EMRI Modeling in Perturbation Theory

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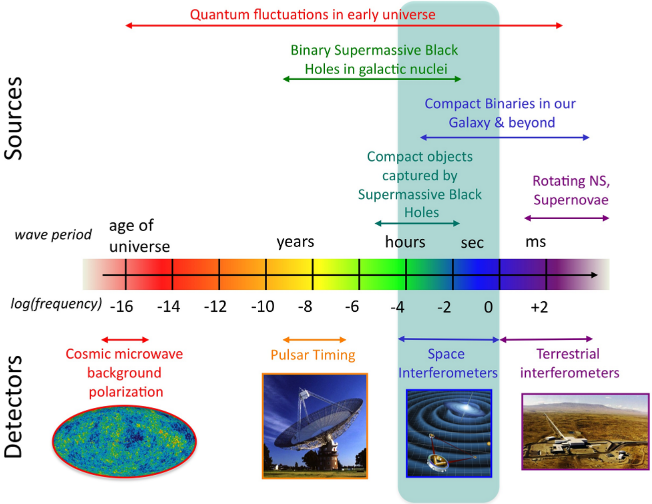
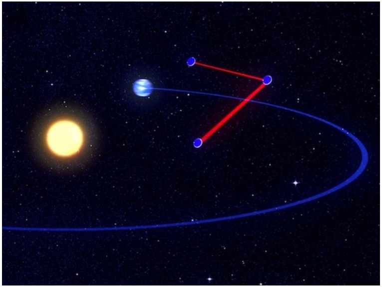
Physical Sources



[NASA]

eLISA

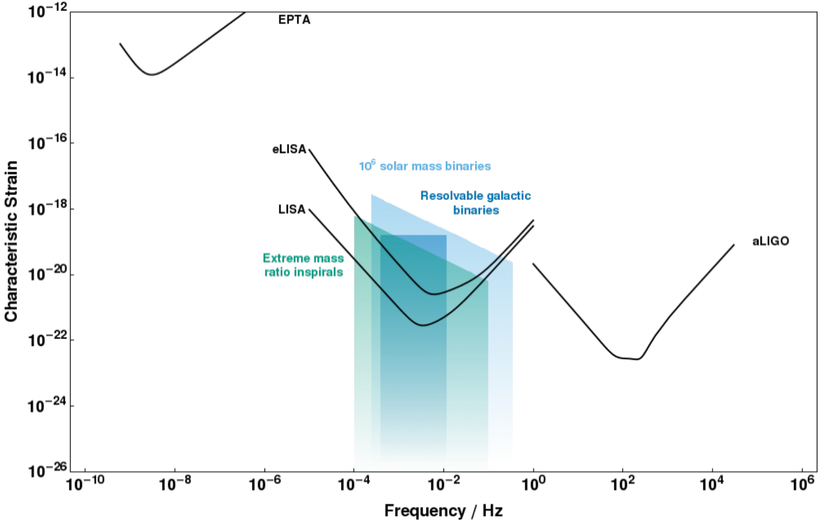
Proposed space-based interferometer and successor to the LISA project.



[AEI]

- low frequency range of 10^{-4} Hz $\leq f \leq 10^{-1}$ Hz
- Two-arm interferometer with 10^{6} km beam length
- Flight in 2028?

eLISA



[Berry et al.]

All Warmed Up

- The EMRI System
- Radiation Background
- First-Order Theory
- Second-Order Theory

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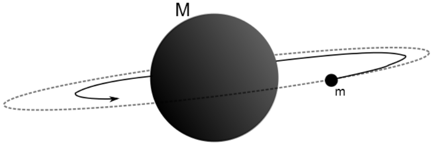
Summary of the EMRI Problem



- Solar mass compact object in orbit about a supermassive black hole
- Mass ratio $m/M \ll 1$
- System loses energy due to radiation; inspiral timescale of $\sim M/m$ orbits

Why Perturbation Theory?

