Hamiltonian analytic treatment of spinning compact binaries in general relativity

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Outline

- Some History on Hamiltonian General Relativity
- Hamiltonian Setting of General Relativity (A)
- Binary Black Hole Spacetimes
- Hamiltonian Setting of General Relativity (B)
- Higher-Order-PN Dynamical Systems
- Spin and Gravity

higher order spin dynamics in collaboration with:
Damour, Hartung, Hergt, Jaranowski, Steinhoff, Wang, Zeng

sometimes: $c = 1$, $G = 1$

Some History on Hamiltonian General Relativity

Dirac 1958-1959

1978 Nelson/Teitelboim: Dirac field

2009 Barausse/Racine/Buonanno: spinning test particles in “Kerr”

Arnowitt/Deser/Misner 1959-1960

1961 Kimura: 1PN

1974 Ohta/Okamura/Kimura/Hiida: 2PN (in part)

1985 GS: 2.5PN; Damour/GS: 2PN

2001 Damour/Jaranowski/GS: 3PN

2009 Steinhoff/GS: self-gravitating spinning particles

2013 Jaranowski/GS: 4PN (in part)

Schwinger 1963

1963 Kibble: Dirac field

Refinements

DeWitt 1967; Regge/Teitelboim 1974

Hamiltonian Setting of General Relativity (A)

stress-energy tensor of ideal fluid:

$$T^{\mu}_{\nu} = (\varrho c^2 + \varrho\epsilon + p)u_\nu u^\mu + p\delta^\mu_\nu, \quad u_\nu u^\nu = -1, \quad u_\nu = g_{\nu\mu}u^\mu$$

$$\epsilon = \epsilon(\varrho, s), \qquad d\epsilon = \frac{p}{\varrho^2}d\varrho + Tds$$

canonical variables:

$$\varrho_* = \sqrt{-g}u^0\varrho, \qquad s, \qquad \pi_i = \frac{1}{c}\sqrt{-g}T^0_i$$

Lie-Poisson brackets:

$$\{\pi_i(\mathbf{x},t), \varrho_*(\mathbf{x}',t)\} = \frac{\partial}{\partial x'^i}[\varrho_*(\mathbf{x}',t)\delta(\mathbf{x}-\mathbf{x}')]$$

$$\{\pi_i(\mathbf{x},t), s(\mathbf{x}',t)\} = \frac{\partial s(\mathbf{x}',t)}{\partial x'^i}\delta(\mathbf{x}-\mathbf{x}')$$

$$\{\pi_i(\mathbf{x},t), \pi_j(\mathbf{x}',t)\} = \pi_i(\mathbf{x}',t)\frac{\partial}{\partial x'^j}\delta(\mathbf{x}-\mathbf{x}') - \pi_j(\mathbf{x},t)\frac{\partial}{\partial x^i}\delta(\mathbf{x}-\mathbf{x}')$$

$$\frac{\partial \varrho_*}{\partial t} = -\partial_i \left(\frac{\delta H}{\delta \pi_i} \varrho_* \right) \quad \Longleftrightarrow \quad \partial_\mu (\sqrt{-g} \varrho u^\mu) = 0$$

$$\frac{\partial s}{\partial t} = -\frac{\delta H}{\delta \pi_i} \partial_i s \qquad \Longleftrightarrow \qquad u^\mu \partial_\mu s = 0$$

$$\frac{\partial \pi_i}{\partial t} = -\partial_j \left(\frac{\delta H}{\delta \pi_j} \pi_i \right) - \partial_i \left(\frac{\delta H}{\delta \pi_j} \right) \pi_j - \partial_i \left(\frac{\delta H}{\delta \varrho_*} \right) \varrho_* + \frac{\delta H}{\delta s} \partial_i s$$

$$\Longleftrightarrow \quad \nabla_\mu \left(\sqrt{-g} T^\mu_i \right) = 0$$

$$\frac{\partial A}{\partial t} = \{A, H\}, \qquad v^i = \frac{\delta H}{\delta \pi_i}, \qquad v^i = c \frac{u^i}{u^0}$$

linear momentum and angular momentum:

$$P_i = \int d^3x \ \pi_i, \qquad J_i = \int d^3x \ \epsilon_{ijk} x^j \pi_k$$

$$\epsilon = p = s = 0 \quad \text{(dusty matter)}$$

point particles:

$$\varrho_* = \sum_a m_a \delta(\mathbf{x} - \mathbf{x}_a), \qquad \pi_i = \sum_a p_{ai} \delta(\mathbf{x} - \mathbf{x}_a), \qquad v_a^i = \frac{dx_a^i}{dt}$$

$$\{x_a^i, p_{aj}\} = \delta_{ij}, \qquad \text{zero otherwise}$$

$$\frac{dp_{ai}}{dt} = -\frac{\partial H}{\partial x_a^i}, \qquad \frac{dx_a^i}{dt} = \frac{\partial H}{\partial p_{ai}}$$

Poincaré algebra

$\{P_i, \mathrm{H}\} = \{J_i, H\} = 0$

$\{J_i, P_j\} = \varepsilon_{ijk}\, P_k\ ,\ \ \{J_i, J_j\} = \varepsilon_{ijk}\, J_k$

$\{J_i, G_j\} = \varepsilon_{ijk}\, G_k$

$\{G_i, H\} = P_i$

$\{G_i, P_j\} = \frac{1}{c^2}\, H\, \delta_{ij}$

$\{G_i, G_j\} = -\frac{1}{c^2}\, \varepsilon_{ijk}\, J_k$

Lorentz boost vector:

$K_i = -t\, P_i + G_i$

$\frac{dK_i}{dt} = \frac{\partial K_i}{\partial t} + \{K_i, H\} = -P_i + \{G_i, H\} = 0$

canonical variables for non-interacting particle

total angular momentum: $\mathbf{J} = \hat{\mathbf{X}} \times \mathbf{P} + \hat{\mathbf{S}}$

Hamiltonian: $H = \sqrt{m^2 + \mathbf{P}^2}$

Lorentz boost: $\mathbf{K} = -\mathrm{t}\mathbf{P} + H\hat{\mathbf{X}} - \frac{1}{H+m}\hat{\mathbf{S}} \times \mathbf{P}$

center-of-energy: $\bar{\mathbf{X}} = \hat{\mathbf{X}} - \frac{1}{(H+m)H}\,\hat{\mathbf{S}} \times \mathbf{P}$

$$\mathbf{K} = -\mathrm{t}\mathbf{P} + H\bar{\mathbf{X}}, \qquad \mathbf{G} = H\bar{\mathbf{X}}$$

center-of-spin: $\hat{\mathbf{X}}$; $\{\hat{X}^i, \hat{X}^j\} = 0$ (Newton-Wigner coordinates)

center-of-energy: $\bar{\mathbf{X}} = \hat{\mathbf{X}} - \frac{1}{(H+m)H}\, \hat{\mathbf{S}} \times \mathbf{P}$

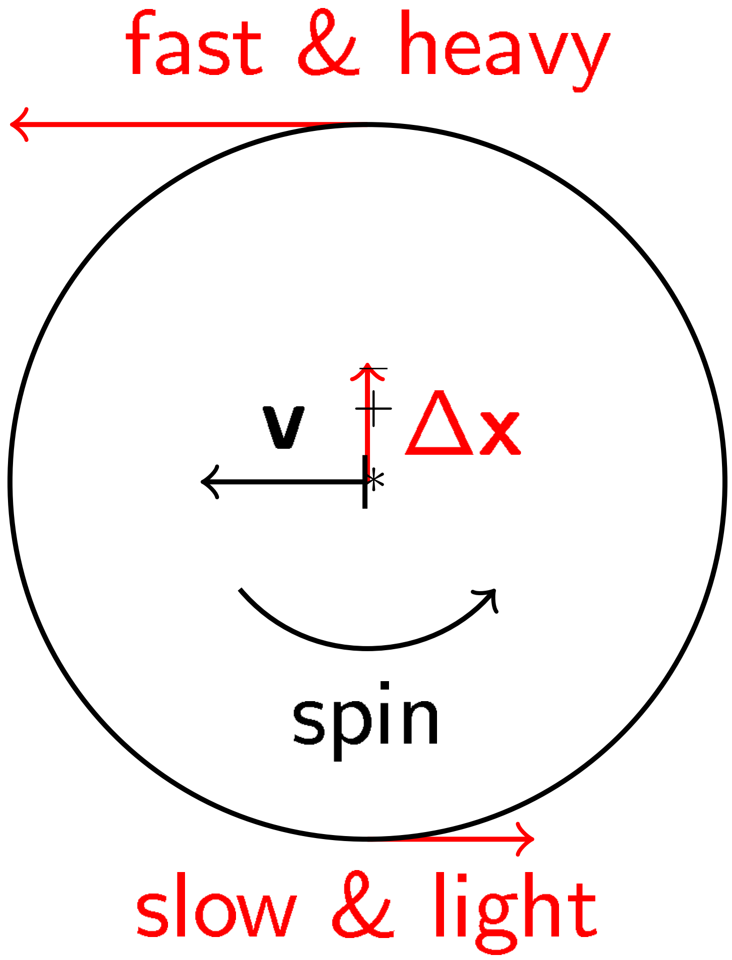
center-of-inertia: $\mathbf{X} = \hat{\mathbf{X}} + \frac{1}{(H+m)m}\, \hat{\mathbf{S}} \times \mathbf{P}$

