A new way to count degrees of freedom in dRGT massive gravity

Recent developments in massive gravity

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C. Deffayet, J. Mourad, GZ, 1207.6338, 1208.4493

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Motivations behind massive gravity

- **Cosmological motivations:** adding a mass term to the graviton modifies gravity on large scales of order $1/m$

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! NOT SO EASY

Massless theory

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- Introduction of an auxiliary non-dynamical metric $f_{\mu\nu}$

Quadratic theory (massless)

- Quadratic order action $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$S^{(2)} = -\frac{M_P^2}{2} \int d^4x \; h^{\mu\nu} \mathcal{E}^{\rho\sigma}_{\mu\nu} h_{\rho\sigma}$$

where

$$\mathcal{E}^{\rho\sigma}_{\mu\nu} \equiv -\frac{1}{2}\Big(\delta^{\rho}_{(\mu}\delta^{\sigma}_{\nu)}\Box - 2\delta^{(\sigma}_{(\mu}\partial_{\nu)}\partial^{\rho)} + \eta^{\rho\sigma}\partial_{\mu}\partial_{\nu} - \eta_{\mu\nu}\eta^{\rho\sigma}\Box + \eta_{\mu\nu}\partial^{\rho}\partial^{\sigma}\Big)$$

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- Possible mass terms

$$h_{\mu\nu}h^{\mu\nu} \quad \text{and} \quad h^2$$

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- Equations of motion

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- Second order equations of motion for a priori 10 degrees of freedom i.e. 10×2 functions of the spatial coordinates can be set as initial conditions
- 4 constraint equations $\partial^\mu h_{\mu\nu} - \gamma\partial_\nu h = 0$ (Bianchi identity)

Fierz-Pauli theory $\gamma = 1$ (1939)

Additional constraint equation

- Constraints $\partial^\mu h_{\mu\nu} - \partial_\nu h = 0$ imply $\partial^\mu \partial^\nu h_{\mu\nu} - \Box h = 0$
- Trace of the equations of motion $2(\partial^\mu \partial^\nu h_{\mu\nu} - \Box h) = -3m^2 h$

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! Can be seen explicitly by considering plane wave solutions $h_{\mu\nu} = \epsilon_{\mu\nu} e^{ik_\rho x^\rho}$ of the equations of motion $\Box h_{\mu\nu} - m^2 h_{\mu\nu} = 0$

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- Even when $\gamma \in]1/4, 1[$ a ghost appears...

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- We need a non-linear theory which reduces to Fierz-Pauli at quadratic order

Non-linear completions of Fierz-Pauli theory

- Einstein-Hilbert action + mass term of the form

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- Due to the absence of a fifth constraint as in Fierz-Pauli theory

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- Vielbein and multi-vielbein reformulation by Hinterbichler, Rosen

The mass terms

The dRGT action

$$S_{dRGT} = M_P^2 \int d^4x \sqrt{-g} R - M_P^2 m^2 \sum_{n=0}^{3} \alpha_n \int d^4x \sqrt{-g} E_n(\sqrt{g^{-1}f})$$

where $f_{\mu\nu}$ is non-dynamical and

$$\begin{aligned} E_0(A) &= 1 \\ E_1(A) &= Tr(A) \\ E_2(A) &= \frac{1}{2}(Tr(A^2) - Tr(A)^2) \\ E_3(A) &= \frac{1}{6}(Tr(A)^3 - 3Tr(A)Tr(A^2) + Tr(A^3)) \end{aligned}$$

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$\rightarrow$ 3 parameter family of non-trivial theories

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→ BD ghost $\implies$ higher order equations of motion for the scalar mode in the decoupling limit

- dRGT massive gravity avoids this by ensuring that the scalar has only second order equations of motion
- Presence of Galileon terms in the decoupling limit action

Vierbein reformulation

- Unpleasant presence of a matrix square root in the action

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Existence of matrix square roots