- Linearize the Einstein equations, allowing for point-like particles.
- The two length scales in the problem pose difficulties for numerical relativity.
- Strong relativistic fields disallow post-Newtonian analyses close to merger.
- Mature models allow for comparisons between theories...

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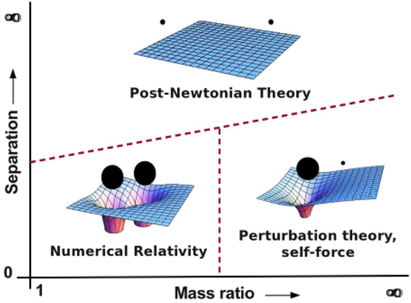
Binary parameter space



[Leor Barack]

- u^t comparisons between pN and PT
- IMRI comparisons between NR and PT

Checkpoint

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Special Radiation

[Dirac]

Develop a relativistic treatment of accelerated charged particles.

- Solve the equations

$$\partial^\nu \partial_\nu A_\mu = 4\pi j_\mu$$
$$\partial^\mu A_\mu = 0$$

- Find a general solution for the field,

$$\begin{aligned} F^{\mu\nu}_{\text{act}} &= F^{\mu\nu}_{\text{adv}} + F^{\mu\nu}_{\text{in}} \\ &= F^{\mu\nu}_{\text{ret}} + F^{\mu\nu}_{\text{out}} \end{aligned}$$

- Define the radiation field as

$$F^{\mu\nu}_{\text{rad}} = F^{\mu\nu}_{\text{out}} - F^{\mu\nu}_{\text{in}}$$

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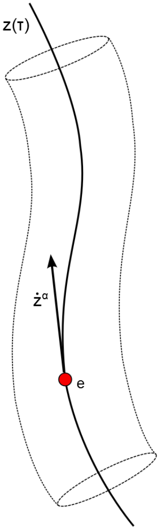

- Examine the fields,

$$F_{\mathrm{S}}^{\mu\nu} \equiv \frac{1}{2}\left(F_{\mathrm{ret}}^{\mu\nu} + F_{\mathrm{adv}}^{\mu\nu}\right)$$
$$F_{\mathrm{R}}^{\mu\nu} \equiv F_{\mathrm{act}}^{\mu\nu} - F_{\mathrm{S}}^{\mu\nu}$$

- $F_{\mathrm{R}}^{\mu\nu}$ is constructed purely from homogeneous solutions and produces the radiation reaction:

$$m\ddot{z}_\mu = e\,\dot{z}_\nu F^{\mathrm{R}}{}_{\mu}{}^{\nu}$$

Electric Radiation in Curved Spacetimes [DeWitt & Brehme]

Solve the equation: $\nabla^2 A_\mu + R_\mu{}^\gamma A_\gamma = -4\pi j_\mu$. Local hyperbolicity of the field equations allows for a Hadamard expansion of the Green's function:

$$G^{\mathrm{sym}}_{\mu\alpha}(x,z) = \frac{1}{8\pi}\left[u_{\mu\alpha}(x,z)\delta(\sigma) - v_{\mu\alpha}(x,z)\theta(-\sigma)\right].$$

Conserving energy-momentum flux through the worldtube provides the equations of motion:

$$m\ddot{z}^\alpha = eF^{\mathrm{in}\ \alpha}{}_\beta \dot{z}^\beta + \frac{2}{3}(\dddot{z}^\alpha - \dot{z}^\alpha \ddot{z}^2) + e^2\dot{z}^\beta \int_{-\infty}^{\tau} (\nabla_\beta v^\alpha{}_{\gamma'} - \nabla^\alpha v_{\beta\gamma'})\dot{z}^{\gamma'}\, \mathrm{d}\tau'$$

- Standard radiation damping terms
- "tail" piece caused by curvature

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Linearize the Einstein Equations

- Assume that the physical spacetime may be expanded perturbatively,

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu},$$

where $g^0_{\mu\nu}$ is the Schwarzschild metric and $h_{\mu\nu} \sim \mathcal{O}(m)$.

