Thermodynamic Origin of Einstein's Equations and the Null Energy Condition

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M. P. and A. Svesko, "Einstein's Equations from the Stretched Future Light Cone," to appear.

M. P., S. Sarkar, and A. Svesko, "A Local First Law of Gravity," to appear.

M. P. and A. Svesko, "Thermodynamic Origin of the Null Energy Condition," arXiv: 1511.06460; PLB.

M. P. and J.-P. van der Schaar, "Derivation of the Null Energy Condition," arXiv: 1406.5163; PRD.

Science or Science Fiction?

Gödel universe

Gott time machine

Morris-Thorne wormhole

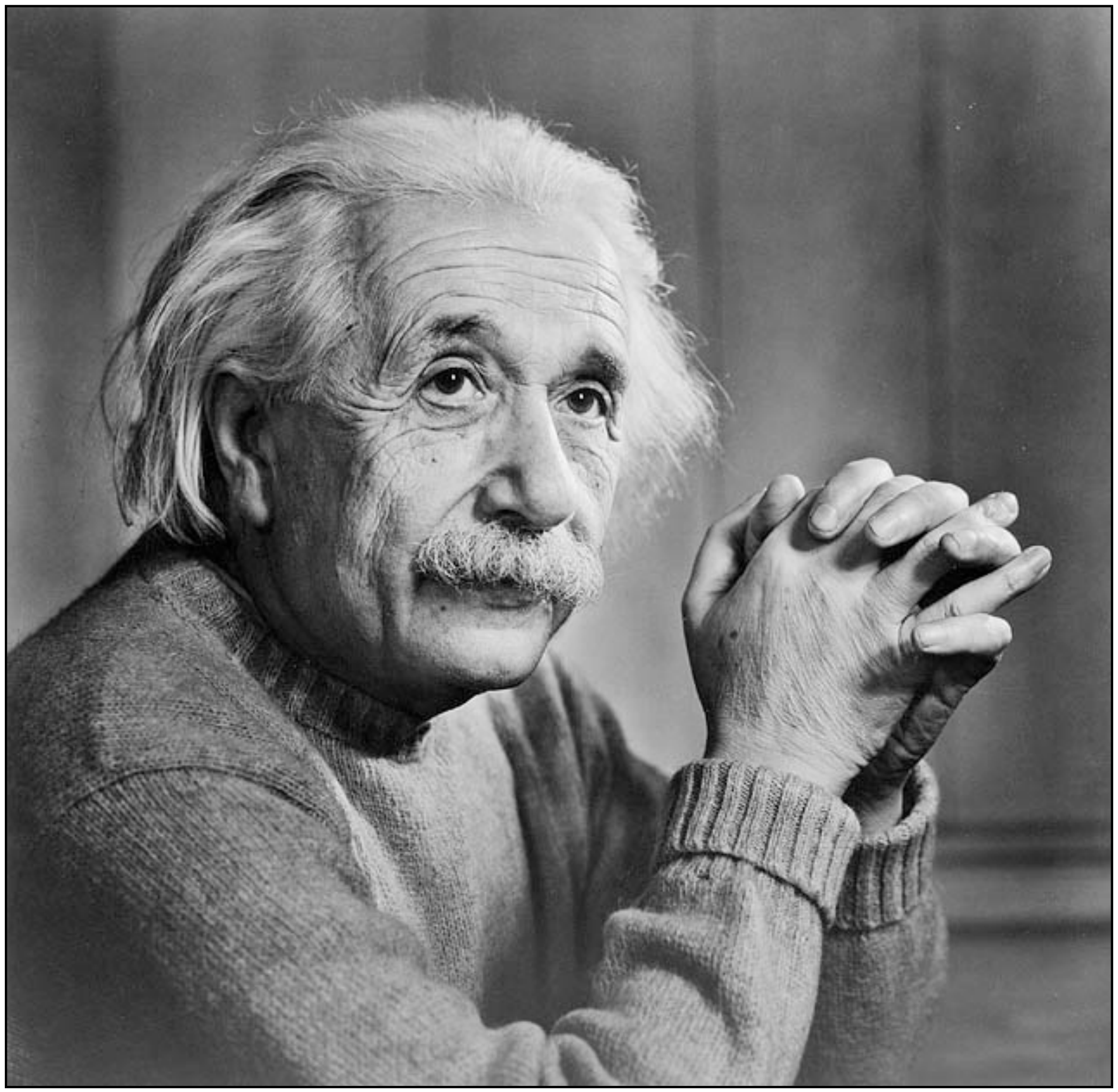
Bouncing cosmologies

Big Rip phantom cosmology

Negative mass black holes

Super-extremal black holes

Einstein on the Gödel Universe



"... cosmological solutions of the gravitation equations have been found by Mr Gödel. It will be interesting to weigh whether these are not to be excluded on physical grounds."

“Physical Grounds” or Ad Hoc Criteria?

No closed causal curves

Stable causality

Global hyperbolicity

Cosmic censorship

Geodesic completeness

Generalized second law of thermodynamics

Energy conditions

The Null Energy Condition

The most basic of the energy conditions is the null energy condition (NEC):

$$T_{\mu\nu} k^{\mu} k^{\nu} \geq 0$$

where k is any light-like vector.

The weak and strong energy conditions both imply the null energy condition.

All Metrics are Solutions of Einstein's Equations

If there are no conditions on the energy-momentum tensor, *any* metric becomes an exact solution to Einstein's equations:

Simply *define* the energy-momentum tensor to equal the Einstein tensor.

Importance of Energy Conditions

Singularity theorems.

Positive energy theorems.

Black hole no-hair theorem.

Laws of black hole mechanics.

Exclusion of traversable wormholes, construction of time machines, creation of a universe in the laboratory, and cosmological bounces.

$$\dot{H} = -(\rho + p) \leq 0$$

Energy Conditions from Matter?

The energy conditions are conventionally viewed as conditions on the energy-momentum tensor for *matter*.

Matter

