The Universe as a dynamical system

From Friedmann, to Bianchi passing by the Jungle

Greco Seminar

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Ensta-ParisTech, Universite Paris Saclay

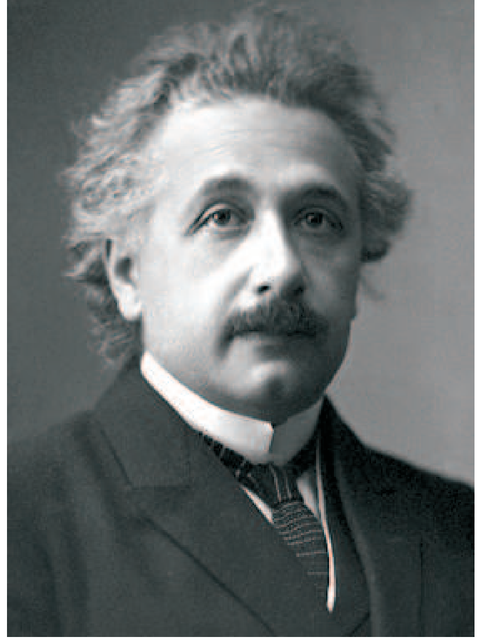
Overview

A Dynamical Universe ?

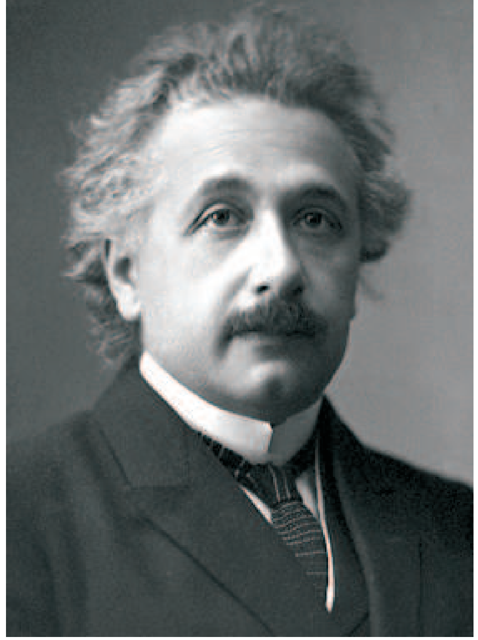
Friedmann Universe

Bianchi Universe

Einstein Legacy



Einstein Legacy



1905 - Special Relativity Principle $\longrightarrow$

The equations of physics
are the same in all
galilean (inertial) frames

$\longleftarrow$ Minkowski : $\mathbb{M}_4 = \mathcal{C}^{\pm} \cup \mathcal{L} \cup \mathcal{A}$

$$S_m = -\int mcds - \int \left(A_\alpha J^\alpha + \frac{1}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta} \right) d\Omega$$

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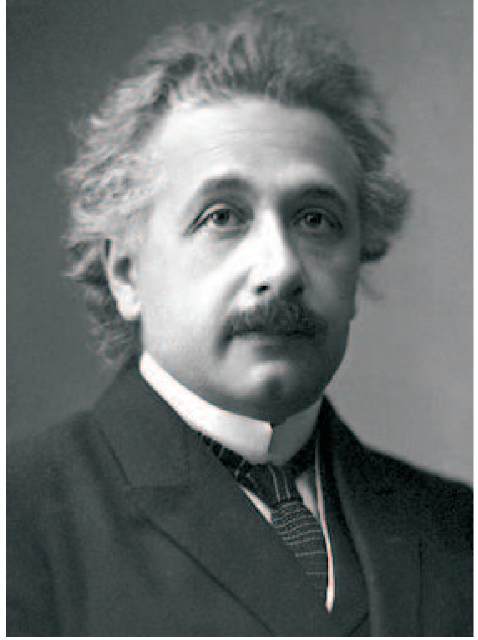

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$\nearrow$

1907 - Equivalence principle $\Rightarrow$ General relativity

We [...] propose the complete equivalence between a gravitationnal field and the acceleration of the corresponding frame

The material content of the universe makes impossible the existence of an inertial frame at the universe scale !

The equations of physics are the same in all frames

We pass from $\mathbb{M}_4$ $[\xi^\alpha]$ to a Riemann manifold $[x^\mu]$ in dimension 4

$$ds^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu$$

The universe becomes dynamical

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1915 - General Relativity $\chi = 8\pi G c^{-4}$

$$S = S_m - \frac{1}{2\chi}\int g^{\mu\nu} R_{\mu\nu}\sqrt{-\mathrm{g}}\,d\Omega$$ with $R_{\mu\nu} = R_{\mu\nu}(g)$ Ricci Tensor

variation of which gives : $$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \chi T_{\mu\nu}$$

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1917 - Homogeneous, static and isotropic Universe (Einstein)

$$ds^2 = -dt^2 + a\left(\frac{dr^2}{1 - Rr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2\right)$$

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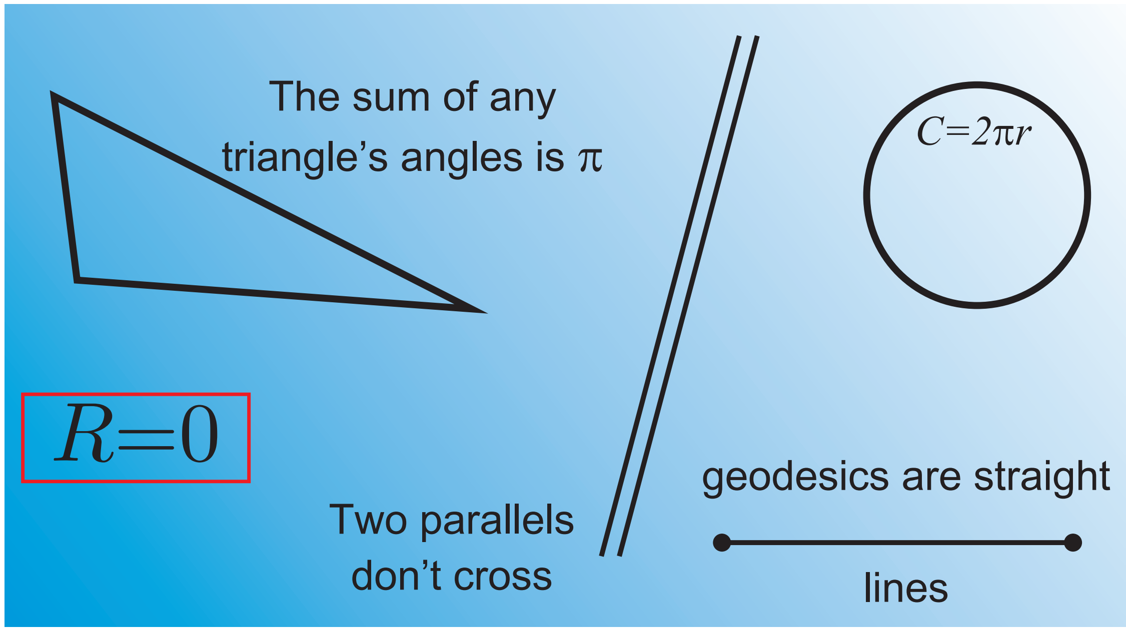

The plane ...

no static solutions

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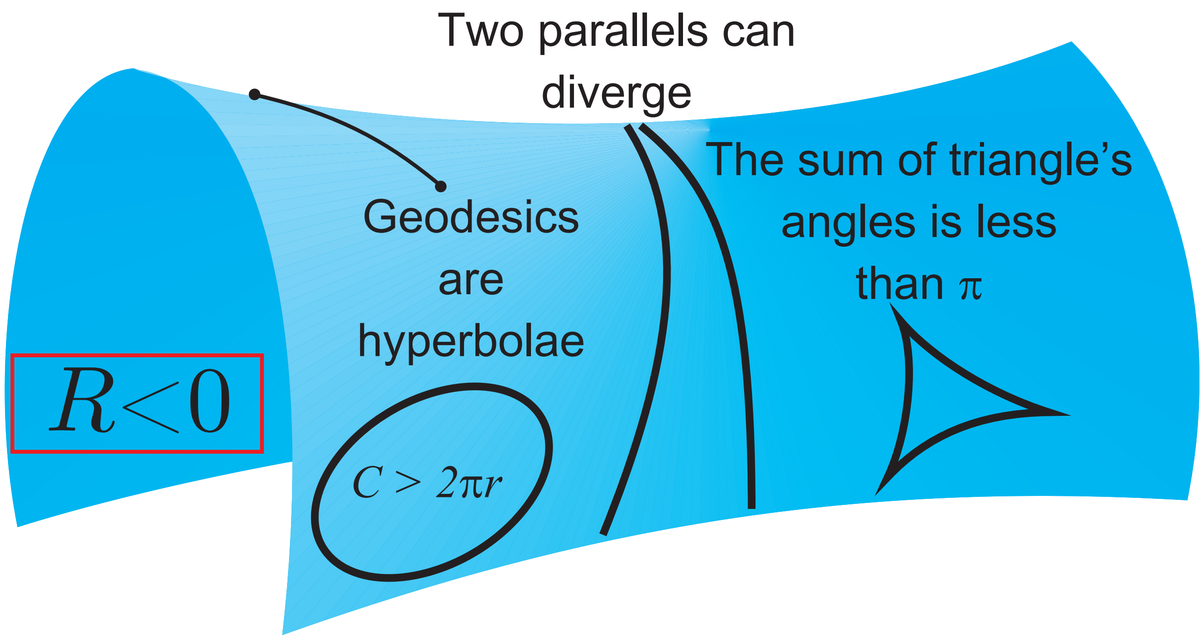

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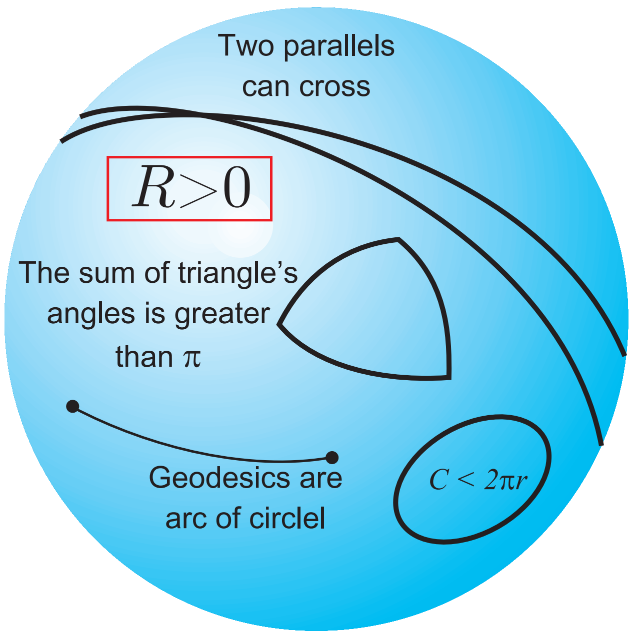

The sphere ...

allows a static solution

$$p = cste,\ \epsilon = cste$$

$$a = cste,\ R = 6/a^2$$

if

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \chi T_{\mu\nu}$$

Λ : Cosmological Const.

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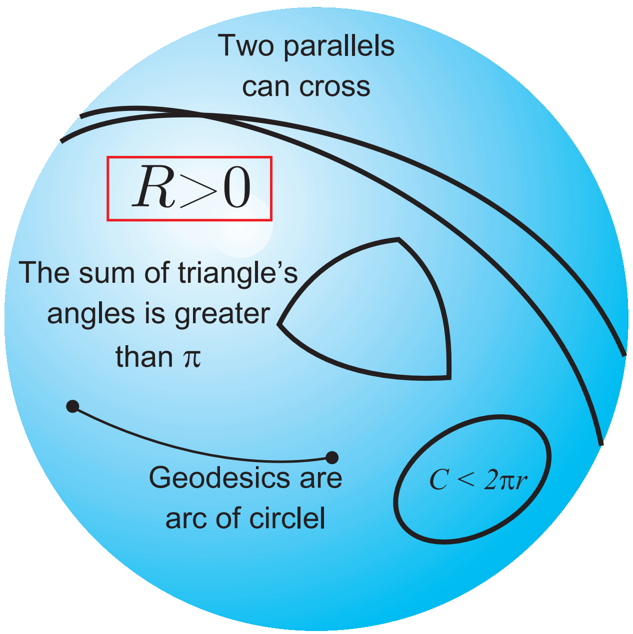

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Einstein Universe is unstable !

$a(t) = a(1 + \delta_a(t))$

$p(t) = p(1 + \delta_p(t))$

$\epsilon(t) = a(1 + \delta_\epsilon(t))$

$\delta_p(t) = \omega\delta_\epsilon(t)$

$a(t)$ diverges if $\omega > -1/3$

Friedmann, Lemaitre and Hubble

- **1922 - 1924 : Alexandre Friedmann**

Friedmann, Lemaitre and Hubble

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$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 + \frac{k}{a^2} = \frac{8\pi G\epsilon}{3} \qquad (F_1)$$

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}\left(\epsilon + 3P\right) \qquad (F_2)$$

$$a^3\frac{dP}{dt} = \frac{d\left[\left(\epsilon + P\right)a^3\right]}{dt} \qquad (F3)$$

Energy impulsion conservation

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Critical Density : $\epsilon_o = \frac{3H^2}{8\pi G} = 1.87847(23) \times 10^{-29}\ h^2 \cdot \mathrm{g} \cdot \mathrm{cm}^{-3}$

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Big controversy with Einstein but Friedmann dies in September '25

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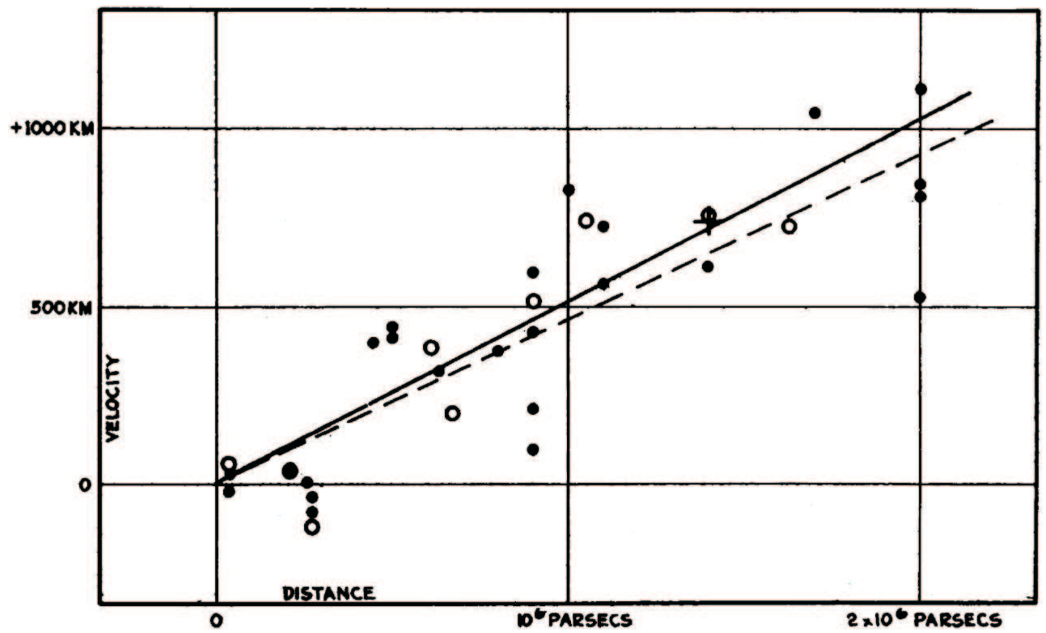
1929 - Hubble : The Universe is expanding !



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A dynamical Universe

Very fun !

