Sharp decay estimates for the Klein Gordon equation on Kerr-AdS

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Introduction

- Astrophysicists and theoretical physicists are interested in the solutions of the Einstein equations.
- The Einstein equations is system of hyperbolic PDEs, for which we can formulate a Cauchy problem.
- We know certain stationary solutions (Minkowski space, de-Sitter space, Anti-de-Sitter (AdS), Schwarzschild spacetime, Kerr spacetime, Kerr-AdS spacetime, etc.).
- Question: which of these solutions are stable, linearly, non-linearly ?

The trivial solutions

We look for the simplest solutions of $Ric(g) = \Lambda g$.

- When $\Lambda = 0$, Minkowski space.
- When $\Lambda > 0$, de-Sitter space.
- When $\Lambda < 0$, Anti-de-Sitter space.

Anti-de-Sitter

Fix $\Lambda < 0$. Consider the manifold $\mathbb{R}^4$ with Lorentzian metric

$$g_{AdS} = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\sigma_{S^2},$$

where $d\sigma_{S^2}$ is standard metric on S^2 and $l^2 = -\frac{3}{\Lambda}$.

$$\Box_{g_{AdS}} \psi = -\left(1 + \frac{r^2}{l^2}\right)^{-1} \psi_{tt} + \frac{1}{r^2} \partial_r \left(r^2 \left(1 + \frac{r^2}{l^2}\right) \psi_r \right) + \frac{1}{r^2} \Delta_{S^2} \psi.$$

The non-linear stability of the trivial solutions

- Minkowski space is non-linearly stable (Christodoulou-Klainerman, 1993, Lindblad-Rodnianski 2003,..).
- de-Sitter is non-linearly stable (Friedrich, 1986).
- Conjecture for Anti-de-Sitter [Dafermos-Holzegel, Anderson]: Instability (Numerics and heuristics of Bizoń-Rostworowski, see also Dias-Horowitz-Santos).

Non-linear stability of spacetimes

For the study of non-linear problems, it is important to keep in mind the following:

- The non-linear structure is important: Linear stability does not imply non-linear stability.
- A linear stability result is only usefull if it leads to a quantitative decay estimate.

Examples:

Consider the following non-linear wave equations:

$$\Box\phi = (\partial_t\phi)^2 \tag{1}$$

$$\Box\phi = (\partial_t\phi)^2 - (\partial_r\phi)^2 \tag{2}$$

$\phi = 0$ is a solution to both equations. However, it is a stable solution for only one of them, which one ?

Answer: the second one (example due to Fritz John), because the non-linearity has a special structure:

$$(\partial_t \phi)^2 - (\partial_r \phi)^2 = \partial_v \phi \cdot \partial_u \phi$$

with

$$v = t - r, \quad u = t + r.$$

These special structure is known as the *null structure.*

Identifying a similar structure in the Einstein equations was key to the proof of the non-linear stability of Minkowski space by Christodoulou-Klainerman.

Quantitative decay estimate is important:

A quantitative decay estimate is an inequality of the form:
For all regular solutions to $\Box\phi = 0$, for all $t \geq 1$,

$$|\phi(t,x)| \leq \frac{1}{t}||\phi_0||$$

where $||\phi_0||$ is a norm depending only on the initial data, for instance the energy of ϕ and of some of its derivatives.

A non-quantitative statement is typically:

there exists no growing mode solutions
or

$$|\phi(t,x)| \to 0, \quad t \to 0.$$

The importance of quantitative decay estimates is that all proofs of non-linear stability for pdes such as the Einstein equations use them all the time.

Here, we shall consider (scalar) linear stability of solutions which are asymptotically Anti-de-Sitter.
Roughly, our results can be summarized as follows:

- We consider a linear equation $(\Box_g + m)(\psi) = 0$ where $\Box_g + m$ is a Klein Gordon operator associated to a Kerr-AdS spacetime (with natural conditions on the parameters).

- We prove that solutions ψ of $(\Box_g + m)(\psi) = 0$ satisfies the following decay estimate

$$E_{1,loc}[\psi](t) \lesssim \frac{1}{\log(2+t)} E_2[\psi](t=0).$$

  where

- $E_{1,loc}(t) =$ "local energy" at time t.

- E_2 second order energy, controls ψ, $\partial\psi$, $\partial^2\psi$ in L^2

Moreover, we prove that the estimate is **sharp**.
The slow decay rate is a consequence of a *stable trapping* phenomenon.