related spin supplementary conditions:

center-of-inertia: $S^{\mu\nu} P_\nu = 0$

center-of-energy: $\bar{S}^{\mu\nu} n_\nu = 0, \qquad n_\mu = (-1, 0, 0, 0)$

center-of-spin: $m\hat{S}^{\mu\nu} n_\nu + \hat{S}^{\mu\nu} P_\nu = 0$



various centers: $\mathbf{X}(*)$, $\bar{\mathbf{X}}(-)$, $\hat{\mathbf{X}}(+)$

many particle systems with interaction (no radiation)

$$\mathbf{P} = \sum_a \mathbf{p}_a$$

$$\mathbf{J} = \sum_a (\mathbf{r}_a \times \mathbf{p}_a + \mathbf{s}_a)$$

$$\mathcal{M}^2 \equiv H^2 - \mathbf{P}^2, \qquad H = \sqrt{\mathcal{M}^2 + \mathbf{P}^2}$$

$$\mathbf{G} = H\hat{\mathbf{X}} - \frac{1}{H + \mathcal{M}}(\mathbf{J} - \hat{\mathbf{X}} \times \mathbf{P}) \times \mathbf{P}$$

$$\{\hat{X}^i, \hat{X}^j\} = \{P^i, P^j\} = 0, \quad \{\hat{X}^i, P^j\} = \delta^{ij}$$

$$\{\mathcal{M}, \hat{X}^j\} = \{\mathcal{M}, P^j\} = \{\mathcal{M}, H\} = 0$$

Binary Black Hole Spacetimes

isolated BH

$$ds^2 = -\left(\frac{1-\frac{Gm}{2rc^2}}{1+\frac{Gm}{2rc^2}}\right)^2 c^2dt^2 + \left(1+\frac{Gm}{2rc^2}\right)^4 \delta_{ij}dx^i dx^j$$

$$= -\left(\frac{1-\frac{Gm}{2Rc^2}}{1+\frac{Gm}{2Rc^2}}\right)^2 c^2dt^2 + \left(1+\frac{Gm}{2Rc^2}\right)^4 \delta_{ij}dX^i dX^j$$

symmetry transformation (inversion): $Rr = \left(\frac{Gm}{2c^2}\right)^2$

$R^2 = X^i X^i, \quad r^2 = x^i x^i$

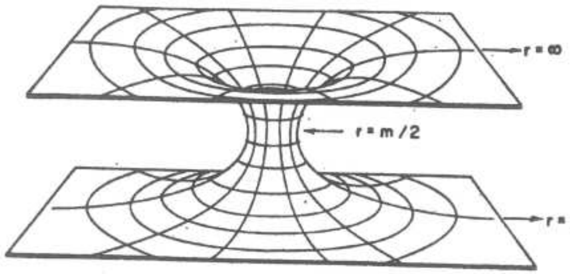


FIG. 1. A two-dimensional analog of the Schwarzschild-Kruskal manifold is shown isometrically imbedded in flat three-space. The figure shows the curvature and topology of the metric

$$ds^2 = (1 + m/2r)^4 (dr^2 + r^2 d\theta^2).$$

The sheets at the top and bottom of the funnel continue to infinity and represent the asymptotically flat regions of the manifold ($r \to 0$, $r \to \infty$).

Brill/Lindquist, JMP 1963

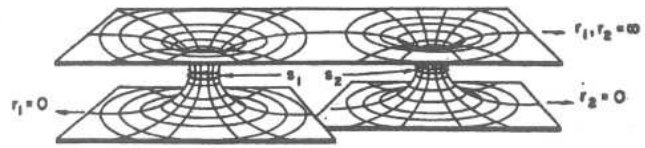


FIG. 2. A two-dimensional analog of the hypersurface of time symmetry of a manifold containing two "throats" is shown isometrically imbedded in flat three-space. The figure illustrates the curvature and topology for a system of two "particles" of equal mass m, and separation large compared to m, described by the metric

$$ds^2 = (1 + m/2r_1 + m/2r_2)^4 ds_F^2.$$

Brill-Lindquist BHs

initial-value metric

$$ds^2 = -\left(\frac{1-\frac{\beta_1 G}{2r_1c^2}-\frac{\beta_2 G}{2r_2c^2}}{1+\frac{\alpha_1 G}{2r_1c^2}+\frac{\alpha_2 G}{2r_2c^2}}\right)^2 c^2dt^2 + \left(1+\frac{\alpha_1 G}{2r_1c^2}+\frac{\alpha_2 G}{2r_2c^2}\right)^4 d\mathbf{x}^2$$

total energy: $$\boxed{E_{ADM} = -\frac{c^4}{2\pi G}\oint_{i_0} ds_i\partial_i\Psi = (\alpha_1+\alpha_2)c^2}$$

$\Psi = 1 + \frac{\alpha_1 G}{2r_1c^2} + \frac{\alpha_2 G}{2r_2c^2}$

inversion map of the three-metric at the throat of black hole 1

$$r_1' r_1 = \left(\frac{\alpha_1 G}{2c^2}\right)^2, \qquad r_1' = |\mathbf{x}'-\mathbf{x}_1|, \qquad r_1 = |\mathbf{x}-\mathbf{x}_1|$$

$$
\begin{aligned}
dl^2 &= \Psi^4 d\mathbf{x}^2 = \left(1 + \frac{\alpha_1 G}{2r_1c^2} + \frac{\alpha_2 G}{2r_2c^2}\right)^4 d\mathbf{x}^2 \\
&= \Psi'^4 d\mathbf{x}'^2 = \left(1 + \frac{\alpha_1 G}{2r_1'c^2} + \frac{\alpha_1\alpha_2 G^2}{4r_2r_1'c^4}\right)^4 d\mathbf{x}'^2
\end{aligned}
$$

$$
\mathbf{r}_2 = \frac{\alpha_1^2 G^2}{4c^4}\frac{\mathbf{r}_1'}{r_1'^2} + \mathbf{r}_{12}, \qquad \mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2
$$

$$
\boxed{m_1 \equiv -\frac{c^2}{2\pi G}\oint_{i_{01}} ds_i'\partial_i'\Psi' = \alpha_1 + \frac{\alpha_1\alpha_2 G}{2r_{12}c^2}}
$$

$$
\Psi' = 1 + \frac{\alpha_1 G}{2r_1'c^2} + \frac{\alpha_1\alpha_2 G^2}{4r_2r_1'c^4}
$$

dynamical approach

$$g_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\rm TT}$$

$$\pi^{ii} = 0, \qquad \pi^{ij} = -\gamma^{1/2}(K^{ij} - \gamma^{ij}K), \quad \pi^i_i = \pi^{ij}h_{ij}^{\rm TT}$$

unique decomposition: $\pi^{ij} = \tilde{\pi}^{ij} + \pi_{\rm TT}^{ij}$

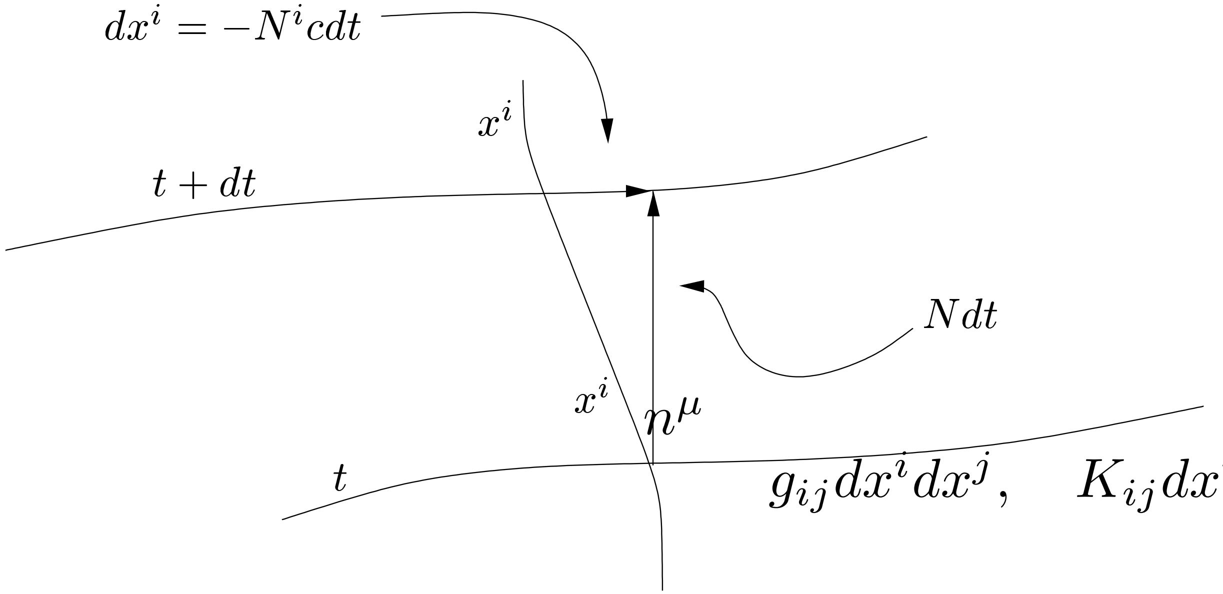
$$\tilde{\pi}^{ij} = \partial_i \pi^j + \partial_j \pi^i - \frac{2}{3}\delta_{ij}\partial_k \pi^k$$