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 + \frac{k}{a^2} = \frac{8\pi G\epsilon}{3} + \frac{\Lambda}{3} \qquad (F_1)$$

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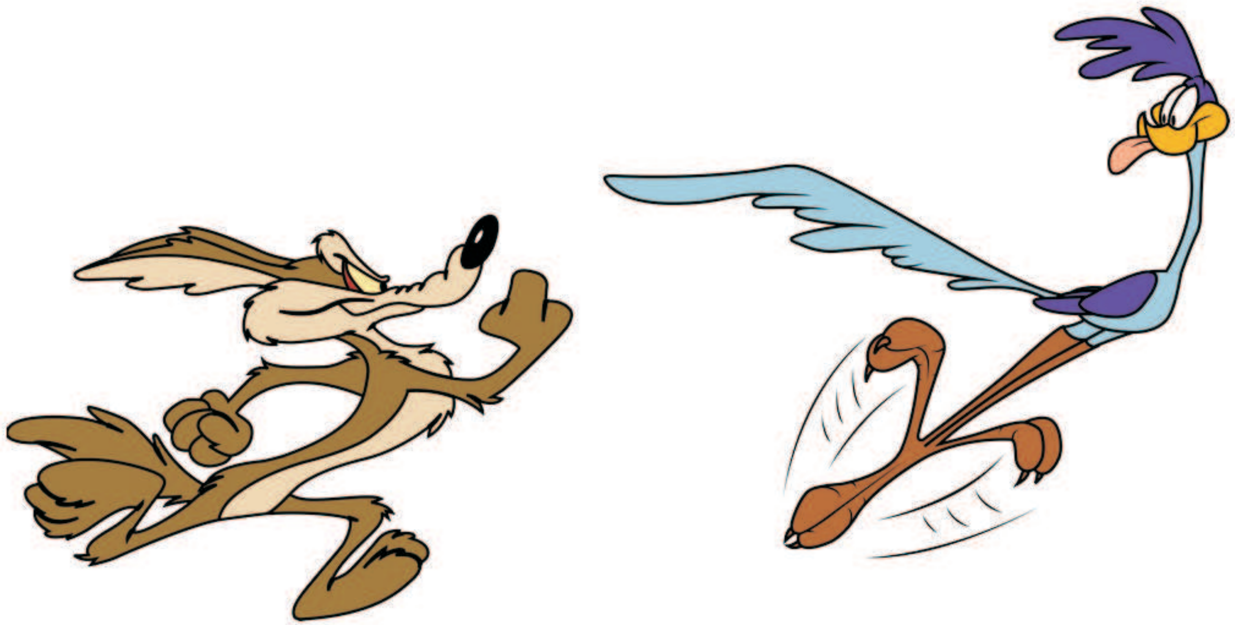
Impulsion-Energy conservation

Friedmann's Universes Dynamics



Predator-Prey, competition

Building...

$$\begin{cases} \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\epsilon}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \\ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\epsilon + 3P\right) + \frac{\Lambda}{3} \\ \dot{\epsilon} = -3H\left(P + \epsilon\right) \end{cases}$$

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, setting

$$H\left(t\right) = \frac{\dot{a}}{a} = \frac{d\left(\ln a\right)}{dt}$$

$$q\left(t\right) = -\frac{\ddot{a}}{a}\frac{1}{H^2} = -\frac{\ddot{a}\, a}{\dot{a}^2}$$

$$\Omega_m\left(t\right) = \frac{8\pi G\epsilon}{3H^2},\ \Omega_k\left(t\right) = -\frac{k}{a^2 H^2}$$

and $\Omega_\Lambda\left(t\right) = \frac{\Lambda}{3H^2}$

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and $\Omega_\Lambda(t) = \frac{\Lambda}{3H^2}$

we obtain

$$\begin{cases} \Omega_m + \Omega_k + \Omega_\Lambda = 1 & (F1.1) \\ \frac{4\pi G}{3H^2}(\epsilon + 3P) = q + \Omega_\Lambda & (F2.1) \\ \dot{\epsilon} = -3H(P+\epsilon) & (F3.1) \end{cases}$$

State equation

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ω	-1	0	$1/3$	$2/3$	1
Kind of Matter	Quantum Vacuum	Incoherent Dust Gas	Photon Ideal Gas	monoatomic Ideal Gas	Stiff matter

$$\omega \in [-1, 1]\ ,\quad \Gamma \in [0, 2]\ ,\quad \gamma \in [-2, 4]$$

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Barotropic Friedmann's Equations :

$$\begin{cases} \Omega_k = 1 - \Omega_m - \Omega_\Lambda \\ q = \frac{\Omega_m\,(1 + 3\omega)}{2} - \Omega_\Lambda \\ (\ln\epsilon)' = -3\,(1 + \omega) \end{cases}$$

$$' = \frac{d}{d\ln a}$$

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$$\begin{cases} \Omega_k = 1 - \Omega_m - \Omega_\Lambda \\ \\ \Omega'_m = \Omega_m \left[(1 + 3\omega)(\Omega_m - 1) - 2\Omega_\Lambda \right] \\ \\ \Omega'_\Lambda = \Omega_\Lambda \left[\Omega_m (1 + 3\omega) + 2(1 - \Omega_\Lambda) \right] \end{cases}$$

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setting $\gamma = 1 + 3\omega$ in the interval $[-2, 4]$

$X' = F_\gamma(X)$ with $X = [\Omega_m, \Omega_\Lambda]^\top$ and $F_\gamma : \left| \begin{array}{ccc} \mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \\ (x, y) & \mapsto & (f_1(x,y), f_2(x,y)) \end{array} \right.$

where

$$\begin{cases} f_1(x,y) = x(\gamma x - 2y - \gamma) \\ f_2(x,y) = y(\gamma x - 2y + 2) \end{cases}$$

Lotka-Volterra like equation

Equilibria

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Equilibrium : $X^* = [x, y]^\top = [\Omega_m, \Omega_\Lambda]^\top$ such that $F_\gamma(X^*) = 0$

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- de Sitter Universe : $X_1^* = [0, 1]^\top$ and $\Omega_k = 0$

If $\dot{a} > 0$ then $a(t) \propto e^{\sqrt{\frac{\Lambda}{3}}t}$

Uncreated Universe in perpetual exponential expansion.

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- Einstein-de Sitter Universe : $X_2^* = [1, 0]^\top$ and $\Omega_k = 0$

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  Big-Bang followed by a decelerated expansion.

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- Milne Universe : $X_3^* = [0, 0]^\top$ and $\Omega_k = 1$

  $k = -a^2H^2$: Hyperbolic universe with $a(t) = \dot{a}_0 t + a_0$

  Linearly expanding Universe since Big-Bang : exotic cosmological models ?

Dynamic is a competition !

$$\begin{cases} \Omega'_m = \Omega_m \left(\gamma\Omega_m - 2\Omega_\Lambda - \gamma\right) \\ \\ \Omega'_\Lambda = \Omega_\Lambda \left(\gamma\Omega_m - 2\Omega_\Lambda + 2\right) \end{cases}$$

Competition between Ω_m and Ω_Λ "referred" by Ω_k ;

3 equilibrium states :

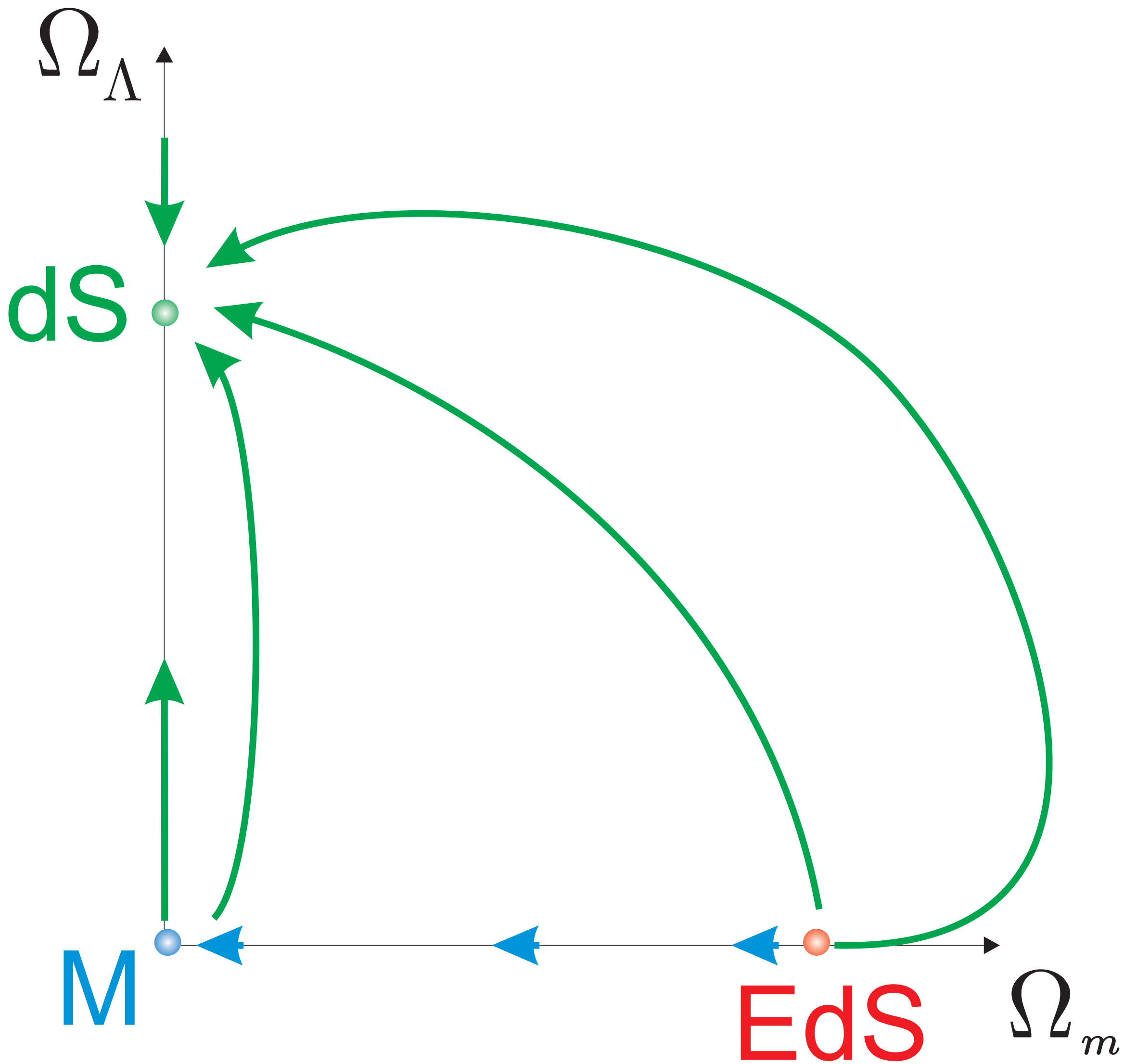
- Matter (EdS) - $\gamma-$Hyperbolic ;
- Curvature (M) - $\gamma-$Hyperbolic ;
- Cosmological Constant (dS) - Stable.

The most competitive is always the Cosmological Constant : $\gamma \in [-2, 4]$.

No Limit Cycle (Bendixon criteria, $\mathrm{div}(F)$ has constant sign on $[0, 1]^2$?

The fate of Friedmann's Universes

If ω of Ω_m is in $]-1,-1/3[$:



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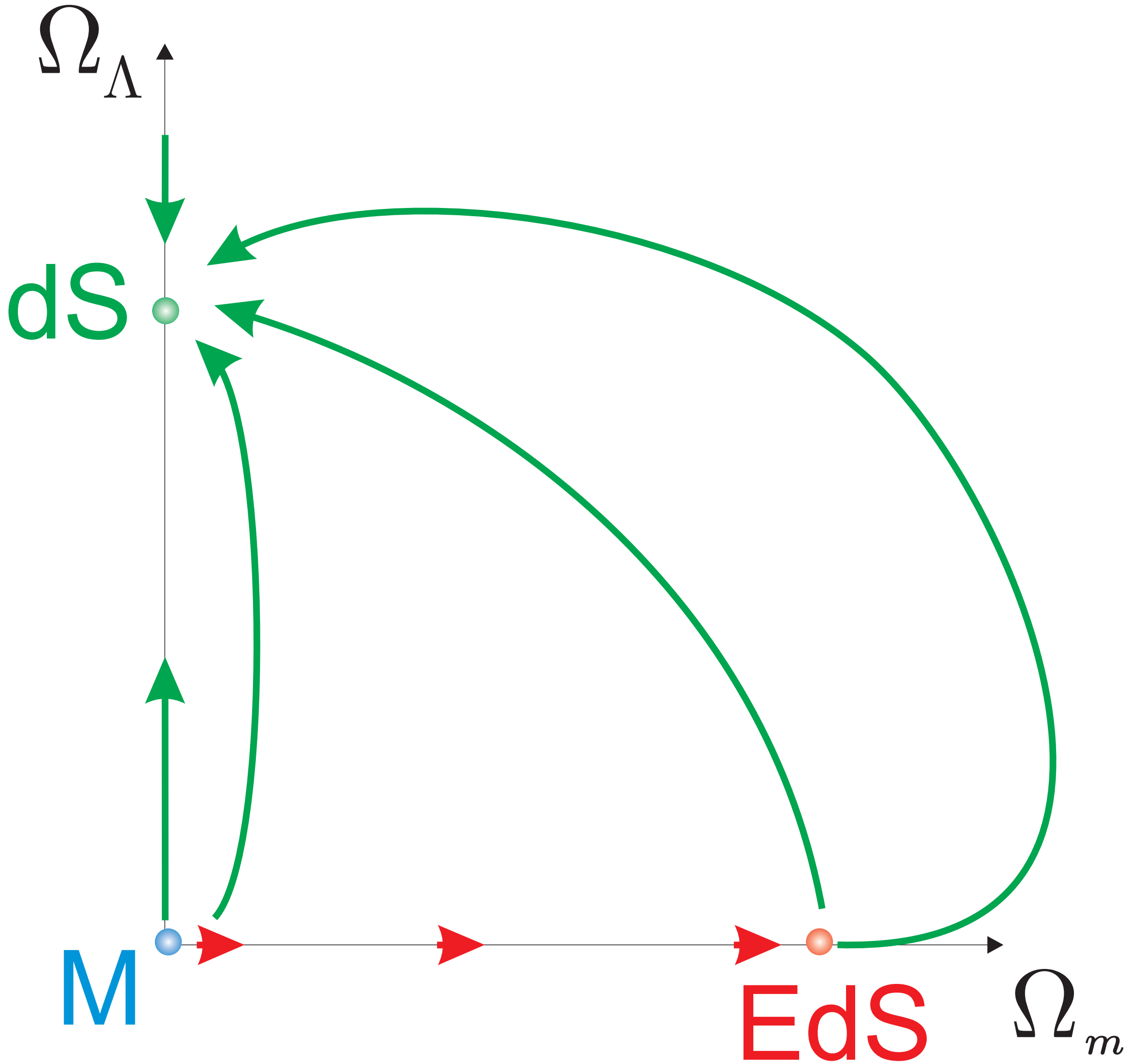

Coupled species : Jungle Universe

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Without any coupling between species (Ω_i) the dynamic is fully **degenerated** :

$$\mathbf{x} = (\Omega_\mathrm{b}, \Omega_\mathrm{d}, \Omega_\mathrm{r}, \Omega_\mathrm{e})^\top \quad , \quad \mathbf{x}' = \mathrm{diag}(\mathbf{x})\,(\mathbf{r} + \mathbb{A}\mathbf{x})$$

with

$$\mathbb{A} = \begin{bmatrix} 1+3\omega_\mathrm{b} & 1+3\omega_\mathrm{d} & 1+3\omega_\mathrm{r} & 1+3\omega_\mathrm{e} \\ 1+3\omega_\mathrm{b} & 1+3\omega_\mathrm{d} & 1+3\omega_\mathrm{r} & 1+3\omega_\mathrm{e} \\ 1+3\omega_\mathrm{b} & 1+3\omega_\mathrm{d} & 1+3\omega_\mathrm{r} & 1+3\omega_\mathrm{e} \\ 1+3\omega_\mathrm{b} & 1+3\omega_\mathrm{d} & 1+3\omega_\mathrm{r} & 1+3\omega_\mathrm{e} \end{bmatrix} \quad \text{and } \mathbf{r} = \begin{bmatrix} -1-3\omega_\mathrm{b} \\ -1-3\omega_\mathrm{d} \\ -1-3\omega_\mathrm{r} \\ -1-3\omega_\mathrm{e} \end{bmatrix}$$

As $\mathrm{rank}(\mathbb{A}) = 1$, equilibria must lie on axes $\mathbf{x}_i = 0$, this is Friedmann's dynamics.