- By defining a variation of the Einstein tensor,

$$G^{(n)}_{\mu\nu}(g, h) \equiv \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n} G_{\mu\nu}(g + \epsilon h)\right]_{\epsilon=0},$$

we may expand $G_{\mu\nu}(g^0 + h)$ about the background g^0:

$$G_{\mu\nu}(g^0 + h) = G^{(1)}_{\mu\nu}(g^0, h) + G^{(2)}_{\mu\nu}(g^0, h) + \cdots,$$

with $G^{(n)}_{\mu\nu}(g^0, h) \sim \mathcal{O}(m^n)$.

Linearize the Einstein Equations

Given a perturbing stress-energy tensor $T_{\mu\nu} \sim \mathcal{O}(m)$, the Einstein equations may be written to first-order in m as

$$G^{(1)}_{\mu\nu}(g^0, h) = 8\pi T_{\mu\nu} + \mathcal{O}(m^2),$$

with

$$\begin{aligned} G^{(1)}_{\mu\nu}(g^0, h) = {} & \nabla^2 h_{\mu\nu} + \nabla_\mu \nabla_\nu h - 2\nabla_{(\mu}\nabla^\alpha h_{\nu)\alpha} \\ & + 2R^0{}_\mu{}^\alpha{}_\nu{}^\beta h_{\alpha\beta} + g^0_{\mu\nu}(\nabla^\alpha\nabla^\beta h_{\alpha\beta} - \nabla^2 h). \end{aligned}$$

Gravitational Radiation Reaction

[Mino et al.] & [Quinn & Wald]

Goal: extend the work done by DeWitt & Brehme to gravity.

Assumptions:

- Point-particle limit is well-defined within the scope of linearized theory
- 4-momentum is proportional to 4-velocity,

$$p^\alpha = m u^\alpha$$

- Particle travels along a geodesic of $g^0_{\mu\nu}$ at lowest order

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$$\Rightarrow T_{\mu\nu} = m \int \dot{z}_\mu(\tau)\dot{z}_\nu(\tau)\delta^{(4)}(x, z(\tau))\, \mathrm{d}\tau$$

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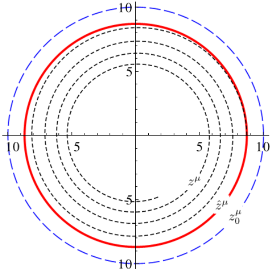

[Adam Pound]

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- Apply the Lorentz gauge to simplify the equations:

$$\nabla^\mu \bar{h}_{\mu\nu} = 0$$

$$\nabla^2 h_{\mu\nu} - 2R^{0\;\alpha\;\;\beta}_{\;\mu\;\;\nu} h_{\alpha\beta} = -16\pi T_{\mu\nu} + \mathcal{O}(m^2)$$

- Extensive local Green's function analysis finds

$$\ddot{z}^\mu = \left(\tfrac{1}{2}\nabla^\mu h^{\text{in}}_{\alpha\beta} - \nabla_\alpha h^{\text{in}\;\mu}_{\;\;\;\beta} - \tfrac{1}{2}\dot{z}^\mu \dot{z}^\gamma \nabla_\gamma h^{\text{in}}_{\alpha\beta}\right)\dot{z}^\alpha \dot{z}^\beta - \tfrac{11}{3} m (\dddot{z}^\mu - \ddot{z}^2 \dot{z}^\mu)$$

$$+ m\dot{z}^\alpha \dot{z}^\beta \int_{-\infty}^{\tau} \left(\frac{1}{2}\nabla^\mu G^-_{\alpha\beta a'b'} - \nabla_\alpha G^{-\;\mu}_{\beta\;\;a'b'} - \frac{1}{2}\dot{z}^\mu \dot{z}^\gamma \nabla_\gamma G^-_{\alpha\beta a'b'}\right)\dot{z}^{a'}\dot{z}^{b'}\, d\tau'$$

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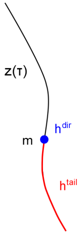
Gravitational Radiation Reaction

[Mino et al.] & [Barack et al.]

