit associated with different kinds of particles as discussed in section 4.

(i) Particles with charge q in an external magnetic field:

$$\gamma(v^{\hat{\phi}2} - v_K^2) = r_0 \zeta_0 v^{\hat{\phi}}, \quad \zeta_0 = qB_0/m. \quad (5.1)$$

(ii) Particles with a magnetic dipole in an external magnetic field:

$$\gamma(v^{\hat{\phi}2} - v_K^2) = -\beta \frac{M}{r_0 N}, \quad \beta = 2\mu B_0/m. \quad (5.2)$$

(iii) Particles with spin in the background geometry:

$$(v^{\hat{\phi}2} - v_K^2) = \frac{3M^2}{r_0^2 N} \hat{s} v^{\hat{\phi}}, \quad \hat{s} = s/(mM). \quad (5.3)$$

(iv) Neutral particles in a given radiation field:

$$N\gamma^3 \left(1 - \frac{v^{\hat{\phi}2}}{v_K^2} \right) = \text{sgn}[\sin \beta_0] \frac{A}{M}, \quad A = \sigma \Phi_0^2 \mathcal{E}^2 / m. \quad (5.4)$$

In all these cases (as well as in cases which are combinations of these), which originate in different contexts, deviations from circular geodesic motion are given by

$$v^{\hat{\phi}} = \pm v_K + \Delta v, \quad (5.5)$$

$$\Delta v_{\xi_0} = \pm \frac{r_0 \xi_0}{2\gamma_K}, \quad \Delta v_{\hat{s}} = \pm \frac{3}{2} \left(\frac{M}{r_0}\right)^{3/2} v_K \hat{s},$$

$$\Delta v_{\mu} = -\frac{\beta}{2\gamma_K} \left(\frac{M}{r_0}\right)^{1/2}, \quad \Delta v_A = \frac{1}{2\gamma_K^3} \left(\frac{M}{r_0}\right)^{1/2} \text{sgn}[\sin \beta_0] \frac{A}{M}.$$

Uncertainty of the measurements

- (i) One cannot distinguish between a particle with a magnetic dipole μ moving on a mean radius r_0 and a one with electric charge $|q| = 2\mu M^{1/2} r_0^{-3/2}$, for any mass m and a magnetic (test) field B_0 .
- (ii) One cannot distinguish between a neutral particle moving on a mean radius r_0 with spin s and a particle having a magnetic dipole μ in a magnetic field B_0 with $|\mu B_0| = (3/2)v_K \gamma_K (s/r_0)$, for any mass m .
- (iii) One cannot distinguish between a spinning particle with spin s on a mean radius r_0 and a charged particle in a magnetic field B_0 with $|qB_0| = 3M^{1/2} r_0^{-5/2} \gamma_K v_K s$. The latter case is complementary to the previous ones. It is then clear that a measurement of the correction to any given geodesic property is not sufficient by itself alone to identify the structure of the particle under consideration. Only combined measurements of different kinds can overcome this ambiguity.
- (iv) One cannot distinguish between a spinning particle with spin s also endowed with a magnetic dipole moment (e.g. a pulsar) and neutral non-spinning and not magnetized geodesic particle.

Effects of friction forces on the motion of objects in smoothly matched interior/exterior spacetimes

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and Sauro Succi^{1,3}

Friction forces in the presence of matter or null fields

PR friction force extended to matter fields

Friction forces in a matter field

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + pg_{\mu\nu}$$

$$f_{(\text{fric})}(U)^{\alpha} = -\sigma P(U)^{\alpha}_{\beta} T^{\beta\mu} U_{\mu}$$

$$\bar{U}^{\alpha} = \gamma(U, u)[\|v(U, u)\|u^{\alpha} + \hat{v}(U, u)^{\alpha}]$$

$$f_{(\text{fric})}(U)^{\alpha} = -\sigma(\rho + p)\gamma(U, u)^2\|v(U, u)\|\bar{U}^{\alpha}$$