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$$\mathbf{x} = (\Omega_\mathrm{b}, \Omega_\mathrm{d}, \Omega_\mathrm{r}, \Omega_\mathrm{e})^\top \quad , \quad \mathbf{x}' = \mathrm{diag}(\mathbf{x})\,(\mathbf{r} + \mathbb{A}\mathbf{x})$$

with

$$\mathbb{A} = \begin{bmatrix} 1+3\omega_\mathrm{b} & 1+3\omega_\mathrm{d} & 1+3\omega_\mathrm{r} & 1+3\omega_\mathrm{e} \\ 1+3\omega_\mathrm{b} & 1+3\omega_\mathrm{d} & 1+3\omega_\mathrm{r} & 1+3\omega_\mathrm{e} \\ 1+3\omega_\mathrm{b} & 1+3\omega_\mathrm{d} & 1+3\omega_\mathrm{r} & 1+3\omega_\mathrm{e} \\ 1+3\omega_\mathrm{b} & 1+3\omega_\mathrm{d} & 1+3\omega_\mathrm{r} & 1+3\omega_\mathrm{e} \end{bmatrix} \quad \text{and } \mathbf{r} = \begin{bmatrix} -1-3\omega_\mathrm{b} \\ -1-3\omega_\mathrm{d} \\ -1-3\omega_\mathrm{r} \\ -1-3\omega_\mathrm{e} \end{bmatrix}$$

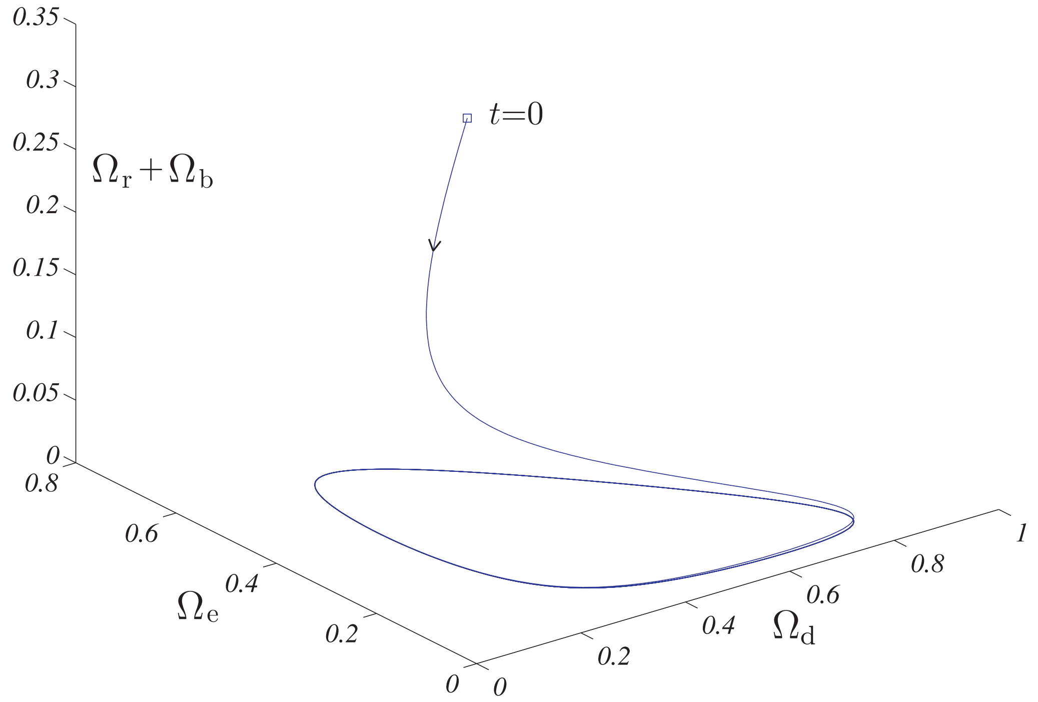
As $\mathrm{rank}(\mathbb{A}) = 1$, equilibria must lie on axes $\mathbf{x}_i = 0$, this is Friedmann's dynamics.

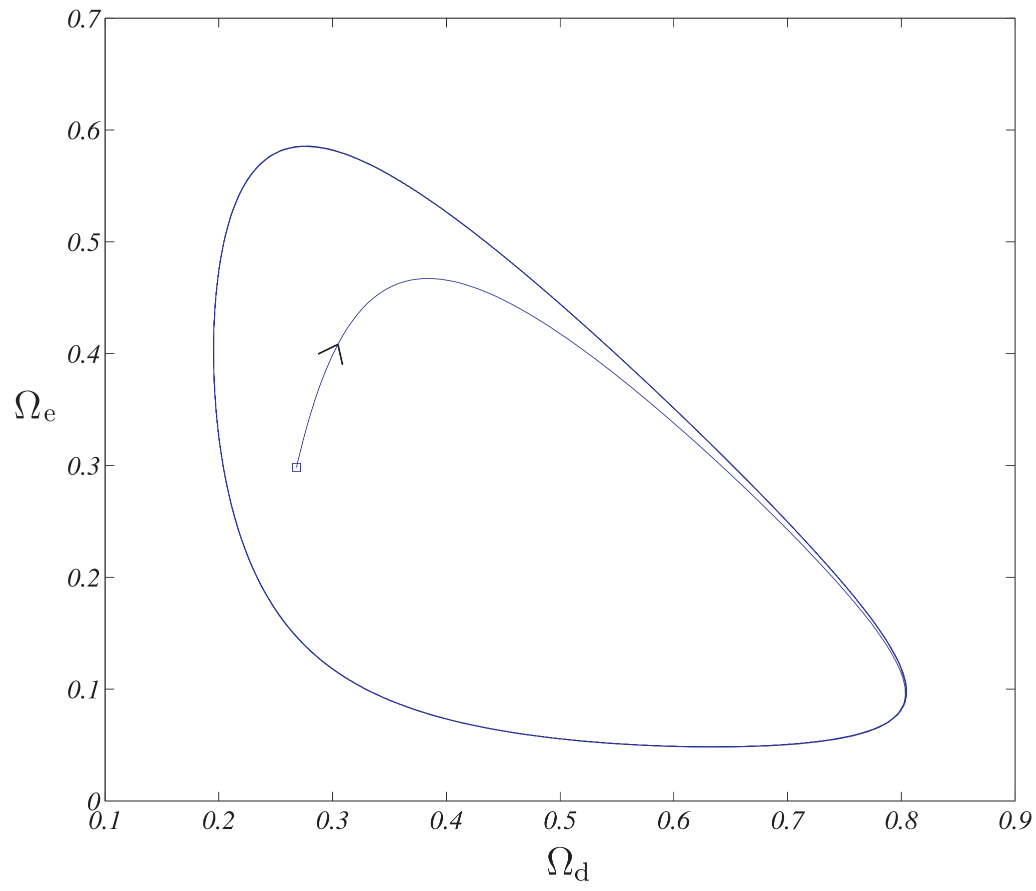
Introducing coupling between any barotropic components of the Universe, the dynamical systems becomes

$$\begin{aligned} x_i &= \Omega_i \\ r_i &= -(1+3\omega_i) \\ A_{ij} &= 1 + 3\omega_j + \varepsilon_{ij} \;\; \text{with } \varepsilon_{ij} = -\varepsilon_{ji} \;\; \text{and } \varepsilon_{ii} = 0 \end{aligned} \tag{1}$$

The matrix $\mathbb{A}$ can have any rank, it can be invertible, equilibria can be everywhere, this is Jungle dynamics. [e.g. Perez et. al., 2014]

Dark coupling...

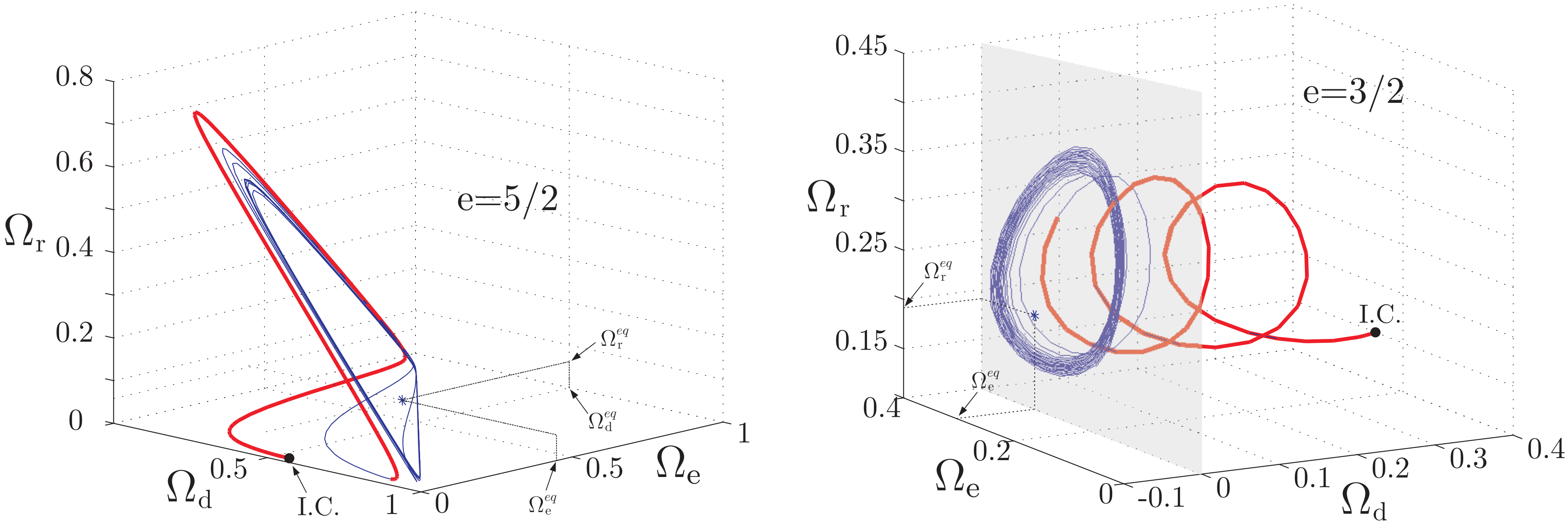




Coupling between dark energy and dark mater with $\varepsilon = 4$.

The radiative components (Ω_r) and the baryonic matter (Ω_b) dilutes and disappears while the dark component converges toward a limit cycle.

Other possibilities...



Evolution of the three coupled density parameters, in the 3D phase space. The beginning of the orbit is overlined. Initial condition is indicated by a black dot. Relevant equilibria are indicated by a star.

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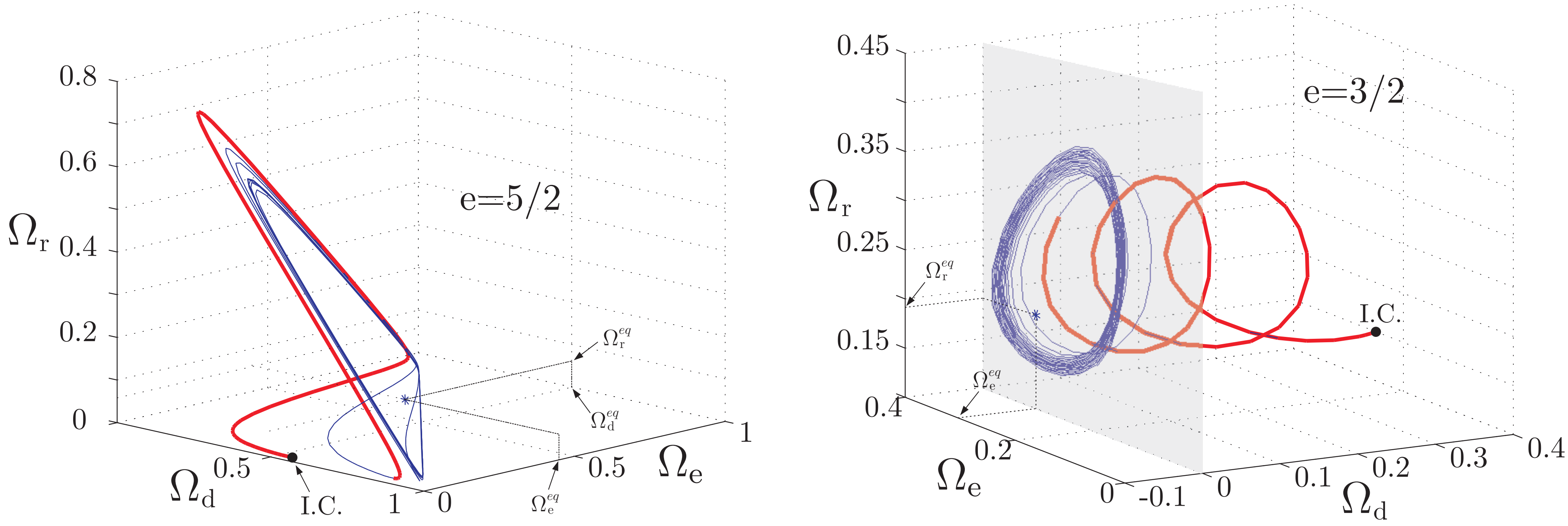

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The interaction term in the continuity equation of a fluid i reads

$$\dot{\rho}_i = -3H\rho_i(1+\omega_i) + \sum_{j=1}^{n} \epsilon_{ij} H \Omega_j \rho_i$$

It actually modifies its equation of state which then describes a barotropic uid with an effective time-dependent barotropic index $\omega_i^{\text{eff}} = \omega_i - \sum_{j=1}^{n} \frac{1}{3} \epsilon_{ij} \Omega_j$