We have a fantastic framework for understanding matter: quantum field theory.

Violation of the Null Energy Condition in QFT

Consider a well-behaved NEC-obeying theory of matter, theory A.

Let theory B have the same action as theory A but with an overall minus sign.

Theory B then violates the NEC.

But theory B is otherwise just as well-behaved as theory A!

Hence the NEC does not follow from QFT.

How the Energy Conditions are Used in Practice

In gravitational theorems, the energy conditions are always used in conjunction with the equations of motion.

$$T_{\mu\nu}k^\mu k^\nu \geq 0 \Rightarrow (R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})k^\mu k^\nu \geq 0 \Rightarrow R_{\mu\nu}k^\mu k^\nu \geq 0$$

Why not just start with a condition on the Ricci tensor?

Perhaps the real null "energy condition" is

$$R_{\mu\nu}k^\mu k^\nu \geq 0$$

We will derive this condition.

Viewpoint in this Talk

We will take

$$R_{\mu\nu}k^{\mu}k^{\nu} \geq 0$$

as a fundamental property of gravity. We will continue to refer to this as the "null energy condition". But note that this is now a condition on geometry.

Classical Gravity

Property 1:

$$G_{\mu\nu} = T_{\mu\nu}$$

Property 2:

$$R_{\mu\nu}k^{\mu}k^{\nu} \geq 0$$

Where Could the NEC Come From?

The geometric form of the NEC does not come from general relativity.

We have argued that the matter form does not come from QFT either.

Hence we should look for the origin of the NEC in theories that contain *both* matter and gravity.

That is, the NEC is a clue to the *unification* of matter and gravity.

Local Holographic Thermodynamics

Premise of Talk

Associate gravitational entropy *locally* to patches of certain null surfaces.

Assume that this entropy arises from the coarse-graining of some dual microscopic statistical-mechanical system.

Emergent Gravity

Jacobson (1995) showed that Einstein's equations followed from the Clausius relation dQ = TdS when applied to "local Rindler horizons".

Our prescription will be somewhat different. We will attribute thermodynamic properties to infinitesimal (patches of) expanding future light cones.

