Self-gravity, Resonances & Orbital diffusion in stellar discs and its WKB limit

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12$^{\text{th}}$ October 2015
IAP - GRECO

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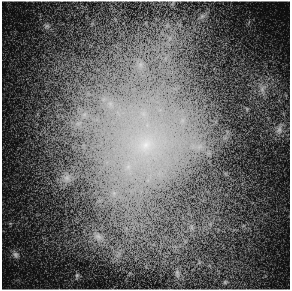
Context

- ΛCDM paradigm for the formation of structures
  Interactions with the circum-galactic medium
  - **Constructive** (e.g. adiabatic gas accretion)
  - **Destructive** (e.g. satellite infall)



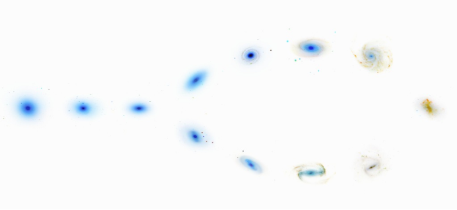
500 kpc

- Recent theoretical works to describe the effects on a system's over cosmic time induced by
  - **External disturbances** (e.g. large structures)
  - **Discreteness noise** (e.g. giant molecular clouds)

What are the respective roles of **nature vs. nurture** on the observed properties of self-gravitating systems?

Why Secular Dynamics ?

What do stable self-gravitating galactic discs do during a Hubble time?



How does a galaxy respond

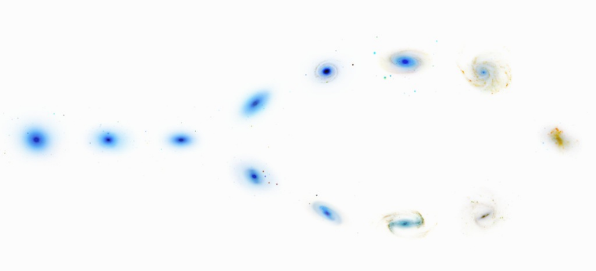
- *to its environment?* Nurture
    Dressed Fokker Planck diffusion
- *to its internal graininess?* Nature
    Balescu-Lenard diffusion

- *Which process matters most on cosmic timescales?*

Of interest for galactic chemodynamics (GAIA), planetesimals, Galactic Center

Powerful quasi-linear theories accounting for non-linear gravity for $t \gg t_{\mathrm{dyn}}$

The key aspects of Secular Dynamics



- **Galactic discs** have secularly five main characteristics
  - Inhomogeneous (= *complex trajectories*)
  - Multi-periodic with short dynamical times (= *phase mixed*)
  - Embedded in a *live* cosmic environment (= *perturbed*)
  - Made of a finite number of particles (= *discrete*)
  - Self-gravitating and cold (= *amplified*)

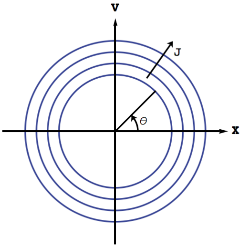
- **Resonant** effects $\Longrightarrow$ **Secular evolution**.

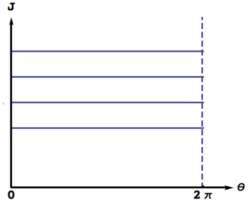
Angle-Actions coordinates

- Goal : Describe the complex physical trajectories using **integrals of motions**
- Action J_i associated to the i^{th} coordinate (q_i, p_i)

$$J_i = \frac{1}{2\pi} \oint_{\gamma_i} \mathrm{d}q_i \, p_i \, .$$

- Consequences : $H(\boldsymbol{q}, \boldsymbol{p}) = H(\boldsymbol{J})$. *Intrinsic* frequencies : $\boldsymbol{\Omega} = \partial H / \partial \boldsymbol{J}$.
- Hamilton's Equations become : $\begin{cases} \boldsymbol{\theta}(t) = \boldsymbol{\theta}_0 + \boldsymbol{\Omega}\, t \;\; \in [0; 2\pi[\, , \\ \boldsymbol{J}(t) = cst. \, . \end{cases}$





Examples of Angle-Actions Coordinates

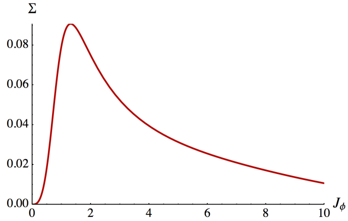
- **Harmonic oscillator** : $\begin{cases} \theta = \tan^{-1}[v/\omega x] \, , \\ J = E/\omega \, . \end{cases}$
- **Homogeneous system in a periodic box** : $\begin{cases} \boldsymbol{x} \sim \boldsymbol{\theta} \, , \\ \boldsymbol{v} \sim \boldsymbol{J} \, . \end{cases}$
- **2D axisymmetric potential** : $\begin{cases} J_r = \dfrac{1}{\pi} \displaystyle\int_{r_{\min}}^{r_{\max}} \mathrm{d}r \, \sqrt{2(E - \Phi(r)) - L_z^2/r^2} \, , \\ J_\phi = L_z \, . \end{cases}$
- **Spherical potentials** : $\begin{cases} J_\phi = L_z \, , \\ J_\theta = L - |L_z| \, , \\ J_r \, . \end{cases}$

Some nice properties :

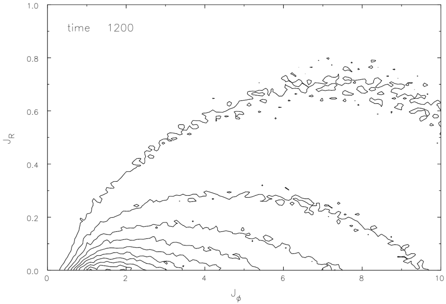
- **Jean's Theorem** : Stationnary DF are of the form $F_0(\boldsymbol{J})$.
- Actions are adiabatic invariants.
- Canonical transformation : $\mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} = \mathrm{d}\boldsymbol{\theta} \, \mathrm{d}\boldsymbol{J}$.
- Diagonalise the linearised Boltzmann equation.
- Label orbits!

An example of secular evolution

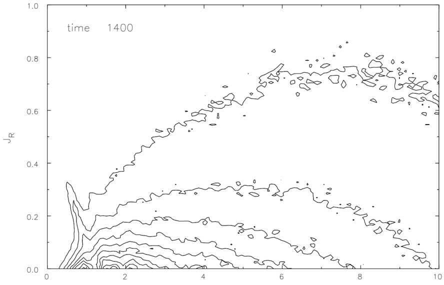
- Sellwood's 2012 numerical experiment
  - Stationnary tapered *Mestel disc* sampled with up to $500M$ particles
  - Evolution with a $N-$Body code
  - Appearance of *transient spiral waves*



Initial DF



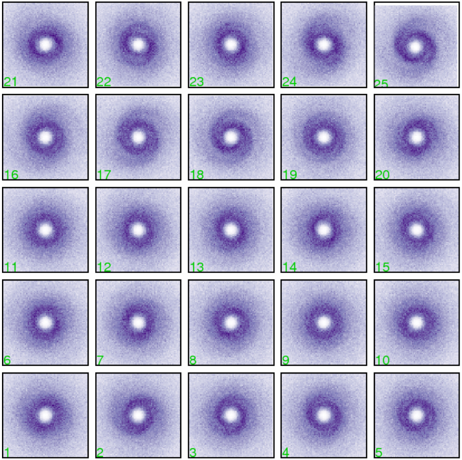
Evolved DF



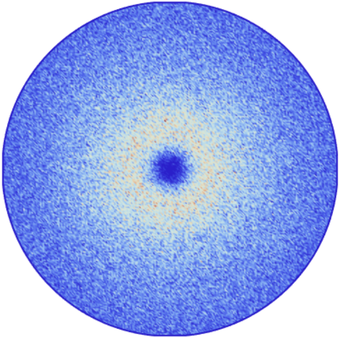
Sellwood (2012)

Secular diffusion in action-space

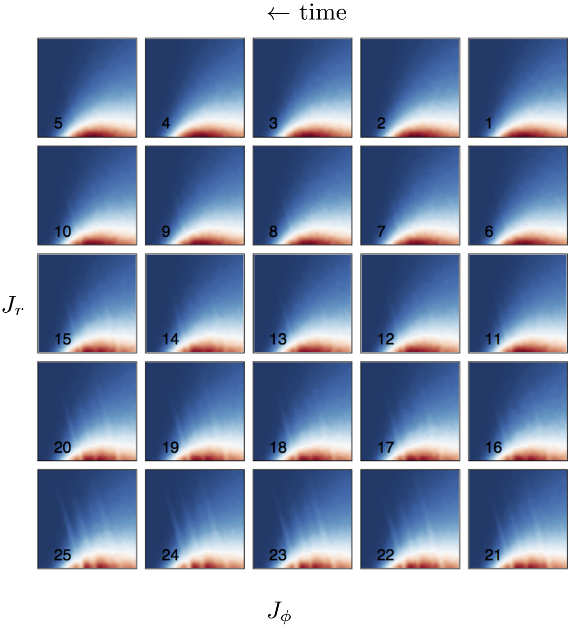
An example of secular evolution

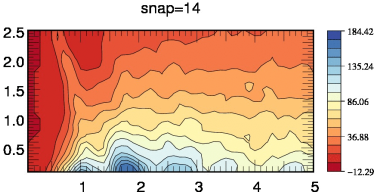


→ time



An example of secular evolution





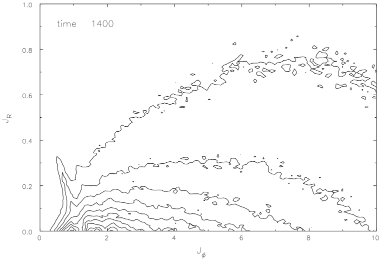
Dressed Fokker-Planck diffusion equation

- **Aim** : Describe the *secular forcing* of a stable collisionless self-gravitating system induced by an external perturbation.

- References
  - *Binney, Lacey (1988)* Angle-Action no self-gravity
  - *Weinberg (1993)* Angle-Action self gravitating sphere
  - *Pichon, Aubert (2006)* Angle-Action self gravitating disc
  - *Fouvry, Pichon, Prunet (2015)* MNRAS 2015 449 (1)
  - *Fouvry, Pichon (2015)* MNRAS 2015 449 (2)
  - *Fouvry, Binney, Pichon (2015)* ApJ 806 117

- This work: self-gravitating WKB limit + explicit quadrature for diffusion/drift coefficient.