Anti-de-Sitter

Fix $\Lambda < 0$. Consider the manifold $\mathbb{R}^4$ with Lorentzian metric

$$g_{AdS} = -\left(1+\frac{r^2}{l^2}\right)dt^2 + \left(1+\frac{r^2}{l^2}\right)^{-1}dr^2 + r^2 d\sigma_{S^2},$$

where $d\sigma_{S^2}$ is standard metric on S^2 and $l^2 = -\frac{3}{\Lambda}$.

$$\Box_{g_{AdS}}\psi = -\left(1+\frac{r^2}{l^2}\right)^{-1}\psi_{tt} + \frac{1}{r^2}\partial_r\left(\left(1+\frac{r^2}{l^2}\right)\psi_r\right) + \frac{1}{r^2}\Delta_{S^2}\psi.$$

Energy spaces for Klein-Gordon equation on Anti-de-Sitter

Consider the r-weighted energy norms

$$\begin{aligned} ||\psi||_{H^{0,-2}_{AdS}} &= \int_{\mathbb{R}^3} r^{-2}\psi^2 r^2 dr d\sigma_{S^2}, \\ ||\psi||_{H^1_{AdS}} &= \int_{\mathbb{R}^3} \left(r^2|\psi_r|^2 + |\nabla\!\!\!/\psi|^2 + |\psi|^2\right) r^2 dr d\sigma_{S^2}. \\ \|\psi\|^2_{H^2_{AdS}} &= \|\psi\|^2_{H^1_{AdS}} \\ &\quad + \int_{\mathbb{R}^3} \left[r^4 \left(\partial_r\partial_r\psi\right)^2 + r^2|\nabla\!\!\!/\partial_r\psi|^2 + |\nabla\!\!\!/\nabla\!\!\!/\psi|^2\right] r^2 dr \sin\theta d\theta d\phi \end{aligned}$$

and define the energy norms

$$\begin{aligned} E_1[\psi] &= ||\partial_t\psi||_{H^{0,-2}_{AdS}} + ||\psi||_{H^1_{AdS}} \\ E_2[\psi] &= ||\partial_{tt}\psi||_{H^{0,-2}_{AdS}} + ||\partial_t\psi||_{H^1_{AdS}} + ||\psi||_{H^2_{AdS}} + \sum_{i=1,2,3} ||\Omega^i\psi||_{H^1_{AdS}} \end{aligned}$$

- g_{AdS} invariant by vector field $T = \partial_t$ in AdS so get conservation of the following energy

$$\int_{t=const} \Big[(1+r^2)^{-1}\psi_t^2 + (1+r^2)\psi_r^2 + |\nabla\!\!\!/ \psi|^2 + m\psi^2 \Big] r^2 dr d\omega.$$

- Note that the conformal wave operator is $\Box_g - \frac{1}{6}R$ which in AdS corresponds to $m = -\frac{2}{l^2}$, i.e. a negative term in the above energy.

- Use Hardy type inequalities to control the m-term

$$\int_{\Sigma_t} \psi^2 r^2 dr d\omega \leq C_H \int_{\Sigma_t} r^4 \psi_r^2 dr d\omega$$

- For any asymptotically AdS spacetime, the equation $\Box_g \psi = m\psi$ is well-posed in the H^k_{AdS} spaces provided that $m > -\frac{9}{4l^2}$. (Breitenlohner-Freedmann, Ishibashi-Wald, Bachelot, Holzegel, Vasy, Warnick).

Wave confinement in AdS

- In AdS, there are periodic finite energy solutions to the wave equation (spectrum of the associated elliptic operator is discrete). So no decay !

- No decay together with the strong nonlinearities in the Einstein equations leads to

  Conjecture 1 (Dafermos-Holzegel, Anderson)**.** *AdS is dynamically unstable.*

  Remark 1: Numerics and heuristics of Bizoń-Rostworowski, see also Dias-Horowitz-Santos.
  Remark 2: Dynamics in AdS may be dependent upon choice of boundary conditions.