First Law

Generalized Einstein Equations

Consider an arbitrary relativistic diffeomorphism-invariant theory of gravity:

$$I = \frac{1}{16\pi} \int d^D x \sqrt{-g} L(g_{ab}, R_{abcd}) + I_{\mathrm{matter}}$$

Define

$$P^{abcd} = \frac{\partial L}{\partial R_{abcd}}$$

Then the equation of motion of classical gravity is

$$P_a^{\ cde} R_{bcde} - 2\nabla^c \nabla^d P_{acdb} - \frac{1}{2} L g_{ab} = 8\pi T_{ab}$$

For example, for Einstein gravity, L = R, and $P^{abcd} = \frac{1}{2}(g^{ac}g^{bd} - g^{ad}g^{bc})$

and this reduces to Einstein's equations.

Goal: Gravitational Equations from the First Law

Our goal is to derive the generalized Einstein equations

$$P_a{}^{cde} R_{bcde} - 2\nabla^c \nabla^d P_{acdb} - \frac{1}{2} L g_{ab} = 8\pi T_{ab}$$

from the first law of thermodynamics, dM = TdS, applied locally.

We need to define M, T, and S.

Local Killing Vectors

Consider an arbitrary point, P, in an arbitrary spacetime.

In the vicinity of P, spacetime is approximately flat.

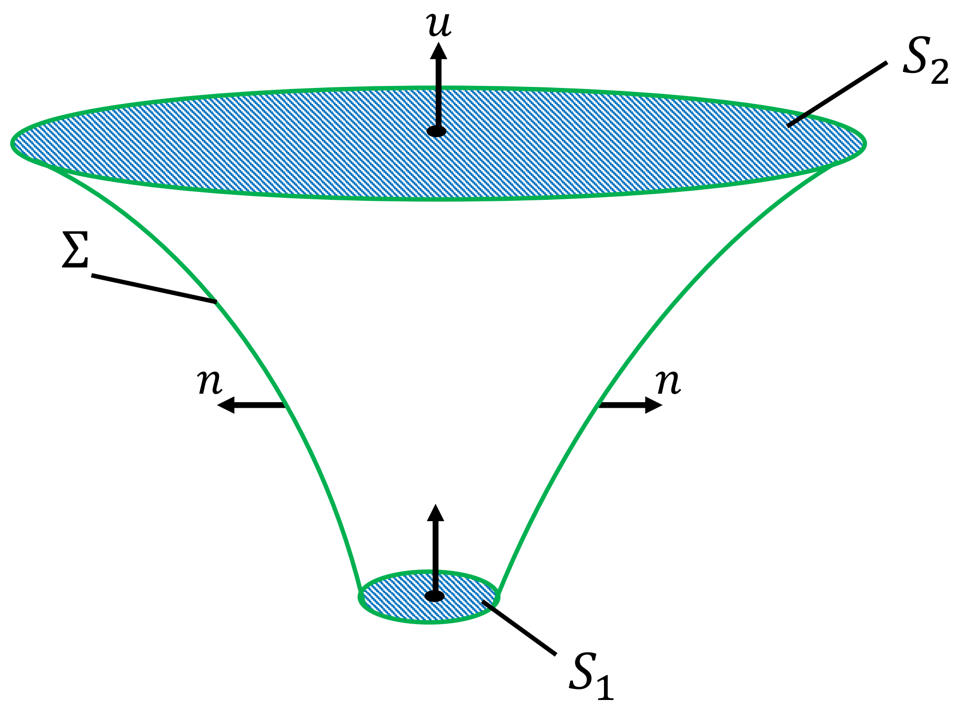
Hence there are local Lorentz symmetries, generated by approximate Killing vectors.

M

For a Killing vector ξ_a

$$\nabla_a (T^{ab}\xi_b) = 0$$

Integrating over a compact region of spacetime, we have



$$M_i \equiv \int_{S_i} d^{D-1}x\sqrt{h}\, u_a T^{ab}\xi_b$$

$$\Delta M = M_2 - M_1 = \int_{\Sigma} d\Sigma_a T^{ab}\xi_b = \Delta Q$$

T

In the vicinity of P, spacetime is approximately flat. The local Lorentz symmetries include boosts. Consider a surface generated by *radial* boosts.

$$\xi^a = r\partial_t^a + t\partial_r^a$$

The surface $\xi^2 = -\alpha^2$ is a timelike de Sitter-like hyperboloid. An observer moving along a boost has a constant proper acceleration, $1/\alpha$.