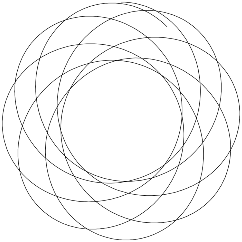
Case of application



- **Self-gravitating responsive system**
  - A typical infinitely thin galactic disc.
- **Source of exterior perturbations**
  - Surrounding dark-matter halo.
  - Intrinsic *internal noise*.
- **Probes of secular evolution**
  - *Temperature* of the disc : typical spread in radial energy ΔJ_r.
  - Radial migration : wandering in action-space (J_ϕ, J_r).
  - *Resonant ridges* in action-space.

The hypothesis of the formalism

- **Collisionless system**
  - Collisionless Bolztmann's Equation $$\boxed{\frac{\partial F}{\partial t} + [F, H] = 0\,.}$$
- **Small perturbations**
  - Perturbation Theory and Quasi-Linear Approach
- **Complex Trajectories in physical space**
  - Use of the angle-actions coordinates $(\boldsymbol{\theta}, \boldsymbol{J})$.
- **Timescales decoupling**
  - Short oscillating timescale = dynamical time.
  - Long secular timescale = *diffusion* in action-space.



Collisionless Boltzmann-Poisson Equation

- **Distribution function** in phase-space $(\boldsymbol{x}, \boldsymbol{v})$: $f(\boldsymbol{x}, \boldsymbol{v}, t)$.
- Hamilton's Equations for one particle $\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} \partial H / \partial \boldsymbol{v} \\ -\partial H / \partial \boldsymbol{x} \end{bmatrix} = \dot{\boldsymbol{w}}$.
- Local conservation of the distribution function (*continuity equation*)

$$\frac{\partial f}{\partial t} + \mathrm{div}(\dot{\boldsymbol{w}} f) = 0 .$$

- *Usual* Hamiltonians are $H = \frac{1}{2}\boldsymbol{v}^2 + \Phi(\boldsymbol{x})$, so that : $\mathrm{div}(\dot{\boldsymbol{w}}) = 0$.
- Collisionless Boltzmann's Equation

$$\boxed{\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} - \frac{\partial \Phi}{\partial \boldsymbol{x}} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0 .}$$

- *Auto-Coherent* Collisionless Boltzmann Equation (= Vlasov Equation) for self-gravitating system adds

$$\boxed{\Delta \Phi = 4 \pi G \int \mathrm{d}\boldsymbol{v} \, f(\boldsymbol{x}, \boldsymbol{v}, t) .}$$

The main bricks of secular diffusion

The distribution function and Hamiltonian of the system are given by

$$\begin{cases} F(\boldsymbol{J},\boldsymbol{\theta},t) = F_0(\boldsymbol{J},t) + f(\boldsymbol{J},\boldsymbol{\theta},t)\,, \\ H(\boldsymbol{J},\boldsymbol{\theta},t) = H_0(\boldsymbol{J}) + \psi^e(\boldsymbol{J},\boldsymbol{\theta},t) + \psi^s(\boldsymbol{J},\boldsymbol{\theta},t)\,. \end{cases}$$

Hypothesis

$$\begin{cases} F_0 \,:\, \text{Secular DF evolving slowly, } \partial F_0/\partial t \ll \partial f/\partial t\,, \\ f \,:\, \text{Fast perturbations variations, } f \ll F_0\,, \\ \psi^e \,:\, \textit{Exterior} \text{ potential perturbations induced by the environment}\,, \\ \psi^s \,:\, \textit{Self} \text{ potential perturbations through the disc response}\,. \end{cases}$$

The decoupling of timescales leads to two equations of evolution

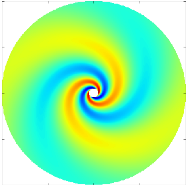
$$\text{Fast timescale :} \qquad \frac{\partial f}{\partial t} + \boldsymbol{\Omega}\cdot\frac{\partial f}{\partial \boldsymbol{\theta}} - \frac{\partial F_0}{\partial \boldsymbol{J}}\cdot\frac{\partial (\psi^e+\psi^s)}{\partial \boldsymbol{\theta}} = 0\,.$$

$$\text{Secular timescale :} \qquad \frac{\partial F_0}{\partial t} = \frac{1}{(2\pi)^d}\frac{\partial}{\partial \boldsymbol{J}}\cdot\left[\int \mathrm{d}\boldsymbol{\theta}\, f\, \frac{\partial\left[\psi^e+\psi^s\right]}{\partial \boldsymbol{\theta}}\right]\,.$$

Potential-Density basis

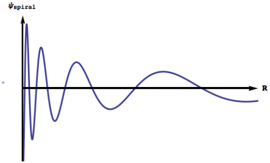
- Aims :
  - Reduce an infinite functionnal space to a finite-dimension space.
  - Solve the *non-local* Poisson equation implicitly, once for all.
- Introduce a *representative* basis of potential functions $(\rho^{(p)}, \psi^{(p)})$.
- **Biorthogonality property**

$$\Delta\psi^{(p)} = 4\pi G\, \rho^{(p)} \quad ; \quad \int \mathrm{d}\boldsymbol{x}\, [\psi^{(p)}(\boldsymbol{x})]^{*} \rho^{(q)}(\boldsymbol{x}) = -\delta_p^q \,.$$



- Examples
  - **Logarithmic spirals** (2D)

$$\begin{cases} \Sigma_{\alpha,k_\phi}(R,\phi) \propto \dfrac{1}{R^{3/2}} e^{i(\alpha \log[R/R_0] + k_\phi \phi)} \,, \\ \psi_{\alpha,k_\phi}(R,\phi) \propto \dfrac{1}{R^{1/2}} e^{i(\alpha \log[R/R_0] + k_\phi \phi)} \,. \end{cases}$$



Taking into account the dressing

How does the system respond to an imposed perturbation ?

- Introducing a *representative* basis of potential functions $\psi^{(p)}$, so that

$$\begin{cases} \psi^{\text{ext}} = \sum_p b_p(t)\, \psi^{(p)} \, . & \text{Imposed } \textbf{exterior perturbation} \\ \psi^{\text{self}} = \sum_p a_p(t)\, \psi^{(p)} \, . & \text{Amplified } \textbf{response of the system} \end{cases}$$

- Non-Markovian **amplification mechanism**

$$\boxed{\boldsymbol{a}(t) = \int_{-\infty}^{t} \mathrm{d}\tau \, \mathbf{M}(t-\tau) \left[\boldsymbol{a}(\tau) + \boldsymbol{b}(\tau)\right] \, ,}$$

where $\mathbf{M}(F_0)$ is the **response matrix** of the system, given by

$$\widehat{\mathbf{M}}_{pq}(\omega) = (2\pi)^d \sum_{\boldsymbol{m}} \int \mathrm{d}\boldsymbol{J} \, \frac{\boldsymbol{m}\cdot\partial F_0/\partial \boldsymbol{J}}{\omega - \boldsymbol{m}\cdot\boldsymbol{\Omega}} \psi_{\boldsymbol{m}}^{(p)\,*}(\boldsymbol{J}) \, \psi_{\boldsymbol{m}}^{(q)}(\boldsymbol{J}) \, ,$$

and $\psi_{\boldsymbol{m}}^{(p)}(\boldsymbol{J}) = \dfrac{1}{(2\pi)^d} \displaystyle\int \mathrm{d}\boldsymbol{\theta} \, \psi^{(p)}[\boldsymbol{x}(\boldsymbol{\theta}, \boldsymbol{J})] \, e^{-i\boldsymbol{m}\cdot\boldsymbol{\Omega}} \, .$

Resonances are at the *intrinsic frequencies* of the system : $\boldsymbol{m}\cdot\boldsymbol{\Omega}$.

The secular diffusion equation

The long term evolution equation is a *Fokker-Planck* equation (no drift term)
Binney–Lacey (1988), Weinberg (1993), Pichon–Aubert (2006)

$$\boxed{\frac{\partial F_0}{\partial t} = \sum_{\boldsymbol{m}} \boldsymbol{m} \cdot \frac{\partial}{\partial \boldsymbol{J}} \left[D_{\boldsymbol{m}}(\boldsymbol{J})\, \boldsymbol{m} \cdot \frac{\partial F_0}{\partial \boldsymbol{J}} \right] .}$$

where $D_{\boldsymbol{m}}(\boldsymbol{J})$ are anisotropic diffusion coefficients given by

$$D_{\boldsymbol{m}}(\boldsymbol{J}) = \frac{1}{2} \sum_{p,q} \psi_{\boldsymbol{m}}^{(p)}(\boldsymbol{J})\, \psi_{\boldsymbol{m}}^{(q)*}(\boldsymbol{J}) \left[[\mathbf{I} - \widehat{\mathbf{M}}]^{-1} \cdot \left\langle \widehat{\boldsymbol{b}} \cdot \widehat{\boldsymbol{b}}^{*\,t} \right\rangle \cdot [\mathbf{I} - \widehat{\mathbf{M}}]^{-1} \right]_{qp} (\omega = \boldsymbol{m} \cdot \boldsymbol{\Omega})$$

$$\sim \frac{\left\langle \left| \psi^{\mathrm{ext}}(\omega) \right|^2 \right\rangle}{\left| \varepsilon(\boldsymbol{m}, \omega) \right|^2} (\omega = \boldsymbol{m} \cdot \boldsymbol{\Omega}) \, .$$

Two behaviors are possible (*perturbation spectrum* vs. *responsiveness*)

$$\begin{cases} \dfrac{1}{|\varepsilon|^2} \gg 1 \, . & \text{The system reacts the same way, whatever the perturbations.} \\ \dfrac{1}{|\varepsilon|^2} \sim 1 \, . & \text{The system is } \textit{shaped} \text{ by the perturbations.} \end{cases}$$

Difficulties of the diffusion equation

- Evolution equation

$$\frac{\partial F_0}{\partial t} = \sum_{\boldsymbol{m}} \boldsymbol{m} \cdot \frac{\partial}{\partial \boldsymbol{J}} \left[D_{\boldsymbol{m}}(\boldsymbol{J}) \, \boldsymbol{m} \cdot \frac{\partial F_0}{\partial \boldsymbol{J}} \right] .$$

- Anisotropic diffusion coefficients, recalling that $\widehat{\mathbf{M}} = \widehat{\mathbf{M}}(F_0)$

$$D_{\boldsymbol{m}}(\boldsymbol{J}) = \frac{1}{2} \sum_{p,q} \psi_{\boldsymbol{m}}^{(p)} \psi_{\boldsymbol{m}}^{(q)*} \left[[\mathbf{I} - \widehat{\mathbf{M}}]^{-1} \cdot \widehat{\mathbf{C}} \cdot [\mathbf{I} - \widehat{\mathbf{M}}]^{-1} \right]_{qp} (\omega = \boldsymbol{m} \cdot \boldsymbol{\Omega}) \, .$$

- **Inhomogeneous system**
  - Introduce explicitly the mapping $(\boldsymbol{x}, \boldsymbol{v}) \mapsto (\boldsymbol{\theta}, \boldsymbol{J})$ (seldomly known)

    Solution : Use the epicyclic approximation for $2D$−disc.