Proposed metric perturbation decomposition: $h_{\mu\nu} = h^{\text{dir}}_{\mu\nu} + h^{\text{tail}}_{\mu\nu}$



- $h^{\text{dir}}_{\mu\nu}$ produces a generalization of the Abraham-Lorentz-Dirac force.
- $h^{\text{dir}}_{\mu\nu}$ and $\nabla_\alpha h^{\text{tail}}_{\mu\nu}$ diverge in the coincidence limit.

"Regularize" the physical spacetime:

$$\tilde{g}_{\mu\nu} = g^0_{\mu\nu} + h^{\text{tail}}_{\mu\nu}$$

$$\Rightarrow F^{\text{self}}_\alpha = F^{\text{tail}}_\alpha \equiv m k_\alpha{}^{\beta\gamma\delta} \left\langle \nabla_\delta \bar{h}^{\text{tail}}_{\beta\gamma} \right\rangle$$

Taking a Step Back

- Arbitrary mass renormalization constant

  $\Rightarrow$ fix with asymptotic matching

- Conflict between point-particle limit and Lorentz gauge:

  consistent analysis requires "gauge relaxation"

$$\nabla^{\mu}\bar{h}_{\mu\nu} = 0 \quad \rightarrow \quad \nabla^{\mu}\bar{h}_{\mu\nu} = \mathcal{O}(m^2)$$

- Neither h^{dir} nor h^{tail} is "physical," i.e. neither is independently a solution to the linearized Einstein equations

  $\Rightarrow \tilde{g}$ is not a "physical" background

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Re-Imagining the Background

[Detweiler & Whiting]

- Green's function used by Mino et al. is locally unique *up to a homogeneous solution* H.
- Choice for H motivated by Dirac's $F^{\mathsf{S}}_{\mu\nu}$ field construction:

$$G^{\mathsf{S}} \equiv G^{\mathsf{sym}} + H$$
$$= \frac{1}{8\pi}\left[u(x,z)\delta(\sigma) + v(x,z)\theta(\sigma)\right]$$

- Tensor perturbation produced, $h^{\mathsf{S}}_{\mu\nu}$, is a local particular solution to the linearized EEs.
- Provides a regular, homogeneous remainder when removed from the retarded perturbation,

$$h^{\mathsf{R}}_{\mu\nu} \equiv h^{\mathsf{ret}}_{\mu\nu} - h^{\mathsf{S}}_{\mu\nu}$$

Re-Imagining the Background

[Detweiler & Whiting]

- Both $h^{\mathrm{S}}_{\mu\nu}$ and $h^{\mathrm{R}}_{\mu\nu}$ are separately solutions to the linearized EEs.

- The force results from the regular, differentiable $h^{\mathrm{R}}_{\mu\nu}$,

$$F^{\alpha}_{\mathrm{self}} = -m\left(g_0^{\alpha\beta} - \dot{z}^{\alpha}\dot{z}^{\beta}\right)\dot{z}^{\gamma}\dot{z}^{\delta}\left(\nabla_{\gamma}h^{\mathrm{R}}_{\delta\beta} - \frac{1}{2}\nabla_{\beta}h^{\mathrm{R}}_{\gamma\delta}\right)$$

- The "regularized" physical metric now consists of the combination of the background and a non-local homogeneous field,

$$\tilde{g}_{\mu\nu} = g^{0}_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$$

Effectively Sourced

[Barack & Goldbourn] & [Vega & Detweiler]

The schematic difference $h^{\mathrm{R}}_{\mu\nu} = h^{\mathrm{ret}}_{\mu\nu} - h^{\mathrm{S}}_{\mu\nu}$ belies the complex regularization techniques required to resolve $h^{\mathrm{R}}_{\mu\nu}$.