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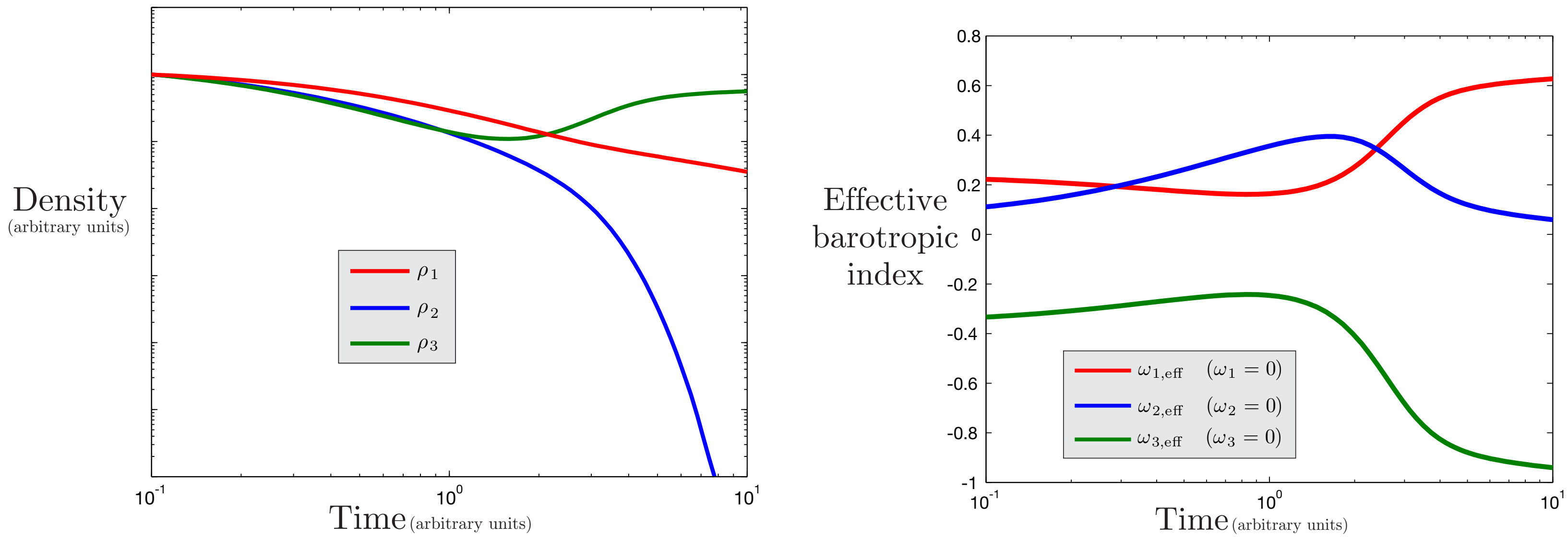
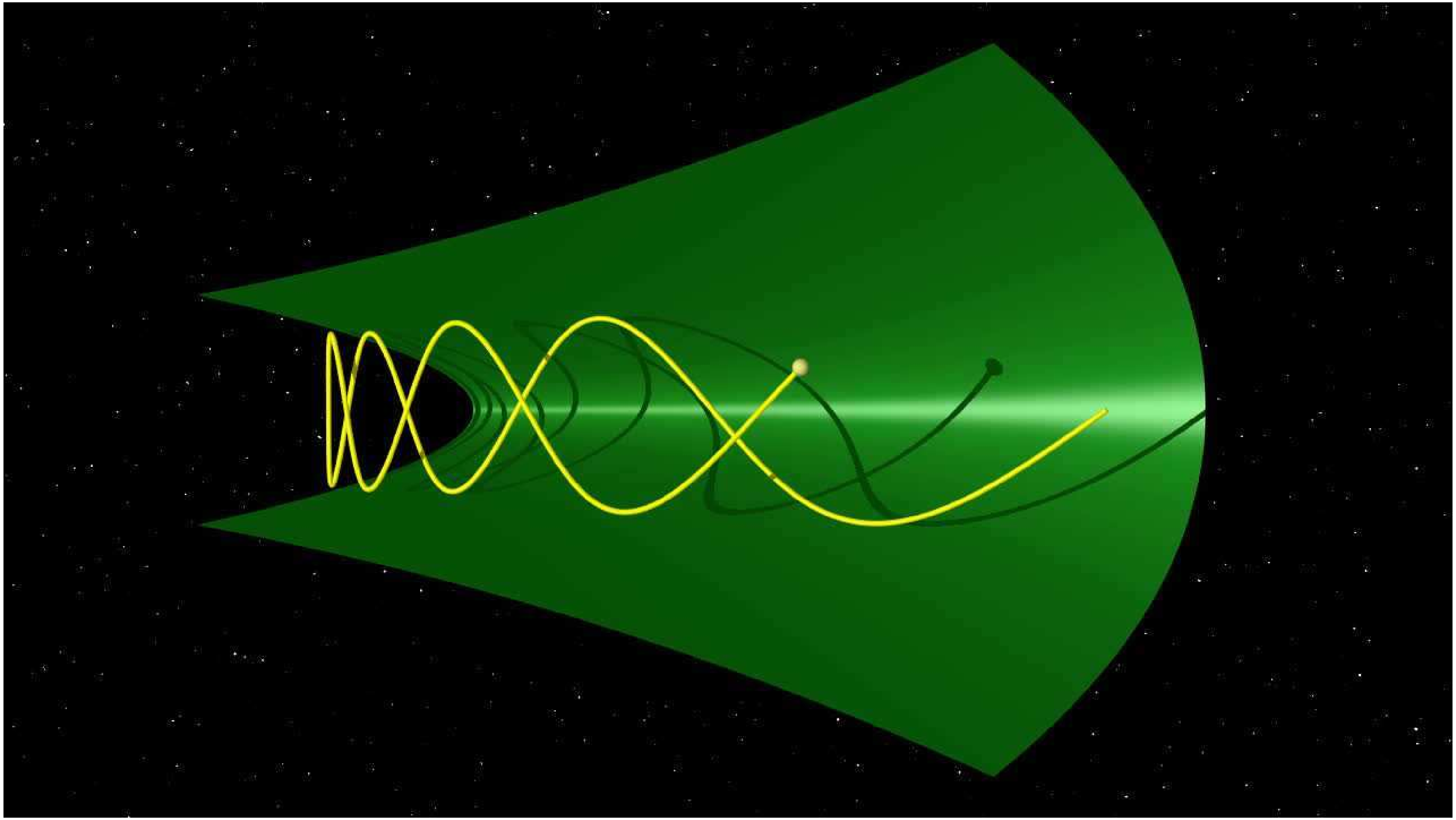
Exemple :



Jungle Interaction $(\epsilon_{12} = -2; \epsilon_{23} = -3; \epsilon_{13} = 0)$ between three dark matter fluids

Bianchi Universes



The Cosmological Billiard

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1969 V. Belinski, L. Khalatnikov & E. Lifchitz : Singularity may be chaotic if Universe is anisotropic !

Homogeneous Manifold in 3+1 dimension

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Synchronous Frame : $ds^2 = \tilde{g}_{ij}\, dx^i\, dx^j - dt^2$, $\mathbb{E} = \Sigma_t,\ \tilde{g}_{ij} = \tilde{g}_{ij}(t)$

Invariant Forms basis $\mathbb{G}$: $e^i_j\, dx^j$

$$C_{ab}^{\ \ c} = \left(\partial_i e^c_j - \partial_j e^c_i\right) e^j_a\, e^i_b \quad \text{(Structure Constants)}$$

$$\sigma_a := e^i_a \partial_i \quad \text{such that} \quad [\sigma_a, \sigma_b] = C_{ab}^{\ \ c}\, \sigma_c$$

The set of $C_{ab}^{\ \ c}$ is a determination of $\mathbb{G}$.

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Decomposition $C_{ab}^{\ \ c} := \varepsilon_{abd}\, N^{dc} + \delta^c_b\, M_a - \delta^c_a\, M_b \ \Rightarrow\ N^{ab}$ symetric

Equivalence Classes of Homogeneous Universes	$\equiv$	Equivalence Classes of N^{ab} and M_b such that $N^{ab} M_b = 0$

$$N^{ab} = \begin{bmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{bmatrix} \qquad M_b = [m, 0, 0]$$

Bianchi's Classification

Class $\mathcal{A}$: $m = 0$, Class $\mathcal{B}$: $m \neq 0$

	n_1	n_2	n_3	m	Model
0 is a triple eigenvalue of N	0	0	0	0	B_{I}
	0	0	0	$\forall$	B_{V}
0 is a double eigenvalue of N	1	0	0	0	B_{II}
	0	1	0	$\forall$	B_{IV}
0 is a simple eigenvalue of N	1	1	0	0	B_{VII_o}
	0	1	1	$\forall$	B_{VII_m}
	1	-1	0	0	B_{VI_o}
	0	1	-1	$\neq 1$	B_{VI_m}
	0	1	-1	1	B_{III}
0 is not an eigenvalue of N	1	1	1	0	B_{IX}
	1	1	-1	0	B_{VIII}

BKL Formalism

(e.g. [Belinski, Khalatnikov et Lifchitz, 69])

$$ds^2 = \tilde{g}_{ij} dx^i dx^j - dt^2 = \sum_{i=1}^{3} e^{A_i(\tau)} dx_i^2 - V^2(\tau) d\tau^2$$

The lapse function is the volume of the universe : $V^2 = e^{A_1+A_2+A_3}$, $dt = V d\tau$

The matter is isotropic & barotropic : $P = (\Gamma - 1)\epsilon \quad \Longrightarrow \quad \epsilon = \epsilon_0 V^{-\Gamma}$

$$\begin{cases} 0 &= E_c + E_p + E_m = H \\ \chi\epsilon_0 \left(2-\Gamma\right) V^{2-\Gamma} &= A_1'' + \left(n_1 e^{A_1}\right)^2 - \left(n_2 e^{A_2} - n_3 e^{A_3}\right)^2 \\ \chi\epsilon_0 \left(2-\Gamma\right) V^{2-\Gamma} &= A_2'' + \left(n_2 e^{A_2}\right)^2 - \left(n_3 e^{A_3} - n_1 e^{A_1}\right)^2 \\ \chi\epsilon_0 \left(2-\Gamma\right) V^{2-\Gamma} &= A_3'' + \left(n_3 e^{A_3}\right)^2 - \left(n_1 e^{A_1} - n_2 e^{A_2}\right)^2 \end{cases}$$

$$E_c = \frac{1}{2} \sum_{i \neq j = 1}^{3} A_i' A_j' \qquad E_p = \sum_{i \neq j=1}^{3} n_i n_j e^{A_i + A_j} - \sum_{i=1}^{3} n_i^2 e^{2A_i}$$

$$E_m = -4\chi\epsilon\, V^2 \qquad {}' = \frac{d}{d\tau} \quad , \quad \chi = \frac{8\pi G}{c^4}$$

Vacuum B_{I} Solution : The fondamental state

In conformal time variable, Spatial Einstein Equations write $A_i'' = 0$ which gives in physical time $e^{A_i} = \lambda_i t^{2k_i/\Omega}$ where $V(t) = \frac{1}{2}\Omega t + \Omega_0$. Time Einstein Equation makes appear a global parameter $u \in [1, +\infty[$

$$\begin{cases} p_1 = k_1/\Omega = -u\left(1+u+u^2\right)^{-1} & \in \left[-\frac{1}{3}, 0\right] \\ p_2 = k_2/\Omega = (1+u)\left(1+u+u^2\right)^{-1} & \in \left[0, \frac{2}{3}\right] \\ p_3 = k_3/\Omega = u\,(1+u)\left(1+u+u^2\right)^{-1} & \in \left[\frac{2}{3}, 1\right] \end{cases}$$

Vacuum B_{I} Universe's metric writes

$$ds^2 = \lambda_1 \mathbf{t}^{\mathbf{2p_1}} dx_1^2 + \lambda_2 \mathbf{t}^{\mathbf{2p_2}} dx_2^2 + \lambda_3 \mathbf{t}^{\mathbf{2p_3}} dx_3^2 - dt^2$$

If $t \to 0$ ($\to$ singularity)

•	:	Exponential Expansion
• •	:	Exponential Contraction
V	:	Linear Contraction

Vacuum B_{I} defines a *Kasner State* characterized by u and Ω

Vacuum B_{II} solution : The idea by BKL...

Vacuum B_{II} dynamics in τ :

$$\begin{cases} A_1'' = -e^{2A_1} \\ A_2'' = +e^{2A_1} \\ A_3'' = +e^{2A_1} \\ e^{2A_1} = A_1'A_2' + A_1'A_3' + A_2'A_3' \end{cases}$$

But in t it appears as a transition between 2 Kasner States :

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But in t it appears as a transition between 2 Kasner States :

When $t \longrightarrow +\infty$		When $t \longrightarrow 0$
$[u, \Omega]$ $(p_1 < p_2 < p_3)$ (●●●) Kasner 1	$\longrightarrow$	$[u-1, \Omega(1-2p_1)]$ (●●●) si $u > 2$ $\left[(u-1)^{-1}, \Omega(1-2p_1)\right]$ (●●●) si $u \leq 2$ Kasner 2

Amazing Bianchi Universes !

Hamiltonian Formalism

e.g. [Misner '70]

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$$E_c = \frac{1}{2} \sum_{i\neq j=1}^{3} A_i' A_j'$$

Diagonalize E_c ...

Hamiltonian Formalism

e.g. [Misner '70]

$$M := \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\mathbf{q} := [q_1 \ q_2 \ q_3]^\top = M \ [A_1 \ A_2 \ A_3]^\top$$

$$\mathbf{p} := [p_1 \ p_2 \ p_3]^\top = M \ [A_1' \ A_2' \ A_3']^\top$$

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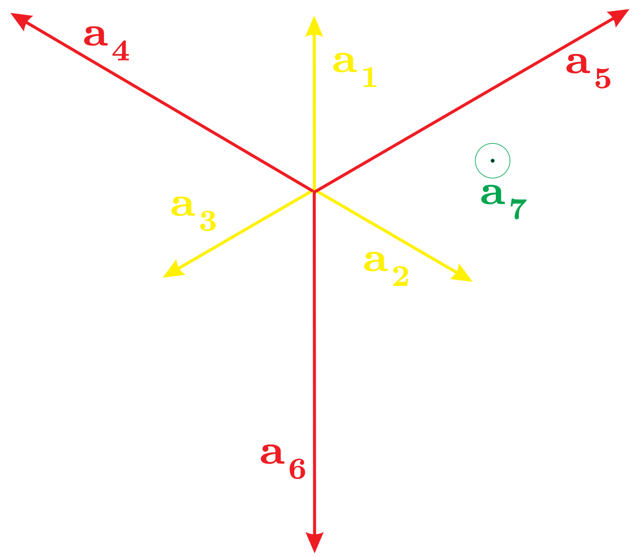
Einstein Equations become Todda-Like

$$q_{1,2}' = -\nabla_{p_{1,2}} H \qquad p_{1,2}' = -\nabla_{q_{1,2}} H$$
$$q_3' = \nabla_{q_3} H \qquad p_3' = -\nabla_{p_3} H$$

with $H = \frac{1}{2}\langle \mathbf{p}, \mathbf{p} \rangle + \sum_{i=1}^{7} k_i e^{(\mathbf{a}_i, \mathbf{q})}$

$$(x, y) := +x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$\langle x, y \rangle := -x_1 y_1 - x_2 y_2 + x_3 y_3$$



$$k_1 := 2n_1 n_2 \qquad k_2 := 2n_1 n_3 \qquad k_3 := 2n_2 n_3$$
$$k_4 := -n_1^2 \qquad k_5 := -n_2^2 \qquad k_6 := -n_3^2$$
$$k_7 = -4\varepsilon_o \chi$$

Integrability

Integrable Differential System $\Longrightarrow$ Regular Solutions (Reciprocally ?)

Two used methods :

- Show that the solution is analytic (formal series)

  Kovalewski-Poincaré Theory (Painlevé)

- Show that the system admits enough first integrals

  Lax Theory (Liouville)

Kovalewski-Poincaré Theory

If $\dfrac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^n$ admits Self-Similar Solution (3S)

$$\tilde{\mathbf{x}} = \left[c_1 (t - t_o)^{-g_1}, \ldots, c_n (t - t_o)^{-g_n}\right]^\top \quad \mathbf{g} \in \mathbb{Z}^n \quad \mathbf{c} \in \mathbb{R}^n$$

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then the linearized system around $\tilde{\mathbf{x}}$ too !

$$\mathbf{z} = \left[k_1 (t - t_o)^{\rho_1 - g_1}, \ldots, k_n (t - t_o)^{\rho_n - g_n}\right]^\top \qquad \rho \in \mathbb{C}^n$$

Kovalewski Exponents : $\{\rho\} =$ Sp $[D\mathbf{f}(\mathbf{x})(\mathbf{c}) + \mathrm{diag}(\mathbf{g})]$

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Kovalewski Exponents : $\{\rho\} = \text{Sp}\left[D\mathbf{f}(\mathbf{x})(\mathbf{c}) + \text{diag}(\mathbf{g})\right]$

Poincaré and Yoshida then show that

$$x_i(t) \propto (t - t_o)^{-g_i} S\left[(t - t_o)^{\rho_1}, ..., (t - t_o)^{\rho_n}\right]$$

$\rho \in \mathbb{Q}^n$ is sufficient for analiticity of $\mathbf{x}(t)$

Kovalewski & Bianchi

e.g. Melnikov’s Team in Moscow, [Gavrilov et al.,94] ,[Pavlov,96] and [Szydlowksi & Besiada,02]

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A new change of variables

$$\{\mathbf{q},\mathbf{p}\} \mapsto \{\mathbf{u},\mathbf{v}\} \quad \text{avec} \quad \left[\begin{array}{l} \mathbf{u} \in \mathbb{R}^7,\ u_{i=1,\ldots,7} := \langle \mathbf{a}_i, \mathbf{p} \rangle \\ \mathbf{v} \in \mathbb{R}^7,\ v_{i=1,\ldots,7} := \exp\left(\mathbf{a}_i, \mathbf{q}\right) \end{array}\right.$$

Kovalewski & Bianchi

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The Bianchi dynamics becomes

$$\forall i = 1, \ldots, 7 \quad \left[\begin{array}{l} v_i' = u_i v_i \\ u_i' = \sum_{j=1}^{7} W_{ij} v_j \end{array}\right. \quad \text{with } W_{ij} := -k_j \left\langle \mathbf{a}_i, \mathbf{a}_j \right\rangle$$

which admits a plenty of 3S : $\tilde{\mathbf{x}} = \left[\lambda t^{-1}, \mu t^{-2}\right]^\top$ for each $[\lambda, \mu] \in \mathbb{R}^7 \times \mathbb{R}^7$ solution of

$$\left[\begin{array}{l} \sum_{j=1}^{7} W_{ij}\, \mu_j = -\lambda_i \\ \lambda_i\, \mu_i = -2\mu_i \end{array}\right.$$

Bianchi's Integrability

[JP & Larena,07]

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4 class of equivalence of Bianchi Universes in Kovalewski sense

Class I : B_{I}	Class II : B_{II} & B_{IV}
Class III : B_{III}, $B_{\text{VI}_{o,a}}$ & $B_{\text{VII}_{o,a}}$	Class IV : B_{VIII} & B_{IX}

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- Vacuum & $\forall\Gamma \in \mathbb{Q}$: $K_I \subset \mathbb{Q}$: Int.
- Vacuum & Stiff matter : $K_{II} \cup K_{III} \subset \mathbb{Q}$: Int.
- Matter with $\Gamma \in \mathbb{Q} \cap [0, \Gamma_o]$: $K_{II} \cup K_{III} \subset \mathbb{Q}$: Int.
- Matter with $\Gamma \in [\Gamma_o, 2[$: $K_{II} \cup K_{III} \subset \mathbb{C}$: Not Int.
- Vacuum & $\forall\Gamma \in [0, 2[$: $K_{IV} \subset \mathbb{C}$: Not Int.

$$\Gamma_o := \frac{11+\sqrt{73}}{3} \approx 0,82$$

$\Gamma = 0$: Scalar Field

$\Gamma = 1$: Dust

$\Gamma = 4/3$: Quantum Id. Gas.($\mu = 0$)

$\Gamma = 5/3$: Classical Id. Gas

$\Gamma = 2$: Stiff Matter

Singularity could be chaotic...