- **Long-range system**
  - Construct basis elements $\psi^{(p)}$ + Invert the response matrix $\widehat{\mathbf{M}}$.

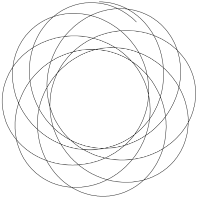
    Solution : Use WKB tighly wound hypothesis: restriction to local resonances.

- **Perturbed system**
  - Perform ensemble average over N-body simulation.

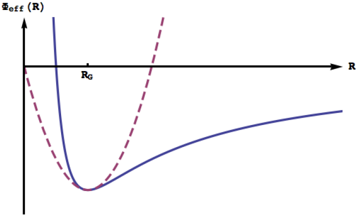
    Solution : Idealised isolated numerical simulation → Poisson shot noise.

The epicyclic approximation

- Real trajectories are complex rosettes (2 oscillations).



- Effective potential : $\ddot{R} = -\mathrm{d}\Phi_{\mathrm{eff}}/\mathrm{d}R$
  with $\Phi_{\mathrm{eff}}(R) = \Phi(R) + J_\phi^2/R^2$.
- Taylor-Expansion for small oscillations



- Circular orbits at the **guiding radius** $R_g(J_\phi)$: $R_g \longleftrightarrow J_\phi$, .
- **Radial harmomic oscillation** at the frequency $\kappa(J_\phi)$: $\boxed{J_r = \frac{1}{2}\kappa A^2 \,.}$
- **Azimuthal rotation** at the frequency $\Omega(J_\phi)$.

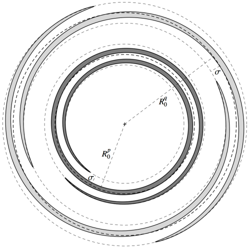
All intrinsic frequencies are function of J_ϕ only

The WKB approach

- WKB approximation = **tightly wound spirals**
- Aim : Restrict ourself to local resonances with an appropriate basis

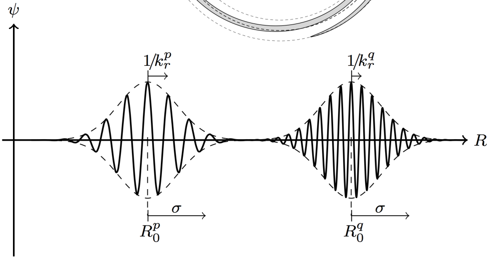
$$\psi^{[k_r,k_\phi,R_0]}(R,\phi) = \mathcal{A}\, e^{i(k_r R + k_\phi \phi)} \exp\left[-\frac{(R-R_0)^2}{2\sigma^2}\right] .$$

- Three indices
  - k_ϕ : azimuthal number (discrete)
  - k_r : radial frequency (continuous)
  - R_0 : *central* radius (continuous)



- σ : Decoupling scale
- It is a **biorthogonal basis** under the assumption

$$R_0 \gg \sigma \gg \frac{1}{k_r}$$



Using the WKB basis

- Initial Conditions : Schwarzschild DF : $F_0(R_g, J_r) = \dfrac{\Omega_\phi \, \Sigma}{\pi \, \kappa \, \sigma_r^2} \exp\left[-\dfrac{\kappa}{\sigma_r^2} J_r\right]$.
- Consequence : Different basis elements $\psi^{[k_r, k_\phi, R_0]}$ *do not interfere.*

- **Diagonalisation of the response matrix**

$$\widehat{\mathbf{M}}_{[k_r^1, k_\phi^1, R_0],[k_r^2, k_\phi^2, R_0]} = \delta_{k_r^1}^{k_r^2} \delta_{k_\phi^1}^{k_\phi^2} \, \lambda_{[k_r^1, k_\phi^1, R_0]}$$

- $\lambda(k_r, k_\phi, R_0)$ represents the *local responsiveness* of the system

$$\boxed{\lambda(k_r, k_\phi, R_0) = \frac{2\pi G \xi \Sigma}{\kappa^2 (1 - s^2)} \mathcal{F}(s, \chi)} \, .$$

$$\begin{cases} * \, s = \frac{\omega - k_\phi \Omega}{\kappa} \, , \\ * \, \chi = \frac{\sigma_r^2 k_r^2}{\kappa^2} \, , \\ * \, \mathcal{F}(s, \chi) \quad \text{(reduction factor)} \, . \end{cases}$$

Kalnajs (65), Lin&Shu(66)

- Next key step : Riemann sum transformations
(Disappearance of the *ad hoc* σ. *Critical sampling* : $\Delta R_0 \Delta k_r = 2\pi$.)

$$\frac{1}{\sigma} \sum_{k_r} \to \int \! \mathrm{d}k_r \quad \text{and} \quad \sigma \sum_{R_0} \to \int \! \mathrm{d}R_0 \, .$$

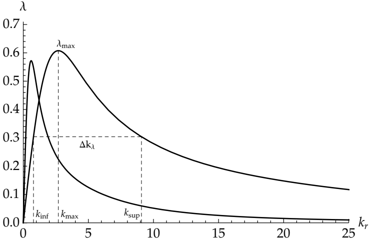
WKB and Secular Diffusion Equation

- Expression of the diffusion coefficients

$$\boldsymbol{D}_{\boldsymbol{m}}(\boldsymbol{J}) = \int \mathrm{d}k_r \, \mathcal{J}_{m_r}^2 \left[\sqrt{\frac{2J_r}{\kappa}} k_r \right] \left[\frac{1}{1-\lambda_{k_r}} \right]^2 \widehat{\mathcal{C}}[m_\phi, \boldsymbol{m}\cdot\boldsymbol{\Omega}, k_r, R_g] \, .$$

Perturbation power spectrum : $\widehat{\mathcal{C}}[m_\phi, \boldsymbol{m}\cdot\boldsymbol{\Omega}, k_r, R_g]$ to be measured

- **Approximation of the small denominators** : $k_r \mapsto \lambda_{k_r}$ is sharp

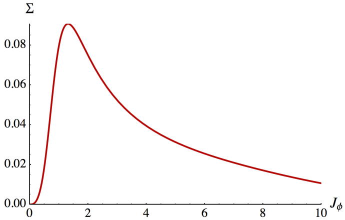


Final expression of the diffusion coefficients

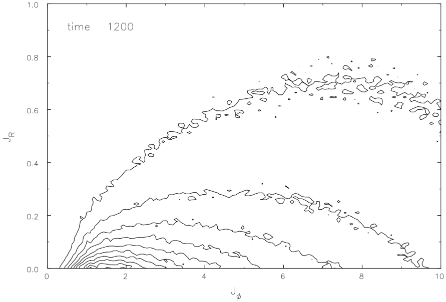
$$\boldsymbol{D}_{\boldsymbol{m}}(\boldsymbol{J}) = \Delta_k \, \mathcal{J}_{m_r}^2 \left[\sqrt{\frac{2J_r}{\kappa}} \, k_{\mathrm{max}} \right] \left[\frac{1}{1-\lambda_{\mathrm{max}}} \right]^2 \widehat{\mathcal{C}}[m_\phi, \boldsymbol{m}\cdot\boldsymbol{\Omega}, k_{\mathrm{max}}, R_g] \, .$$

WKB - Case of application

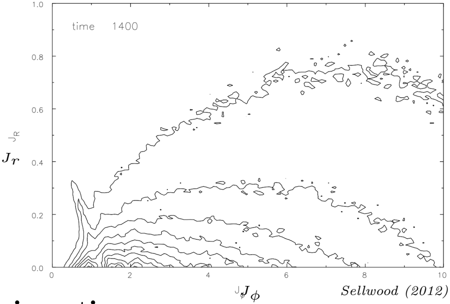
- Sellwood's 2012 numerical experiment
  - Stationnary tapered *Mestel disc* sampled with up to $500M$ particles
  - Evolution with a $N-$Body code
  - Appearance of *transient spiral waves*



Initial DF



Evolved DF



Sellwood (2012)

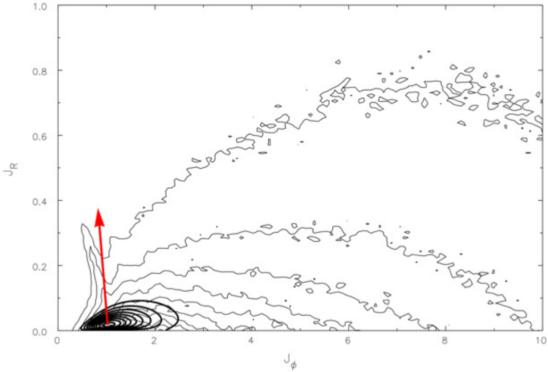
Secular diffusion in action-space

WKB - Case of application

- Secular diffusion flux density

$$\boxed{\boldsymbol{\mathcal{F}}_{\mathrm{tot}} = \sum_{\boldsymbol{m}} \boldsymbol{m}\, \boldsymbol{m} \cdot \frac{\partial F_0}{\partial \boldsymbol{J}}\, D_{\boldsymbol{m}}(\boldsymbol{J})} \qquad \begin{cases} * \ \boldsymbol{m} & (\textit{resonances}), \\ * \ \boldsymbol{m} \cdot \partial F_0 / \partial \boldsymbol{J} & (\textit{inhomogeneity}), \\ * \ D_{\boldsymbol{m}}(\boldsymbol{J}) & (\textit{susceptibility}). \end{cases}$$