$\pi_{\rm TT}^{ij} c^3/16\pi G$: canonical conjugate to $h_{ij}^{\rm TT}$

3+1 splitting of spacetime

$$n^\mu = (1, -N^i)/N$$

$$n_\mu = (-N, 0, 0, 0)$$



$$g_{ij} dx^i dx^j, \quad K_{ij} dx^i dx^j$$

$$K_{ij} = -N\Gamma^0_{ij} = -N g^{0\mu}(g_{i\mu,j} + g_{j\mu,i} - g_{ij,\mu})/2$$

$$\boxed{ds^2 = -(N c dt)^2 + g_{ij}(dx^i + N^i c dt)(dx^j + N^j c dt)}$$

Hamilton and momentum constraints

$$g^{1/2}R - \frac{1}{g^{1/2}}\left(\pi^i_j\pi^j_i - \frac{1}{2}\pi^i_i\pi^j_j\right) = \frac{16\pi G}{c^3}\sum_a \left(m_a^2c^2 + \gamma^{ij}p_{ai}p_{aj}\right)^{1/2}\delta_a$$

$$2\sqrt{-g}G^{\mu\nu}n_\mu n_\nu = \frac{16\pi G}{c^4}\sqrt{-g}T^{\mu\nu}n_\mu n_\nu$$

$$-2\partial_j\pi^j_i + \pi^{kl}\partial_i g_{kl} = \frac{16\pi G}{c^3}\sum_a p_{ai}\delta_a$$

$$2\sqrt{-g}G^\mu_i n_\mu = \frac{16\pi G}{c^4}\sqrt{-g}T^\mu_i n_\mu$$

$\mathcal{H} = 0$ (Hamilton constraint) and $\mathcal{H}_i = 0$ (momentum constraint)

$$-\left(1+\frac{1}{8}\ \phi\right)\Delta\phi=\frac{16\pi G}{c^2}\sum_a m_a\delta_a \qquad (h_{ij}^{\mathrm{TT}}=0=p_{ai})$$

$$\phi=\frac{4G}{c^2}\left(\frac{\alpha_1}{r_1}+\frac{\alpha_2}{r_2}\right)$$

$$\alpha_a=m_a-\frac{m_a+m_b}{2}+\frac{c^2r_{ab}}{G}\left(\sqrt{1+\frac{m_a+m_b}{c^2r_{ab}/G}+\left(\frac{m_a-m_b}{2c^2r_{ab}/G}\right)^2}-1\right)$$

$$\boxed{H_{\mathrm{BL}}=(\alpha_1+\alpha_2)\ c^2=(m_1+m_2)\ c^2-G\ \tfrac{\alpha_1\alpha_2}{r_{12}}}$$

Metric in d-dimensional conformally flat space:

$$g_{ij} = \left(1 + \frac{1}{4}\frac{d-2}{d-1}\phi\right)^{\frac{4}{d-2}} \delta_{ij}$$

$$\phi = \frac{4G}{c^2}\frac{\Gamma(\frac{d-2}{2})}{\pi^{\frac{d-2}{2}}}\left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}}\right)$$

$$\Psi = 1 + \frac{1}{4}\frac{d-2}{d-1}\phi$$

$$-\Delta^{-1}\delta = \frac{\Gamma((d-2)/2)}{4\pi^{d/2}} r^{2-d}$$

$$\Psi = 1 + \frac{G(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}} \left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}} \right)$$

$$\left(1 + \frac{G(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}} \left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}} \right) \right) \alpha_1\delta_1 = m_1\delta_1$$

$1 < d < 2$

$$\left(1 + \frac{G(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}} \frac{\alpha_2}{r_{12}^{d-2}} \right) \alpha_1\delta_1 = m_1\delta_1$$

Hamiltonian Setting of General Relativity (B)

Independent field variables

3 CC: $g_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\rm TT}, \qquad [3g_{ij,j} - g_{jj,i}] = 0$

1 CC: $\pi^{ii} = 0, \quad \pi^{ij} = -\gamma^{1/2}(K^{ij} - \gamma^{ij}K), \qquad \pi_i^i = 2\gamma^{1/2}K = \pi^{ij}h_{ij}^{\rm TT}$

unique decomposition: $\pi^{ij} = \tilde{\pi}^{ij} + \pi_{\rm TT}^{ij}$

$$\tilde{\pi}^{ij} = \partial_i \pi^j + \partial_j \pi^i - \frac{2}{3}\delta_{ij}\partial_k \pi^k$$

$\pi_{\rm TT}^{ij} c^3/16\pi G$: canonical conjugate to $h_{ij}^{\rm TT}$

$$g^{1/2}R = \frac{1}{g^{1/2}}\left(\pi^i_j\pi^j_i - \frac{1}{2}\pi^i_i\pi^j_j\right) + \frac{16\pi G}{c^3}\sum_a\left(m_a^2c^2 + \gamma^{ij}p_{ai}p_{aj}\right)^{1/2}\delta_a$$

$(G^{00} = \frac{8\pi G}{c^4}T^{00})$

$$-2\partial_j\pi^j_i + \pi^{kl}\partial_i g_{kl} = \frac{16\pi G}{c^3}\sum_a p_{ai}\delta_a \qquad (G^0_i = \frac{8\pi G}{c^4}T^0_i)$$

ADM Hamiltonian

$$H\left[x^i_a, p_{ai}, h^{\rm TT}_{ij}, \pi^{ij}_{\rm TT}\right] = -\frac{c^4}{16\pi G}\int d^3x\ \Delta\phi\left[x^i_a, p_{ai}, h^{\rm TT}_{ij}, \pi^{ij}_{\rm TT}\right]$$

Routh functional

$$R\left[x^i_a, p_{ai}, h^{\rm TT}_{ij}, \partial_t h^{\rm TT}_{ij}\right] = H - \frac{c^3}{16\pi G}\int d^3x\ \pi^{ij}_{\rm TT}\partial_t h^{\rm TT}_{ij}$$

$$\frac{\delta \int R(t')dt'}{\delta h_{ij}^{\rm TT}(x^k,t)} = 0, \quad \dot{p}_{ai} = -\frac{\partial R}{\partial x_a^i}, \quad \dot{x}_a^i = \frac{\partial R}{\partial p_{ai}}$$

on-field-shell Routh functional:

$$R_{\rm on}(t) = R\left[x_a^i, p_{ai}, h_{ij}^{\rm TT}[x_a^k, p_{ak}], \partial_t h_{ij}^{\rm TT}[x_a^k, p_{ak}]\right]$$

$$\dot{p}_{ai}(t) = -\frac{\delta \int R_{\rm on}(t')dt'}{\delta x_a^i(t)}, \qquad \dot{x}_a^i(t) = \frac{\delta \int R_{\rm on}(t')dt'}{\delta p_{ai}(t)}$$

$$\frac{\delta \int R_{\rm on}(t')dt'}{\delta z(t)} = \frac{\partial R_{\rm on}}{\partial z(t)} - \frac{d}{dt}\frac{\partial R_{\rm on}}{\partial \dot{z}(t)} + ..., \quad z = (x_a^i, p_{ai})$$

Post-Newtonian expansions

$$R\left[x_a^i, p_{ai}, h_{ij}^{\mathrm{TT}}, \partial_t h_{ij}^{\mathrm{TT}}\right] - Mc^2 = \sum_{n=0}^{\infty} \left(\frac{1}{c^2}\right)^n R_n\left[x_a^i, p_{ai}, \hat{h}_{ij}^{\mathrm{TT}}, \partial_t \hat{h}_{ij}^{\mathrm{TT}}\right]$$

$$h_{ij}^{\mathrm{TT}} = \tfrac{G}{c^4} \hat{h}_{ij}^{\mathrm{TT}}$$

$$\left(\Delta - \frac{\partial_t^2}{c^2}\right) h_{ij}^{\mathrm{TT}} = \frac{G}{c^4} \sum_{n=0}^{\infty} \left(\frac{1}{c^2}\right)^n D_{nij}^{\mathrm{TT}}[x, x_a(t), p_a(t), \hat{h}^{\mathrm{TT}}(t), \partial_t \hat{h}^{\mathrm{TT}}(t)]$$

Higher-Order-PN Dynamical Systems

4PN binary BH conservative dynamics

$$H(t) = m_1c^2 + m_2c^2 + H_N + \frac{1}{c^2}H_{[1PN]}$$

$$+ \frac{1}{c^4}H_{[2PN]} + \frac{1}{c^6}H_{[3PN]} + \frac{1}{c^8}H_{[4PN]} + \ldots$$

$$+ \frac{1}{c^5}H_{[2.5PN]}(t) + \frac{1}{c^7}H_{[3.5PN]}(t) + \ldots$$

$\hat{H} = (H - Mc^2)/\mu, \qquad \mu = m_1m_2/M, \qquad M = m_1 + m_2$

$\nu = \mu/M, \qquad 0 \le \nu \le 1/4$

test-body case: $\nu = 0$, equal-mass case: $\nu = 1/4$

CMF: $\mathbf{p}_1 + \mathbf{p}_2 = 0, \qquad \mathbf{p} \equiv \mathbf{p}_1/\mu,$

