Equivalence Between the Covariant and Bardeen Perturbation Formalisms

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09 December 2013

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Introduction and Motivation

- Linear cosmological perturbation theory is well known and understood, however, the gauge freedom makes its interpretation complicated.
- There are two main approaches to the perturbations theory, Bardeen's through the metric perturbation analyses and the covariant formalism.
- In both formalisms it is important to understand the conditions for linearity.
- Beyond first order the gauge problem escalates.
- It is important to understand the exact relation between both formalisms to address more complicated problems.
- Here we are discussing our paper S. D. P. Vitenti, F. T. Falciano, and N. Pinto-Neto, *ArXiv e-prints* (2013), arXiv: 1311.6730 [astro-ph.CO]. The references to other works can be found there.

Objectives

- Compare the two Gauge Invariant (GI) approaches, Bardeen's and covariant.
- Describe the Bardeen's approach covariantly introducing the concept of mixed and pure tensors.
- Show that the gauge freedom in the metric approach is close related to the foliation freedom in the covariant approach.
- Obtain tensors which reduce to Bardeen's variables at first order.
- Demonstrate that the Bardeen's variables appear naturally in both approaches when we impose foliation and gauge independence.

A brief recollection of the gauge dependence

General covariance is not equivalent to the gauge dependency in perturbation theory.

- The perturbation gauge freedom is often defined heuristically as an infinitesimal coordinate transformation **keeping the functional form of the background quantities fixed**, i.e., one defines a perturbation on a tensor $Q(x)$ as

$$\delta Q(x) = Q(x) - \overline{Q}(x);$$

- Under a change of coordinate generated by B^μ we obtain

$$\delta Q(x) \to Q(x) + \pounds_B Q(x) - \overline{Q}(x) \approx \delta Q(x) + \pounds_B Q(x).$$

- This approach fixes the background objects functional form and mixes the coordinate freedom with the perturbations gauge freedom.

A formal description

- All background objects are defined in the background manifold M_{bg}.
- The family λ of diffeomorphisms $\Upsilon_\lambda : \mathrm{I} \times M_{bg} \to M$, introduces a bijective tensor map from M_{bg} to M as Υ^*_λ.
- Through this bijective map, we can define the background objects in M as $\overline{Q} \equiv \Upsilon^*_0 Q_{bg}$.

Perturbative Hypothesis

Given a background metric manifold M_{bg}, there is a diffeomorphism Υ_0 such that

$$\delta g_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu},$$

is small in a well defined sense.

- Given Υ_0, there are many families of diffeomorphisms Υ_λ such that for a small λ we have:

$$\delta g^{(\lambda)}{}_{\mu\nu} = g_{\mu\nu} - (\bar{g}_{\mu\nu} + \pounds_A \bar{g}_{\mu\nu}) = \delta g_{\mu\nu} - 2\overline{\nabla}_{(\mu} A_{\nu)}. \qquad (1)$$

- As long as $2\overline{\nabla}_{(\mu} A_{\nu)}$ is small in the same sense of $\delta g_{\mu\nu}$ the perturbations remain small.

The description above has some advantages:

- Both background and perturbed tensors transform as usual under a coordinate transformation.
- Does not fix any functional form or introduce any preferred coordinate system.
- Allows the introduction of a covariant positive definite norm to measure the size of the perturbations.

Given a background and a physical tensor respectively $\bar{T}$ and T, we have the following transformation under the change of diffephormism:

$$\begin{aligned} \bar{T} &\longrightarrow \bar{T} + \pounds_A \bar{T}, \\ T &\longrightarrow T. \end{aligned}$$

Combining with a small coordinate transformation:

$$\begin{aligned} \bar{T} &\longrightarrow \bar{T} + \pounds_{A+B} \bar{T}, \\ T &\longrightarrow T + \pounds_B T \approx T + \pounds_B \bar{T}. \end{aligned}$$