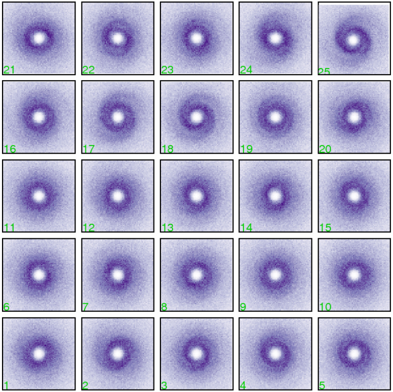
- Mimic *intrinsic noise* due to the discrete sampling : $\boxed{\left|\psi^{\mathrm{ext}}\right|^2 \propto \Sigma(R_g)\,.}$



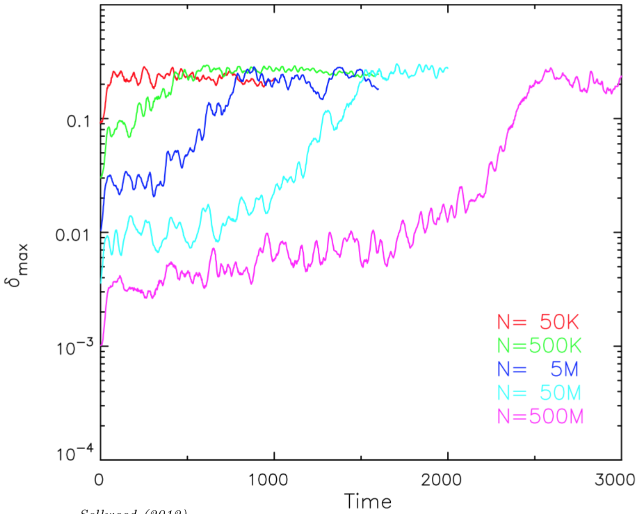
$$\begin{cases} * \ \text{Dominant ILR.} \\ * \ \text{Start position of the ridge.} \\ * \ \textit{Super}\text{-diffusion} : \sigma(J^s_{\mathrm{ILR}}) \propto t\,. \end{cases}$$

Conclusion - Dressed Fokker-Planck

- *Rich* framework to describe *forced* secular evolution
  - Source of perturbation via $\langle |\psi^{\text{ext}}|^2 \rangle$.
  - Self-gravity via ξ and λ .
  - Susceptibility via $D_{\boldsymbol{m}}(\boldsymbol{J})$.
  - Inhomogeneity via $\partial F_0/\partial \boldsymbol{J}$.
  - Temperature via σ_r^2 .
  - Physical structure via T_{inner} .
  - Dynamical structure via $\boldsymbol{J} \mapsto \boldsymbol{\Omega}(\boldsymbol{J})$.



What about finite−N effects?



Sellwood (2012)

Collisional dynamics?

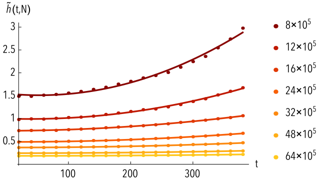
- Run ensemble-averaged $N-$body simulations with varying N.
- *Distance* to the initial DF

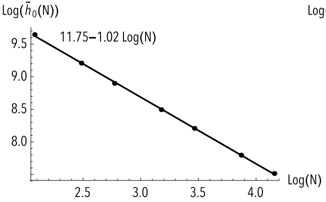
$$\tilde{h}(t,N) = \left\langle \int \! \mathrm{d}\boldsymbol{J} \, \left[F(t,\boldsymbol{J},N) - \langle F(t\!=\!0,\boldsymbol{J},N)\rangle \right]^2 \right\rangle$$

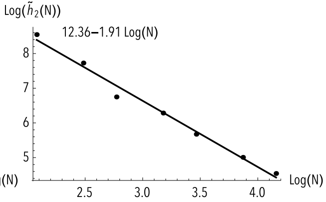
- Initial behavior

$$\tilde{h}(t,N) \simeq \tilde{h}_0(N) + \tilde{h}_1(N)\, t + \tilde{h}_2(N)\, \frac{t^2}{2} \Longrightarrow \begin{cases} \tilde{h}_0(N) \propto 1/N \; (\textbf{Poisson shot noise}) \\ \tilde{h}_1(N) = 0 \\ \tilde{h}_2(N) \propto 1/N^2 \; (\textbf{Collisional scaling}) \end{cases}$$

- Measurements in $N-$body simulations







The Balescu-Lenard kinetic equation

- **Aim** : Describe the secular evolution of a discrete *collisional* inhomogeneous self-gravitating system driven by *finite*$-N$ effects.

- References
  - *Balescu (1960)*, *Lenard (1960)* : Plasma
  - *Weinberg (1998)* : Jean's swindle
  - *Heyvaerts (2010)* : Angle-Action: BBGKY
  - *Chavanis (2012)* : Angle-Action: Klimontovitch
  - *Fouvry, Pichon, Chavanis (2015)* : $2D$ WKB limit - A&A ...

- Novelty: first implementation in the WKB self-gravitating limit : simple quadrature for diffusion and drift.

Liouville's Equation

- System of N identical interacting particles, $\boldsymbol{w} = (\boldsymbol{x}, \boldsymbol{v})$.
- Hamiltonian of the system : $H_N = \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{v}_i^2 + \sum_{i<j}^{N} U(\boldsymbol{x}_i - \boldsymbol{x}_j)\,.$
- Individual dynamics governed by Hamilton's equation

$$\frac{\mathrm{d}\boldsymbol{x}_i}{\mathrm{d}t} = \frac{\partial H_N}{\partial \boldsymbol{v}_i} \quad ; \quad \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\frac{\partial H_N}{\partial \boldsymbol{x}_i}\,.$$

- $N-$body DF $f^{(N)}(\boldsymbol{w}_1, ..., \boldsymbol{w}_N, t)$ governed by Liouville's equation

$$\begin{aligned} 0 &= \frac{\partial f^{(N)}}{\partial t} + \mathrm{div}\Big[\dot{\boldsymbol{w}}\, f^{(N)}\Big] \quad \text{continuity equation} \\ &= \frac{\partial f^{(N)}}{\partial t} + \sum_{i=1}^{N} \left\{ \boldsymbol{v}_i \cdot \frac{\partial f^{(N)}}{\partial \boldsymbol{x}_i} + \boldsymbol{F}_i \cdot \frac{\partial f^{(N)}}{\partial \boldsymbol{v}_i} \right\} \\ &= \frac{\partial f^{(N)}}{\partial t} + \sum_{i=1}^{N} \left\{ \frac{\partial H_N}{\partial \boldsymbol{v}_i} \cdot \frac{\partial f^{(N)}}{\partial \boldsymbol{x}_i} - \frac{\partial H_N}{\partial \boldsymbol{x}_i} \cdot \frac{\partial f^{(N)}}{\partial \boldsymbol{v}_i} \right\} \\ &= \frac{\partial f^{(N)}}{\partial t} + \Big[f^{(N)}, H_N \Big]\,. \end{aligned}$$

- **Exact and reversible equation but in a $6ND$ phase-space.**

BBGKY Hierarchy

- Reduced DF in **$6nD$ phase space**

$$f_n(\boldsymbol{w}_1, ..., \boldsymbol{w}_n, t) = \frac{N!}{(N-n)!} \int \mathrm{d}\boldsymbol{w}_{n+1} \, \mathrm{d}\boldsymbol{w}_N \, f^{(N)}(\boldsymbol{w}_1, ..., \boldsymbol{w}_N, t) \, .$$

- Reduced $n-$body Hamiltonian

$$H_n = \frac{1}{2} \sum_{i=1}^{n} \boldsymbol{v}_i^2 + \sum_{i<j\leq n} U_{i,j} \, .$$

- $n^{\mathrm{th}}-$BBGKY equation for f_n

$$\boxed{\frac{\partial f_n}{\partial t} + [f_n, H_n] = \sum_{i=1}^{n} \int \mathrm{d}\boldsymbol{x}_{n+1} \, \mathrm{d}\boldsymbol{v}_{n+1} \frac{\partial U_{i,n+1}}{\partial \boldsymbol{x}_i} \cdot \frac{\partial f_{n+1}}{\partial \boldsymbol{v}_i} \, .}$$

- Content
  - $n-$body dynamics : Liouville's equation and $(n+1)^{\mathrm{th}}$ order *collision term.*
  - Exact hierarchy of equation : a truncation is needed.

From BBGKY to Vlasov

- Two-body correlation function

$$\boxed{f_2(\boldsymbol{w}_1, \boldsymbol{w}_2) = f_1(\boldsymbol{w}_1)\, f_1(\boldsymbol{w}_2) + g_2(\boldsymbol{w}_1, \boldsymbol{w}_2)\,.}$$

- **BBGKY$-n=1$ equation**

$$\frac{\partial f_1}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial f_1}{\partial \boldsymbol{x}_1} - \frac{\partial f_1}{\partial \boldsymbol{v}_1} \cdot \frac{\partial}{\partial \boldsymbol{x}_1} \left[\int \mathrm{d}\boldsymbol{w}_2\, U_{1,2} f_1(\boldsymbol{w}_2) \right] = \int \mathrm{d}\boldsymbol{w}_2 \frac{\partial U_{1,2}}{\partial \boldsymbol{x}_1} \cdot \frac{\partial g_2}{\partial \boldsymbol{v}_1}\,.$$

- Separable system with no particle correlation : $g_2 = 0\,.$

$$\begin{cases} \displaystyle\int \mathrm{d}\boldsymbol{v}_2\, f_1(\boldsymbol{x}_2, \boldsymbol{v}_2, t) = \rho(\boldsymbol{x}_2, t)\,, \\ \displaystyle\int \mathrm{d}\boldsymbol{x}_2\, \rho(\boldsymbol{x}_2, t) U(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \Phi(\boldsymbol{x}_1, t)\,. \end{cases} \Longrightarrow \boxed{\frac{\partial f_1}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial f_1}{\partial \boldsymbol{x}_1} - \frac{\partial \Phi}{\partial \boldsymbol{x}_1} \cdot \frac{\partial f_1}{\partial \boldsymbol{v}_1} = 0\,.}$$

- We recover **Vlasov equation** for an uncorrelated system of N particles to describe the **collisionless** evolution.