$p_r = (\mathbf{n} \cdot \mathbf{p}), \qquad \mathbf{q} \equiv (\mathbf{x}_1 - \mathbf{x}_2)/GM, \qquad \mathbf{n} = \mathbf{q}/|\mathbf{q}|$

$$\hat{H}_N = \frac{p^2}{2} - \frac{1}{q}$$

$$\hat{H}_{[1PN]} = \frac{1}{8}(3\nu - 1)p^4 - \frac{1}{2}[(3+\nu)p^2 + \nu p_r^2]\frac{1}{q} + \frac{1}{2q^2}$$

$$\begin{aligned}
\hat{H}_{[2PN]} &= \frac{1}{16}(1 - 5\nu + 5\nu^2)p^6 \\
&+ \frac{1}{8}[(5 - 20\nu - 3\nu^2)p^4 - 2\nu^2 p_r^2 p^2 - 3\nu^2 p_r^4]\frac{1}{q} \\
&+ \frac{1}{2}[(5 + 8\nu)p^2 + 3\nu p_r^2]\frac{1}{q^2} - \frac{1}{4}(1 + 3\nu)\frac{1}{q^3}
\end{aligned}$$

$$
\begin{aligned}
\hat{H}_{[3PN]} &= \frac{1}{128}(-5+35\nu-70\nu^2+35\nu^3)p^8 \\
&+ \frac{1}{16}[(-7+42\nu-53\nu^2-5\nu^3)p^6+(2-3\nu)\nu^2 p_r^2 p^4 \\
&+ 3(1-\nu)\nu^2 p_r^4 p^2 - 5\nu^3 p_r^6]\frac{1}{q} \\
&+ [\frac{1}{16}(-27+136\nu+109\nu^2)p^4+\frac{1}{16}(17+30\nu)\nu p_r^2 p^2 \\
&+ \frac{1}{12}(5+43\nu)\nu p_r^4]\frac{1}{q^2} \\
&+ [\left(-\frac{25}{8}+\left(\frac{1}{64}\pi^2-\frac{335}{48}\right)\nu-\frac{23}{8}\nu^2\right)p^2 \\
&+ \left(-\frac{85}{16}-\frac{3}{64}\pi^2-\frac{7}{4}\nu\right)\nu p_r^2]\frac{1}{q^3}+[\frac{1}{8}+\left(\frac{109}{12}-\frac{21}{32}\pi^2\right)\nu]\frac{1}{q^4}
\end{aligned}
$$

3PN

Jaranowski/GS ('98)[in part], Damour/Jaranowski/GS ('01)

Blanchet/Faye ('01)[in part], Blanchet/Damour/Esposito-Farèse ('04)
[harmonic gauge, point masses]

Itoh/Futamase ('03) [harmonic gauge, surface integrals]

Foffa/Sturani ('11) [Effective Field Theory]

$$
\begin{aligned}
\hat{H}_{[4PN]} \quad &= \quad \left(\frac{7}{256} - \frac{63}{256}\nu + \frac{189}{256}\nu^2 - \frac{105}{128}\nu^3 + \frac{63}{256}\nu^4\right) p^{10} \\
&+ \quad \left\{ \frac{45}{128}p^8 - \frac{45}{16}p^8\nu + \left(\frac{423}{64}p^8 - \frac{3}{32}p_r^2p^6 - \frac{9}{64}p_r^4p^4\right)\nu^2 \right. \\
&+ \quad \left(-\frac{1013}{256}p^8 + \frac{23}{64}p_r^2p^6 + \frac{69}{128}p_r^4p^4 - \frac{5}{64}p_r^6p^2 + \frac{35}{256}p_r^8\right)\nu^3 \\
&+ \quad \left.\left(-\frac{35}{128}p^8 - \frac{5}{32}p_r^2p^6 - \frac{9}{64}p_r^4p^4 - \frac{5}{32}p_r^6p^2 - \frac{35}{128}p_r^8\right)\nu^4 \right\} \frac{1}{q} \\
&+ \quad \left\{ \frac{13}{8}p^6 + \left(-\frac{791}{64}p^6 + \frac{49}{16}p_r^2p^4 - \frac{889}{192}p_r^4p^2 + \frac{369}{160}p_r^6\right)\nu \right. \\
&+ \quad \left(\frac{4857}{256}p^6 - \frac{545}{64}p_r^2p^4 + \frac{9475}{768}p_r^4p^2 - \frac{1151}{128}p_r^6\right)\nu^2 \\
&+ \quad \left.\left(\frac{2335}{256}p^6 + \frac{1135}{256}p_r^2p^4 - \frac{1649}{768}p_r^4p^2 + \frac{10353}{1280}p_r^6\right)\nu^3 \right\} \frac{1}{q^2}
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & \left\{ \frac{105}{32} p^4 + \left[C_{41} + \left(\frac{237}{40} p^4 - \frac{1293}{40} p_r^2 p^2 + \frac{97}{4} p_r^4 \right) \ln \frac{q}{s} \right] \nu \right. \\
+ \quad & \left. C_{42}\, \nu^2 + \left(-\frac{553}{128} p^4 - \frac{225}{64} p_r^2 p^2 - \frac{381}{128} p_r^4 \right) \nu^3 \right\} \frac{1}{q^3} \\
+ \quad & \left\{ \frac{105}{32} p^2 + \left[C_{21} + \left(\frac{233}{40} p^2 - \frac{29}{6} p_r^2 \right) \ln \frac{q}{s} \right] \nu + C_{22}\, \nu^2 \right\} \frac{1}{q^4} \\
+ \quad & \left\{ -\frac{1}{16} + \left[c_{01} + \frac{21}{20} \ln \frac{q}{s} \right] \nu + c_{02}\, \nu^2 \right\} \frac{1}{q^5}
\end{aligned}
$$

$$
\begin{aligned}
C_{42} &= \left(-\frac{1189789}{28800} + \frac{18491}{16384}\pi^2\right) p^4 + \left(-\frac{127}{3} - \frac{4035}{2048}\pi^2\right) p_r^2 p^2 \\
&+ \left(\frac{57563}{1920} - \frac{38655}{16384}\pi^2\right) p_r^4 \\
C_{22} &= \left(\frac{672811}{19200} - \frac{158177}{49152}\pi^2\right) p^2 + \left(-\frac{21827}{3840} + \frac{110099}{49152}\pi^2\right) p_r^2 \\
c_{02} &= -\frac{1256}{45} + \frac{7403}{3072}\pi^2
\end{aligned}
$$

$$
\begin{aligned}
{\color{red}C_{41}} &= {\color{red}c_{411}} p^4 + {\color{red}c_{412}} p_r^2 p^2 + {\color{red}c_{413}} p_r^4 \\
{\color{red}C_{21}} &= {\color{red}c_{211}} p^2 + {\color{red}c_{212}} p_r^2 \\
{\color{red}c_{01}} &= {\color{red}c_{01}}
\end{aligned}
$$

ISCO

$$H = H(\mathbf{p}, \mathbf{r}), \qquad p^2 = p_r^2 + j^2/r^2, \qquad p_r = (\mathbf{p}\cdot\mathbf{r})/r$$

circular orbits: $p_r = 0, \qquad p^2 = j^2/r^2, \qquad H = H(j, r)$

circular motion: $\frac{\partial}{\partial r} H(j, r) = 0 \rightarrow H(j)$

orbital frequency: $\omega = \frac{dH(j)}{dj} \rightarrow H(\omega)$

ISCO: $\boxed{\frac{dH(\omega)}{d\omega} = 0}$ or, alternatively $\frac{\partial^2}{\partial r^2} H(j, r) = 0$

$$\text{SBH:} \quad E(x) = \frac{1-2x}{(1-3x)^{1/2}} - 1$$

$$= -\frac{1}{2}x + \frac{3}{8}x^2 + \frac{27}{16}x^3 + \frac{675}{128}x^4 + \frac{3969}{256}x^5 + \ldots$$

$$E(x) \equiv \frac{H(x) - mc^2}{mc^2}, \qquad x = \left(\frac{GM\omega}{c^3}\right)^{2/3}$$

circular orbits:

$$\omega_{\rm circ} = \omega_{\rm radial} + \omega_{\rm periastron} = 2\pi \frac{1+k}{P}, \quad x = \left(\frac{GM\omega_{\rm circ}}{c^3}\right)^{2/3}$$

$$c^2 E_{4PN} \equiv \hat{H}_N + c^{-2}\hat{H}_{[1PN]} + c^{-4}\hat{H}_{[2PN]} + c^{-6}\hat{H}_{[3PN]} + c^{-8}\hat{H}_{[4PN]}$$

$$\begin{aligned} E_{4PN}(x) &= -\frac{x}{2} + \left(\frac{3}{8} + \frac{1}{24}\nu\right) x^2 + \left(\frac{27}{16} - \frac{19}{16}\nu + \frac{1}{48}\nu^2\right) x^3 \\ &+ \left(\frac{675}{128} + \left(-\frac{34445}{1152} + \frac{205}{192}\pi^2\right)\nu + \frac{155}{192}\nu^2 + \frac{35}{10368}\nu^3\right) x^4 \end{aligned}$$