Define

$$T = \frac{1}{2\pi\alpha}$$

S

Wald found an expression for black hole entropy in general theories of gravity:

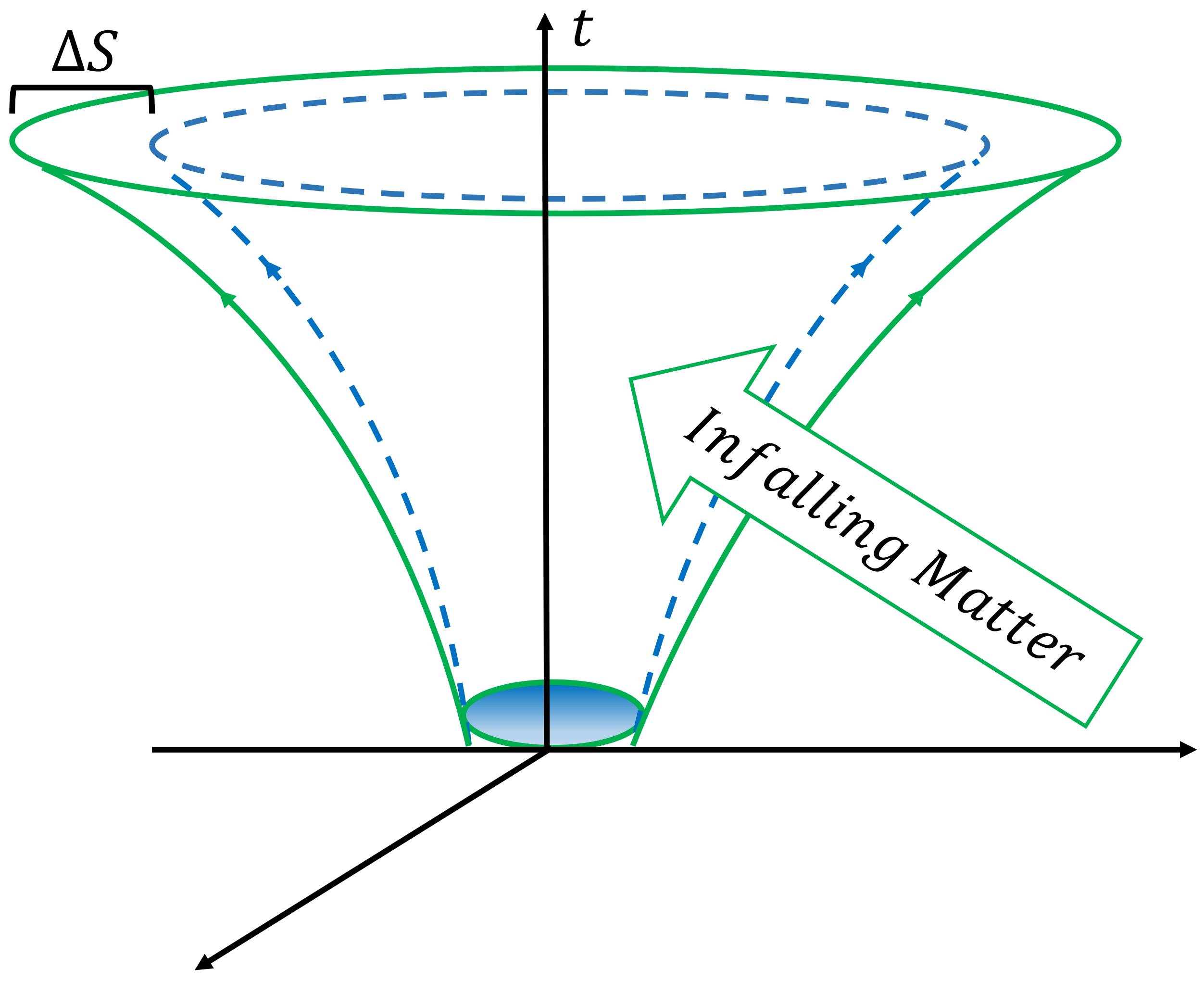
$$S = \frac{\alpha}{8} \int_S dS_{ab} J^{ab}$$

where $J^{ab} = -2P^{abcd}\nabla_c \xi_d + 4\xi_d \left(\nabla_c P^{abcd}\right)$

is the “Noether potential” for a timelike Killing vector, ξ_a .