Wave confinement in AdS II

- This can be understood in a *compactification* of the problem. Ex: take ψ spherically symmetric solution, let $r^\star = arctan\frac{r}{l}$ and $u = r\psi$ then u solves

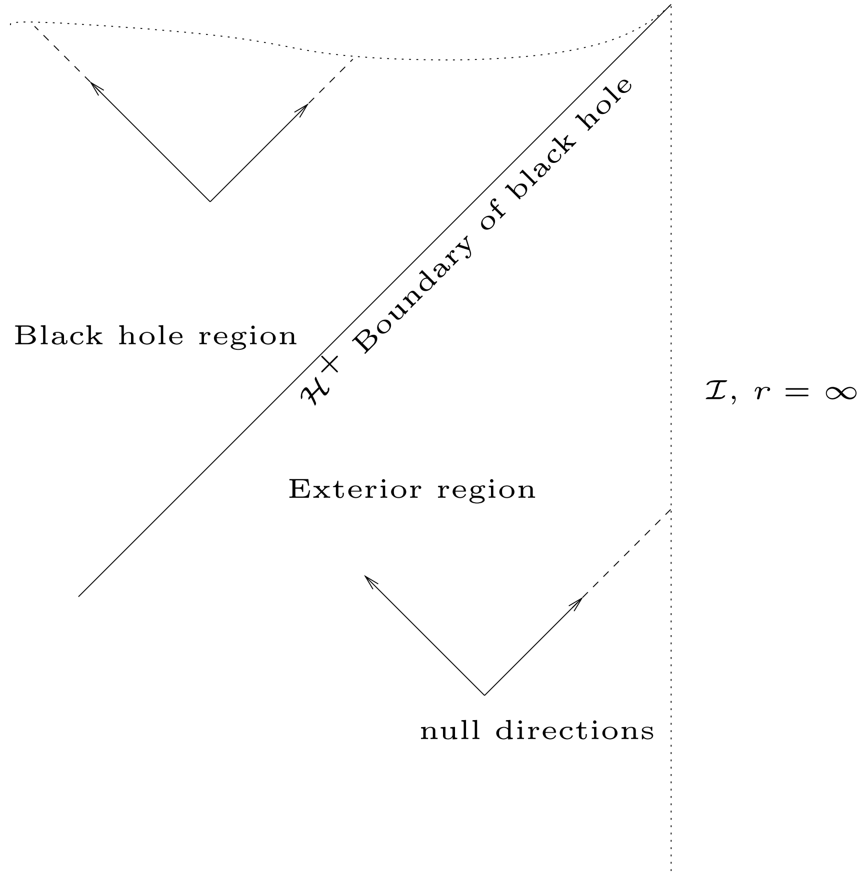
$$u_{tt} - u_{r^\star r^\star} + V(r^\star)u = 0$$

  in a strip $0 \leq r^\star \leq \pi/2$ with Dirichlet data at both boundaries.

- In other words, no radiation can escape through infinity.

- Use vector field method using $T = \partial_t$ and commutation by T: $H^{1,s}_{AdS}$ norms for $s > 0$ can be propagated by the equations, i.e. stronger norms than the energy norm are propagated.

Scalar waves in asymptotically AdS black holes



The Schwarzschild-AdS metrics

Let $M, l > 0$ and consider the metric

$$ds^2 = -(1-\mu)dt^2 + (1-\mu)^{-1}dr^2 + r^2 d\sigma^2_{S^2}$$

- where $(1-\mu) = 1 - \frac{2M}{r} + \frac{r^2}{l^2}$,
- $M > 0, l = \infty$ corresponds to the Schwarzschild metric,
- $1-\mu$ has one real root denoted $r_+ > 0$, which depends on M and l.
- The black hole exterior+horizon is $\mathcal{R} = \mathbb{R}_t \times [r_+, \infty) \times S^2$.
- The wave operator is

$$\Box_g \psi := -(1-\mu)^{-1}\psi_{tt} + r^{-2}\partial_r(r^2(1-\mu)\psi_r) + r^{-2}\Delta_{S^2}\psi,$$

The Kerr-AdS black holes

- Let $M > 0, l > 0$ and let a be a real number such that $|a| \leq l$.
- Schematically, the Kerr-AdS metric takes the form

$$g = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dtd\phi,$$

  where all coefficients depend on r and θ only and g_{rr} is singular at some $r_+ > 0$.
- As before, $\mathcal{R} = [r_+, \infty) \times S^2$.