Stokes Law and Poiseuille's Law

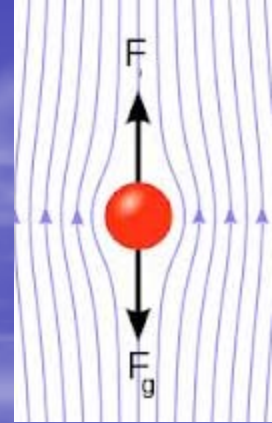
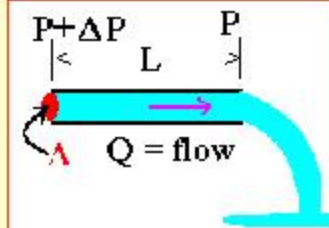
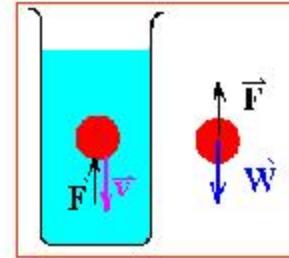
Consequences of Viscosity

- Stokes' law gives viscous retarding force to motion of a sphere through a fluid:

$$F = 6\pi R v \eta$$

- Poiseuille's Law gives volume flow rate of a fluid through a pipe (Q in m³/sec).

$$Q = \pi R^4 \Delta P / (8 \eta L)$$



$$f_{(\text{Stokes})} = 6\pi R v \rho V$$

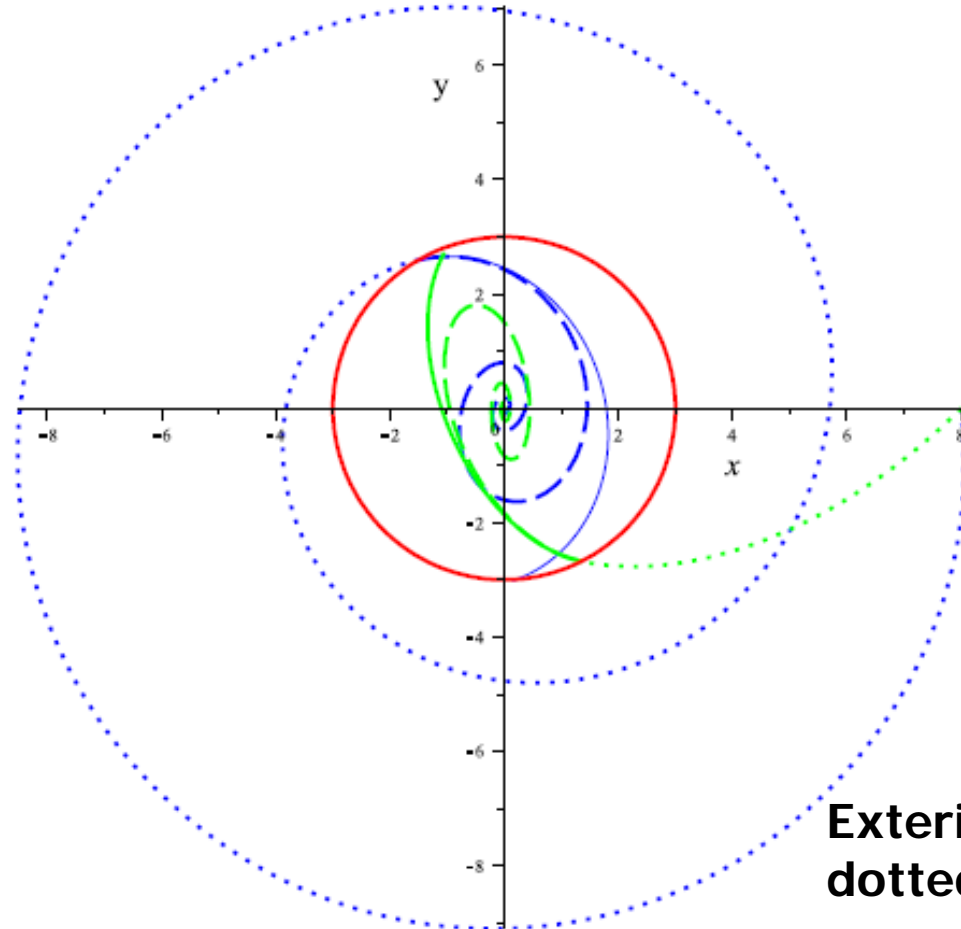


η

$$f_{(\text{fric})} = \sigma c \gamma^2 \left(\rho + \frac{p}{c^2} \right) \frac{V}{c} \sim \sigma \gamma^2 (\rho + p) V$$

We immediately recognize in equation (2.15) the same structure as in equation (2.14), with the 'expected' relativistic corrections: (i) the γ factor (which reduces to 1 at small speed) and (ii) the group $(\rho + p/c^2)V$ replacing the classical one (ρV), since in the relativistic regime the pressure becomes comparable with the mass–energy density (as familiar from the equation of relativistic hydrostatic equilibrium, discussed in many textbooks). As a consequence, the identification $6\pi R v \leftrightarrow \sigma c$ follows.