Gauge Invariance – Stewart and Walker Lemma

Given the rules above, the perturbation transform as:

$$\delta T \equiv T - \bar{T} \to \delta T - \pounds_A \bar{T}.$$

Fixed background

We recuperate the heuristic description by setting $B^\mu = -A^\mu$,

$$\begin{aligned} \bar{T} &\longrightarrow \bar{T}, \\ T &\longrightarrow T + \pounds_B T \approx T - \pounds_A \bar{T}. \end{aligned}$$

- Given the formulation above, it is clear that for $\pounds_A \bar{T} = 0$ we have GI tensors. The conditions to enable this were studied by Stewart and Walker (SW) and are known as SW Lemma.
- This prescription generates GI objects at first order.

Stewart and Walker Lemma exceptions

Within the SW Lemma there is a hidden assumption that is sometimes overlooked.

- The perturbation δT is defined as the difference $T - \bar{T}$.
- The conditions in the SW Lemma are sufficient only if the tensor T is defined solely in terms of quantities from the physical manifold, i.e. defined independently of M_{bg} and Υ_0.
- If in the definition of T we also use any background tensor, then even if the background tensor $\bar{T}$ is a simple constant its perturbation might not be gauge invariant.

Pure and Mixed Tensors

We propose the following classification:

- Any tensor that is defined strictly in terms of objects from a single manifold we shall call a pure tensor.
- A tensor that involves objects from both manifolds in its definition we shall call a mixed tensor.

Pure and Mixed Perturbations

We extend the classification for the perturbations:

- A perturbation of a tensor shall be called a pure perturbation if it is defined as the difference of two pure tensors.
- A mix perturbation is defined as the difference of a mixed with a pure tensor.

Now it becomes clear that the SW Lemma applies **only** for pure perturbations and not for mixed ones.

Background 3+1 projections

- Consider a Friedmann-Lemaître-Robertson-Walker (FLRW) metric as the background metric.
- Related to this metric, there is a preferred geodesic vector field $\bar{v}^{\mu}$ which defines the maximally symmetric spatial hyper-surfaces.
- We can decompose physical tensors in M by projecting them with respect to this preferred vector field $\bar{v}^{\mu}$.

Metric perturbations

We can project the physical metric itself using the background foliation $\bar{v}^\mu$.

- We define the mixed scalar $\mathcal{P} \equiv g_{\mu\nu}\bar{v}^\mu\bar{v}^\nu/2$.
- The background version of this tensor is simply $\overline{\mathcal{P}} \equiv \bar{g}_{\mu\nu}\bar{v}^\mu\bar{v}^\nu/2 = -1/2$.

Thus, the mixed perturbation associated with these projections reads

$$\phi \equiv \mathcal{P} - \overline{\mathcal{P}}.$$

Comparison with the coordinate dependent approach

Note that ϕ has been defined in a globally covariant manner. To compare with the usual coordinate dependent approach we define a coordinate system in which $\bar{v}^\mu = \delta^\mu{}_0$ and, therefore,

$$g_{00} = \bar{g}_{00} + 2\phi = -1 + 2\phi \quad \rightarrow \quad \phi = \frac{1}{2}\delta g_{00} = \frac{1}{2}\left(g_{00} - \bar{g}_{00}\right).$$

Some remarks:

- The commonly used metric perturbation ϕ is simply the mixed perturbation associated with $\mathcal{P}$.
- This is an example of a perturbation that violates the SW Lemma. The background tensor $\overline{\mathcal{P}}$ is a simple constant, the mixed perturbation ϕ is a covariant scalar but it is not GI.
- Any coordinate dependent perturbation can be redefined in a covariant manner as done for ϕ.

The transformation rule for ϕ under a gauge transformation.