More precisely,

$$g_{KAdS} = \frac{\Sigma}{\Delta_-}dr^2 + \frac{\Sigma}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \left(r^2+a^2\right)^2 - \Delta_- a^2 \sin^2\theta}{\Xi^2\Sigma} \sin^2\theta d\phi^2$$
$$-2\frac{\Delta_\theta \left(r^2+a^2\right) - \Delta_-}{\Xi\Sigma} a \sin^2\theta \; d\phi dt - \frac{\Delta_- - \Delta_\theta a^2 \sin^2\theta}{\Sigma} dt^2$$

with

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta_\pm = \left(r^2+a^2\right)\left(1+\frac{r^2}{l^2}\right) \pm 2Mr$$

$$\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta, \qquad \Xi = 1 - \frac{a^2}{l^2}\,.$$

Moreover, r_+ is the largest real root of $\Delta_-(r)$.

In Schwarzschild(-AdS) and Kerr(-AdS), the coordinates

- (t, r, ω) are singular at the horizon,
- As is usual, we can introduce another coordinate system $(t^*, r, \tilde{\omega})$, with g regular at $\partial\mathcal{R} = \{r = r_+\}$.

Problem: Prove quantitative decay for solutions of $\Box_g \psi = m\psi$ pour $(\psi, \psi_t) \in H^k_{AdS} \times H^{k-1}_{AdS}$.

On Schwarzschild/Kerr, huge litterature (Tataru-Tohaneanu, Tohaneanu, Dafermos-Rodnianski, Blue-Sterbenz, Andersson-Blue, Donninger-Schlag-Soffer...).

Also other results concerning tails of of mode solutions (Price, Gundlach-Price-Pulin, Andersson, Barack, Barack-Ori, Donninger-Schlag-Soffer,..)

Idem on Schwarzschild-de-Sitter, Kerr-de-Sitter (Dafermos-Rodnianksi, Bony-Häfner, Melrose-Sa Barreto-Vasy, Vasy, Dyatlov, ..)

For Schwarzschild-AdS or Kerr-AdS, uniform boundedness results (Holzegel 2009, Holzegel-Warnick 2012) if $|a|$ not too large compared to r_+.

Log decay of Klein-Gordon waves in Kerr-AdS

We prove

Theorem 1 (Holzegel-J.S., 2011-2013)**.** *Let* ψ *be a solution in* H^2_{AdS} *of* $\Box_g \psi = m\psi$ *in* $(\mathcal{R}, g)$, *g metric of a Kerr-AdS spacetime such that* $|a|l < r_+^2$, $m \geq -\frac{9}{4l^2}$. *Let* $R > r_+$. *Then, for all* $t \geq 0$,

$$E_{1,loc}[\psi](t) := \left(||\psi||_{H^1_{AdS,\{r \geq R\}}} + ||\psi_t||_{H^{0,-2}_{AdS,\{r \geq R\}}} \right)(t) \leq \frac{C}{\log(2+t)} E_2(\psi)(t=0),$$

where $C > 0$ *is some universal constant depending only on the parameters* (a, l, M, m). *Moreover, the estimate is* ***sharp***.

Remark 1: If $|a|l > r_+^2$, then it is conjectured that not even boundedness of solutions hold! (cf Shlapentokh-Rothman, Cardoso-Dias).

Remark 2: Initially range of parameters smaller (cf recent work of Holzegel-Warnick).

Remark 3: Lower bounds actually holds without restrictions on a.

Sharpness

Let SCH^2_{AdS} be the set of solutions with finite second energy $E_2(\psi)$. Let $t^\star_0 \geq 0$ be fixed and define for any non-zero ψ and $t^\star \geq 0$

$$Q\left[\psi\right](t^\star) := \log(2+t^\star)\left[\frac{E_{1,loc}(\psi)(t)}{E_2(\psi)(t^\star_0)}\right]^{\frac{1}{2}} .$$

Then there exists a universal constant $C > 0$ such that

$$\limsup_{t^\star \to +\infty} \sup_{\psi \in SCH^2_{AdS}, \psi \neq 0} Q\left[\psi\right](t^\star) > C > 0.$$

Equivalently, sharpness means that the statement

There exists a function $t \to \delta(t)$ such that $\delta(t) \to 0$ as $t \to +\infty$ and such that for all solutions ψ, we have the estimate

$$\left(||\psi||_{H^1_{AdS,\{r \geq R\}}} + ||\psi_t||_{H^{0,-2}_{AdS,\{r \geq R\}}} \right)(t) \leq \frac{\delta(t)}{\log(2+t)} E_2(\psi)(t=0),$$

is false.