→ Examples of particle interaction with a massive gas

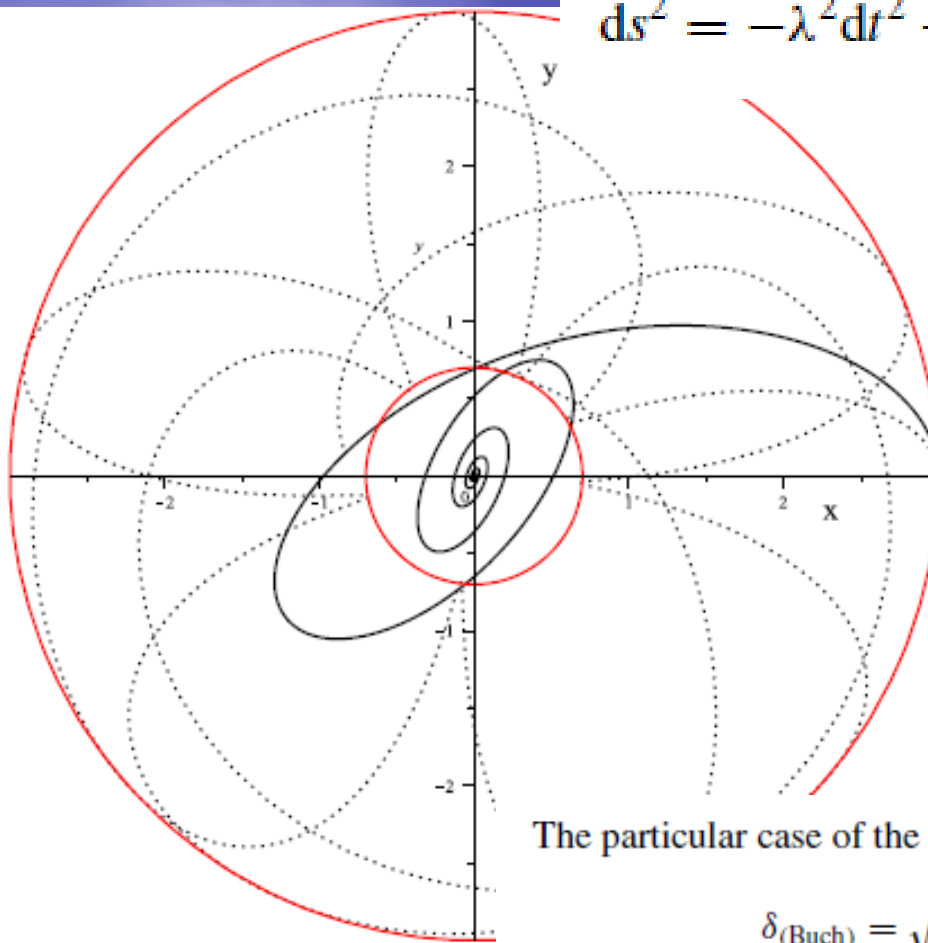


Schwarzschild interior and exterior solution matched

Exterior (geodesic) motion:
dotted

Schwarzschild interior and exterior matched: orbits. The plot shows the motion of a massive particle coming from $r_0 = 8M$ and $\phi_0 = 0$, with initial velocity $v_0 = 0.4$ and different values of initial inclination α_0 . The particle follows a geodesic motion in the exterior region and a non-geodesic, or scattered motion, in the interior region. In particular, the exterior motion is represented with a dotted line, while the interior with a dashed or a continuous line, for the scattered or geodesic, respectively. The boundary of the gas cloud is