The standard technique is mode-sum regularization,

$$h^{\mathrm{R}}_{\mu\nu}(z) = \sum_{\ell} h^{\mathrm{R},\ell}_{\mu\nu}(z) = \lim_{x\to z} \sum_{\ell} \left[h^{\mathrm{ret},\ell}_{\mu\nu}(x) - h^{\mathrm{S},\ell}_{\mu\nu}(x) \right]$$

Effectively Sourced

[Barack & Goldbourn] & [Vega & Detweiler]

What if we can't integrate for $h^{\mathrm{ret}}_{\mu\nu}$?

Effectively Sourced

[Barack & Goldbourn] & [Vega & Detweiler]

Consider the following minimally-coupled scalar charge problem:

$$\nabla^2 \psi^{\mathsf{ret}} = -4\pi q \int_\gamma \delta^{(4)}(x - z(\tau))\, \mathrm{d}\tau$$

- The self-force as described by $F_\alpha = q \nabla_\alpha \psi^{\mathsf{ret}}$ will require regularization.

- Consider a local expansion of the following singular field:

$$\psi^{\mathsf{S}} = \frac{q}{\rho} + \mathcal{O}(\rho^3/\mathcal{R}^4) \quad \text{as} \quad \rho \to 0$$

As an approximation, we truncate the expansion of ψ^{S},

$$\psi^{\mathcal{P}} \equiv \frac{q}{\rho}$$

Effectively Sourced

[Barack & Goldbourn] & [Vega & Detweiler]

- Locally, then, we see

$$\nabla^2\psi^{\mathcal{P}} = -4\pi q \int_\gamma \delta^{(4)}(x - z(\tau))\, \mathrm{d}\tau + \mathcal{O}(\rho/\mathcal{R}^4) \quad \text{as} \quad \rho \to 0$$

- Define a “window” function W, such that $W = 1 + \mathcal{O}(\rho/\mathcal{R}^4)$ locally and vanishes sufficiently far away from the worldline.

- We may then define a residual field,

$$\psi^{\mathcal{R}} \equiv \psi^{\mathsf{ret}} - W\psi^{\mathcal{P}},$$

and we observe that

$$\begin{aligned}\nabla^2\psi^{\mathcal{R}} &= -\nabla^2(W\psi^{\mathcal{P}}) - 4\pi q \int_\gamma \delta^{(4)}(x - z(\tau))\, \mathrm{d}\tau \\ &\equiv S_{\mathsf{eff}}\end{aligned}$$

Effectively Sourced

[Barack & Goldbourn] & [Vega & Detweiler]

So what's the point?

- We are left to solve the following equation:

$$\nabla^2\psi^{\mathcal{R}} = S_{\text{eff}}$$

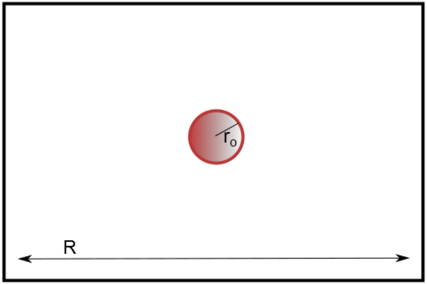
- Close to the particle, $\psi^{\mathcal{R}} \approx \psi^{\text{R}}$
- In the wave zone, $\psi^{\mathcal{R}} = \psi^{\text{ret}}$

A Simple Example

Imagine an object of small radius r_o with spherically symmetric charge density $\rho(r)$ and associated potential φ^{act}.

- Place this object inside a conducting box with size $R \gg r_o$.
- Assume also that the object is at rest at the origin, which eliminates radiation; thusly, the field φ^{act} will satisfy

$$\nabla^2 \varphi^{act} = -4\pi\rho.$$



- Our goal: solve for φ^{act} numerically, given $\rho(r)$, and the force exerted on the object due to φ^{act}.
- With $R \gg r_o$, we must utilize two vastly different length scales in the computation.