The Wald entropy reduces to A/4 for Einstein gravity.

Change in Entropy due to Infalling Matter



$$\Delta S = \Delta S_{\rm tot} - \Delta S_0$$

Variation of Wald Entropy

Apply the Wald entropy formula to our infinitesimal compact surface:

$$S = \frac{\alpha}{8}\int_S dS_{ab} J^{ab} = -\frac{\alpha}{4}\int_S dS_{ab}(P^{abcd}\nabla_c \xi_d - 2\xi_d \nabla_c P^{abcd})$$

Then, using Stokes' theorem, the change in Wald entropy is

$$\Delta S_{\mathrm{tot}} = \frac{\alpha}{4}\int_\Sigma d\Sigma_a \nabla_b (P^{abcd}\nabla_c \xi_d - 2\xi_d \nabla_c P^{abcd})$$

That is

$$\Delta S_{\mathrm{tot}} = \frac{\alpha}{4}\int_\Sigma d\Sigma_a \left[-\nabla_b (P^{adbc} + P^{acbd})\nabla_c \xi_d + P^{abcd}\nabla_b \nabla_c \xi_d - 2\xi_d \nabla_b \nabla_c P^{abcd}\right]$$

The term in parenthesis drops out by symmetry in indices c and d.

Killing's Identity

If ξ_a were a true Killing vector, it would satisfy Killing's identity:

$$\nabla_b \nabla_c \xi_d = R^e_{bcd} \xi_e$$

The failure of Killing's identity to hold gives an extra term in the integral. But this extra term evaluates precisely to ΔS_0, i.e. the increase in the entropy due to the inherent expansion of the hyperboloid.

That is,

$$\Delta S_{\text{tot}} = +\frac{\alpha}{4} \int_\Sigma d\Sigma_a \left[P^{abcd} R^e_{bcd} \xi_e - 2\xi_d \nabla_b \nabla_c P^{abcd} \right] + \Delta S_0$$

First Law of Gravity

Therefore the Clausius relation says

$$\Delta Q = T\Delta S = T(\Delta S_{\mathrm{tot}} - \Delta S_0)$$

where $\Delta Q = \int_\Sigma d\Sigma_a T^{ab}\xi_b$ and $T = \dfrac{1}{2\pi\alpha}$

and

$$\Delta S = \frac{\alpha}{4}\int_\Sigma d\Sigma_a (P^{acde}R^b_{cde} - 2\nabla_c\nabla_d P^{acdb}\xi_b)$$

Putting this all together (and using the Bianchi identity) we find Einstein's equations:

$$P_a^{\ cde}R_{bcde} - 2\nabla^c\nabla^d P_{acdb} - \frac{1}{2}Lg_{ab} = 8\pi T_{ab}$$

Second Law

From the Second Law to the NEC

Quote a result about near-equilibrium thermodynamic systems obeying the second law.

Attribute thermodynamic properties to future light cones.

Show that null congruences corresponding to near-equilibrium thermodynamic systems exist at every point.

Obtain the null energy condition from the second law.