Elements of proof of decay

Typical elements in analysis of wave equations on black hole spacetimes

- Red-shift
- Superradiance
- Trapping

Red-shift

Consider first Schwarzschild-AdS.

- g is invariant by $T = \partial_t \to$ conservation of a T energy.
- But T becomes null at $\mathcal{H}^+$.
- Hence, the energy

$$\int_{t=const} \left(\psi_t^2 + \frac{1}{r}(r - r_+)\psi_r^2 + \ ... \right) ... \, r^2 dr d\omega.$$

degenerate at r_+.

- For Kerr-AdS, the energy density can even be negative! (superradiance)
- However, near r_+ can construct multiplier (Rodnianski-Dafermos, Holzegel in AdS case with $m < 0$)

The trapping: the geodesic flow on Kerr-AdS

- is integrable (cf Carter constant).
- If $a = 0$, there exists null geodesics orbiting around $r = 3M$.
- For $a \neq 0$, there still exists periodic null geodesics in a neighbourdhood (of size a) of $r = 3M$.
- But, viewed in $T\mathcal{M}^*$, this behaviour is unstable. (the trapped set is of positive codimension.)
- In asymptotically flat Kerr, this is all the trapping, but in the asymptotically AdS, there is also a trapping at infinity !

Elements of the proof for decay

- Give yourself a frequency cutt-off. Decompose ψ into a high-low frequency $\psi = \psi_{\leq L} + \psi_{>L}$.
- Note that this will be a spacetime frequency decomposition.
- Prove a multiplier estimate on $\psi_{\leq L}$ of the form

$$\int_t ||\psi_{\leq L}||^2_{H^1_{AdS, r \geq R}} \leq e^{CL} E_1(\psi)$$

- For $\psi_{\geq L}$, we would like a Poincaré type inequality

$$||\psi_{>L}||^2_{H^1_{AdS}} \leq \frac{1}{L} E_1(\psi).$$

  However, because of spacetime frequency decompostion, we can only get a spacetime type of Poincaré inequality of the type

$$\int_0^\tau ||\psi_{>L}(t')||^2_{H^1_{AdS}} dt' \leq \frac{1}{L} E_1(\psi)\tau.$$

- Then interpolate.

1-d reduction: Kerr-AdS case

Use carter seperation of variables (and some rescaling of ψ) to obtain an equation of form

$$\omega^2 u = -\frac{d^2 u}{(dr^\star)^2} + \left(\lambda_{km}(a\omega)V(r^\star) + m^2 W(r^\star) + \omega m U(r^\star) + R(r^\star)\right) u.$$

Here the $\lambda_{km}(a\omega)$ are angular frequencies corresponging to the eigenvalues of (modified)-oblate-spheroidal operator.

(modified)-oblate-spheroidal-harmonics

The $Q(\omega)_{S^2}$ operator is defined by

$$\begin{aligned}-Q\left(\omega\right)f \quad &= \quad \frac{1}{\sin\theta}\partial_\theta\left(\Delta_\theta \sin\theta \partial_\theta f\right) + \frac{\Xi^2}{\Delta_\theta}\frac{1}{\sin^2\theta}\partial^2_{\tilde{\phi}} f \\ &\quad + \Xi\frac{a^2\omega^2}{\Delta_\theta}\cos^2\theta f - 2ia\omega\frac{\Xi}{\Delta_\theta}\frac{a^2}{l^2}\cos^2\theta\ \partial_{\tilde{\phi}} f\ ,\end{aligned}$$

where $\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta$ and $\Xi = 1 - \frac{a^2}{l^2}$.

Eigenvalues of $Q(\omega)^2_S$ denoted by $\lambda_{km}(\omega)$.

Eingenfunctions $S_{km}(\omega)$.

Lemma 1 (estimates for the λ_{km})**.** $\lambda_{km} + a^2\omega^2 \geq |m|(m+1)$.

Superradiance ?

- Recall that g_{tt} is not always negative.
- This means that the natural conserved energy associated to the invariance of g by ∂_t is a priori not coercive.
- However, g is also invariant by ∂_ϕ and there is a special combination of the type $K = \partial_t + C(a, M, l)\partial_\phi$ such the conserved energy associated to K is coercive in $r > r_+$ (and degenerate near r_+), provided that $|a|l < r_+^2$.
- The vector field K is called the Hawking-Real vectorfield.