drawn as a circle. The orbits are presented for different initial inclinations, corresponding to $\alpha_0 = 3$ and $\alpha_0 = 4$. It is observed that the test particle subject to these initial conditions can exit from the region where the gas is present, only in the geodesic case. Otherwise, it reaches the center of the configuration in $r = 0$, and 'sits' there forever.

$$ds^2 = -\lambda^2 dt^2 + \Lambda^2 (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)$$



$$\lambda \equiv \lambda(r) = A \frac{1 - k\delta}{1 + k\delta}$$

$$\Lambda \equiv \Lambda(r) = \frac{(1 + k\delta)^2}{1 + \frac{r^2}{a^2}}$$

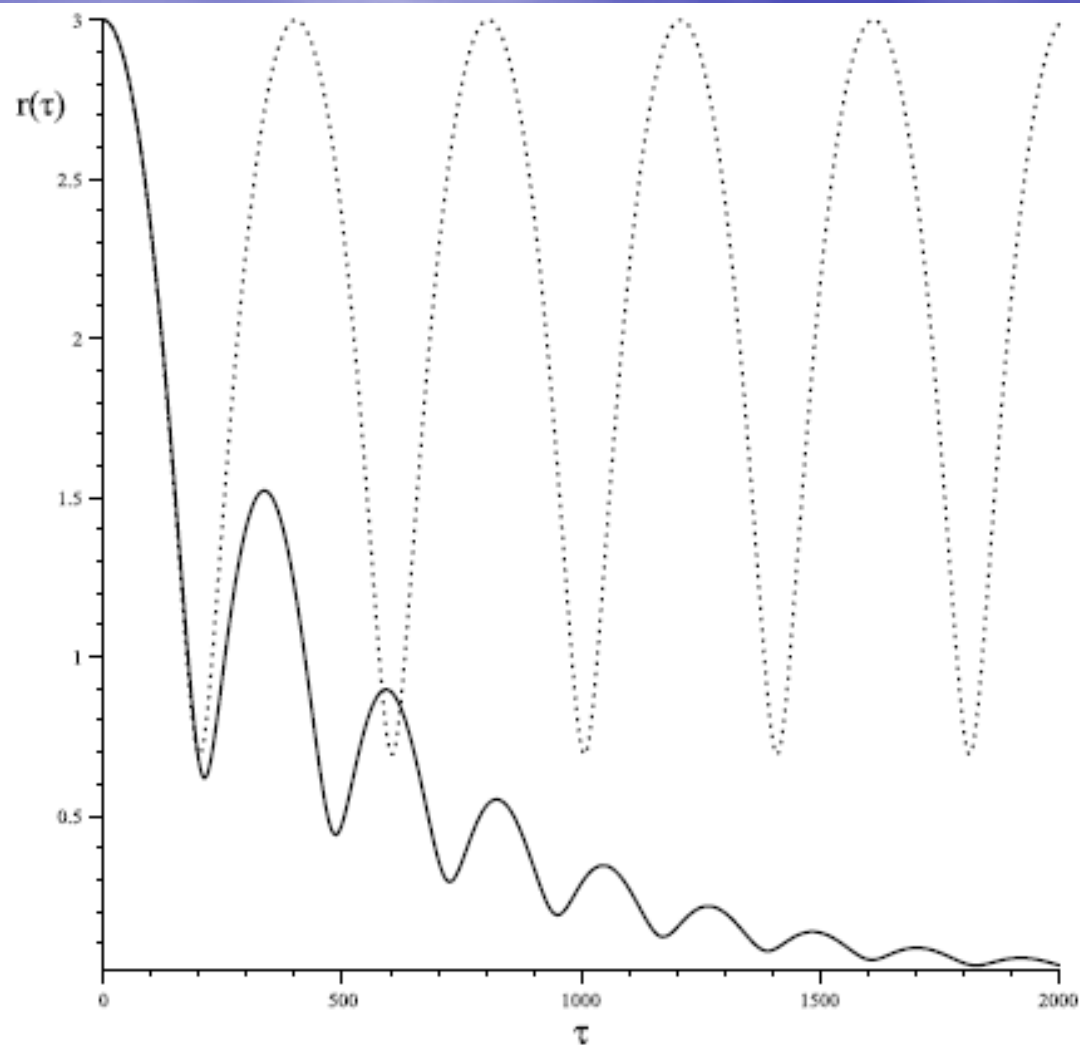
$$\delta \equiv \delta(r) = \sqrt{\frac{1 + \frac{r^2}{a^2}}{1 + b\frac{r^2}{a^2}}}$$