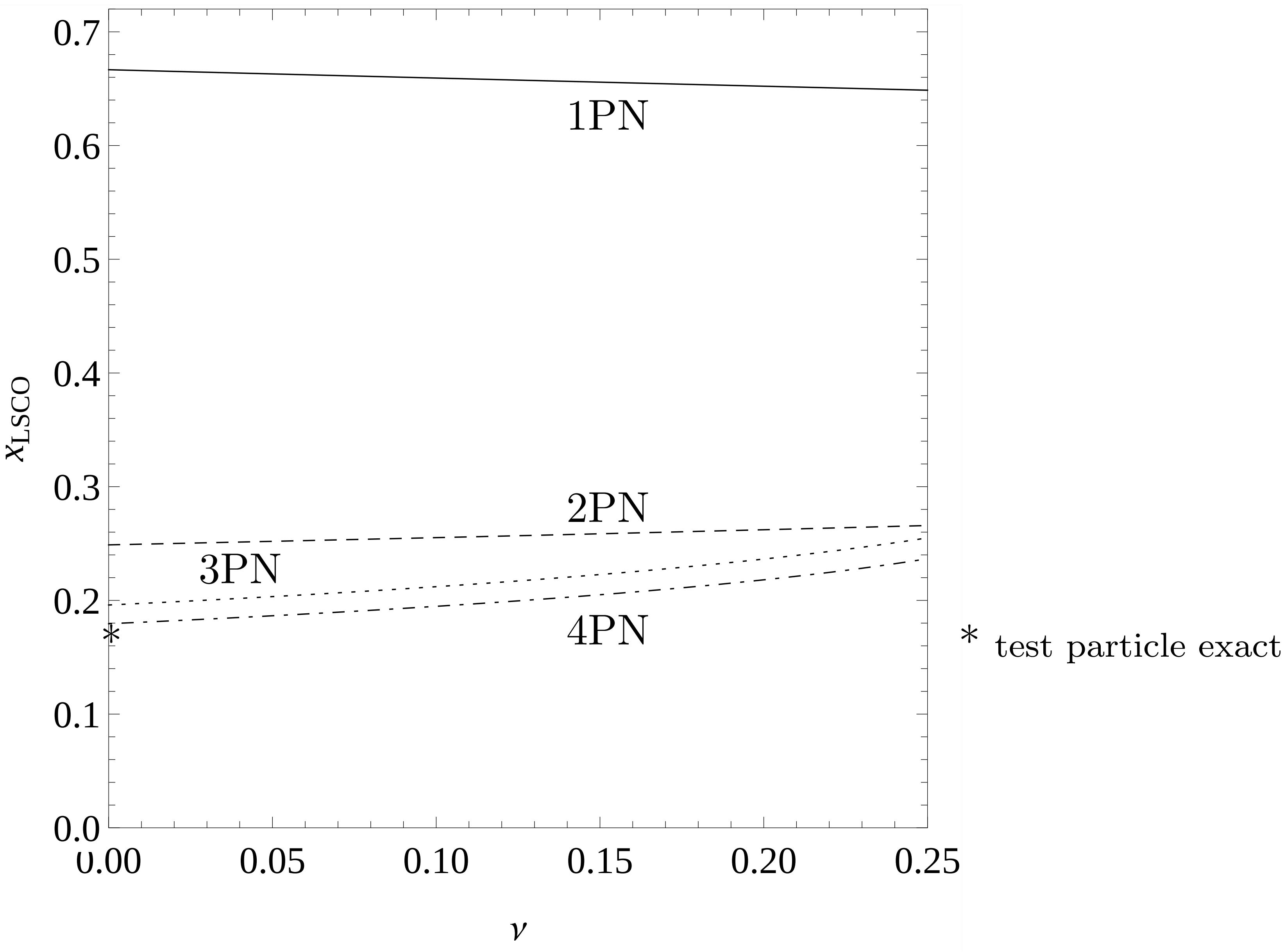
$$-\tfrac{1}{2}\left(-\tfrac{3960}{128} + [c_1 + \tfrac{448}{15}\ln x]\nu + \left(-\tfrac{498449}{3456} + \tfrac{3157}{576}\pi^2\right)\nu^2 + \tfrac{301}{1728}\nu^3 + \tfrac{77}{31104}\nu^4\right) x^5$$

Damour ('10)[ln$x$], Blanchet/Detweiler/Le Tiec/Whiting ('10)[ln$x$]

Jaranowski/GS ('12)[ln$x$, ν^3, ν^4], ('13)[ν^2], Foffa/Sturani ('13) [ln$x$, ν^3, ν^4]

$$c_1 = -\tfrac{123671}{5760} + \tfrac{9037}{1536}\pi^2 + \tfrac{1792}{15}\ln 2 + \tfrac{896}{15}\gamma = 153.88...$$

Bini/Damour ('13), Le Tiec/Blanchet/Whiting ('12) [numerical value]



0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
$x_{\rm LSCO}$
1PN
2PN
3PN
4PN
0.00
0.05
0.10
0.15
0.20
0.25
ν
* test particle exact

Dynamical invariants

radial action $i_r(E,j)$:

$$i_r(E,j) = \frac{1}{2\pi}\oint dr\ p_r(E,j,r), \quad (\hat{H} = E)$$

phase of revolution Φ (periastron advance k):

$$\frac{\Phi}{2\pi} = 1 + k = -\frac{\partial}{\partial j} i_r(E,j)$$

orbital period P:

$$\frac{P}{2\pi GM} = \frac{\partial}{\partial E} i_r(E,j)$$

periastron advance at 3pN:

$$
\begin{aligned}
k &= \frac{1}{c^2}\frac{3}{j^2}\left\{1+\frac{1}{c^2}\left[\frac{5}{4}(7-2\nu)\frac{1}{j^2}+\frac{1}{2}(5-2\nu)\,E\right]\right.\\
&+ \left.\frac{1}{c^4}\left[a_1(\nu)\frac{1}{j^4}+a_2(\nu)\frac{E}{j^2}+a_3(\nu)\,E^2\right]\right\}
\end{aligned}
$$

orbital period at 3pN:

$$
\begin{aligned}
\frac{P}{2\pi GM} &= \frac{1}{(-2E)^{3/2}}\left\{1-\frac{1}{c^2}\frac{1}{4}(15-\nu)E\right.\\
&+ \frac{1}{c^4}\left[\frac{3}{2}(5-2\nu)\frac{(-2E)^{3/2}}{j}-\frac{3}{32}(35+30\nu+3\nu^2)\,E^2\right]\\
&+ \left.\frac{1}{c^6}\left[a_2(\nu)\frac{(-2E)^{3/2}}{j^3}-3a_3(\nu)\frac{(-2E)^{5/2}}{j}+a_4(\nu)\,E^3\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
a_1(\nu) &= \frac{5}{2}\left(\frac{77}{2}+\left(\frac{41}{64}\pi^2-\frac{125}{3}\right)\nu+\frac{7}{4}\nu^2\right)\\
a_2(\nu) &= \frac{105}{2}+\left(\frac{41}{64}\pi^2-\frac{218}{3}\right)\nu+\frac{45}{6}\nu^2\\
a_3(\nu) &= \frac{1}{4}(5-5\nu+4\nu^2)\\
a_4(\nu) &= \frac{5}{128}(21-105\nu+15\nu^2+5\nu^3)
\end{aligned}
$$

orbital motion at 2PN:

$$
\begin{aligned}
r &= a_r(1 - e_r \cos u) \\
\frac{2\pi}{P}(t - t_0) &= u - e_t \sin u + F_{v-u}(v - u) + F_v \sin v + ... \\
\frac{2\pi}{\Phi}(\phi - \phi_0) &= v + G_{2v}\sin(2v) + G_{3v}\sin(3v) + ...
\end{aligned}
$$

$$
v = 2\arctan\left[\sqrt{\frac{1+e_\phi}{1-e_\phi}}\tan\frac{u}{2}\right]
$$

2.5PN binary BH (orbital) dissipative dynamics

$$\frac{1}{c^5} H_{[2.5PN]}(t) = \frac{2G}{5c^5} \frac{d^3 Q_{ij}(t)}{dt^3} \left(\frac{p_{1i} p_{1j}}{m_1} + \frac{p_{2i} p_{2j}}{m_2} - \frac{G m_1 m_2}{r_{12}} \right)$$

$$Q_{ij}(t) = \sum_{a=1,2} m_a (x_a^i x_a^j - \frac{1}{3} \mathbf{x}_a^2 \delta_{ij})$$