A Simple Example

- Given $\rho(r) = \text{const.}$ and total charge q, then we may write the particular solution as

$$\varphi^{\mathrm{S}}(r) = \begin{cases} \dfrac{q}{2r^3}(3r_o^2 - r^2) & r < r_o \\ \dfrac{q}{r} & r > r_o \end{cases}$$

  and it is clear to see that $\nabla^2 \varphi^{\mathrm{S}} = -4\pi\rho$.

- Define $\varphi^{\mathrm{R}} \equiv \varphi^{act} - \varphi^{\mathrm{S}}$

- Thus

$$\nabla^2 \varphi^{\mathrm{R}} = -\nabla^2 \varphi^{\mathrm{S}} - 4\pi\rho = 0$$

A Simple Example

- We may re-formulate the problem to solve for the homogeneous field,

$$\nabla^2\varphi^{\mathrm{R}} = 0$$

subject to more complicated boundary conditions,

$$\varphi^{\mathrm{R}} = -\varphi^{\mathrm{S}} \quad \text{on the box.}$$

- The force on the object may be found

$$\begin{aligned} \mathbf{F} &= -\int \rho(r)\nabla\varphi^{act}\,\mathrm{d}^3x \\ &= -\int \rho(r)\nabla\varphi^{\mathrm{R}}\,\mathrm{d}^3x \\ &\to -q\nabla\varphi^{\mathrm{R}}|_{r=0} \quad \text{in the point-particle limit} \end{aligned}$$

A Simple Example: Part 2

Consider again the problem of the charged object in the conducting box. Here, we wish to solve the problem while maintaining the boundary conditions.

Define a window function with the following properties:

- $W(r) = 1$ in a region which includes at least the entire source.
- $W(r) = 0$ for $r > r_W$, where $r_o < r_W < R$.
- $W(r)$ is C^∞ and changes only over a large length scale $\sim r_W$

A Simple Example: Part 2

The puncture and residual fields are then constructed such that,

$$\nabla^2 \varphi^{\mathcal{P}} \approx -4\pi\rho$$
$$\varphi^{\mathcal{R}} = \varphi^{\text{act}} - W\varphi^{\mathcal{P}}$$

and we arrive with the following equation,

$$\nabla^2 \varphi^{\mathcal{R}} = -\varphi^{\mathcal{P}} \nabla^2 W - 2\nabla W \cdot \nabla \varphi^{\mathcal{P}}$$
$$\equiv S_{\text{eff}}$$

and the original boundary condition

$$\varphi^{\mathcal{R}} = 0 \quad \text{on the box.}$$

Validity of Assumptions

Recall the assumptions made:

- Point-particle limit is well-defined within the scope of linearized theory
- 4-momentum is proportional to 4-velocity,

$$p^\alpha = mu^\alpha$$

- Particle travels along a geodesic of $g^0_{\mu\nu}$ at lowest order

Validity of Assumptions

Is the point-particle limit valid?

Rigorous Verification

[Gralla & Wald]

Utilize a regular expansion of the physical spacetime, chosen to be a "properly" scaling 1-parameter family of spacetimes.

- Regular expansion of the metric:

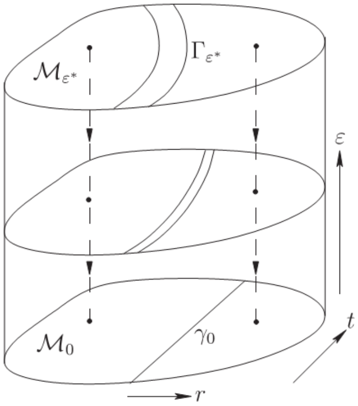
$$g_{\mu\nu}(x,\epsilon) = g^0_{\mu\nu}(x) + \epsilon h^{(1)}_{\mu\nu}(x) + \epsilon^2 h^{(2)}_{\mu\nu}(x) + \cdots$$