A necessary and sufficient condition for a real matrix A to admit a real square root is the following: for every negative eigenvalue λ, the number of identical Jordan blocks (in the Jordan decomposition of A) associated with λ must be even

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$$g_{\mu\nu} = \eta_{AB} E^A{}_\mu E^B{}_\nu \quad \text{and} \quad g^{\mu\nu} = \eta^{AB} e_A{}^\mu e_B{}^\nu$$
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- Symmetry condition $e_A{}^\mu L_{B\mu} = e_B{}^\mu L_{A\mu}$ implies

$$\sqrt{g^{-1}f}^\mu{}_\nu = e_A{}^\mu L^A{}_\nu$$

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$$S_{dRGT} = M_P^2 \int \Omega^{AB} \wedge E^*_{AB} - M_P^2 m^2 \sum_{n=0}^{3} \beta_n \int L^{A_1} \wedge \cdots \wedge L^{A_n} \wedge E^*_{A_1 \ldots A_n}$$

where

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? Choosing vierbeins obeying the symmetry condition

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First result

An invertible matrix M can be decomposed as $M = \Lambda.S$ with Λ being a Lorentz transformation matrix and S a symmetric matrix if and only if the matrix $\eta M^t \eta M$ admits a real square root which can be written as a product of η by a symmetric matrix

An incomplete equivalence

Second result

The symmetric vierbein condition can be imposed via local Lorentz transformations if and only if

(*i*) the matrix $g^{-1}f$ admits a real square root γ

(*ii*) $f\gamma$ is symmetric

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! YES in dimensions 2, 3, 4

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HYP: $g^{-1}f$ has real square roots

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- Negative eigenvalues: only five possible Jordan forms

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\begin{aligned}
J_1 &= diag(-u,-u,-v,-v) \\
J_2 &= diag(-u,-u,v,w) \quad \text{(only one which can occur)} \\
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$\rightarrow$ J_2 : existence a real square root with the desired symmetry

The vierbein action as a starting point

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$$S_{dRGT} = M_P^2 \int \Omega^{AB} \wedge E^*_{AB} - M_P^2 m^2 \sum_{n=0}^{3} \beta_n \int L^{A_1} \wedge \cdots \wedge L^{A_n} \wedge E^*_{A_1 \ldots A_n}$$

where L^A is the non-dynamical one-form dx^A and

$$E^*_{A_1 \ldots A_n} \equiv \frac{1}{(4-n)!} \varepsilon_{A_{n+1} \ldots A_4} E^{A_{n+1}} \wedge \cdots \wedge E^{A_4}$$

$$\Omega^{AB} \equiv d\omega^{AB} + \omega^A{}_C \wedge \omega^{CB} \quad \text{(Curvature two-form)}$$

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! Theory with a priori 16 degrees of freedom

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- Einstein three-form

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→ Equations of motion: $G_A = t_A$ or, in components, $G_{AB} = t_{AB}$

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  - Only β_0 and β_1 are non-zero
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- Counter-example: if only β_0 and β_2 an antisymmetric vierbein condition is also compatible with the constraints

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- Specific cases
  - Only β_0 and β_1 are non-zero

$$\omega^B{}_{A\mu} {e_B}^\mu = 0$$

  - Only β_0 and β_2 are non-zero (with the symmetry condition enforced)

$$\omega^B{}_{C\mu} {e_B}^\mu {e_A}^C + \omega^B{}_{A\mu} {e_C}^\mu {e_B}^C - \omega^B{}_{A\mu} {e_B}^\mu {e_C}^C = 0$$

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ADDITIONAL CONSTRAINT

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- Other cases not treatable in the same manner...

Recovering Fierz-Pauli

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- Symmetry condition verified because $t^{AB}_{(1)} \propto \mathrm{Tr}(e^{(1)})\eta^{AB} - e^{AB}_{(1)}$
- Relationship to the metric formalism well defined $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

Recovering Fierz-Pauli

- Change of variable formulas

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$\rightarrow$ The constraints are independent

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$\rightarrow$ It should be easier to do than in the metric formalism