In frequency space: need to combine the frequency associated to t and the frequency associated to ϕ. For instance, Helmhotz equation in the form

$$(\omega - Cm)^2 u = -u'' + V(\omega, m, k, r, \theta)u.$$

Superradiance can lead to unboundedness: Growing mode solutions have been constructed for some Klein-Gordon equation on Kerr (Yakov Shlapentokh-Rothman 2012, cf also Cardoso-Dias).

The frequency sets

Let $L > 0$ be a large number. We first do a high-low frequency decomposition:

1. The high frequency set is $\{|\omega - Cm|^2 + \lambda_{km}(\omega) > L\}$

2. The low frequency set is $\{|\omega - Cm|^2 + \lambda_{km}(\omega) \leq L\}$

The low frequency set must also be decomposed to single out the almost stationary frequency set

$$\{|\omega - Cm|^2 \leq L^{1/2}\}$$

We then construct multipliers for all low frequencies.

Quasimodes

To probe the decay of solutions, there is a well known technique in semi-classical analysis, which is the construction of the so-called *quasimodes*.

- A quasimode is an *approximate* solution ψ_ℓ

$$(\Box_g + m)\,\psi_\ell = F_\ell.$$

- A quasimode is periodic in time (like a mode solution) $\psi_\ell = e^{i\omega_\ell t}\varphi_\ell(r,\theta,\phi)$.
- A quasimode is (typically) localized in space.
- Finally, the error F_ℓ goes to zero as ℓ (the frequency scale) goes to infinity.

Quasimodes and sharpness of the main estimate

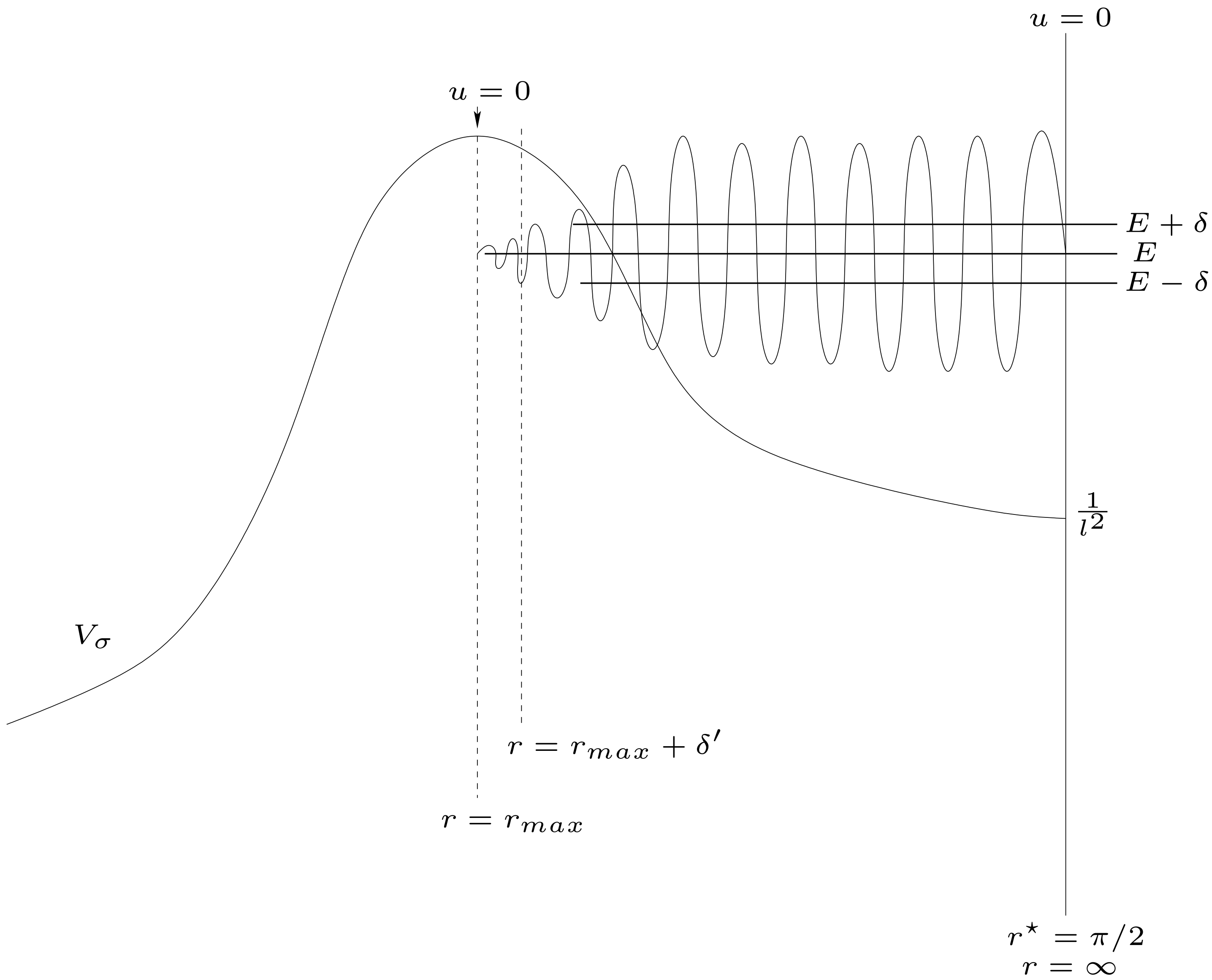
- Using the so-called Duhamel Formula, the existence of quasimodes translates into lower bounds for the decay estimate.
- If rate of decay of F_ℓ is polynomial in $1/\ell$, then we get that solutions cannot decay faster than a certain polynomial in $1/t$.
- If rate of decay of F_ℓ is of type $e^{-C\ell}$, then we get that solutions cannot decay faster than $(\log t)^{-1}$.
- Quasimodes are also strongly related to the quasi-normal modes. Many results in math litterature (cf Tang-Zworski) of type: existence of quasimodes implies existence of quasi-normal-modes with similar frequencies.
- This has been done for Schwarzschild-AdS (Gannot 2012).
- Cf Numerical work on quasinormal modes for AdS black holes (Festuccia-Liu..)