Multipole expansion in far zone

$$h_{ij}^{\mathrm{TT}}(\mathbf{x},t) = \frac{G}{c^4}\frac{P_{ijkm}(\mathbf{n})}{r}\sum_{l=2}^{\infty}\left\{\left(\frac{1}{c^2}\right)^{\frac{l-2}{2}}\frac{4}{l!}\,\mathrm{M}^{(l)}_{kmi_3\ldots i_l}(t-\frac{r_*}{c})\;N_{i_3\ldots i_l}\right.$$

$$\left. + \left(\frac{1}{c^2}\right)^{\frac{l-1}{2}}\frac{8l}{(l+1)!}\,\epsilon_{pq(k}\,\mathrm{S}^{(l)}_{m)pi_3\ldots i_l}(t-\frac{r_*}{c})\;n_q\;N_{i_3\ldots i_l}\right\}$$

$$\mathrm{M}_{ij}(t-\frac{r_*}{c}) = \widehat{\mathrm{M}}_{ij}\left(t-\frac{r_*}{c}\right)$$

$$+ \frac{2Gm}{c^3}\int_0^{\infty}dv\;\ln\left(\frac{v}{2b}\right)\widehat{\mathrm{M}}^{(2)}_{ij}(t-\frac{r_*}{c}-v)+O(1/c^4),$$

$$r_* = r + \frac{2Gm}{c^2}\ln\left(\frac{r}{cb}\right)+O(1/c^3)$$

Luminosity and energy loss:

$$\mathcal{L}(t) = \frac{c^3}{32\pi G} \oint_{\mathrm{FZ}} (\partial_t h_{ij}^{\mathrm{TT}})^2 r^2 d\Omega$$

$$\begin{aligned}
\mathcal{L} &= \frac{G}{5c^5} \sum_{n=0}^{\infty} \left(\frac{1}{c^2} \right)^n \hat{\mathcal{L}}_n \\
&= \frac{G}{5c^5} \left\{ \mathrm{M}_{ij}^{(3)} \mathrm{M}_{ij}^{(3)} + \frac{1}{c^2} \left[\frac{5}{189} \mathrm{M}_{ijk}^{(4)} \mathrm{M}_{ijk}^{(4)} + \frac{16}{9} \mathrm{S}_{ij}^{(3)} \mathrm{S}_{ij}^{(3)} \right] \right. \\
&+ \left. \frac{1}{c^4} \left[\frac{5}{9072} \mathrm{M}_{ijkm}^{(5)} \mathrm{M}_{ijkm}^{(5)} + \frac{5}{84} \mathrm{S}_{ijk}^{(4)} \mathrm{S}_{ijk}^{(4)} \right] \right\}
\end{aligned}$$

$$- < \frac{d\mathcal{E}(t)}{dt} > \; = \; < \mathcal{L}(t - r/c) >$$

Spin and Gravity

tetrad field e_a^μ: $e_a^\mu e_{b\mu} = \eta_{ab}, \qquad e_{a\mu} e_{b\nu} \eta^{ab} = g_{\mu\nu} = g_{\nu\mu}$

local LT: $e_a'^\mu = L^b{}_a e_b^\mu, \qquad L^a{}_c \eta_{ab} L^b{}_d = \eta_{cd}$

linear connection ω_μ^{ab}: $D_\mu \phi \equiv \partial_\mu \phi + \frac{1}{2} \omega_\mu^{ab} G_{[ab]} \phi$

local LT: $\omega_\mu'^{ab} = L^a{}_c L^b{}_d \omega_\mu^{cd} + L^a{}_d \partial_\mu L^{bd}, \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}$

inf. local LT: $\delta\phi = \delta\xi^{ab} G_{[ab]} \phi$

curvature tensor $R^{ab}{}_{\mu\nu}$: $D_\mu D_\nu \phi - D_\nu D_\mu \phi = R^{ab}{}_{\mu\nu} G_{[ab]} \phi$

$$R^{ab}{}_{\mu\nu} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\nu^{ac} \omega_\mu^{bd} \eta_{cd} - \omega_\mu^{ac} \omega_\nu^{bd} \eta_{cd}$$

Lagrangian for gravity

$$\mathcal{L}_G = \frac{1}{16\pi} \det(e_\gamma^c) e_a^\mu e_b^\nu R^{ab}{}_{\mu\nu}(\omega) + \partial_\mu \mathcal{C}^\mu$$

vacuum Einstein equations:

$$0 = \frac{\delta \mathcal{L}_G}{\delta e_a^\mu} e_{a\nu} \equiv 2\det(e_\gamma^c)(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)$$

$$0 = \frac{\delta \mathcal{L}_G}{\delta \omega_\mu^{ab}} \quad \Rightarrow \quad \omega_\mu^{ab} = \omega_\mu^{ab}(e, \partial_\nu e) \qquad \text{no torsion !}$$

Lagrangian for spinning objects

$$\mathcal{L}_M = \int d\tau \left[\left(p_\mu - \frac{1}{2} S_{ab}\, \omega_\mu{}^{ab} \right) \frac{dz^\mu}{d\tau} + \frac{1}{2} S_{ab} \frac{d\theta^{ab}}{d\tau} \right] \delta_{(4)}$$

$$\mathcal{L}_C = \int d\tau \left[\lambda_1^a p^b S_{ab} + \lambda_{2[i]} \Lambda^{[i]a} p_a - \frac{\lambda_3}{2} (p^2 + m^2) \right] \delta_{(4)}$$

$$d\theta^{ab} = \Lambda_C^{\ a} d\Lambda^{Cb} = -d\theta^{ba}$$

resulting covariant equations of motion:

$$\frac{DS_{ab}}{D\tau} = 0, \qquad \frac{DS_{\mu\nu}}{D\tau} = 0$$

$$\frac{Dp_\mu}{D\tau} = -\frac{1}{2} R_{\mu\rho ab} u^\rho S^{ab} = -\frac{1}{2} R_{\mu\rho\alpha\beta} u^\rho S^{\alpha\beta}$$

$$u^\mu \equiv \frac{dz^\mu}{d\tau} = \lambda_3 p^\mu, \qquad p^b S_{ab} = 0, \quad p^\beta S_{\alpha\beta} = 0$$

$$\sqrt{-g} T^{\mu\nu} = \int d\tau \left[\lambda_3 p^\mu p^\nu \delta_{(4)} + \left(u^{(\mu} S^{\nu)\alpha} \delta_{(4)} \right)_{||\alpha} \right]$$

Canonical setting

$$ds^2 = -(Ncdt)^2 + g_{ij}(dx^i + N^i cdt)(dx^j + N^j cdt)$$

$$H = \int d^3x (N\mathcal{H} - N^i \mathcal{H}_i) + \frac{c^4}{16\pi G} \oint_{i^0} d^2 s_i (g_{ij,j} - g_{jj,i})$$

$$N|_{i^0} = 1 + \mathcal{O}(1/r), \quad N^i|_{i^0} = \mathcal{O}(1/r), \quad g_{ij} = \delta_{ij} + \mathcal{O}(1/r)$$

If the constraints $\mathcal{H} = 0$ and $\mathcal{H}_i = 0$ are fulfilled and adapted coordinate conditions are applied, then

$$H = \frac{c^4}{16\pi G} \oint_{i^0} d^2 s_i (g_{ij,j} - g_{jj,i}) \equiv H_{\mathrm{ADM}}$$

solution of the matter constraints

$$n^\mu = (1, -N^i)/N, \qquad n_\mu = (-N, 0, 0, 0)$$

$$\lambda_3 : \quad np \equiv n^\mu p_\mu = -\sqrt{m^2 + \gamma^{ij} p_i p_j} \qquad \gamma^{ik} g_{kj} = \delta^i_j$$

$$\lambda_1 : \quad nS_i \equiv n^\mu S_{\mu i} = \frac{p_k \gamma^{kj} S_{ji}}{np}$$

$$\lambda_2 : \quad \Lambda^{[j](0)} = \Lambda^{[j](i)} \frac{p_{(i)}}{p^{(0)}}, \quad \Lambda^{[0]a} = -\frac{p^a}{m}$$

time gauge for the tetrads

$$e^{\mu}_{(0)} = n^{\mu}, \qquad e^{0}_{(0)} = \frac{1}{N}, \qquad e^{i}_{(0)} = -\frac{N^i}{N}$$

$$g_{ij} = e^{(m)}_{i} e_{(m)j}$$

$$\mathcal{L}_{MC} = -N\mathcal{H}^{\text{matter}} + N^i \mathcal{H}^{\text{matter}}_i$$

$$\mathcal{H}^{\text{matter}} = -np\delta - K^{ij}\frac{p_i n S_j}{np}\delta - (nS^k\delta)_{;k}$$

$$\mathcal{H}^{\text{matter}}_i = (p_i + K_{ij} n S^j)\delta + \left(\frac{1}{2}\gamma^{mk} S_{ik}\delta + \delta^{(k}_i \gamma^{l)m}\frac{p_k n S_l}{np}\delta\right)_{;m}$$

transformation to canonical matter variables

$$z^i = \hat{z}^i - \frac{nS^i}{m - np}, \quad nS_i = -\frac{p_k \gamma^{kj} \hat{S}_{ji}}{m}$$

$$S_{ij} = \hat{S}_{ij} - \frac{p_i nS_j}{m - np} + \frac{p_j nS_i}{m - np}$$

$$\lambda^{[i](j)} = \hat{\lambda}^{[i](k)} \left(\delta_{kj} + \frac{p_{(k)} p^{(j)}}{m(m - np)} \right)$$

$$P_i = p_i + K_{ij} n S^j + \hat{A}^{kl} e_{(j)k} e^{(j)}_{l,i} - \left(\frac{1}{2} S_{kj} + \frac{p_{(k} n S_{j)}}{np} \right) \Gamma^{kj}{}_i$$

$$g_{ik} g_{jl} \hat{A}^{kl} = \frac{1}{2} \hat{S}_{ij} + \frac{m p_{(i} n S_{j)}}{np(m - np)}$$

$$S^{ab} S_{ab} = \hat{S}_{(i)(j)} \hat{S}_{(i)(j)} = 2 \hat{S}_{(i)} \hat{S}_{(i)} = 2 s^2 = \text{const}$$