Bianchi's Billiards

e.g. [Jantzen,82] , [Uggla,97]

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Setting $d\tilde{t} = V^{1/3}dt$ et $m = V^{4/3}$ the dynamics becomes

$$\begin{cases} \dfrac{dq_{1,2}}{d\tilde{t}} = \dfrac{p_{1,2}}{m} = \dfrac{\partial E}{\partial q_{1,2}} \\[2ex] \dfrac{dp_{1,2}}{d\tilde{t}} = -\dfrac{\partial \xi}{\partial q_{1,2}} = \dfrac{\partial E}{\partial p_{1,2}} \end{cases}$$

with

$$E = \frac{p_1^2 + p_2^2}{2m} - \xi\left(q_1, q_2\right) = \frac{(dV/dt)^2}{V^{2/3}}$$

Pour $t \to 0$ $\left| \begin{array}{l} E \to +\infty \\ m \to 0 \end{array} \right.$

$$\xi\left(q_1, q_2\right) = \sum_{i=1}^{7} k_i e^{(\pi(\mathbf{a}_i), \mathbf{q})}$$, $\mathbf{q} \in \mathbb{R}^2$, π : Projector on $(\mathbf{e}_1, \mathbf{e}_2)$

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Bianchi Dynamics $\Leftrightarrow$ Dynamics of 2D decreasing mass particle with an increasing energy in the potential well ξ

The Cosmological Billiard

The Cosmological Billiard

"Isolated" Dynamics : $\dfrac{d^2y}{dx^2} = -k^2 e^y$ with $y(0) = 0 = \left.\dfrac{dy}{dx}\right|_{x=0}$.

$$y(x) = \ln\left[1 - \mathrm{th}^2\left(\frac{kx}{\sqrt{2}}\right)\right] = -2\ln\left[\mathrm{ch}\left(\frac{kx}{\sqrt{2}}\right)\right] \approx \begin{array}{l} \pm\sqrt{2}kx + 2\ln 2 \\ \text{when } x \to \pm\infty \end{array}$$

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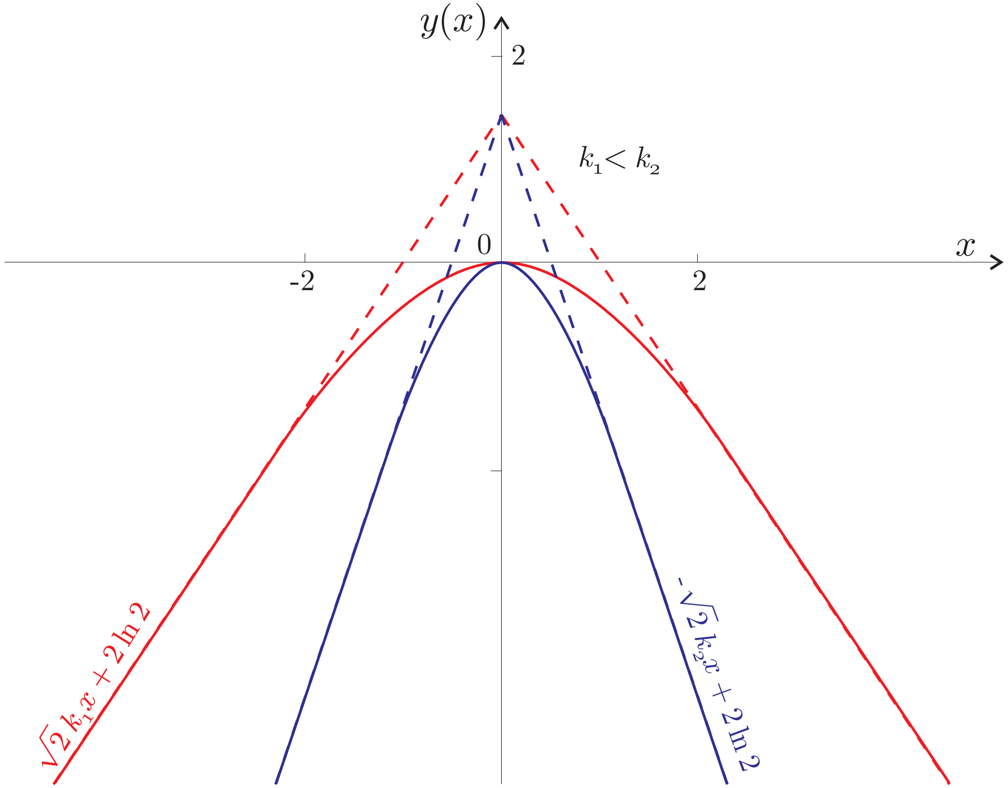
when $x \to \pm\infty$

Rebound on 1 cushion



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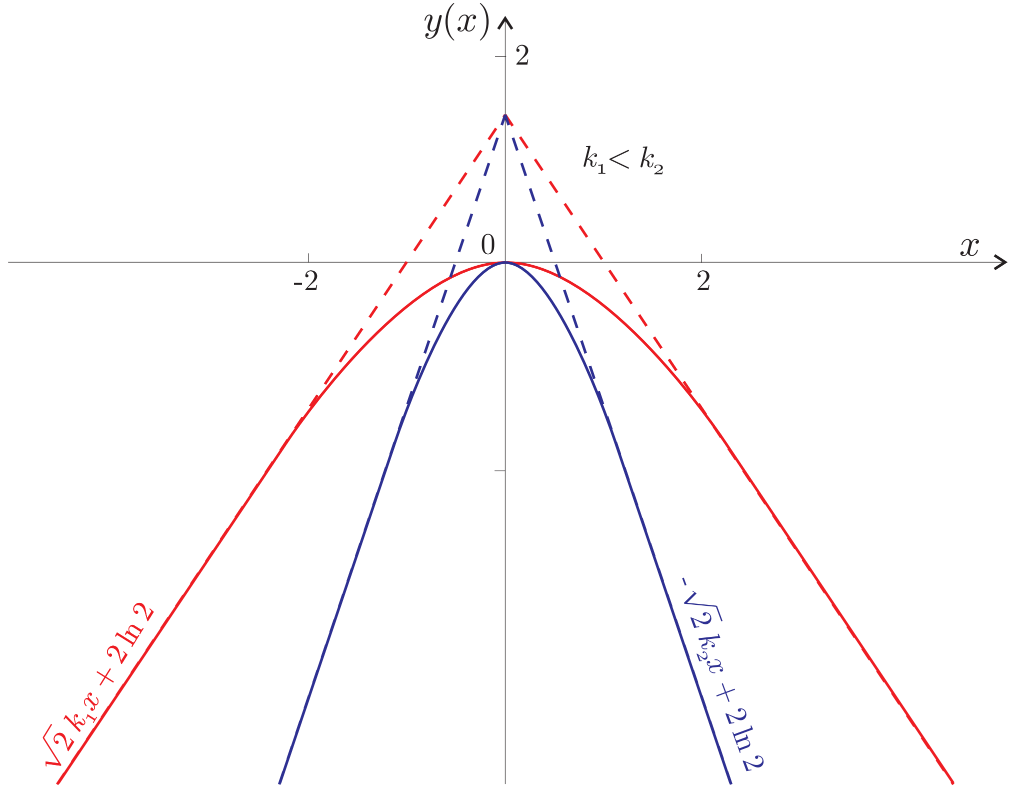
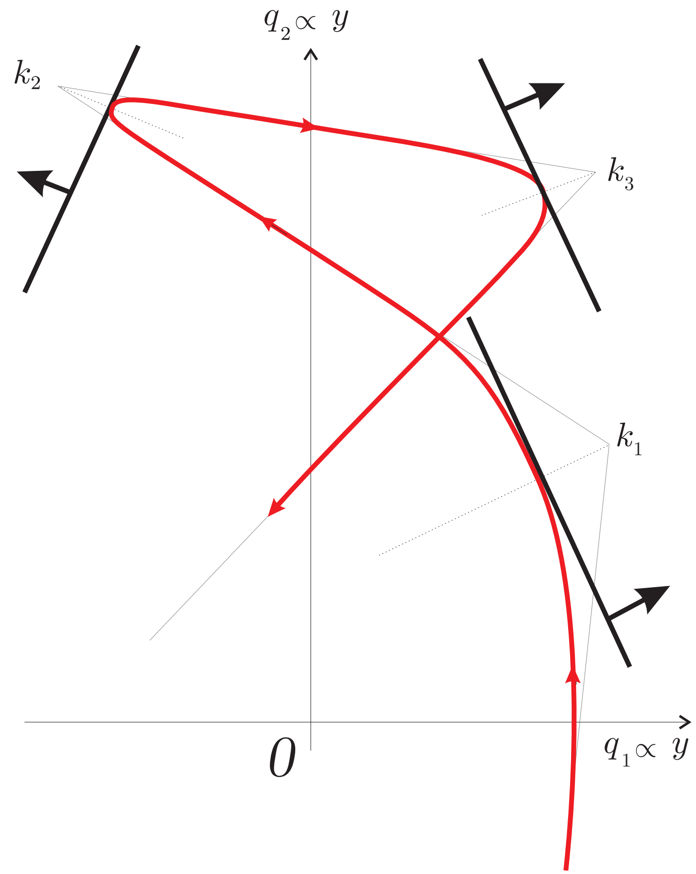
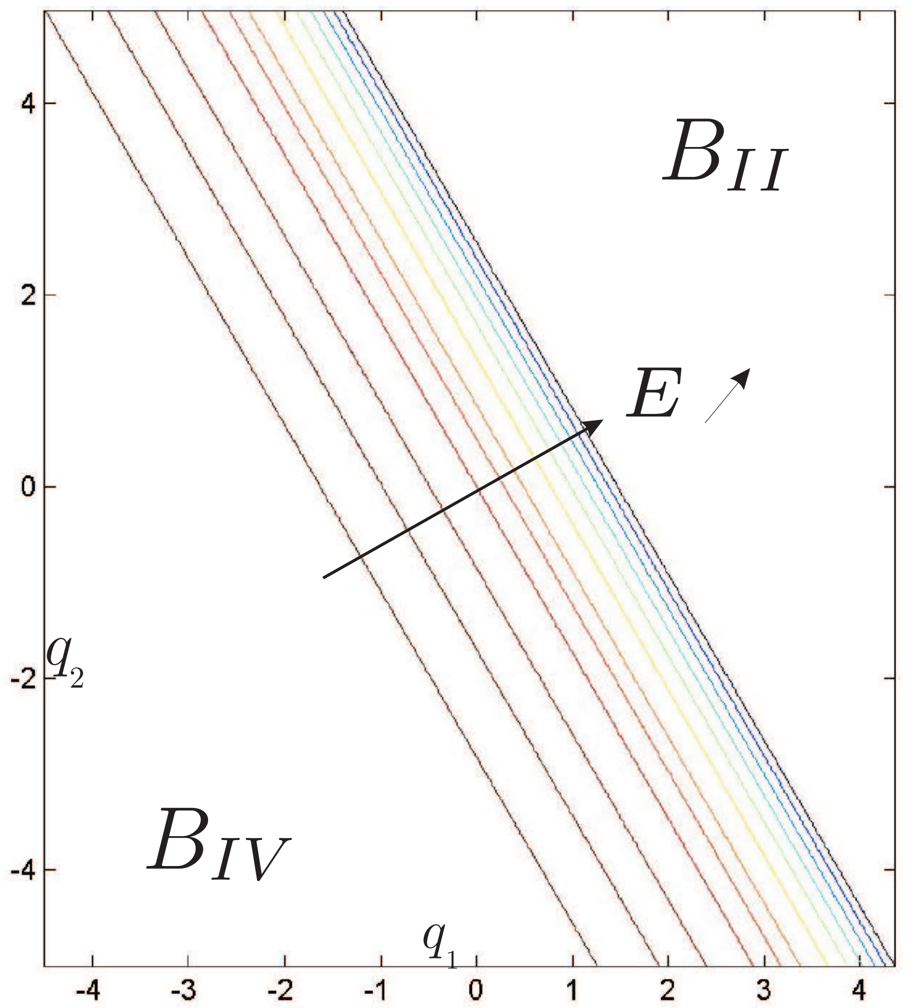


Several cushions ...