- First, decompose the gauge transformation vector field as

$$A^\mu = \mathcal{A}\bar{v}^\mu + \mathsf{A}^\mu, \quad \mathsf{A}^\mu = \bar{\gamma}\left[A^\mu\right],$$

with $\bar{\gamma}\left[A^\mu\right]$ being the projection with respect to $\bar{\gamma}_{\mu\nu} = \bar{g}_{\mu\nu} + \bar{v}_\mu\bar{v}_\nu$.

- Using that $\bar{v}_\mu$ is geodesic, i.e., $\bar{a}_\mu \equiv \overline{\nabla}_{\bar{v}}\bar{v}_\mu = 0$ and defining the notation $\dot{T} \equiv \bar{\gamma}\left[\pounds_{\bar{v}}T\right]$.
- We have

$$\mathcal{P} \to \frac{\bar{v}^\mu\bar{v}^\nu\left(g_{\mu\nu} - \pounds_A\bar{g}_{\mu\nu}\right)}{2} = \mathcal{P} + \dot{\mathcal{A}}, \quad \Rightarrow \quad \phi \to \phi + \dot{\mathcal{A}},$$

as expected.

- In the above expression the only hypothesis made is that the background foliation is geodesic.
- Apart from this, this rule gives the general transformation for ϕ in an arbitrary background.

FLRW Background

The other metric perturbations are much more complicated and hence from, here on, we shall restrict ourselves to a FLRW background. Therefore, the background is assumed to be described by:

$$\bar{\mathcal{K}}_{\mu\nu} \equiv \bar{\nabla}_\mu \bar{v}_\nu,$$

$$\bar{\mathcal{K}}_{\mu\nu} = \frac{\bar{\Theta}}{3}\bar{\gamma}_{\mu\nu}, \qquad \bar{\mathcal{R}}_{\mu\nu} = 2\bar{K}\bar{\gamma}_{\mu\nu},$$

$$\bar{D}_\mu \bar{\Theta} = 0 = \bar{D}_\mu \bar{K},$$

with $\bar{\mathcal{K}}_{\mu\nu}$, $\bar{\Theta}$ and $\bar{\mathcal{R}}_{\mu\nu}$ being respectively the extrinsic curvature, expansion factor and the spatial Ricci tensor, and $\bar{D}_\mu$ is the spatial covariant derivative of the background.

Metric perturbations

Decomposing $\delta g_{\mu\nu}$ with respect to the FLRW foliation we have:

$$\delta g_{\mu\nu} = 2\phi \bar{v}_\mu \bar{v}_\nu + 2B_{(\mu}\bar{v}_{\nu)} + 2C_{\mu\nu},$$

where

$$\phi \equiv \frac{1}{2}\delta g_{\bar{v}\bar{v}}, \quad B_\mu \equiv -\bar{\gamma}\left[\delta g_{\bar{v}\mu}\right], \quad C_{\mu\nu} \equiv \frac{1}{2}\bar{\gamma}\left[\delta g_{\mu\nu}\right].$$

Covariant definition

The tensors B_μ and $C_{\mu\nu}$ can be define in a covariant manner through the four tensors

$$\mathcal{P}_\mu = \bar{\gamma}\left[g_{\mu\bar{v}}\right], \quad \overline{\mathcal{P}}_\mu = \bar{\gamma}\left[\bar{g}_{\mu\bar{v}}\right] = 0,$$
$$\mathcal{P}_{\mu\nu} = \frac{\bar{\gamma}\left[g_{\mu\nu}\right]}{2}, \quad \overline{\mathcal{P}}_{\mu\nu} = \frac{\bar{\gamma}\left[\bar{g}_{\mu\nu}\right]}{2} = \frac{\bar{\gamma}_{\mu\nu}}{2},$$

such that

$$B_\mu = \mathcal{P}_\mu - \overline{\mathcal{P}}_\mu, \quad C_{\mu\nu} = \mathcal{P}_{\mu\nu} - \overline{\mathcal{P}}_{\mu\nu}.$$

- The decomposition above is a general and does not involves any perturbative hypothesis.
- It is obtained without assuming anything about $\delta g_{\mu\nu}$ or $\bar{g}_{\mu\nu}$.
- In principle, we can always rewrite Einstein's equations in terms of ϕ, B_μ and $C_{\mu\nu}$ and obtain non-linear second order equations of motion for these variables that encode the same information as those written in terms of $g_{\mu\nu}$.