From BBGKY to Balescu-Lenard

- Taking into account two-body correlations but truncation at the order $1/N$ (*i.e.* $g_3 \equiv 0$).
- **BBGKY$-n=2$ equation**

$$\frac{\partial g_2(1,2)}{\partial t} + \left[\boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{x}_1} + \boldsymbol{v}_2 \cdot \frac{\partial}{\partial \boldsymbol{x}_2}\right] g_2(1,2)$$
$$- \left[\int \mathrm{d}\boldsymbol{x}_3 \, \mathrm{d}\boldsymbol{v}_3 \, \frac{\partial U_{1,3}}{\partial \boldsymbol{x}_1} f_1(3) \cdot \frac{\partial}{\partial \boldsymbol{v}_1} + \int \mathrm{d}\boldsymbol{x}_3 \, \mathrm{d}\boldsymbol{v}_3 \, \frac{\partial U_{2,3}}{\partial \boldsymbol{x}_2} f_1(3) \cdot \frac{\partial}{\partial \boldsymbol{v}_2}\right] g_2(1,2)$$
$$- \left[\int \mathrm{d}\boldsymbol{x}_3 \, \mathrm{d}\boldsymbol{v}_3 \, \frac{\partial U_{1,3}}{\partial \boldsymbol{x}_1} g_2(2,3)\right] \cdot \frac{\partial f_1(1)}{\partial \boldsymbol{v}_1} - \left[\int \mathrm{d}\boldsymbol{x}_3 \, \mathrm{d}\boldsymbol{v}_3 \, \frac{\partial U_{2,3}}{\partial \boldsymbol{x}_2} g_2(1,3)\right] \cdot \frac{\partial f_1(2)}{\partial \boldsymbol{v}_2}$$
$$= \frac{\partial U_{1,2}}{\partial \boldsymbol{x}_1} \cdot \left[\frac{\partial}{\partial \boldsymbol{v}_1} - \frac{\partial}{\partial \boldsymbol{v}_2}\right] f_1(1) f_1(2)$$

- Complex to solve for f_1 and g_2, especially in inhomogeneous systems.
- But VERY symmetric.

Inhomogeneous Landau equation

- **Adiabatic approximation** in angle-actions coordinates

$$F(\boldsymbol{x},\boldsymbol{v}) = F(\boldsymbol{J},t)\,.$$

- Neglecting collective effects (*i.e.* self-gravitating amplification), we obtain the **inhomogeneous Landau equation** (Chavanis 2007, 2010)

$$\begin{aligned}\frac{\partial F}{\partial t} =& \pi(2\pi)^d \mu \frac{\partial}{\partial \boldsymbol{J}_1} \cdot \bigg[\sum_{\boldsymbol{m}_1,\boldsymbol{m}_2} \boldsymbol{m}_1 \int \mathrm{d}\boldsymbol{J}_2\, \delta_{\mathrm{D}}(\boldsymbol{m}_1\cdot\boldsymbol{\Omega}_1 - \boldsymbol{m}_2\cdot\boldsymbol{\Omega}_2) \\ & \times \left| A_{\boldsymbol{m}_1,\boldsymbol{m}_2}(\boldsymbol{J}_1,\boldsymbol{J}_2)\right|^2 \left[\boldsymbol{m}_1\cdot\frac{\partial}{\partial \boldsymbol{J}_1} - \boldsymbol{m}_2\cdot\frac{\partial}{\partial \boldsymbol{J}_2}\right] F(\boldsymbol{J}_1,t)\, F(\boldsymbol{J}_2,t)\bigg]\,.\end{aligned}$$

- *Bare* susceptibility coefficients:

$$A_{\boldsymbol{m}_1,\boldsymbol{m}_2}(\boldsymbol{J}_1,\boldsymbol{J}_2) = \frac{1}{(2\pi)^4}\int \mathrm{d}\boldsymbol{\theta}_1\, \mathrm{d}\boldsymbol{\theta}_2\, \frac{-G}{|\boldsymbol{x}(\boldsymbol{\theta}_1,\boldsymbol{J}_1) - \boldsymbol{x}(\boldsymbol{\theta}_2,\boldsymbol{J}_2)|}\, e^{-i(\boldsymbol{m}_1\cdot\boldsymbol{\theta}_1 - \boldsymbol{m}_2\cdot\boldsymbol{\theta}_2)}\,.$$

Taking into account the dressing

How does the system respond to an imposed perturbation ?

- Introducing a *representative* basis of potential functions $\psi^{(p)}$, so that

$$\begin{cases} \psi^{\text{ext}} = \sum_p b_p(t)\, \psi^{(p)} \,. & \text{Imposed } \textbf{exterior perturbation} \\ \psi^{\text{self}} = \sum_p a_p(t)\, \psi^{(p)} \,. & \text{Amplified } \textbf{response of the system} \end{cases}$$

- Non-Markovian **amplification mechanism**

$$\boxed{\boldsymbol{a}(t) = \int_{-\infty}^{t} \mathrm{d}\tau\, \mathbf{M}(t-\tau)\, [\boldsymbol{a}(\tau) + \boldsymbol{b}(\tau)] \,,}$$

where $\mathbf{M}(F_0)$ is the **response matrix** of the system, given by

$$\widehat{\mathbf{M}}_{pq}(\omega) = (2\pi)^d \sum_{\boldsymbol{m}} \int \mathrm{d}\boldsymbol{J}\, \frac{\boldsymbol{m}\cdot\partial F_0/\partial \boldsymbol{J}}{\omega - \boldsymbol{m}\cdot\boldsymbol{\Omega}}\, \psi_{\boldsymbol{m}}^{(p)\,*}(\boldsymbol{J})\, \psi_{\boldsymbol{m}}^{(q)}(\boldsymbol{J}) \,,$$

and $\psi_{\boldsymbol{m}}^{(p)}(\boldsymbol{J}) = \dfrac{1}{(2\pi)^d} \displaystyle\int \mathrm{d}\boldsymbol{\theta}\, \psi^{(p)}[\boldsymbol{x}(\boldsymbol{\theta}, \boldsymbol{J})]\, e^{-i\boldsymbol{m}\cdot\boldsymbol{\Omega}}$.

Resonances are at the *intrinsic frequencies* of the system : $\boldsymbol{m}\cdot\boldsymbol{\Omega}$.

Inhomogeneous Balescu-Lenard equation

- **Inhomogeneous Balescu-Lenard equation**
  Heyvaerts (2010), Chavanis (2012)

$$\frac{\partial F}{\partial t} = \pi(2\pi)^d \mu \frac{\partial}{\partial \boldsymbol{J}_1} \cdot \bigg[\sum_{\boldsymbol{m}_1, \boldsymbol{m}_2} \boldsymbol{m}_1 \int \mathrm{d}\boldsymbol{J}_2 \, \frac{\delta_{\mathrm{D}}(\boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_1 - \boldsymbol{m}_2 \cdot \boldsymbol{\Omega}_2)}{|\mathcal{D}_{\boldsymbol{m}_1, \boldsymbol{m}_2}(\boldsymbol{J}_1, \boldsymbol{J}_2, \boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_2)|^2} \left[\boldsymbol{m}_1 \cdot \frac{\partial}{\partial \boldsymbol{J}_1} - \boldsymbol{m}_2 \cdot \frac{\partial}{\partial \boldsymbol{J}_2} \right] F(\boldsymbol{J}_1, t) \, F(\boldsymbol{J}_2, t) \bigg] .$$

- *Dressed* susceptibility coefficients

$$\frac{1}{\mathcal{D}_{\boldsymbol{m}_1, \boldsymbol{m}_2}(\boldsymbol{J}_1, \boldsymbol{J}_2, \omega)} = \sum_{p,q} \psi^{(p)}_{\boldsymbol{m}_1}(\boldsymbol{J}_1) [\mathbf{I} - \widehat{\mathbf{M}}(\omega)]^{-1} \psi^{(q)\,*}_{\boldsymbol{m}_2}(\boldsymbol{J}_2) .$$

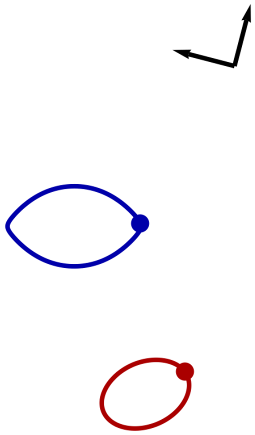
- Written as an anistropic Fokker-Planck equation

$$\frac{\partial F}{\partial t} = \sum_{\boldsymbol{m}_1} \frac{\partial}{\partial \boldsymbol{J}_1} \cdot \left[\boldsymbol{m}_1 \left(A_{\boldsymbol{m}_1}(\boldsymbol{J}_1) \, F(\boldsymbol{J}_1) + D_{\boldsymbol{m}_1}(\boldsymbol{J}_1) \, \boldsymbol{m}_1 \cdot \frac{\partial F}{\partial \boldsymbol{J}_1} \right) \right] .$$

- Content
  - Resonance condition : $\delta_{\mathrm{D}}(\boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_1 - \boldsymbol{m}_2 \cdot \boldsymbol{\Omega}_2)$.
  - Modes with $|\mathcal{D}_{\boldsymbol{m}_1, \boldsymbol{m}_2}| \ll 1$ very efficient.