Cushions's form of B_{II} billiard

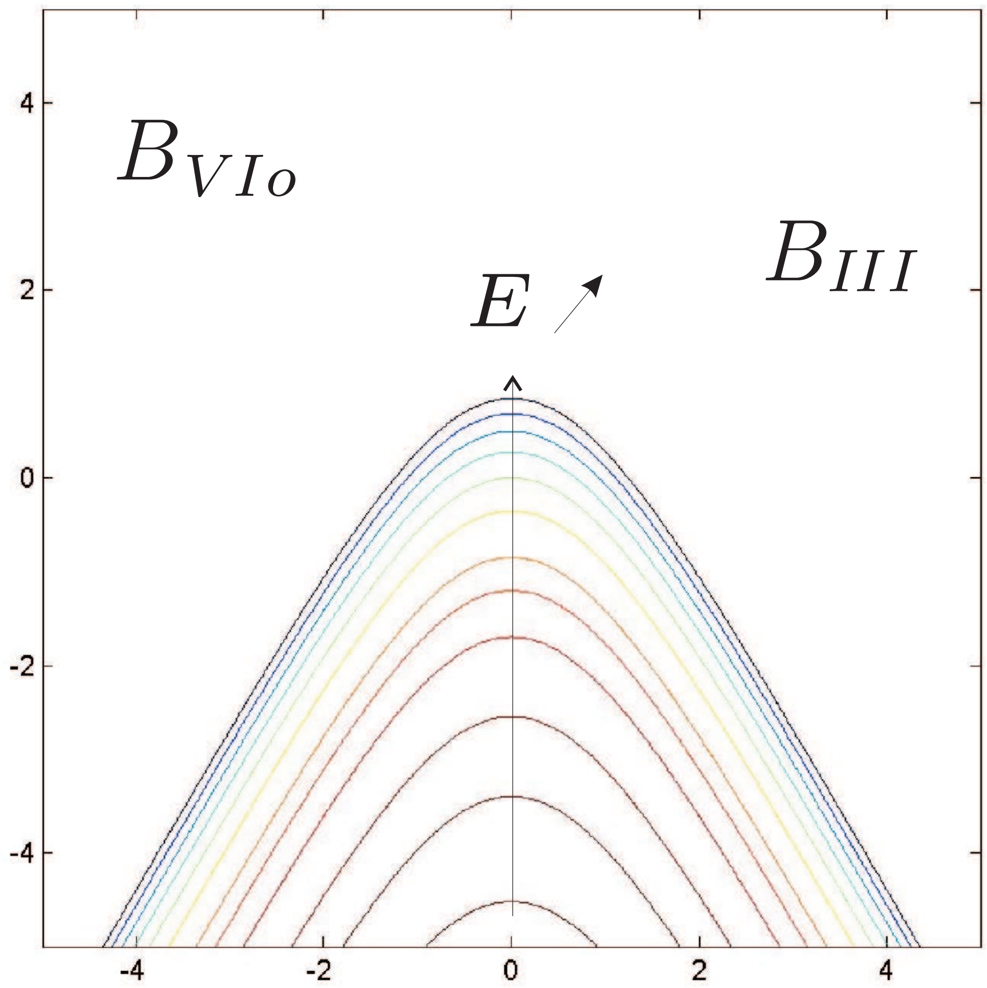
$$n_1 = 1, n_2 = n_3 = 0 : \xi(q_1, q_2) = -e^{\frac{\sqrt{6}}{3} q_2 + \sqrt{2} q_1}$$



Isocontours $\xi = E$

Cushions' form of B_{III} billiard

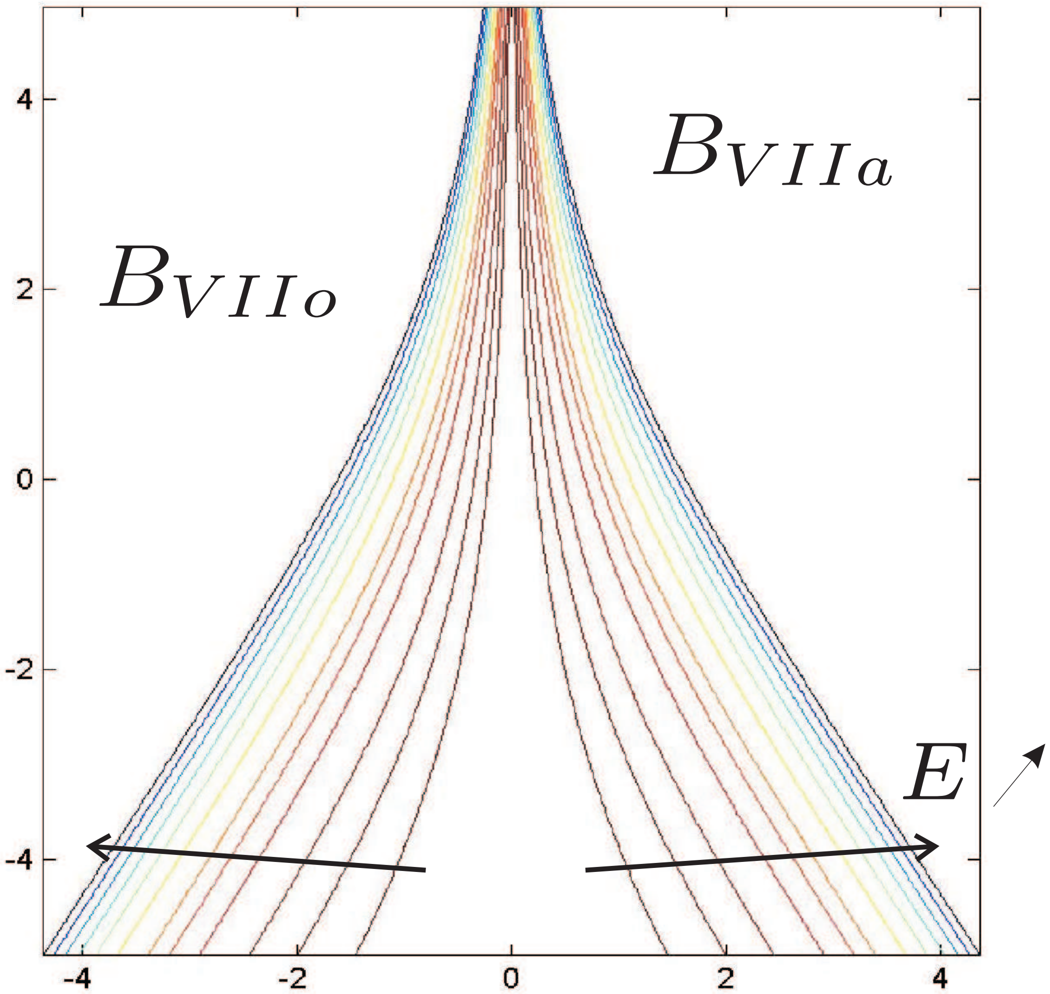
$$n_1 = 1, n_2 = -1, n_3 = 0 : \xi(q_1, q_2) = -e^{\frac{\sqrt{6}}{3}q_2 + \sqrt{2}q_1} - e^{\frac{\sqrt{6}}{3}q_2 - \sqrt{2}q_1} - 2e^{\frac{\sqrt{6}}{3}q_2}$$



Isocontours $\xi = E$

Cushions' form of B_{VII} billiard

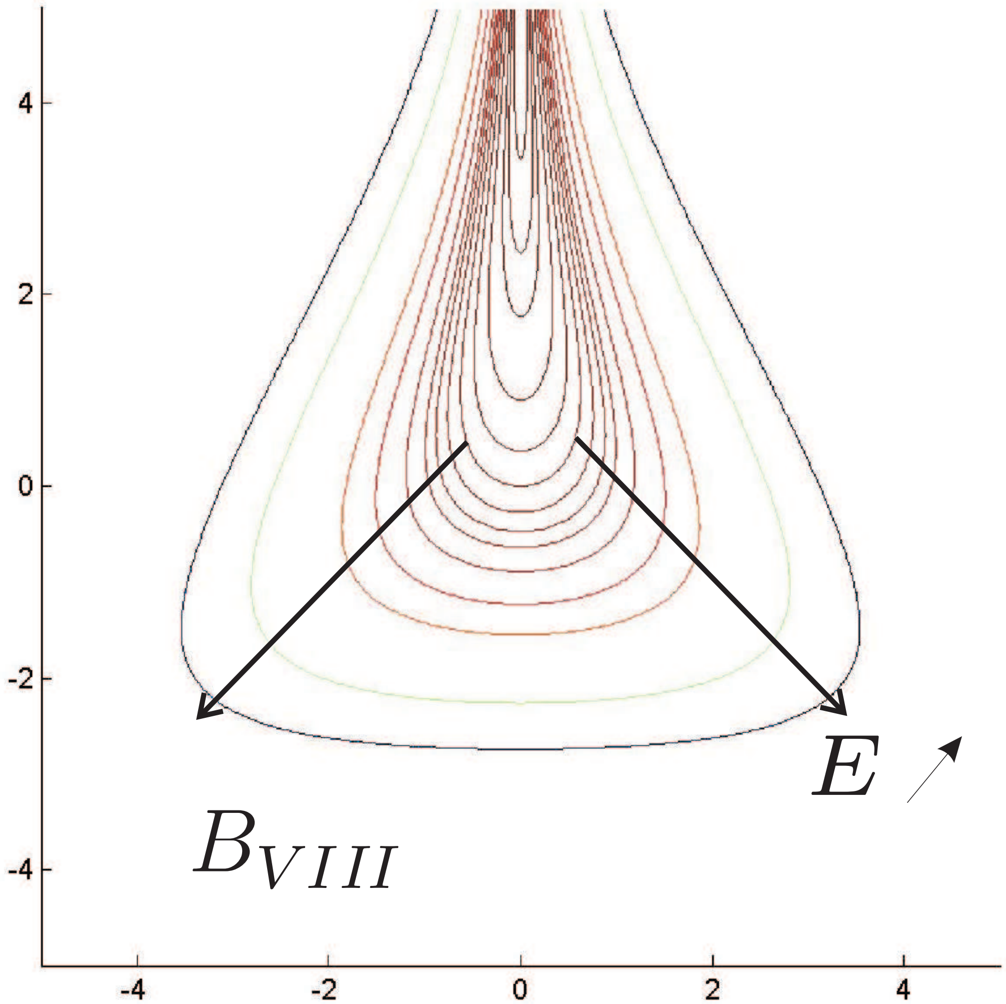
$$n_1 = 1, n_2 = 1, n_3 = 0 : \xi(q_1, q_2) = -e^{\frac{\sqrt{6}}{3}q_2 + \sqrt{2}q_1} - e^{\frac{\sqrt{6}}{3}q_2 - \sqrt{2}q_1} + 2e^{\frac{\sqrt{6}}{3}q_2}$$



Isovaleurs $\xi = E$

Cushions' form of B_{VIII} billiard

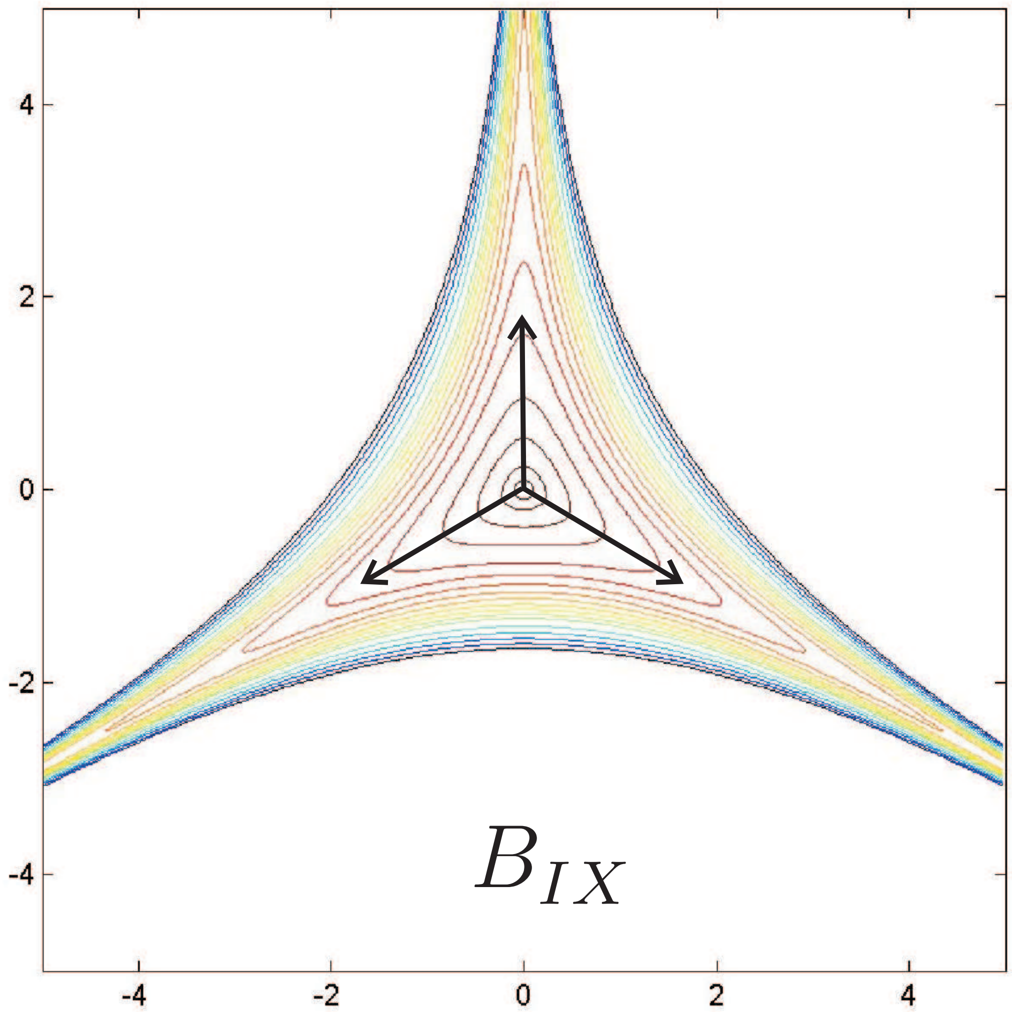
$$n_1 = 1, n_2 = 1, n_3 = -1 : \xi(q_1, q_2) = \cdots !$$



Isocontours $\xi = E$

Cushions' form of B_{IX} billiard

$$n_1 = 1, n_2 = 1, n_3 = 1 : \xi(q_1, q_2) = \cdots!$$



Isocontours $\xi = E$

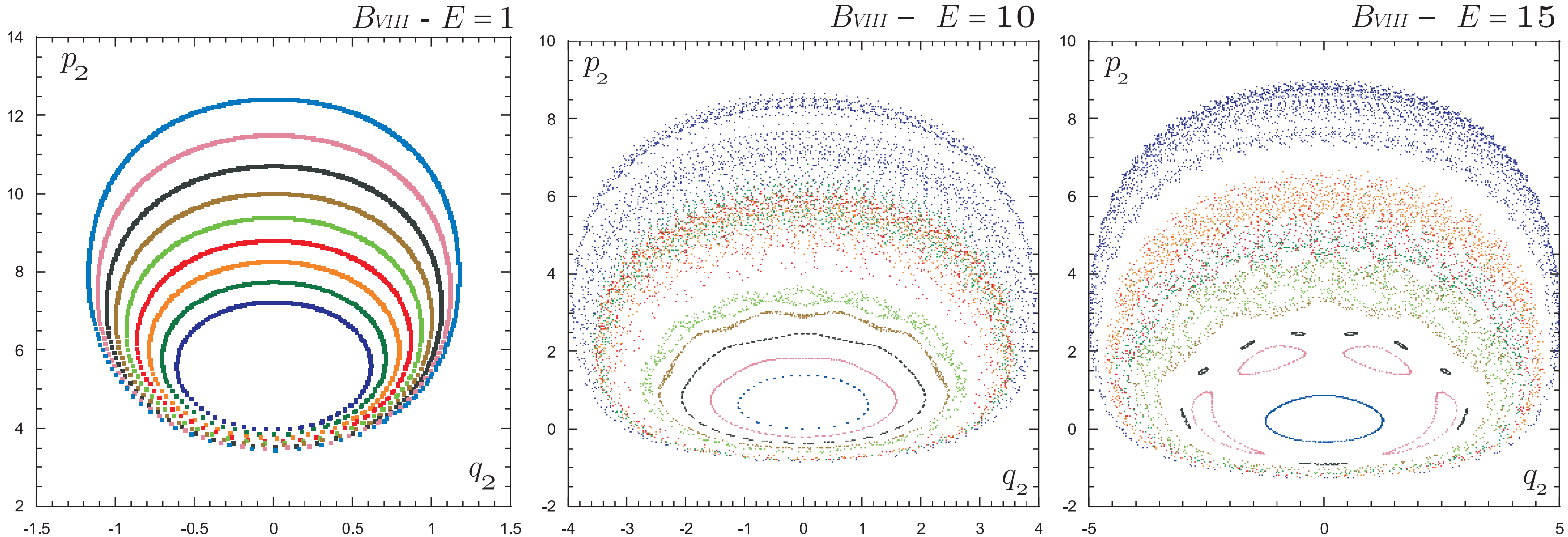
A few numerics ...

Dynamics in B_{IX} not easy !

... but understandable !