Onsager Theory

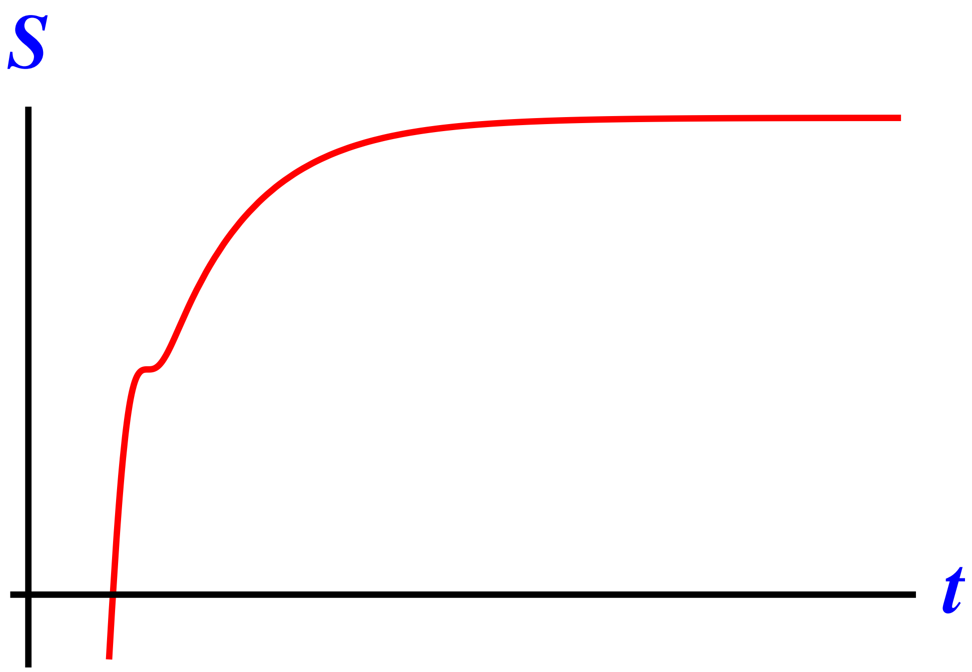
Near-equilibrium thermodynamic systems approaching internal equilibrium via the second law obey not only

$$\dot{S} \geq 0$$

but also

$$\ddot{S} \leq 0$$

G. Falkovich and A. Fouxon, New J. Phys. 6 (2004) 50.



Entropy and Area

As before, attribute Bekenstein-Hawking entropy to infinitesimal patches of expanding future light cones (perhaps more generally, to all non-contracting infinitesimal patches of future-directed null congruences).

Then

$$\begin{aligned} S &= \frac{A}{4} \\ \dot{S} &= \frac{A}{4}\theta \\ \ddot{S} &= \frac{A}{4}\left(\dot{\theta} + \theta^2\right) \end{aligned}$$

Near-Equilibrium Null Congruences Are Everywhere

Consider an arbitrary point, p, in spacetime.

Let $R_{\mu\nu}(p)k^\mu k^\nu = C$ where C is some constant.

We can find explicit solutions of the Raychaudhuri equation which have

$$\theta \approx -C\lambda \qquad \dot{\theta} \approx -C$$

For small enough λ, we have

$$\theta^2 \ll |\dot{\theta}|$$

so this congruence is near equilibrium.

Null Energy Condition from Thermodynamics

The infinitesimal patch obeys the Raychaudhuri equation:

$$\dot{\theta} = -\frac{1}{2}\theta^2 - R_{\mu\nu}k^\mu k^\nu$$

Rewriting that equation we find, for near-equilibrium congruences, that

$$\begin{aligned} R_{\mu\nu}k^\mu k^\nu &= -\dot{\theta} - \frac{1}{2}\theta^2 \\ &= -\left(\dot{\theta} + \theta^2\right) + \frac{1}{2}\theta^2 \\ &= -\frac{1}{S}\ddot{S} + \frac{1}{2S^2}\dot{S}^2 \\ &\geq 0 \end{aligned}$$

This is precisely the geometric form of the NEC.

M. P. and A. Svesko, "Thermodynamic Origin of the Null Energy Condition," arXiv: 1511.06460.

The First and Second Laws of Gravity

First law:

$$G_{\mu\nu} = T_{\mu\nu}$$

Second law:

$$R_{\mu\nu} k^\mu k^\nu \geq 0$$

Stringy Origin of the Null Energy Condition

Worldsheet String Theory

Polyakov action

$$S[X^\mu, h_{ab}] = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} \left(g_{\mu\nu}(X) h^{ab} \partial_a X^\mu \partial_b X^\nu + \alpha' \Phi(X) R^{(2)} \right)$$