Scalar, vector and tensor decomposition

For the following results we are considering the perturbations up to first order. It is convenient to decompose the perturbations in terms of the scalar, vector and tensor (SVT) decomposition, i.e.,

$$
\begin{aligned}
B_\mu &= \overline{D}_\mu \mathcal{B} + \mathtt{B}_\mu, \\
C_{\mu\nu} &= \psi \bar{\gamma}_{\mu\nu} - \overline{D}_\mu \overline{D}_\nu \mathcal{E} + \overline{D}_{(\nu} \mathtt{F}_{\mu)} + W_{\mu\nu},
\end{aligned}
$$

where $\overline{D}_\mu \mathtt{B}^\mu = \overline{D}_\mu \mathtt{F}^\mu = \overline{D}_\mu W^\mu{}_\nu = W_\mu{}^\mu = 0$.

Hyper-surface choice

Apart from the background slicing it is useful to define a $3+1$ splitting also in the physical manifold. Thus, we introduce an arbitrary global timelike vector field v^{μ},

- Since we are interested in foliations "close" to the background slicing, we assume that v^{μ} is such that $\delta v_{\mu} \equiv v_{\mu} - \bar{v}_{\mu}$ is of the same order of $\delta g_{\mu\nu}$.
- The normalization of v_{μ} requires that $\delta v_{\mu}\bar{v}^{\mu} = \phi$.
- We define the spatial projection of δv_{μ} as $\mathsf{v}_{\mu} \equiv \bar{\gamma}\left[\delta v_{\mu}\right]$. Then,

$$\delta v_{\mu} = -\phi\bar{v}_{\mu} + \mathsf{v}_{\mu}.$$

- All freedom in choosing a foliation "close" to the background slicing is contained in v_{μ}.
- We decompose v_{μ} as $\mathsf{v}_{\mu} = \overline{D}_{\mu}\mathcal{V} + \mathsf{V}_{\mu}$, where $\overline{D}_{\mu}\mathsf{V}^{\mu} = 0$.

Background folitation

- In principle the choice of spatial hyper-surface in M is arbitrary.
- We can also use the background slicing itself to foliate M.
- To use this vector field we must first normalize it with respect to the metric $g_{\mu\nu}$, which gives

$$\frac{\bar{v}_\mu}{\sqrt{-\bar{v}_\alpha \bar{v}_\beta g^{\alpha\beta}}} \approx (1-\phi)\bar{v}_\mu.$$

- This expression is simply the general slicing described above setting $\mathsf{v}_\mu = 0$.
- By choosing $\mathsf{v}_\mu = 0$ we are in fact changing the nature of v_μ from pure to a mixed tensor.
- Any pure tensor in M that depends on the vector field v_μ automatically becomes a mixed tensor since v_μ itself becomes dependent on background quantities.
- Perturbations with $\mathsf{v}_\mu = 0$ do not satisfy the SW Lemma.
- Pure perturbations have one extra variable with respect to mixed perturbations, which is precisely the quantity v_μ.

Acceleration of v^μ

Notation

We shall introduce a small circle above each tensor if they are pure tensors and maintain the mixed tensors without any symbol.

The pure perturbation of the acceleration is

$$\delta\mathring{a}_\mu = \overline{D}_\mu\dot{\mathcal{V}} + \dot{\mathsf{V}}_\mu - \overline{D}_\mu\phi.$$

- The perturbation $\delta\mathring{a}_\mu$ is GI, which is expected from SW Lemma.
- The mixed counterpart $(\mathsf{v}_\mu = 0)$ $\delta a_\mu = -\overline{D}_\mu\phi$ is not GI.
- The special combination of the spatial foliation $\mathcal{V}$ and ϕ is what makes $\delta\mathring{a}_\mu$ GI.