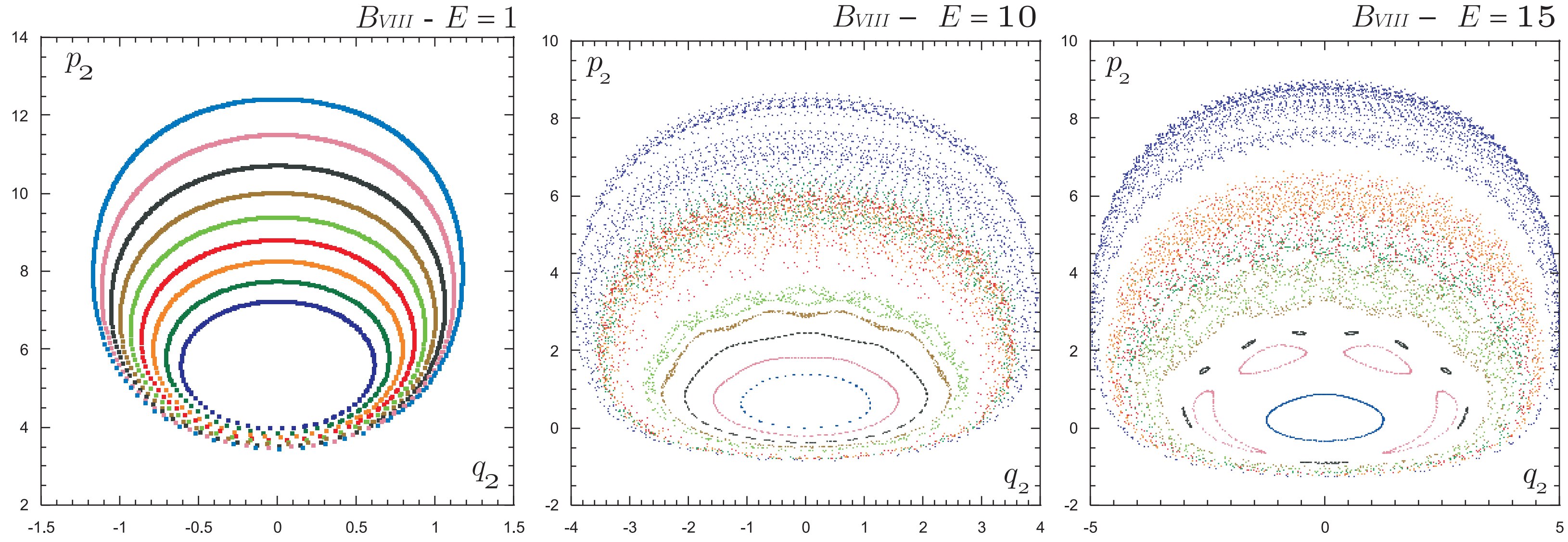
Poincaré's Sections

Poincaré's Sections

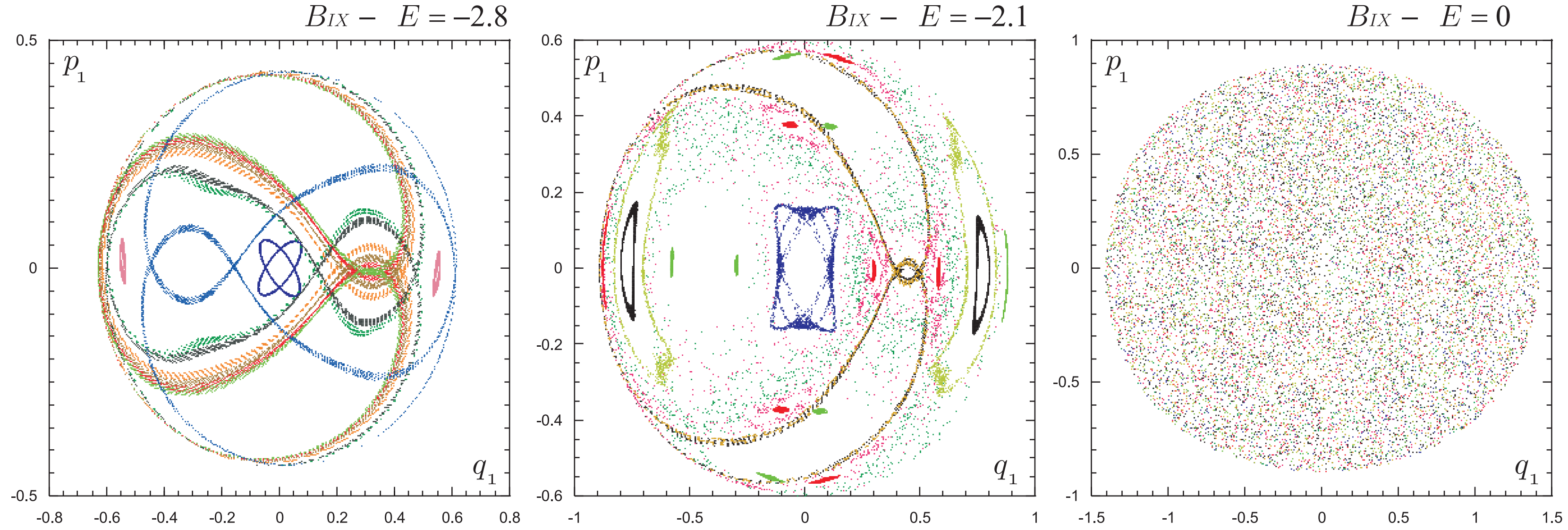


⟶ toward singularity ⟶

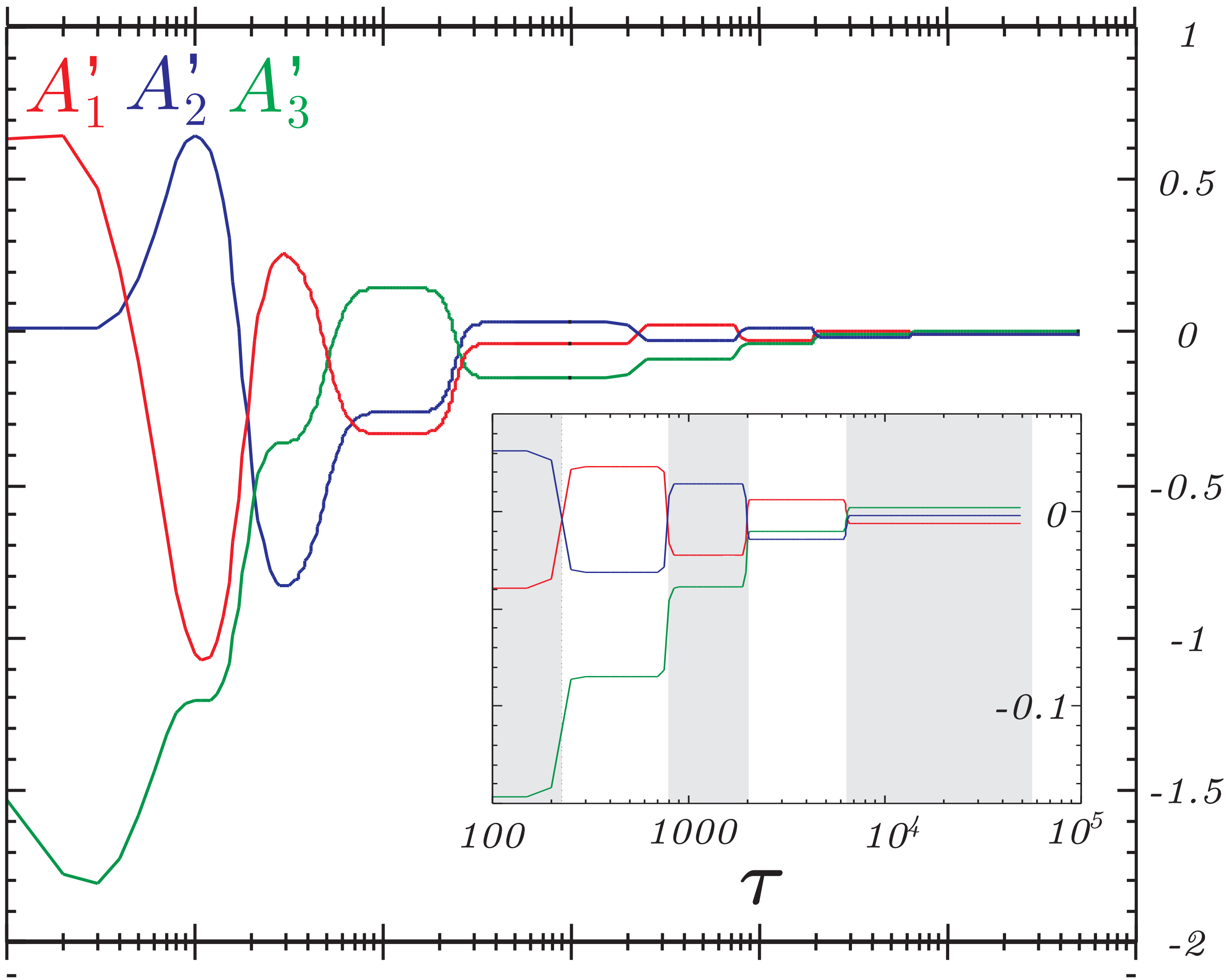
Poincaré's Sections



⟶ toward singularity ⟶



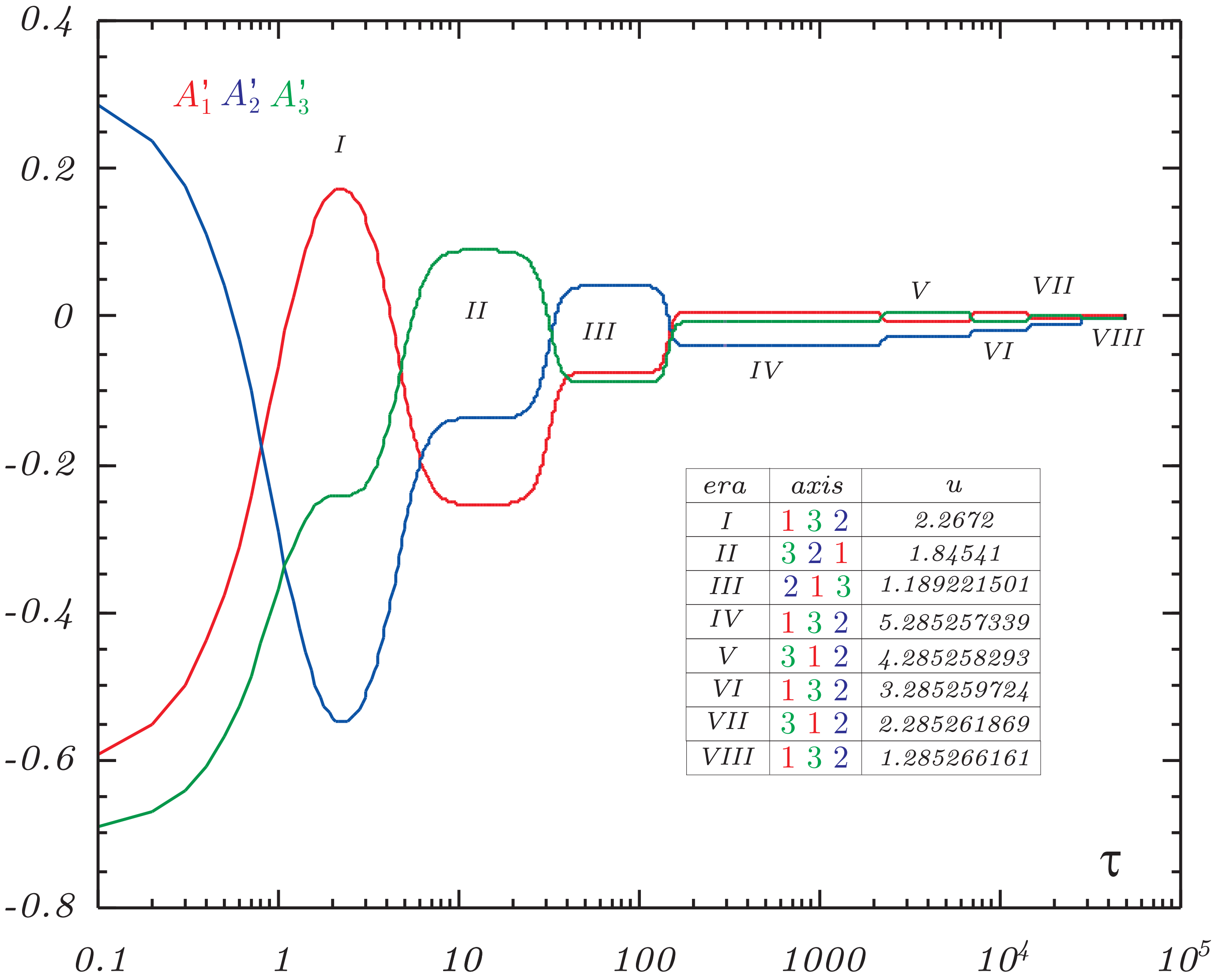
Vacuum B_{IX} BKL Dynamics



era	axis	u
I	● ● ●	2.31
II	● ● ●	1.272
III	● ● ●	3.714502573
IV	● ● ●	2.714620352
V	● ● ●	1.714827418
VI	● ● ●	1.399961514
VII	● ● ●	2.497215986

Random IC's

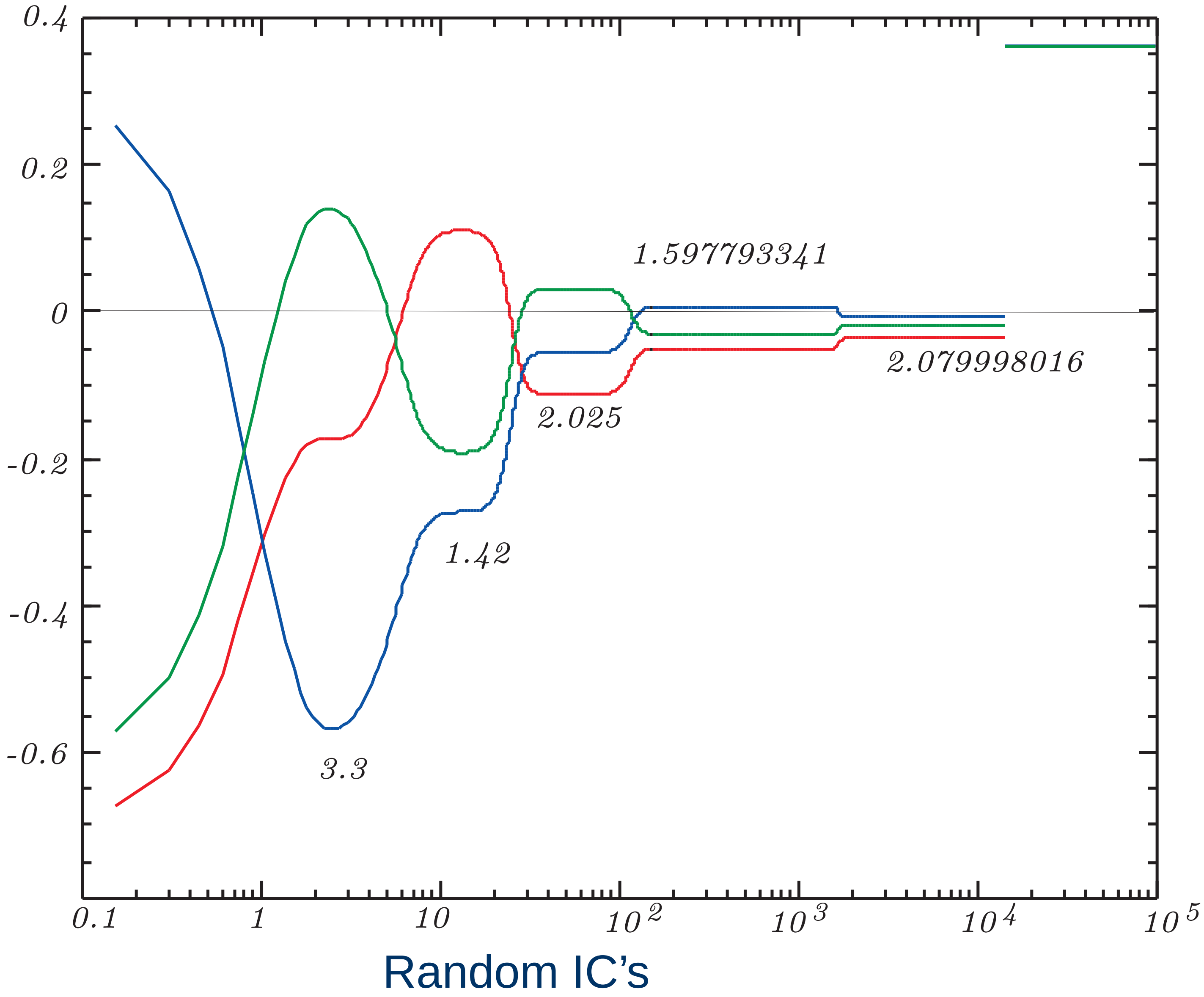
$\omega = 1/3$ B_{IX} BKL Dynamics



era	axis	u
I	1 3 2	2.2672
II	3 2 1	1.84541
III	2 1 3	1.189221501
IV	1 3 2	5.285257339
V	3 1 2	4.285258293
VI	1 3 2	3.285259724
VII	3 1 2	2.285261869
VIII	1 3 2	1.285266161