$X^\mu(\sigma, \tau)$	spacetime coordinates of the string
$h_{ab}(\sigma, \tau)$	worldsheet metric
$R^{(2)}$	two-dimensional Ricci scalar of worldsheet
$g_{\mu\nu}(X)$	spacetime metric
$\Phi(X)$	dilaton

Worldsheet string theory = two-dimensional conformal field theory of D fields + two-dimensional gravity

Strings in Minkowski Space

For a string propagating in flat space, the worldsheet CFT is free.

$$S[X^\mu, h_{ab}] = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} \left(\eta_{\mu\nu} h^{ab} \partial_a X^\mu \partial_b X^\nu + \lambda \alpha' R^{(2)} \right)$$

The equation of motion for the worldsheet metric is the 2D Einstein's equation:

$$-\frac{\lambda}{2\pi} \left(R^{(2)}_{ab} - \frac{1}{2} R^{(2)} h_{ab} \right) = T_{ab}$$

As the Einstein tensor vanishes in 2D, the *worldsheet* stress tensor must be 0.

Spacetime Null Vectors from the String Worldsheet

Consider the Virasoro constraint:

$$T_{ab} = 0 \Rightarrow \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}h_{ab}(\partial X)^2 = 0$$

In light cone coordinates, this becomes

$$\eta_{\mu\nu}\partial_+ X^\mu \partial_+ X^\nu = 0$$

In other words, $k^\mu \equiv \partial_+ X^\mu$ is a *spacetime* null vector field:

$$\eta_{\mu\nu}k^\mu k^\nu = 0$$

Strings in Curved Spacetime

For a string propagating in curved space, the one-loop effective action is

$$S[X_0^\mu, h_{ab}] = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{ab}\partial_a X_0^\mu \partial_b X_0^\nu(\eta_{\mu\nu}+C_\epsilon\alpha' R_{\mu\nu}(X_0)) - \frac{1}{4\pi}\int d^2\sigma\sqrt{-h}C_\epsilon\Phi(X_0)R^{(2)}$$

Virasoro constraint in light-cone coordinates:

$$0 = \partial_+ X_0^\mu \partial_+ X_0^\nu \left(\eta_{\mu\nu} + C_\epsilon\alpha' R_{\mu\nu} + 2C_\epsilon\alpha'\nabla_\mu\nabla_\nu\Phi\right)$$

At zeroeth order in α' we recover our original equation:

$$\eta_{\mu\nu}k^\mu k^\nu = 0$$

But at first order in α' we miraculously discover that

$$k^\mu k^\nu(R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi) = 0$$

Einstein-Frame Metric

The spacetime metric appearing in the string action is the string-frame metric.

It is related to the usual ("Einstein-frame") metric by scaling:

$$g_{\mu\nu} = e^{+\frac{4\Phi}{D-2}} g^E_{\mu\nu}$$

Transforming the Virasoro condition to the Einstein-frame metric, we find

$$0 = k^\mu k^\nu (R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi) = k^\mu k^\nu \left(R^E_{\mu\nu} - \frac{4}{D-2} \nabla^E_\mu \Phi \nabla^E_\nu \Phi \right)$$

That is

$$R^E_{\mu\nu} k^\mu k^\nu = +\frac{4}{D-2} (k^\mu \nabla^E_\mu \Phi)^2 \geq 0$$

This is precisely the desired geometric form of the null energy condition!

Summary of Worldsheet Derivation

The null energy condition is not about matter or energy at all.

The geometric form of the “null energy condition” comes from string theory.

The origin of the null energy condition lies in gravity -- 2D gravity!

The ``physical ground” for the null energy condition is two-dimensional diffeomorphism invariance.

M. P. and J.-P. van der Schaar, “Derivation of the Null Energy Condition,” arXiv: 1406.5163; Phys. Rev. D.

The Dual Origins of Gravity

We have seen that the null energy condition, when viewed as a property of geometry, can be derived in two different ways, each based on a principle.

Similarly, Einstein's equations can also be derived in two different ways.

But why are there *two* derivations?

Local Holography?

Closed String Worldsheet	Spacetime	Holographic Thermodynamics
Weyl invariance	Einstein's equations	First law
Diffeomorphism invariance	Null energy condition	Second law