Kinematic Variables

The other kinematic variables related to v^μ are given by:

- Shear:

$$\delta\mathring{\sigma}_{\mu\nu} = \overline{D}_{\langle\mu}\overline{D}_{\nu\rangle}\left(\mathcal{S}+\mathcal{V}\right) + \overline{D}_{(\mu}\mathsf{S}_{\nu)} + \overline{D}_{(\mu}\mathsf{V}_{\nu)} + \dot{W}_{\mu}{}^{\alpha}\bar{\gamma}_{\alpha\nu},$$

$$\mathcal{S} \equiv \left(\mathcal{B} - \dot{\mathcal{E}} + \frac{2}{3}\overline{\Theta}\mathcal{E}\right), \quad \mathsf{S}^\mu \equiv \mathtt{B}^\mu + \dot{\mathtt{F}}^\mu.$$

- Expansion factor:

$$\delta\mathring{\Theta} = \delta\Theta + \overline{D}^2\mathcal{V}, \qquad \delta\Theta = \overline{D}^2\mathcal{S} + \overline{\Theta}\phi + 3\dot{\psi}.$$

  In this case both quantities are gauge dependent since $\overline{\Theta}$ is not constant.

- Traceless spatial Ricci tensor:

$$r_{\mu\nu} \equiv \mathcal{R}_{\langle\mu\nu\rangle}, \quad \mathcal{R}_{(\mu\nu)} = r_{\mu\nu} + \frac{\mathcal{R}\gamma_{\mu\nu}}{3},$$

$$\delta\mathring{r}_{\mu\nu} = -\overline{D}_{\langle\mu}\overline{D}_{\nu\rangle}\left(\psi + \frac{\overline{\Theta}}{3}\mathcal{V}\right) - \frac{\overline{\Theta}}{3}\overline{D}_{(\mu}\mathsf{V}_{\nu)} - (\overline{D}^2 - 2\overline{K})W_{\mu\nu}.$$

GI variables from scalars

- Given a scalar field φ, Its perturbations under a gauge transformation change as

$$\delta\varphi \to \delta\varphi - \pounds_{A}\bar{\varphi} = \delta\varphi - \mathcal{A}\dot{\bar{\varphi}},$$

  where we are assuming that the background version of φ is homogeneous in the hyper-surfaces.
- The gradient $D_\mu\varphi$ at first order is expressed as

$$D_\mu\varphi \approx \mathsf{v}_\mu\dot{\bar{\varphi}} + \overline{D}_\mu\delta\varphi = \overline{D}_\mu\left(\delta\varphi + \mathcal{V}\dot{\bar{\varphi}}\right) + \dot{\bar{\varphi}}\mathsf{V}_\mu$$

- The particular combination $\delta\varphi + \mathcal{V}\dot{\bar{\varphi}}$ is GI.
- This pure perturbation compare spatial gradients defined in different spatial sections that causes the appearance of the foliation dependent field v_μ.
- This is another almost general rule, i.e. GI tensors constructed in the usual covariant approach will depend on the choice of spatial foliation. There are some few exceptions to this rule, such as the Weyl tensor or, at first order, its projections that defines its electric and magnetic parts.

GI and foliation independent variables from scalars

In the case of scalar fields, it is possible to build gauge and foliation invariant tensors.

- Defining the tensor

$$\mathcal{M}_{\mu\nu} \equiv D_{\langle\mu}D_{\nu\rangle}\varphi - (\pounds_{v}\varphi)\sigma_{\mu\nu}.$$

- That at first order is given by

$$\delta\mathcal{M}_{\mu\nu} = \overline{D}_{\langle\mu}\overline{D}_{\nu\rangle}\left(\delta\varphi - \dot{\bar{\varphi}}\mathcal{S}\right).$$

- The particular combination $\delta\varphi - \dot{\bar{\varphi}}\mathcal{S}$ is also GI, but it is also independent of $\mathcal{V}$, thus, it is Foliation Independent FI.
- In fact, since it is independent of v_{μ} this expression is the same for the mixed perturbations.
- This is equivalent to the usual Bardeen approach, where one uses $\mathcal{S} = a(B - E')$ (in Mukhanov's notation) to form the GI variables.