$$\hat{\lambda}^{(i)}_{[k]} \hat{\lambda}^{[k](j)} = \delta_{ij}$$

$$d\hat{\theta}^{(i)(j)} \equiv \hat{\lambda}^{(i)}_{[k]} d\hat{\lambda}^{[k](j)} = -d\hat{\theta}^{(j)(i)}$$

adding Lagrangian of gravity

$$\hat{\mathcal{L}}_{MK} = P_i \dot{\hat{z}}^i \delta + \frac{1}{2} \hat{S}_{(i)(j)} \dot{\hat{\theta}}^{(i)(j)} \delta$$

$$\hat{\mathcal{L}}_{GK} = \hat{A}^{ij} e_{(k)i} e^{(k)}_{j,0} \delta$$

$$\hat{\mathcal{L}}_{GK} + \mathcal{L}_G = \frac{1}{8\pi} [\pi^{ij} + 8\pi \hat{A}^{ij} \delta] e_{(k)i} e^{(k)}_{j,0} \; + \; \mathcal{L}_{GC} - \frac{1}{16\pi} \mathcal{E}_{i,i}$$

$$\mathcal{E}_i = g_{ij,j} - g_{jj,i}$$

total energy: $E = \frac{1}{16\pi} \oint d^2 s_i \; \mathcal{E}_i = \frac{1}{16\pi} \int d^3 x \; \mathcal{E}_{i,i}$

$$\mathcal{L}_{GC} = -N\mathcal{H}^{\text{field}} + N^i\mathcal{H}_i^{\text{field}}$$

$$\mathcal{H}^{\text{field}} = -\frac{1}{16\pi\sqrt{\gamma}}\left[\gamma R + \frac{1}{2}\left(g_{ij}\pi^{ij}\right)^2 - g_{ij}g_{kl}\pi^{ik}\pi^{jl}\right]$$

$$\mathcal{H}_i^{\text{field}} = \frac{1}{8\pi}g_{ij}\pi^{jk}{}_{;k}$$

$$\pi^{ij} = \sqrt{\gamma}(\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})K_{kl} \qquad \gamma \equiv \det(g_{ij})$$

spatially symmetric time gauge for the tetrads

$$e_{(k)i}e^{(k)}_{j,\mu} = B^{kl}_{ij}g_{kl,\mu} + \frac{1}{2}g_{ij,\mu}$$

$$e_{(i)j} = e_{ij} = e_{ji}$$

$$e_{ij}e_{jk} = g_{ik} \qquad e_{ij} = \sqrt{(g_{kl})}$$

$$2B^{ij}_{kl} = e_{mk}\frac{\partial e_{ml}}{\partial g_{ij}} - e_{ml}\frac{\partial e_{mk}}{\partial g_{ij}}$$

$$\pi^{ij}_{\text{can}} = \pi^{ij} + 8\pi\hat{A}^{(ij)}\delta + 16\pi B^{ij}_{kl}\hat{A}^{[kl]}\delta$$

spacetime-coordinates conditions

$$3g_{ij,j} - g_{jj,i} = 0, \qquad \pi^{ii}_{\text{can}} = 0$$

$$g_{ij} = \Psi^4 \delta_{ij} + h^{\text{TT}}_{ij}, \qquad \pi^{ij}_{\text{can}} = \tilde{\pi}^{ij}_{\text{can}} + \pi^{ij\text{TT}}_{\text{can}}$$

transverse traceless: $h^{\text{TT}}_{ii} = \pi^{ii\text{TT}}_{\text{can}} = h^{\text{TT}}_{ij,j} = \pi^{ij\text{TT}}_{\text{can},j} = 0$

$$\tilde{\pi}^{ij}_{\text{can}} = V^i_{\text{can},j} + V^j_{\text{can},i} - \frac{2}{3}\delta_{ij} V^k_{\text{can},k}$$

constraints: $\mathcal{H}^{\text{field}} + \mathcal{H}^{\text{matter}} = 0, \qquad \mathcal{H}^{\text{field}}_i + \mathcal{H}^{\text{matter}}_i = 0$

total action in canonical form

$$W = \frac{1}{16\pi}\int d^4x\,\pi^{ij\mathrm{TT}}_{\mathrm{can}} h^{\mathrm{TT}}_{ij,0} + \int dt\left[P_i \dot{\hat{z}}^i + \frac{1}{2}\hat{S}_{(i)(j)}\dot{\hat{\theta}}^{(i)(j)} - E\right]$$

Hamiltonian: $E \equiv H_{\mathrm{ADM}} = -\frac{1}{2\pi}\int d^3x\;\Delta\Psi[\hat{z}^i, P_i, \hat{S}_{(i)(j)}, h^{\mathrm{TT}}_{ij}, \pi^{ij\mathrm{TT}}_{\mathrm{can}}]$

$$\{\hat{z}^i, P_j\} = \delta_{ij}, \qquad \{\hat{S}_{(i)}, \hat{S}_{(j)}\} = \epsilon_{ijk}\hat{S}_{(k)}$$

$$\{h^{\mathrm{TT}}_{ij}(\mathbf{x},t), \pi^{kl\mathrm{TT}}_{\mathrm{can}}(\mathbf{x}',t)\} = 16\pi\delta^{\mathrm{TT}kl}_{ij}\delta(\mathbf{x}-\mathbf{x}')$$

spin-gravity interaction ($\mathbf{S} \equiv \hat{\mathbf{S}}$)

leading order spin orbit

$$H^{\mathrm{LO}}_{\mathrm{SO}} = \frac{G}{c^2} \sum_a \sum_{b \neq a} \frac{1}{r_{ab}^2} (\mathbf{S}_a \times \mathbf{n}_{ab}) \cdot \left[\frac{3m_b}{2m_a} \mathbf{p}_a - 2\mathbf{p}_b \right]$$

leading order spin(1)-spin(2)

$$H^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{S}_2} = \frac{G}{c^2} \sum_a \sum_{b \neq a} \frac{1}{2r_{ab}^3} \left[3(\mathbf{S}_a \cdot \mathbf{n}_{ab})(\mathbf{S}_b \cdot \mathbf{n}_{ab}) - (\mathbf{S}_a \cdot \mathbf{S}_b) \right]$$

leading order spin(1) spin(1)

$$H^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{S}_1} = \frac{G}{c^2} \frac{1}{2r_{12}^3} \left[3(\mathbf{S}_1 \cdot \mathbf{n}_{12})(\mathbf{S}_1 \cdot \mathbf{n}_{12}) - (\mathbf{S}_1 \cdot \mathbf{S}_1) \right]$$

$$
\begin{aligned}
H_{\mathrm{SO}}^{\mathrm{NLO}} &= \frac{G}{c^4 r^2}\Bigg[-((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})\Bigg[\frac{5m_2\mathbf{p}_1^2}{8m_1^3}+\frac{3(\mathbf{p}_1\cdot\mathbf{p}_2)}{4m_1^2}\\
&- \frac{3\mathbf{p}_2^2}{4m_1m_2}+\frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})}{4m_1^2}+\frac{3(\mathbf{p}_2\cdot\mathbf{n}_{12})^2}{2m_1m_2}\Bigg]\\
&+ ((\mathbf{p}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12})\left[\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+\frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})}{m_1m_2}\right]\\
&+ ((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{p}_2)\left[\frac{2(\mathbf{p}_2\cdot\mathbf{n}_{12})}{m_1m_2}-\frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})}{4m_1^2}\right]\Bigg]\\
&+ \frac{G^2}{c^4r^3}\Bigg[-((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})\left[\frac{11m_2}{2}+\frac{5m_2^2}{m_1}\right]\\
&+ ((\mathbf{p}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12})\left[6m_1+\frac{15m_2}{2}\right]\Bigg]+(1\leftrightarrow 2)
\end{aligned}
$$

$$
\begin{aligned}
H^{\mathrm{NLO}}_{\mathrm{S}_1\mathrm{S}_2} &= (G/2m_1m_2c^4r^3)[3((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_2\times\mathbf{S}_2)\cdot\mathbf{n}_{12})/2 \\
&+ 6((\mathbf{p}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1\times\mathbf{S}_2)\cdot\mathbf{n}_{12}) \\
&- 15(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \\
&- 3(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{p}_2) + 3(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) \\
&+ 3(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) + 3(\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \\
&+ 3(\mathbf{S}_2\cdot\mathbf{p}_2)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) - 3(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \\
&+ (\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{p}_2) - (\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{p}_1)/2 + (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)/2] \\
&+ (3/2m_1^2r^3)[-((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1\times\mathbf{S}_2)\cdot\mathbf{n}_{12}) \\
&+ (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{p}_1\cdot\mathbf{n}_{12})] \\
&+ (3/2m_2^2r^3)[-((\mathbf{p}_2\times\mathbf{S}_2)\cdot\mathbf{n}_{12})((\mathbf{p}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12}) \\
&+ (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})] \\
&+ (6G^2(m_1+m_2)/c^4r^4)[(\mathbf{S}_1\cdot\mathbf{S}_2) - 2(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})]
\end{aligned}
$$