Random IC's

$\omega = 1$ B_{IX} BKL Dynamics



Attractors

e.g. [Cornish&Lewin,97]

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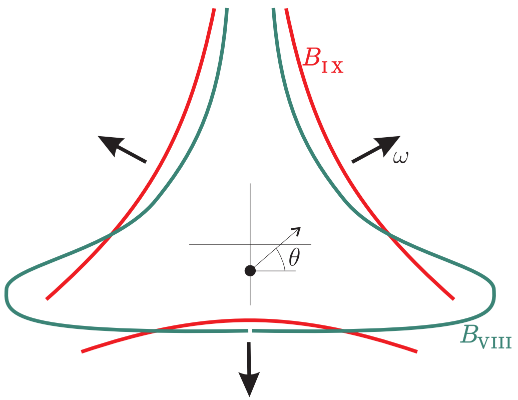
IC's : (θ_o, ω_o)



Stop when $u > u_e = 8$

● : $p_1 \rightleftharpoons x_1$

● : $p_1 \rightleftharpoons x_2$

● : $p_1 \rightleftharpoons x_3$

Attractors

e.g. [Cornish&Lewin,97]

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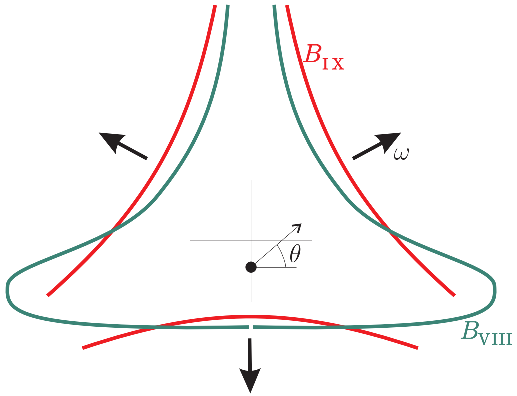

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- ● : $p_1 \rightleftharpoons x_2$
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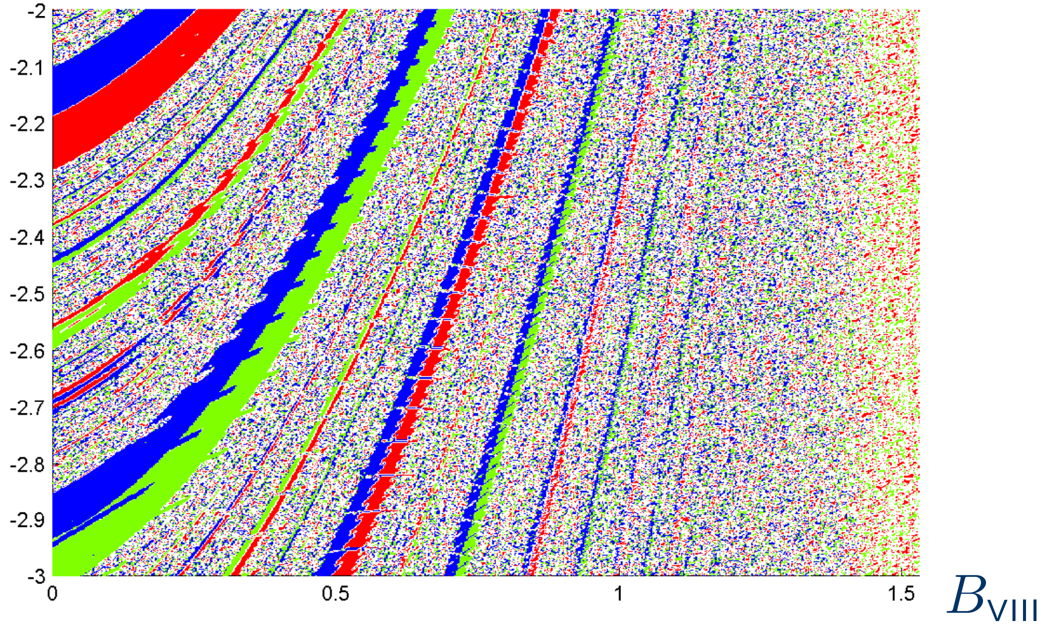

Attractors

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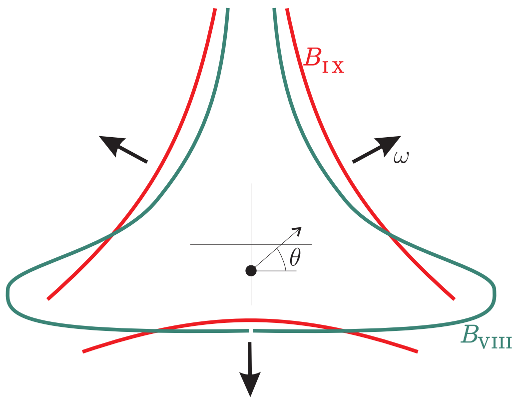
IC's : (θ_o, ω_o)



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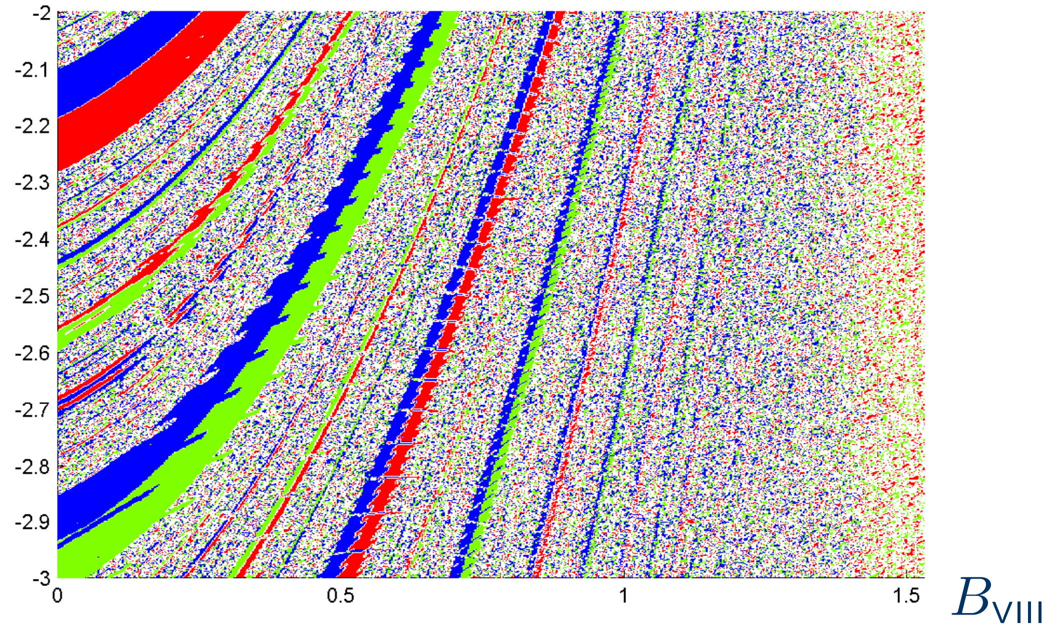
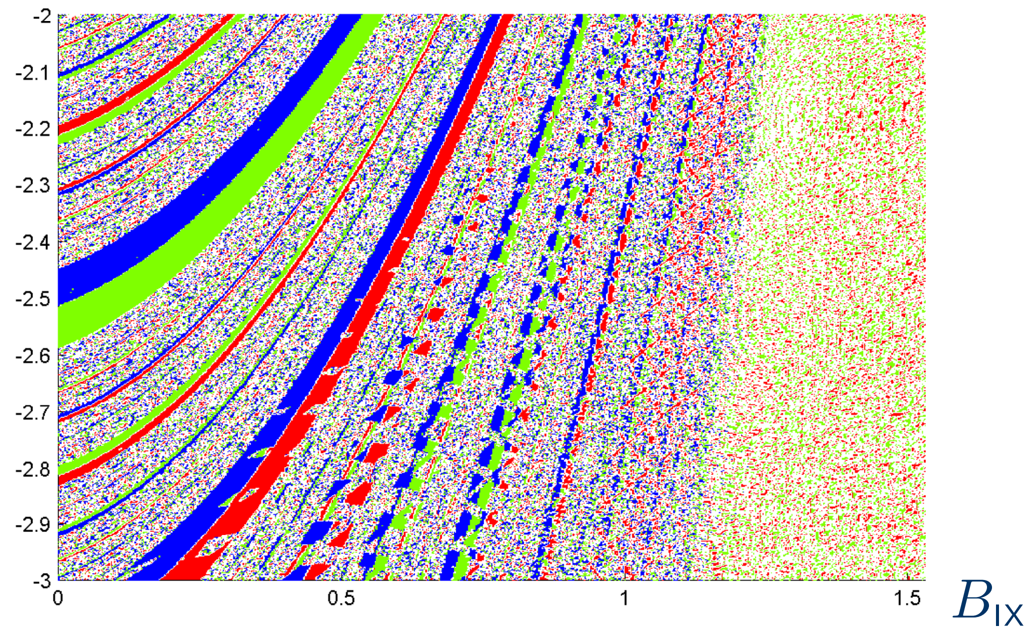
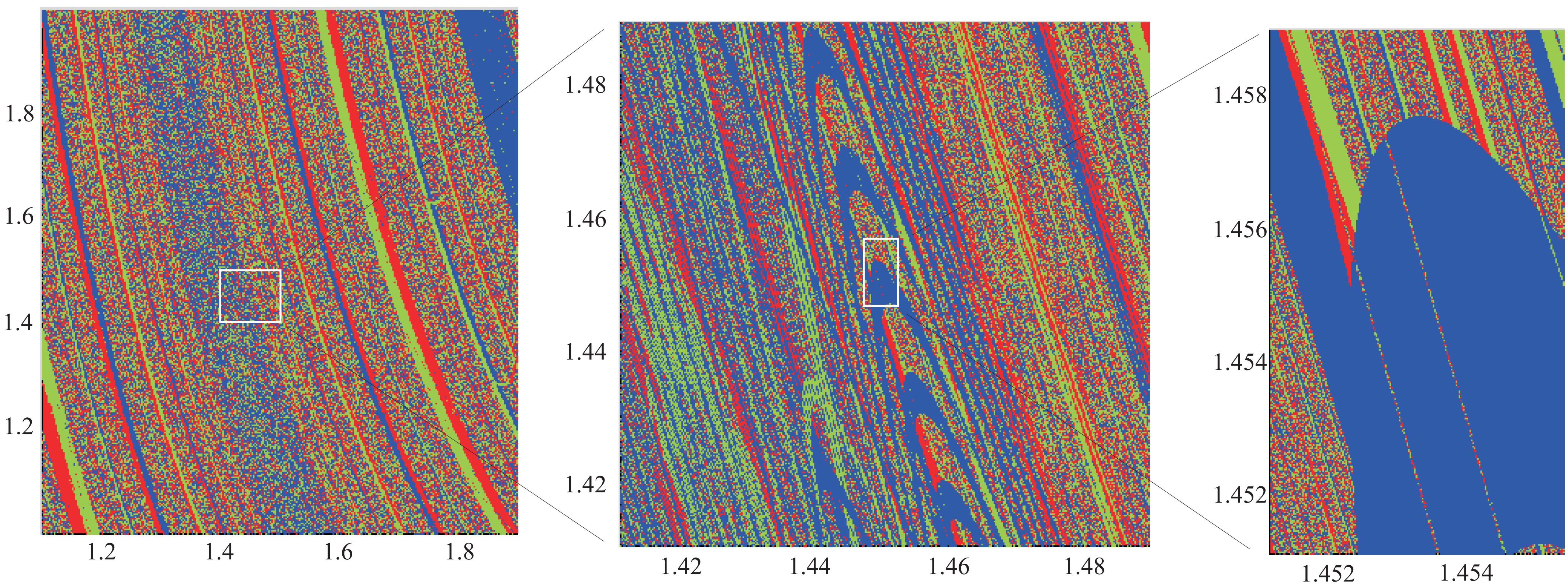
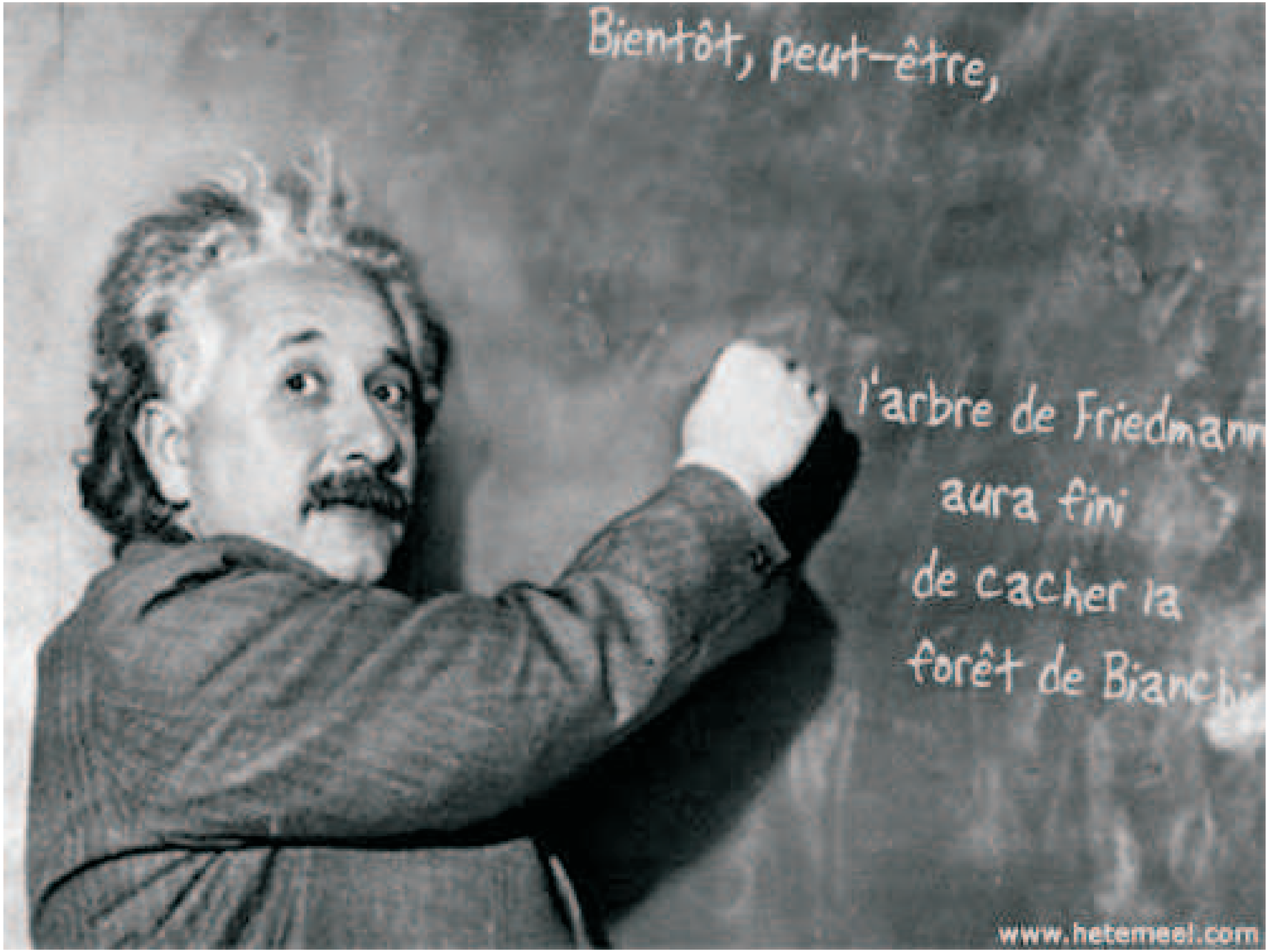




The B_{IX} Fractal



Conclusion



Bientôt, peut-être,
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