We present a comparison table for pure and mixed tensors, where we define the following quantities:

$$T_{\mu\nu} = \mathring{\rho} v_\mu v_\nu + 2 v_{(\mu} \mathring{q}_{\nu)} + \mathring{p} \gamma_{\mu\nu} + \mathring{\Pi}_{\mu\nu},$$

$$X_\mu = \kappa D_\mu \mathring{\rho}, \qquad Y_\mu = \kappa D_\mu \mathring{p}, \qquad Z_\mu \equiv D_\mu \mathring{\Theta},$$

Pure tensors	Mixed tensors
$\mathring{a}_\mu \approx \overline{D}_\mu \left(\dot{\mathcal{V}} - \phi \right)$	$a_\mu \approx \overline{D}_\mu \left(-\dot{\mathcal{A}} - \phi \right)$
$\mathring{\sigma}_{\mu\nu} \approx \overline{D}_{\langle\mu} \overline{D}_{\nu\rangle} \left(\mathcal{S} + \mathcal{V} \right)$	$\sigma_{\mu\nu} \approx \overline{D}_{\langle\mu} \overline{D}_{\nu\rangle} \left(\mathcal{S} - \mathcal{A} \right)$
$\mathring{r}_{\mu\nu} \approx -\overline{D}_{\langle\mu} \overline{D}_{\nu\rangle} \left(\psi + \frac{\overline{\Theta}}{3} \mathcal{V} \right)$	$r_{\mu\nu} \approx -\overline{D}_{\langle\mu} \overline{D}_{\nu\rangle} \left(\psi - \frac{\overline{\Theta}}{3} \mathcal{A} \right)$
$\mathring{X}_\mu = \kappa \overline{D}_\mu \left(\delta\rho + \mathcal{V} \dot{\bar{\rho}} \right)$	$X_\mu = \kappa \overline{D}_\mu \left(\delta\rho - \mathcal{A} \dot{\bar{\rho}} \right)$
$\mathring{Y}_\mu = \kappa \overline{D}_\mu \left(\delta p + \mathcal{V} \dot{\bar{p}} \right)$	$Y_\mu = \kappa \overline{D}_\mu \left(\delta p - \mathcal{A} \dot{\bar{p}} \right)$
$\mathring{Z}_\mu = \overline{D}_\mu \left(\delta\Theta + \overline{D}^2 \mathcal{V} + \mathcal{V} \dot{\overline{\Theta}} \right)$	$Z_\mu = \overline{D}_\mu \left(\delta\Theta - \overline{D}^2 \mathcal{A} - \mathcal{A} \dot{\overline{\Theta}} \right)$
$\mathring{q}_\mu = (\bar{\rho} + \bar{p}) \overline{D}_\mu \left(\mathcal{U} - \mathcal{V} \right)$	$q_\mu = (\bar{\rho} + \bar{p}) \overline{D}_\mu \left(\mathcal{U} + \mathcal{A} \right)$
$\gamma \left[v_\mu \right] = 0$	$\bar{\gamma} \left[v_\mu \right] = \overline{D}_\mu \left(\mathcal{V} + \mathcal{A} \right)$

Comparison of the methods

- If we choose $\mathcal{A} = -\mathcal{V}$ we obtain exactly the same values for the variables.
- The interpretation is that we can make a gauge transformation in a way that the background induced hypersurface matches the physical hyper-surface $\bar{\Sigma} = \Sigma$.
- This does not mean that the mixed perturbations become GI, a further gauge transformation would make $\bar{\Sigma} \neq \Sigma$ again.
- At least at first order this result shows that the ambiguity in choosing a hypersurface in the usual covariant method is equivalent to the gauge choice in the metric perturbation approach.
- In this sense, to start with a hypersurface choice in the covariant approach is equivalent to start with a gauge choice.