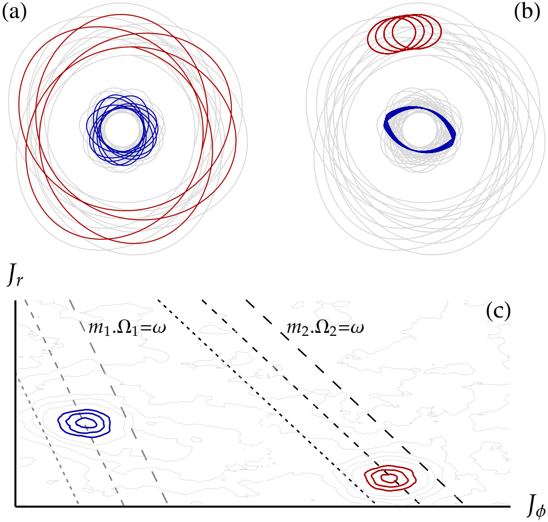
Resonant stellar encounters

- The **resonance condition** : $\delta_{\mathrm{D}}(\boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_1 - \boldsymbol{m}_2 \cdot \boldsymbol{\Omega}_2)$
  leads to *distant resonant encounters*



Resonant stellar encounters

Resonance condition : $\delta_{\mathrm{D}}(\boldsymbol{m}_1\cdot\boldsymbol{\Omega}_1-\boldsymbol{m}_2\cdot\boldsymbol{\Omega}_2)$



WKB Balescu-Lenard equation

- Diffusion equation

$$\frac{\partial F}{\partial t} = \pi(2\pi)^d \mu \frac{\partial}{\partial \boldsymbol{J}_1} \cdot \bigg[\sum_{\boldsymbol{m}_1, \boldsymbol{m}_2} \boldsymbol{m}_1 \int \mathrm{d}\boldsymbol{J}_2 \, \frac{\delta_{\mathrm{D}}(\boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_1 - \boldsymbol{m}_2 \cdot \boldsymbol{\Omega}_2)}{|\mathcal{D}_{\boldsymbol{m}_1, \boldsymbol{m}_2}(\boldsymbol{J}_1, \boldsymbol{J}_2, \boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_2)|^2} \left[\boldsymbol{m}_1 \cdot \frac{\partial}{\partial \boldsymbol{J}_1} - \boldsymbol{m}_2 \cdot \frac{\partial}{\partial \boldsymbol{J}_2} \right] F(\boldsymbol{J}_1, t) \, F(\boldsymbol{J}_2, t) \bigg] .$$

- Dressed susceptibility coefficients

$$\frac{1}{\mathcal{D}_{\boldsymbol{m}_1, \boldsymbol{m}_2}(\boldsymbol{J}_1, \boldsymbol{J}_2, \omega)} = \sum_{p,q} \psi^{(p)}_{\boldsymbol{m}_1}(\boldsymbol{J}_1) [\mathbf{I} - \widehat{\mathbf{M}}(\omega)]^{-1} \psi^{(q)\,*}_{\boldsymbol{m}_2}(\boldsymbol{J}_2) \, .$$

- **Inhomogeneous system**
  - Introduce explicitly the mapping $(\boldsymbol{x}, \boldsymbol{v}) \mapsto (\boldsymbol{\theta}, \boldsymbol{J})$ (seldomly known)

    Solution : Use the epicyclic approximation for $2D-$disc.

- **Long-range system**
  - Construct basis elements $\psi^{(p)}$ + Invert the response matrix $\widehat{\mathbf{M}}$.

    Solution : Use **WKB tighly wound** hypothesis.

- **Resonance condition**
  - Handle the non-trivial resonance condition $\delta_{\mathrm{D}}(\boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_1 - \boldsymbol{m}_2 \cdot \boldsymbol{\Omega}_2)$

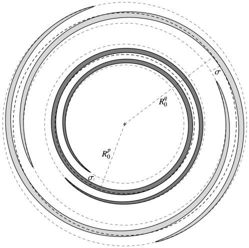
    Solution : Limitation to **local resonances** via the WKB approximation

The WKB approach

- WKB approximation = **tightly wound spirals**
- Aim : Restrict ourselves to local resonances with an appropriate basis

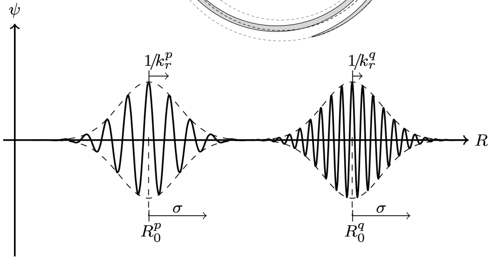
$$\psi^{[k_r, k_\phi, R_0]}(R, \phi) = \mathcal{A}\, e^{i(k_r R + k_\phi \phi)} \, \exp\left[- \frac{(R - R_0)^2}{2\sigma^2} \right] .$$

- Three index
  - k_ϕ : azimuthal number (discrete)
  - k_r : radial frequency (continuous)
  - R_0 : *central* radius (continuous)



- σ : Decoupling scale
- It is a **biorthogonal basis** under the assumption

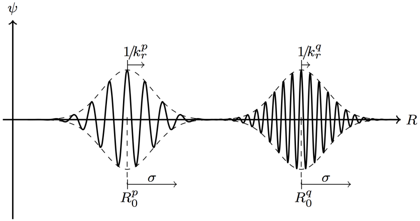
$$R_0 \gg \sigma \gg \frac{1}{k_r}$$

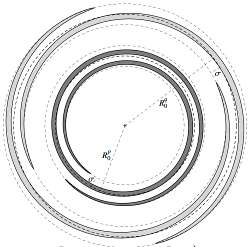


WKB Balescu-Lenard

- Using our **WKB basis**

$$\psi^{[k_r,k_\phi,R_0]}(R,\phi) = \mathcal{A}\, e^{i(k_r R + k_\phi \phi)}\, \exp\left[-\frac{(R-R_0)^2}{2\sigma^2}\right] .$$





- Restriction to exactly local resonances : $\delta_{\mathrm{D}}(\boldsymbol{m}_1\cdot\boldsymbol{\Omega}_1 - \boldsymbol{m}_2\cdot\boldsymbol{\Omega}_2)$

$$\begin{cases} \boldsymbol{m}_1\cdot\boldsymbol{\Omega}_1(R_1) - \boldsymbol{m}_2\cdot\boldsymbol{\Omega}_2(R_2) = 0 \\ |R_1 - R_2| \leq (\text{few})\sigma \end{cases} \Longrightarrow \begin{cases} \boldsymbol{m}_2 = \boldsymbol{m}_1 \,, \\ R_2 = R_1 \,. \end{cases}$$

- Diagonalisation of the response matrix $\widehat{\mathbf{M}}(\omega)$

$$\widehat{\mathbf{M}}_{[k_r^1,k_\phi^1,R_0],[k_r^2,k_\phi^2,R_0]} = \delta_{k_r^1}^{k_r^2}\delta_{k_\phi^1}^{k_\phi^2}\, \lambda_{[k_r^1,k_\phi^1,R_0]} \,.$$

WKB Balescu-Lenard

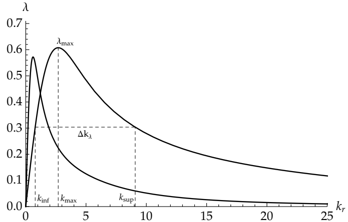
- WKB susceptibility coefficients : $1/\mathcal{D}_{\boldsymbol{m}_1,\boldsymbol{m}_2} = \sum_{p,q} \psi^{(p)}_{\boldsymbol{m}_1} [\mathbf{I} - \widehat{\mathbf{M}}(\omega)]_{pq} \psi^{(q)*}_{\boldsymbol{m}_2}$.

$$\bigg| \frac{1}{\mathcal{D}_{\boldsymbol{m}_1,\boldsymbol{m}_1}(\boldsymbol{J}_1, R_1, J_r^2, \omega)} \bigg|^2 = \bigg[\frac{1}{2\pi} \frac{G}{R_1} \int_{1/\sigma_k}^{+\infty} \!\! \mathrm{d}k_r \frac{1}{k_r} \frac{1}{1 - \lambda_{k_r}(R_1, \omega)} H(R_1, J_r^1, J_r^2) \bigg]^2 .$$

- Two possible behaviors
  - **App. of the small denominators** : $k_r \mapsto \lambda_{k_r}$ is *sharp*. Amplification.
  - **App. of the dominant scale** : $k_r \mapsto \lambda_{k_r}$ *flat*. Strong collisions.

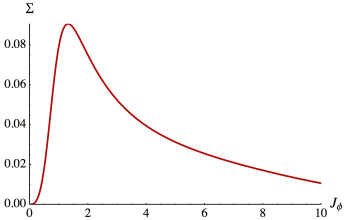
- *Dressed* WKB susceptibility coefficients

$$\bigg| \frac{1}{\mathcal{D}_{\boldsymbol{m}_1,\boldsymbol{m}_1}(R_1, J_r^1, R_1, J_r^2, \omega)} \bigg|^2 = \frac{1}{4\pi^2} \frac{G^2}{R_1^2} \frac{(\Delta k_\lambda)^2}{k_{\mathrm{max}}^2}$$
$$\times \bigg[\frac{1}{1 - \lambda_{\mathrm{max}}} \bigg]^2 \mathcal{J}^2_{m_1^r} \bigg[\sqrt{\frac{2J_r^1}{\kappa_1}} k_{\mathrm{max}} \bigg] \mathcal{J}^2_{m_1^r} \bigg[\sqrt{\frac{2J_r^2}{\kappa_1}} k_{\mathrm{max}} \bigg] .$$

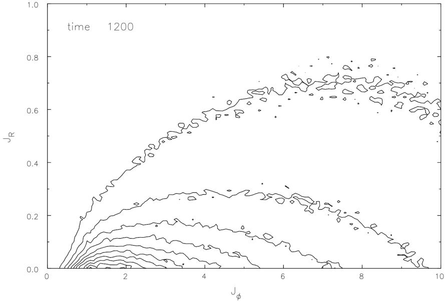


WKB-BL - Case of application

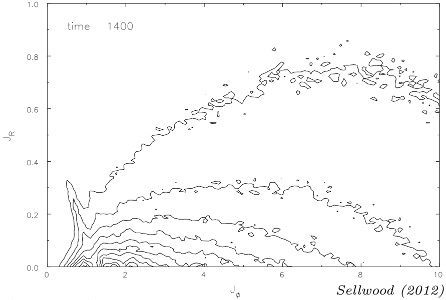
- Sellwood's 2012 numerical experiment
  - Stationnary tapered *Mestel disc* sampled with up to $500M$ particles
  - Evolution with a $N-$Body code
  - Appearance of *transient spiral structures*



Initial DF



Evolved DF



Sellwood (2012)