The particular case of the Buchdahl metric is obtained in the limit $b = 0$, i.e.

$$\delta_{(\text{Buch})} = \sqrt{1 + \frac{r^2}{a^2}}$$

Unbounded Buchdahl metric: orbits. The plot shows the comparison between geodesic (dotted) and scattered motion in the unbounded Buchdahl space, for the initial conditions $v_0 = 0.2$, $r_0 = 3M$, $\phi_0 = 0$ and $\alpha_0 = 0$. In particular, the particle undergoing scattered motion reaches the center of the configuration, while the one moving along a geodesic exhibits a periodic motion between two specific values of the radial coordinate (solid circles). Here $\tilde{\sigma}/M = 10^2$.

Pant-Sah and Buchdal metrics: examples of spacetimes admitting a perfect fluid source



Unbounded Buchdahl metric: radial coordinate. The plot shows the radial coordinate as a function of the proper time, for the case of the unbounded Buchdahl space, for both geodesic (dotted) and scattered motion. This conveys a dynamic picture of the situation shown in the previous figure. Here $\tilde{\sigma}/M = 10^2$.

Conclusions

- PR effect has its natural generalization in GR
- The main feature of the GR discussion concerns the existence of equilibrium solutions (stable or unstable) which may play a role in the formation of certain structures, like accretion disks
- The discussion of PR effect has contributed to the study of general friction forces in GR

Comparison with observed data never is a easy task

- (Small) deviations from theoretical previsions can be explained by assuming the existence of (small) friction forces (caveat: the equivalence principle).

Further developments:

- More realistic models for photon emission by finite-size or point-like sources
- Applications in cosmology