Gauge and Foliation Invariant Variables

- We have shown above that pure and mixed perturbations are foliation and gauge dependent.
- This arbitrariness can be avoided in both cases by looking for invariant tensors combinations.
- Since the freedom in both cases are close related, any GI combination will be automatically be FI.

The usual Bardeen potential Ψ is given by

$$\Psi \equiv \psi - \frac{\overline{\Theta}}{3}\mathcal{S}.$$

Comparing it with the objects in the table above it is easy to show that (for the scalar part)

$$\mathcal{T}_{\mu\nu} = -r_{\mu\nu} - \frac{\Theta}{3}\sigma_{\mu\nu} \approx \overline{D}_{\langle\mu}\overline{D}_{\nu\rangle}\Psi.$$

- The tensor $\mathcal{T}_{\mu\nu}$ can be interpreted as a GI combination of two gauge dependent tensors (mixed) or a FI combination of two foliation dependent tensors (pure). In this sense this tensor is both GI and FI.

The next Bardeen potential Φ, defined by

$$\Phi \equiv \phi + \dot{\mathcal{S}},$$

can be obtained through the tensor,

$$\alpha_{\mu\nu} \equiv D_{\langle\mu} a_{\nu\rangle} + a_{\langle\mu} a_{\nu\rangle},$$

and finally

$$\mathcal{J}_{\mu\nu} \equiv (\pounds_v \sigma)_{\langle\mu\nu\rangle} - \alpha_{\mu\nu} \approx \overline{D}_{\langle\mu} \overline{D}_{\nu\rangle} \Phi.$$

- These two tensors are close related to the electric projection of the Weyl tensor, i.e.,

$$\begin{aligned} E_{\mu\nu} &= \frac{1}{2}\left(\frac{\Theta}{3}\sigma_{\mu\nu} - (\pounds_v \sigma)_{\langle\mu\nu\rangle} + r_{\mu\nu} + \alpha_{\mu\nu}\right), \\ &= -\frac{1}{2}\left(\mathcal{T}_{\mu\nu} + \mathcal{J}_{\mu\nu}\right) \approx -\overline{D}_{\langle\mu} \overline{D}_{\nu\rangle} \frac{1}{2}\left(\Phi + \Psi\right). \end{aligned}$$

- This should be expected, the Weyl tensor is null in a FLRW background, this makes his projections automatically GI and FI at first order.

Conclusions

- We have shown that the foliation and gauge freedom are close related.
- In this sense, to work in the covariant formalism with a given hypersurface choice is equivalent to the metric perturbation formalism with a given gauge choice.
- We can use the gauge freedom to adjust the background foliation to a physically defined hypersurface. In this case, the kinetic variables defined for $\bar{v}^{\mu}$ become the physical observables.
- The usual Bardeen's variables appear naturally in both formalisms, thus, for the gravitational sector they are the natural GI and FI variables.

Perspectives

- We are working in finding a consistent way to define GI/FI necessary conditions for the validity of the perturbative analyses.
- This is crucial in two cases:
  - the evaluation of the linear approximation in bouncing models,
  - the study of the perturbations during the structure formation and the possible backreaction.
- A number of works which extend the GI analyses to higher order are based on the transformation rules of pure tensors, extending the covariant analyses to higher orders. Since mixed tensors can have a much more involved transformation rule, this opens a new way to find GI/FI variables in higher orders.
- We are working in writing the complete non-linear equations for the metric projections ϕ, B_μ and $C_{\mu\nu}$. This will allow us to find higher order equations of motion in a systematic way and to find necessary and sufficient condition for the validity of the perturbative series.