Existence of quasimodes: the Schwarzschild-AdS case

After seperation of variable, we get equation of type

$$-u_\ell'' \frac{1}{\ell(\ell+1)} + V_\sigma u_\ell = \frac{\omega^2}{\ell(\ell+1)} u_\ell \tag{3}$$

for a potential $V_\sigma(r)$,


$u = 0$
$u = 0$
$E + \delta$
E
$E - \delta$
$\frac{1}{l^2}$
V_σ
$r = r_{max} + \delta'$
$r = r_{max}$
$r^\star = \pi/2$
$r = \infty$

- To construct quasimodes, we first construct a sequence of solutions $(u_\ell)_{\ell \in \mathbb{N}}$ to an eigenvalue problem with Dirichlet boundary conditions at $r = 3M$.
- The u_ℓ are solutions to

$$-u_\ell'' \frac{1}{\ell\,(\ell+1)} + V_\sigma u_\ell = \kappa_\ell u_\ell$$

with the κ_ℓ converging to any fixed $E \leq V_{\max}$ as $\ell \to +\infty$.

- In a the region where $V_\sigma \geq \kappa_\ell$, we show that the solutions becomes exponentially small as $\ell \to +\infty$. These are the so-called *Agmon estimates* which in quantum mechanics are used to quantify how small is the tunnel-effect.

- We then defined our quasimodes as follows. For each ℓ, define

$$\omega_\ell^2 = \kappa_\ell . \ell(\ell + 1)$$

and

$$\psi_\ell = e^{i\omega_\ell t} \chi(r) r u_\ell S_{\ell\, 0}(\theta, \phi),$$

where $S_{\ell\, 0}(\theta, \phi)$ is a spherical harmonic with angular momentum number ℓ and $\chi(r)$ is cuttoff function with is 1 for $r \geq 3M + \delta$ and 0 for $r \leq 3M$, for some small enough $\delta > 0$.

- Then ψ_ℓ is a solution to the Klein-Gordon equation on Schwarzschild-AdS apart in a small strip of size δ, where the cuttoff function is not constant.

- In this strip, it satisfies

$$(\Box_g + m)\, \psi_\ell = F_\ell,$$

with the error being exponentially small in ℓ as $\ell \to +\infty$.