The general relativistic Poynting-Robertson effect. The case of a radiating ring

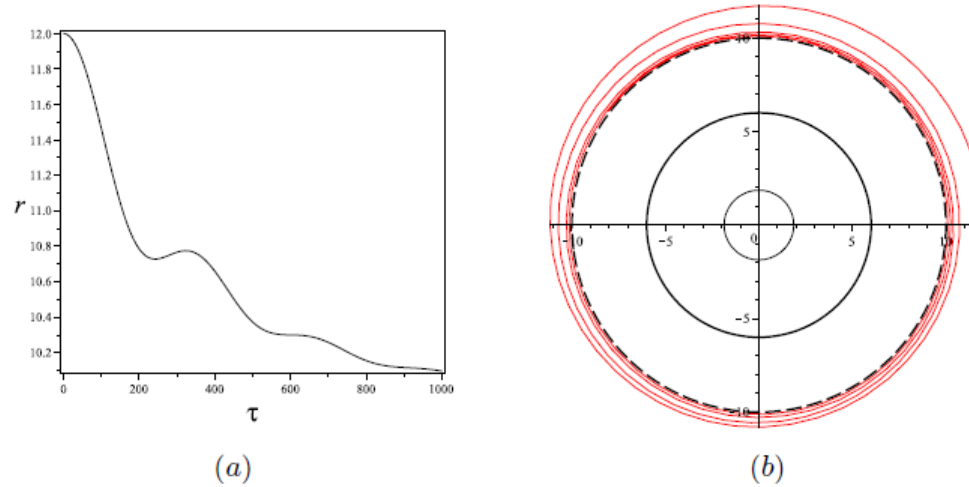


FIG. 1: [Exact $T^{\mu\nu}$, co-rotating emitter] The behavior r as a function of τ is shown for the choice of parameters $a/M = 0.5$, $A/M \approx 2.62$, $r_e/M = 6$ and initial conditions $r(0)/M = 12$, $\alpha(0) = 0 = \phi(0)$, $\nu(0) = 0.25$. The specific intensity turns out to be $I_e \approx 1.35 \times 10^{-3} Am/\sigma$. The critical orbit at $r_0/M = 10$ with $\nu_0^{\hat{\phi}} \approx 0.28$ is reached after few revolutions.

$$I_o = g^4 I_e$$

$$T^{\hat{t}\hat{t}} = \int I_o d\hat{\Omega}, \quad T^{\hat{t}\hat{a}} = \int I_o \hat{\nu}^{\hat{a}} d\hat{\Omega}, \quad T^{\hat{a}\hat{b}} = \int I_o \hat{\nu}^{\hat{a}} \hat{\nu}^{\hat{b}} d\hat{\Omega}$$

$$T^{\alpha\beta} \sim \sum \Phi^2 K^\alpha K^\beta$$

Bibliography (essential)

Poynting J H 1903 *Phil. Trans. R. Soc.* **203** 525

Robertson H P 1937 *Mon. Not. R. Astron. Soc.* **97** 423

Wyatt S P and Whipple F L 1950 *Astrophys. J.* **111** 134 (misprint in equation (2): $-a\theta/r \rightarrow -a\dot{\theta}/r^2$)

Kimura H, Okamoto H and Mukai T 2002 *Icarus* **157** 349

Srikanth R 1999 *Icarus* **140** 231

Eddington A S 1926 *The Internal Constitution of Stars* (Cambridge: Cambridge University Press)

Guess A W 1962 *Astrophys. J.* **135** 855

Abramowicz M A, Ellis G F R and Lanza A 1990 *Astrophys. J.* **361** 470

Lamb F K and Miller M C 1995 *Astrophys. J.* **439** 828

Miller M C and Lamb F K 1996 *Astrophys. J.* **470** 1033

Miller M C and Lamb F K 1993 *Astrophys. J.* **413** L43

Stephani H, Kramer D, MacCallum M A H, Hoenselaers C and Herlt E 2003 *Exact Solutions of Einstein's Field Equations* 2nd edn (Cambridge: Cambridge University Press)

Bini D, Carini P and Jantzen R T 1997 *Int. J. Mod. Phys. D* **6** 1

Bini D, Carini P and Jantzen R T 1997 *Int. J. Mod. Phys. D* **6** 143

Bini D, de Felice F and Jantzen R T 1999 *Class. Quantum Grav.* **16** 2105

Jantzen R T, Carini P and Bini D 1992 *Ann. Phys., NY* **215** 1

Merloni A, Vietri M, Stella L and Bini D 1999 *Mon. Not. R. Astron. Soc.* **304** 155

Ohanian Hans C and Ruffini R 1994 *Gravitation and Spacetime* 2nd edn (New York: W W Norton)

Thanks for your kind attention!