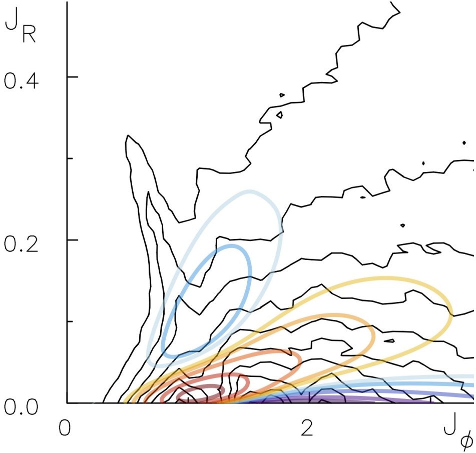
Secular diffusion in action-space

WKB-BL - Case of application

- **Total flux density**

$$\boldsymbol{\mathcal{F}}_{\text{tot}} = \sum_{\boldsymbol{m}} \boldsymbol{m} \left[A_{\boldsymbol{m}}(\boldsymbol{J})\, F(\boldsymbol{J}) + D_{\boldsymbol{m}}(\boldsymbol{J})\, \boldsymbol{m} \cdot \frac{\partial F}{\partial \boldsymbol{J}} \right] \quad \Longrightarrow \quad \frac{\partial F}{\partial t} = \text{div}(\boldsymbol{\mathcal{F}}_{\text{tot}})\,.$$

- Predicted contours for $\text{div}(\boldsymbol{\mathcal{F}}_{\text{tot}})$



$$\begin{cases} \textit{Red} : \text{div}(\boldsymbol{\mathcal{F}}_{\text{tot}}) < 0\,, \\ \textit{Blue} : \text{div}(\boldsymbol{\mathcal{F}}_{\text{tot}}) > 0\,. \end{cases}$$

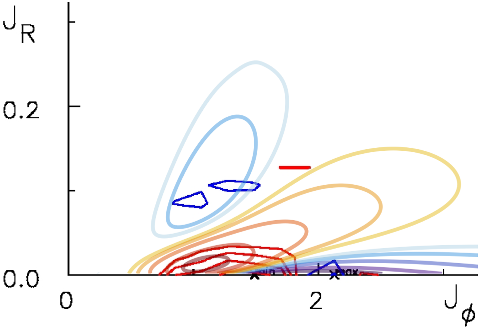
WKB-BL - Diffusion timescale

- Correctly normalised BL equation

$$\boxed{\frac{\partial F}{\partial t} + L[F] = \frac{1}{N} C_{\mathrm{BL}}[F]\,.} \qquad \begin{cases} L = \boldsymbol{\Omega}\cdot\dfrac{\partial}{\partial \boldsymbol{\theta}}\,, & (advection) \\ C_{\mathrm{BL}}[F]\,. & (collisions) \end{cases}$$

- Normalised time $\tau = t/N$: $\dfrac{\partial F}{\partial \tau} = C_{\mathrm{BL}}[F]\,.$
- Comparison with S12 simulation

$$\frac{\Delta\tau_{\mathrm{S12}}}{\Delta\tau_{\mathrm{WKB}}} \simeq 10^{-3}\,.$$



- Secular diffusion too early to be finite$-N$ effects only.
  - **Swing amplification?**

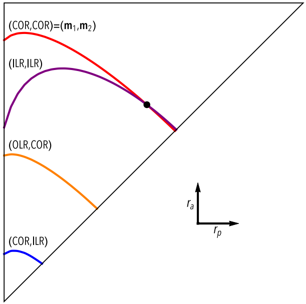
Conclusions - WKB Balescu-Lenard

Interest of WKB - Balescu-Lenard

- **First** application of Balescu-Lenard in astrophysics.
- Powerful formalism with no *ad hoc* assumptions or fine-tuning.
- Validation of $N-$body codes on secular timescales.

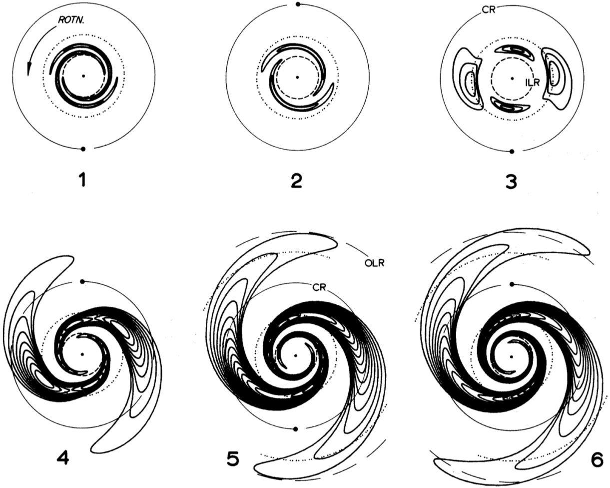
Extensions

- Non-WKB Matrix method
  + comparison to N-body experiments.
- Extension to thick discs.
- Applications to other physical systems
  - Relaxation in the galactic center.
  - Mass segregation in protoplanetary discs.
  - Molecular Clouds and metallicity gradients.



Fouvry, Pichon, Chavanis (2015) : A&A ...

Taking into account *swing amplification*?



Toomre (1981)

- **Use *global* Balescu-Lenard equation**

Difficulties of the Balescu-Lenard equation

- Diffusion equation

$$\frac{\partial F}{\partial t} = \pi(2\pi)^d \mu \frac{\partial}{\partial \boldsymbol{J}_1} \cdot \bigg[\sum_{\boldsymbol{m}_1, \boldsymbol{m}_2} \boldsymbol{m}_1 \int \mathrm{d}\boldsymbol{J}_2 \, \frac{\delta_{\mathrm{D}}(\boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_1 - \boldsymbol{m}_2 \cdot \boldsymbol{\Omega}_2)}{|\mathcal{D}_{\boldsymbol{m}_1, \boldsymbol{m}_2}(\boldsymbol{J}_1, \boldsymbol{J}_2, \boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_2)|^2} \left[\boldsymbol{m}_1 \cdot \frac{\partial}{\partial \boldsymbol{J}_1} - \boldsymbol{m}_2 \cdot \frac{\partial}{\partial \boldsymbol{J}_2} \right] F(\boldsymbol{J}_1, t) \, F(\boldsymbol{J}_2, t) \bigg] \, .$$

- Dressed susceptibility coefficients

$$\frac{1}{\mathcal{D}_{\boldsymbol{m}_1, \boldsymbol{m}_2}(\boldsymbol{J}_1, \boldsymbol{J}_2, \omega)} = \sum_{p,q} \psi^{(p)}_{\boldsymbol{m}_1}(\boldsymbol{J}_1) [\mathbf{I} - \widehat{\mathbf{M}}(\omega)]^{-1} \psi^{(q)\,*}_{\boldsymbol{m}_2}(\boldsymbol{J}_2) \, .$$

- **Inhomogeneous system**
  - Introduce explicitly the mapping $(\boldsymbol{x}, \boldsymbol{v}) \mapsto (\boldsymbol{\theta}, \boldsymbol{J})$ (seldomly known)

Solution : Implement the **explicit action mapping** Tremaine&Weinberg (84).

- **Long-range system**
  - Construct basis elements $\psi^{(p)}$ + Invert the response matrix $\widehat{\mathbf{M}}$.

    Solution : Use *global* basis elements and numerically compute $\widehat{\mathbf{M}}$.

- **Resonance condition**
  - Handle the non-trivial resonance condition $\delta_{\mathrm{D}}(\boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_1 - \boldsymbol{m}_2 \cdot \boldsymbol{\Omega}_2)$

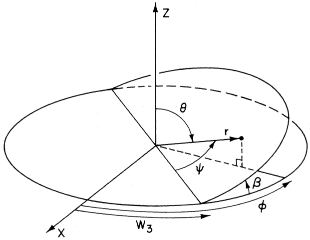
    Solution : Find the **resonant critical lines** and integrate along them.

How to determine $(\boldsymbol{x}, \boldsymbol{v}) \mapsto (\boldsymbol{\theta}, \boldsymbol{J})$?

- Fourier transform (in angles) of the basis elements

$$\psi^{(p)}_{\boldsymbol{m}}(\boldsymbol{J}) = \frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{\theta}\; \psi^{(p)}_{\boldsymbol{m}}(\boldsymbol{x}(\boldsymbol{\theta}, \boldsymbol{J}))\; e^{-i\boldsymbol{m}\cdot\boldsymbol{\theta}}\,.$$

- For $2D$ axisymmetric systems $\psi(r)$
  explicit mapping (Tremaine&Weinberg (84))



- **Explicit canonical angles**

$$\begin{cases} \theta_1 = \Omega_1 \displaystyle\int_{\mathcal{C}_1} \mathrm{d}r\, \frac{1}{\sqrt{2(E-\psi_0(r))-L^2/r^2}}\,, \\ \theta_2 - \psi = \displaystyle\int_{\mathcal{C}_1} \mathrm{d}r\, \frac{\Omega_2 - J_2/r^2}{\sqrt{2(E-\psi_0(r))-L^2/r^2}}\,. \end{cases}$$

- Cumbersome evaluation but can made numerically.

How to compute $\widehat{\mathbf{M}}$?