$$
\begin{aligned}
H^{\mathrm{NLO}}_{\mathrm{S}_1 S_1} = \frac{G}{c^4 r^3} \Bigg[& - \frac{5m_2}{4m_1^3} \left(\mathbf{p}_1 \cdot \mathbf{S}_1\right)^2 + \frac{m_2}{m_1^3} \mathbf{p}_1^2 \mathbf{S}_1^2 - \frac{21 m_2}{8 m_1^3} \left(\mathbf{p}_1 \cdot \mathbf{n}\right)^2 \mathbf{S}_1^2 \\
& - \frac{3m_2}{8m_1^3} \mathbf{p}_1^2 \left(\mathbf{S}_1 \cdot \mathbf{n}\right)^2 + \frac{15 m_2}{4 m_1^3} \left(\mathbf{p}_1 \cdot \mathbf{n}\right) \left(\mathbf{S}_1 \cdot \mathbf{n}\right) \left(\mathbf{p}_1 \cdot \mathbf{S}_1\right) - \frac{3}{4 m_1 m_2} \mathbf{p}_2^2 \mathbf{S}_1^2 \\
& + \frac{9}{4 m_1 m_2} \mathbf{p}_2^2 \left(\mathbf{S}_1 \cdot \mathbf{n}\right)^2 - \frac{1}{4m_1^2} \left(\mathbf{p}_1 \cdot \mathbf{p}_2\right) \mathbf{S}_1^2 - \frac{9}{4m_1^2} \left(\mathbf{p}_1 \cdot \mathbf{p}_2\right) \left(\mathbf{S}_1 \cdot \mathbf{n}\right)^2 \\
& + \frac{3}{2m_1^2} \left(\mathbf{p}_1 \cdot \mathbf{S}_1\right) \left(\mathbf{p}_2 \cdot \mathbf{S}_1\right) - \frac{3}{2m_1^2} \left(\mathbf{p}_1 \cdot \mathbf{n}\right) \left(\mathbf{p}_2 \cdot \mathbf{S}_1\right) \left(\mathbf{S}_1 \cdot \mathbf{n}\right) \\
& - \frac{3}{2m_1^2} \left(\mathbf{p}_2 \cdot \mathbf{n}\right) \left(\mathbf{p}_1 \cdot \mathbf{S}_1\right) \left(\mathbf{S}_1 \cdot \mathbf{n}\right) + \frac{15}{4m_1^2} \left(\mathbf{p}_1 \cdot \mathbf{n}\right) \left(\mathbf{p}_2 \cdot \mathbf{n}\right) \mathbf{S}_1^2 \\
& - \frac{15}{4m_1^2} \left(\mathbf{p}_1 \cdot \mathbf{n}\right) \left(\mathbf{p}_2 \cdot \mathbf{n}\right) \left(\mathbf{S}_1 \cdot \mathbf{n}\right)^2 \Bigg] \\
& - \frac{G^2 m_2}{2 c^4 r^4} \left[5 \left(1 + \frac{m_2}{m_1}\right) \left(\left(\mathbf{S}_1 \cdot \mathbf{n}\right)^2 - \mathbf{S}_1^2\right) + 4 \left(1 + \frac{2m_2}{m_1}\right) \left(\mathbf{S}_1 \cdot \mathbf{n}\right)^2 \right]
\end{aligned}
$$

SO

NLO: Tagoshi/Ohashi/Owen('01), Faye/Blanchet/Buonanno('06), Damour/Jaranowski/GS('08), Steinhoff/Hergt/GS('08), Levi('10), Porto('10)

NNLO: Hartung/Steinhoff('11), Marsat/Bohé/Faye/Blanchet('13)

(N/2)NNLO: Wang/Will('07), Steinhoff/Wang('10)

S1S2

NLO: Steinhoff/Hergt/GS('08), Porto/Rothstein('06, '08, '10), Levi('10)

NNLO: Hartung/Steinhoff('11), Levi('12)

(N/2)NNLO: Zeng/Will('07), Wang/Steinhoff/Zeng/GS('11)

S1S1

NLO[black holes]: Steinhoff/Hergt/GS('08), Porto/Rothstein('08, '10)

NLO[neutron stars]: Porto/Rothstein('08, '10), Steinhoff/Hergt/GS('10)

NLO center-of-mass:

$$\mathbf{G}_{\mathrm{SO}}^{\mathrm{NLO}} = -\sum_a \frac{\mathbf{P}_a^2}{8m_a^3}(\mathbf{P}_a \times \mathbf{S}_a)$$

$$+ \sum_a \sum_{b\neq a} \frac{m_b}{4m_a r_{ab}} \left[((\mathbf{P}_a \times \mathbf{S}_a)\cdot \mathbf{n}_{ab}) \frac{5\mathbf{x}_a + \mathbf{x}_b}{r_{ab}} - 5(\mathbf{P}_a \times \mathbf{S}_a) \right]$$

$$+ \sum_a \sum_{b\neq a} \frac{1}{r_{ab}} \left[\frac{3}{2}(\mathbf{P}_b \times \mathbf{S}_a) - \frac{1}{2}(\mathbf{n}_{ab} \times \mathbf{S}_a)(\mathbf{P}_b \cdot \mathbf{n}_{ab}) - ((\mathbf{P}_a \times \mathbf{S}_a)\cdot \mathbf{n}_{ab}) \frac{\mathbf{x}_a + \mathbf{x}_b}{r_{ab}} \right]$$

$$\mathbf{G}_{\mathrm{S1S2}}^{\mathrm{NLO}} = \frac{1}{2} \sum_a \sum_{b\neq a} \left\{ [3(\mathbf{S}_a \cdot \mathbf{n}_{ab})(\mathbf{S}_b \cdot \mathbf{n}_{ab}) - (\mathbf{S}_a \cdot \mathbf{S}_b)] \frac{\mathbf{x}_a}{r_{ab}^3} + (\mathbf{S}_b \cdot \mathbf{n}_{ab}) \frac{\mathbf{S}_a}{r_{ab}^2} \right\}$$

$$
\begin{aligned}
H_{\text{con}} \quad &= \quad H_N + H_{1PN} + H_{2PN} + H_{3PN} + H_{4PN} \\
&+ \quad H^{\text{LO}}_{SO} + H^{\text{LO}}_{S_1S_2} + H^{\text{LO}}_{S_1^2} + H^{\text{LO}}_{S_2^2} \\
&+ \quad H^{\text{NLO}}_{SO} + H^{\text{NLO}}_{S_1S_2} + H^{\text{NLO}}_{S_1^2} + H^{\text{NLO}}_{S_2^2} \\
&+ \quad H^{\text{NNLO}}_{SO} + H^{\text{NNLO}}_{S_1S_2} \\
&+ \quad H^{\text{LO}}_{p_1S_2^3} + H^{\text{LO}}_{p_2S_1^3} + H^{\text{LO}}_{p_1S_1S_2^2} + H^{\text{LO}}_{p_2S_2S_1^2}
\end{aligned}
$$

$$
\mathcal{H}^{\text{matter}} = m_1 \left(1 - \frac{1}{2} \left(\mathbf{a}_1 \cdot \partial_1 \right)^2 \right) \delta_1 + \frac{1}{2} \mathbf{p}_1 \cdot \left(\mathbf{a}_1 \times \partial_1 \right) \delta_1 + (1 \leftrightarrow 2)
$$

$$
\mathcal{H}^{\text{matter}}_i = p_{1i} \delta_1 + \frac{m_1}{2} \left(\mathbf{a}_1 \times \partial_1 \right)_i \left(1 - \frac{1}{6} \left(\mathbf{a}_1 \cdot \partial_1 \right)^2 \right) \delta_1 + (1 \leftrightarrow 2)
$$

$$
\mathbf{S}_1 = \mathbf{a}_1 m_1, \quad \mathbf{S}_2 = \mathbf{a}_2 m_2
$$

Results for equal masses, circular orbits, and aligned spins:

$$H_{\rm spin} = H_{\rm S_1O} + H_{\rm S_2O} + H_{\rm S_1^2} + H_{\rm S_2^2} + H_{\rm S_1S_2} + H_{\rm S^3} + H_{\rm S^4} + \dots$$

LO NLO NNLO

$$H_{\rm S_1O} = S_1 L \left\{ \frac{7}{8r^3} + \frac{3}{r^4}\left[-1 + \frac{5}{16}\frac{L^2}{r}\right] + \frac{1}{64r^5}\left[401 - \frac{751}{8}\frac{L^2}{r} - \frac{25}{16}\frac{L^4}{r^2}\right] + \dots \right\}$$

$$H_{\rm S_1^2} = S_1^2 \left\{ -\frac{C_{ES^2}}{8r^3} + \frac{1}{16r^4}\left[6C_{ES^2} + 5 - \frac{17C_{ES^2} - 11}{4}\frac{L^2}{r}\right] + \dots \right\}$$

$$H_{\rm S_1S_2} = S_1 S_2 \left\{ -\frac{1}{4r^3} + \frac{1}{2r^4}\left[3 - \frac{7}{8}\frac{L^2}{r}\right] + \frac{1}{64r^5}\left[-271 - 238\frac{L^2}{r} + \frac{45}{8}\frac{L^4}{r^2}\right] + \dots \right\}$$

$$H_{\rm S^3} = \frac{5L}{64r^5}(S_1 + S_2)^3 + \dots$$

$$H_{\rm S^4} = -\frac{3}{128r^5}(S_1 + S_2)^4 + \dots$$

yet only known
for black holes