In Kerr, we want to apply the same technique (in axisymmetry) but the eigenvalue equation becomes non-linear

$$-u_\ell'' \frac{1}{\mu_\ell\left(a^2\omega^2\right)} + V_\sigma u_\ell = \frac{\omega^2}{\mu_\ell\left(a^2\omega^2\right)} u_\ell. \tag{4}$$

The operator now depends on ω^2 but ω^2 is constructed from the eigenvalue!

Solution: consider the eigenvalue κ_ℓ as function of a and ω and use the implicit function theorem (IFT). $\kappa_\ell(a = 0, \omega)$ is then the Schwarzschild-AdS eigenvalue found earlier. (actually, we also need to modify the Schwarzschild-AdS operator).

- Then, for small a, one can use the IFT to construct κ_ℓ, ω_ℓ and u_ℓ.
- To go from small a to any a (such that $|a| \leq l$), we prove global estimates (in $|a|$) for all relevant functions in the application of the IFT.
- For instance, we prove an estimate from below on $\frac{\partial \kappa_\ell}{\partial a}$.
- Thus, we each ℓ, we get the existence of κ_ℓ, ω_ℓ and u_ℓ as before.
- The Agmon estimates can be carried over as before provided we still have $\kappa_\ell(a) \leq V_{\max}$.
- This follows from monotonicity argument: κ_ℓ is decreasing with $|a|$ (modulo lower order terms).

Theorem 2 (Quasimodes for Kerr-AdS). *Let $(g, \mathcal{R})$ denote the black hole exterior of a Kerr-AdS spacetime, with mass $M > 0$, angular momentum per unit mass a and cosmological constant $\Lambda = -\frac{3}{l^2}$. Assume that the parameters satisfy $\alpha < \frac{9}{4}$, $|a| < l$. Then, for $\delta > 0$ sufficiently small, there exists a family of non-zero functions $\psi_\ell \in H^k_{AdS}$ for any $k \geq 0$ such that*

1. $\psi_\ell\,(t, r, \theta, \varphi) = e^{i\omega_\ell t}\varphi_\ell(r, \theta)$ *(axisymmetric and time-periodic),*
2. $0 < c < \frac{\omega_\ell^2}{\ell(\ell+1)} < C$, *for constants c and C independent of ℓ (uniform bounds on the frequencies),*
3. *for all $t^\star \geq t_0^\star$, for all $k \geq 0$,* $|| \left(\Box_g + \frac{\alpha}{l^2}\right)\psi_\ell ||_{H^k_{Ads}(\Sigma_{t^\star})} \leq C_k e^{-C_k \ell} ||\psi_\ell||_{H^0_{AdS}(\Sigma_{t_0^\star})}$, *for some $C_k > 0$ independent of ℓ (approximate solutions to the wave equation),*
4. *the support of $F_\ell := \left(\Box_g + \frac{\alpha}{l^2}\right)\psi_\ell$ is contained in $\{r_{max} \leq r \leq r_{max} + \delta\}$ (spatial localization of the error),*
5. *the support of $\varphi_\ell(r, \theta)$ is contained in $\{r \geq r_{max}\}$ (spatial localization of the solution).*

A non-linear model problem: spherically symmetric Einstein-Klein-Gordon-system

The Einstein-Klein-Gordon system:

$$\begin{aligned} Ric(g) - \frac{1}{2}Rg + \Lambda g &= 8\pi T[\psi], \\ \Box_g \psi &= m\psi, \end{aligned} \tag{5}$$

where $T[\psi]$ is

$$T[\psi] = d\psi \otimes d\psi - \frac{1}{2} g \left(g(\nabla\psi, \nabla\psi) + m\psi^2 \right).$$

Local existence in H^2_{AdS} (for ψ) and some continuation criterion of solutions are known for this system (Holzegel-J.S. 2011).

Remark 1: spherically symmetric solutions to the $Ric(g) = \Lambda g$ are either AdS or Schwarzschild-AdS, i.e. no spherically-symmetric dynamics in the vacuum, hence the coupled system.

Stability of Schwarzschild-AdS for the spherically-symmetric Einstein-Klein-Gordon system

Theorem 3 (Holzegel, J.S. 2011)**.** *Asymptotic and orbital stability of Schwarzschild-AdS hold.*

Our analysis contains:

- Integrated decay types estimate controlling $\int_t ||\psi||_{H^1_{AdS,\{r \geq R\}}}$.
- Pointwise decay estimate for ψ.
- Bootstrap argument to propagate "good" geometrical properties of Schwarzschild-AdS.