- Response matrix (**Resonant poles + Integral over actions**)

$$\widehat{\mathbf{M}} = (2\pi)^3 \sum_{\boldsymbol{m}} \int \mathrm{d}\boldsymbol{J} \, \frac{\boldsymbol{m}\cdot\partial F/\partial \boldsymbol{J}}{\omega - \boldsymbol{m}\cdot\boldsymbol{\Omega}} \, \psi_{\boldsymbol{m}}^{(p)*}(\boldsymbol{J}) \, \psi_{\boldsymbol{m}}^{(q)}(\boldsymbol{J}) \, .$$

- Use the good *action coordinates*

$$J_r = \int_{r_p}^{r_a} \mathrm{d}r \, \sqrt{2(E - \psi_0(r)) - L^2/r^2} \, .$$

- An orbit is completely specified by $(r_p, r_a) \leftrightarrow (E, L) \leftrightarrow (J_r, J_\phi)$.

$$\widehat{\mathbf{M}} \sim \sum_{\boldsymbol{m}} \int \mathrm{d}r_p \mathrm{d}r_a \, \frac{g(r_p, r_a)}{h(r_p, r_a)} \, .$$

- Truncate the $(r_p, r_a)-$space to *cross smoothly* the poles

$$\widehat{\mathbf{M}} \sim \sum_{\boldsymbol{m}} \sum_{i} \aleph(\boldsymbol{m}, i) \, .$$

- Validation via known **unstable modes**: Zang (76), Evans&Read (98)

Dealing with non-local resonances

- **Non-local resonance condition**

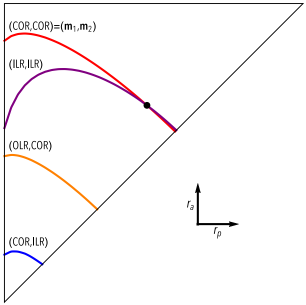
$$\boxed{\text{BL} \sim \int \mathrm{d}\boldsymbol{J}_2 \, \delta_{\mathrm{D}}(\boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_1 - \boldsymbol{m}_2 \cdot \boldsymbol{\Omega}_2) \, G(\boldsymbol{J}_2) \, .}$$

- Use good *action coordinates* : (r_p, r_a)

$$\text{BL} \sim \int \mathrm{d}r_a \, \mathrm{d}r_p \, \delta_{\mathrm{D}}(\boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_1 - \boldsymbol{m}_2 \cdot \boldsymbol{\Omega}_2) \, G(r_p, r_a) \, .$$

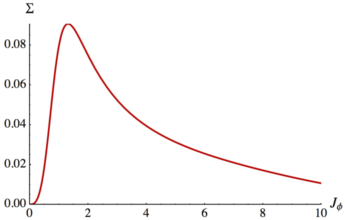
- For fixed $\boldsymbol{J}_1$, $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ identification of **critical resonant lines**, along which to integrate

$$\int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x} \, f(\boldsymbol{x}) \, \delta_{\mathrm{D}}(g(\boldsymbol{x})) = \int_{g^{-1}(0)} \mathrm{d}\sigma \, \frac{f(\boldsymbol{x})}{\nabla g(\boldsymbol{x})} \, .$$



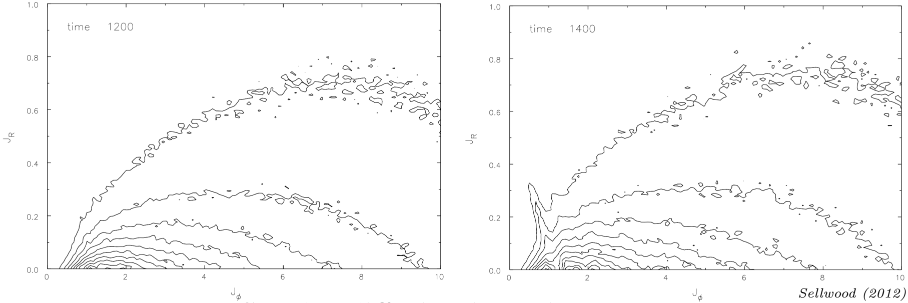
Global-BL - Case of application

- Sellwood's 2012 numerical experiment
  - Stationnary tapered *Mestel disc* sampled with up to $500M$ particles
  - Evolution with a $N-$Body code
  - Appearance of *transient spiral structures*



Initial DF

Evolved DF



Sellwood (2012)

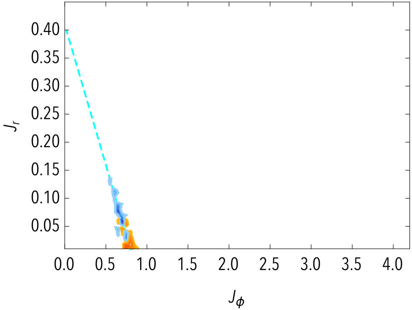
Secular diffusion in action-space

Global-BL - Case of application

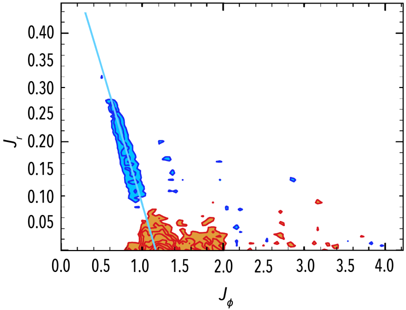
- **Total flux density**

$$\boldsymbol{\mathcal{F}}_{\rm tot} = \sum_{\boldsymbol{m}} \boldsymbol{m} \bigg[A_{\boldsymbol{m}}(\boldsymbol{J})\, F(\boldsymbol{J}) + D_{\boldsymbol{m}}(\boldsymbol{J})\, \boldsymbol{m} \cdot \frac{\partial F}{\partial \boldsymbol{J}} \bigg] \implies \frac{\partial F}{\partial t} = \mathrm{div}(\boldsymbol{\mathcal{F}}_{\rm tot})\,.$$

- Predicted contours for $\mathrm{div}(\boldsymbol{\mathcal{F}}_{\rm tot})$: $\begin{cases} \textit{Red} : \mathrm{div}(\boldsymbol{\mathcal{F}}_{\rm tot}) < 0\,, \\ \textit{Blue} : \mathrm{div}(\boldsymbol{\mathcal{F}}_{\rm tot}) > 0\,. \end{cases}$



Balescu-Lenard



Sellwood (2012)

Global-BL - Diffusion timescale

- Correctly normalised BL equation

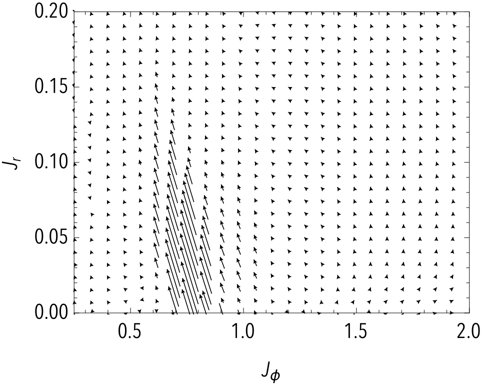
$$\boxed{\frac{\partial F}{\partial t} + L[F] = \frac{1}{N} C_{\mathrm{BL}}[F]\,.} \qquad \begin{cases} L = \boldsymbol{\Omega}\cdot \dfrac{\partial}{\partial \boldsymbol{\theta}}\,, & (\textit{advection}) \\ C_{\mathrm{BL}}[F]\,. & (\textit{collisions}) \end{cases}$$

- Normalised time $\tau = t/N$: $\dfrac{\partial F}{\partial \tau} = C_{\mathrm{BL}}[F]\,.$

- Comparison with S12 simulation

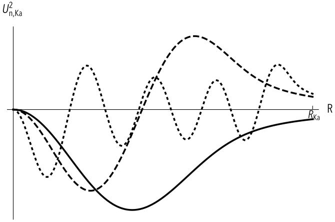
$$\frac{\Delta\tau_{\mathrm{S12}}}{\Delta\tau_{\mathrm{BL}}} \sim \mathcal{O}(1)\,.$$

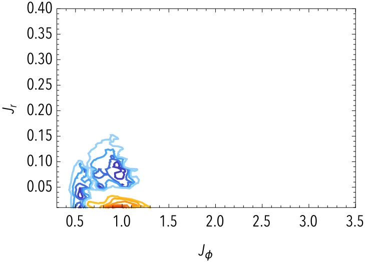
- **Secular diffusion with appropriate timescales.**



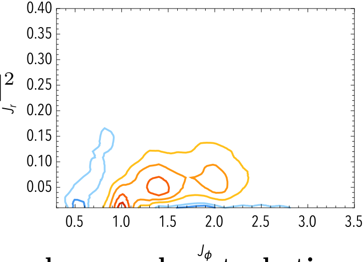
Global-BL - Role of *swing amplification*

- The *swing amplification* driving mechanism for the resonant ridge
  - **Narrow resonant ridge** along the ILR
  - ***Fast*** secular evolution
- Removing **loosely wound** basis elements





- Turning off **collective effects** $1/|\mathcal{D}|^2 \to |A|^2$



- Key mechanism

  Self-gravitating amplification of loosely wound perturbations

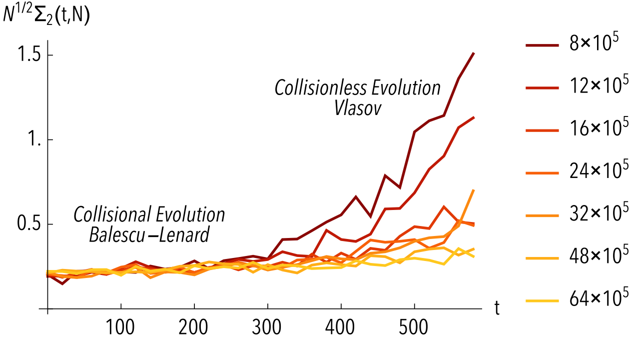
From Balescu-Lenard to Vlasov

- Strength of **non-axisymmetric features**

$$\Sigma_2(t,N) = \left\langle \int_{R_{\rm inf}}^{R_{\rm sup}} \mathrm{d}R\, R\, \mathrm{d}\phi\, \Sigma_{\rm star}(t,N,R,\phi)\, e^{-i2\phi} \right\rangle .$$

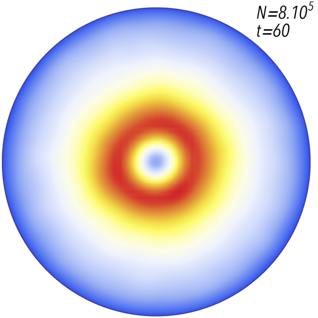
- Late-time evolution: **Transition**

Collisional Balescu-Lenard $\Longrightarrow$ **Collisionless Vlasov**

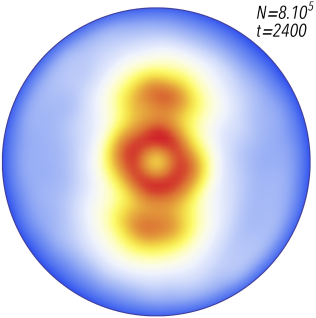


From Balescu-Lenard to Vlasov

- Dynamical **phase transition** in physical space



Initial times



Late times

CONCLUSIONS

- **Quasilinear theory** (both collisionless or collisional) able to approach the complex interplay between nature and nurture driving the secular evolution of self-gravitating systems.
- Novel theory of **WKB** (tightly wound) limit in the context of both dressed Fokker Planck and Balescu-Lenard kinetic theories: explicit quadrature for diffusion/drift coefficients.
- First implementations of **Balescu-Lenard** in stellar dynamics.
- **Prospects**:
  - Disc thickening
  - Galactic centre
  - Globular clusters
  - Radial Migration
  - Cusp-Core transition