- Solve the perturbation equations external to the body:

$$G^{(1)}_{\mu\nu}(g^0, h^{(1)}) = 0$$
$$G^{(1)}_{\mu\nu}(g^0, h^{(2)}) = -G^{(2)}_{\mu\nu}(g^0, h^{(1)})$$

Rigorous Verification

[Gralla & Wald]



[Adam Pound]

- Expand the worldine about a remnant geodesic in the limit $\epsilon \to 0$:

$$z^\mu(\tau, \epsilon) = z_0^\mu(\tau) + \epsilon Z^\mu(\tau) + \cdots$$

- Derive equations of motion for Z^μ at first order.

Rigorous Verification

[Gralla & Wald]

- Makes no assumptions about the body metric.
- Involves no gauge relaxation.
- *Derives* the point-particle approximation.
- Extendable to arbitrary perturbation order.

However:

- Remains only a *local* solution.
- Analysis fails after de-phasing timescale.

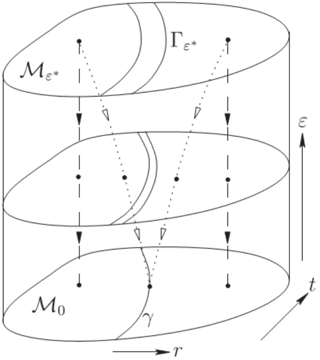
A Self-Consistent Approach

[Pound]

A similar approach, taking motivation from Kates' developed singular perturbation theory in GR.

- Singular expansion of the metric:

$$g_{\mu\nu}(x,\epsilon) = g^0_{\mu\nu}(x) + \epsilon h^{(1)}_{\mu\nu}(x;\gamma) + \epsilon^2 h^{(2)}_{\mu\nu}(x;\gamma) + \cdots$$



[Adam Pound]

- The source moves on z^μ, a worldline which faithfully tracks the body's bulk motion.
- Requires use of coordinates centered on the unspecified worldline.

A Self-Consistent Approach

[Pound]

- Makes no assumptions about the body metric.
- Involves no gauge relaxation.
- *Derives* the point-particle approximation.
- Extendable to arbitrary perturbation order.
- Valid for long timescales.

However:

- Remains only a *local* solution.
- At present, no clear implementation method.

The Final Stretch

- The EMRI System
- Radiation Background
- First-Order Theory
- Second-Order Theory

Motivating the Work

First-order solutions fail to correctly model an EMRI system over the timescales required for evolution.

- The timescale of an inspiral is roughly the radiation-reaction time

$$t_{\text{in}} \sim \frac{M}{m}$$

- The error incurred in the geodesic deviation vector scales locally as

$$\delta Z^{\mu} \sim \left(\frac{m}{M}\right)^2 t^2$$

- After the radiation-reaction time, the deviation to the worldline is no longer a perturbation!

Second-Order Mess

In general, we may write down the second-order equations:

- Express the full physical spacetime to second-order,

$$g_{\mu\nu} = g^{0}_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu},$$

  where $h^{(n)}_{\mu\nu} \sim \mathcal{O}(m^n)$.

- The full second-order problem may be expressed as,

$$G_{\mu\nu}(g^0 + h^{(1)} + h^{(2)}) = 8\pi T_{\mu\nu}(\gamma_0 + \gamma_{1R}) + \mathcal{O}(m^3).$$

- Now expand completely:

$$\begin{aligned} G^{(1)}_{\mu\nu}(g^0, h^{(2)}) = {} & 8\pi T_{\mu\nu}(\gamma_0 + \gamma_{1R}) - 8\pi T_{\mu\nu}(\gamma_0) \\ & - G^{(2)}_{\mu\nu}(g^0, h^{(1)}) + \mathcal{O}(m^3). \end{aligned}$$

Second-Order Mess

- The terms on the RHS introduce several problems to the analysis:

$$G^{(1)}_{\mu\nu}(g^0, h^{(2)}) = 8\pi T_{\mu\nu}(\gamma_0 + \gamma_{1R}) - 8\pi T_{\mu\nu}(\gamma_0) - G^{(2)}_{\mu\nu}(g^0, h^{(1)}) + \mathcal{O}(m^3).$$

- Written out explicitely, it is clear that the non-linearities in $G^{(2)}_{ab}(g, h)$ pose conceptual difficulties.

$$\begin{aligned} 2G^{(2)}_{ab}(g^0, h) = {} & h^{cd}\nabla_a\nabla_b h_{cd} + \frac{1}{2}(\nabla_a h^{cd})(\nabla_b h_{cd}) - \frac{1}{2}C^d(2\nabla_{(a}h_{b)d} - \nabla_d h_{ab}) - 2h^{cd}\nabla_d\nabla_{(a}h_{b)c} + h^{cd}\nabla_c\nabla_d h_{ab} \\ & - h_{ab}\nabla_c\nabla_d h^{cd} + h_{ab}\nabla^2 h - (\nabla^d h_{ac})(\nabla^c h_{bd}) + (\nabla^d h_{ac}(\nabla_d h_b{}^c) + g_{ab}\Big[\frac{1}{2}h^{cd}\nabla_c C_d + \frac{1}{4}C^d C_d \\ & - \frac{1}{2}h^{cd}\nabla_c\nabla_d h - h^{cd}\nabla^2 h_{cd} + h^{cd}\nabla^e\nabla_d h_{ce} + \frac{1}{2}(\nabla_d h_{ce})(\nabla^e h^{cd}) - \frac{3}{4}(\nabla_e h_{cd})(\nabla^e h^{cd})\Big] \end{aligned}$$

$$C_d \equiv 2\nabla^c h_{cd} - \nabla_d h_c{}^c$$

A General Approach to a Solution

[Gralla] & [Pound]

Adapt the puncture method to the second-order problem.

- Assume we know the singular fields "well enough" within the region close to m,

$$h^{1\mathrm{S}}_{\mu\nu} = h^{1\mathcal{P}}_{\mu\nu} + \mathcal{O}(mx^4/r\mathcal{R}^4)$$
$$h^{2\mathrm{S}}_{\mu\nu} + h^{2\mathrm{S}\dagger}_{\mu\nu} = h^{2\mathcal{P}}_{\mu\nu} + \mathcal{O}(m^2x^4/r^2\mathcal{R}^4)$$

- The non-linearities are then smoothed out within the order of the approximation,

$$G^{(2)}_{\mu\nu}(g^0, h^{1\mathrm{S}}) + G^{(1)}_{\mu\nu}(g^0, h^{2\mathrm{S}}) \approx G^{(2)}_{\mu\nu}(g^0, h^{1\mathcal{P}}) + G^{(1)}_{\mu\nu}(g^0, h^{2\mathcal{P}}) \rightarrow \mathcal{O}(m^3)$$

A General Approach to a Solution

[Gralla] & [Pound]

- We may find the approximate solution, given by the residual field

$$h^{\mathcal{R}} \approx h^{\mathrm{R}}.$$

- h^{R} is the desired Detweiler-Whiting field and sources the second-order self-force:

$$\frac{D^2 z^\mu}{d\tau^2} = \frac{1}{2}\left(g^{\mu\nu} + \dot{z}^\mu \dot{z}^\nu\right)\left(g_\nu{}^\rho - h^{\mathrm{R}}_\nu{}^\rho\right)\left(\nabla_\rho h^{\mathrm{R}}_{\sigma\lambda} - 2\nabla_\lambda h^{\mathrm{R}}_{\rho\sigma}\right)\dot{z}^\sigma \dot{z}^\lambda + \mathcal{O}(m^3)$$

Conclusions

- Gravitational self-force formalism has a long, stimulating history
- Recent developments have overcome certain dubious assumptions
- A second-order formalism is established

At present, both UF and Southhampton are working toward circular orbits in Schwarzschild at second-order.