Spatial averaging in relativistic general-fluid inhomogeneous cosmologies

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and in collaboration with **Xavier Roy** and **Asta Heinesen**

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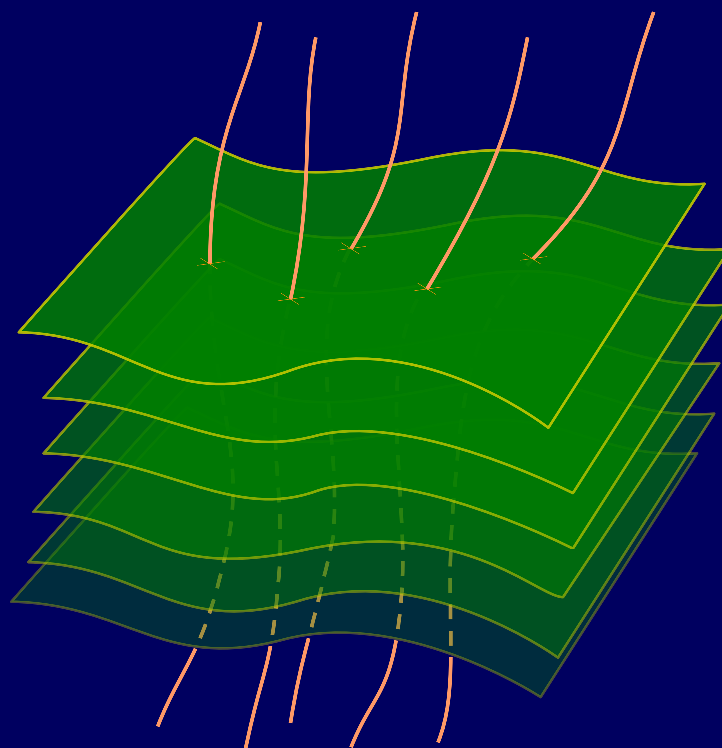

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Introduction: averaging in inhomogeneous cosmology

I – General spatial foliations

- Towards tilted hypersurfaces
- Buiding a foliation — Lapse and Shift
- The fluid flow
- Foliation and fluid — Tilt
- Domain propagation

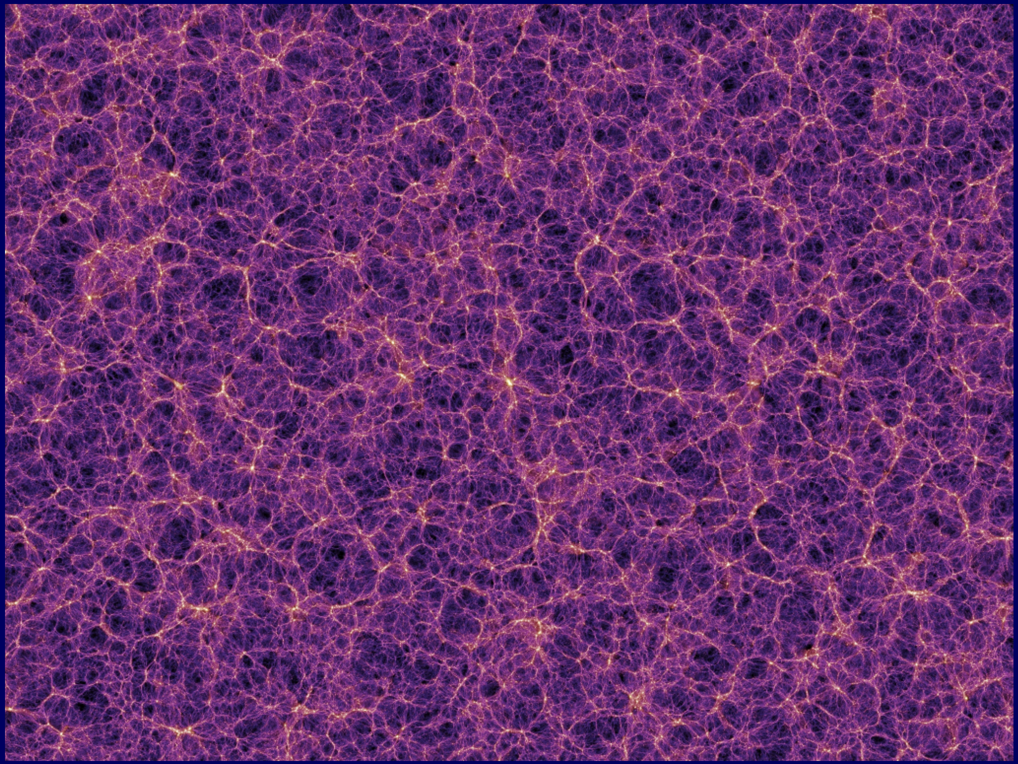
II – Averaging: geometric approach

- Averaging operator and commutation rule
- Averaged Einstein equations and backreaction terms
- Manifestly covariant form — Window function
- Consequences of a geometric approach

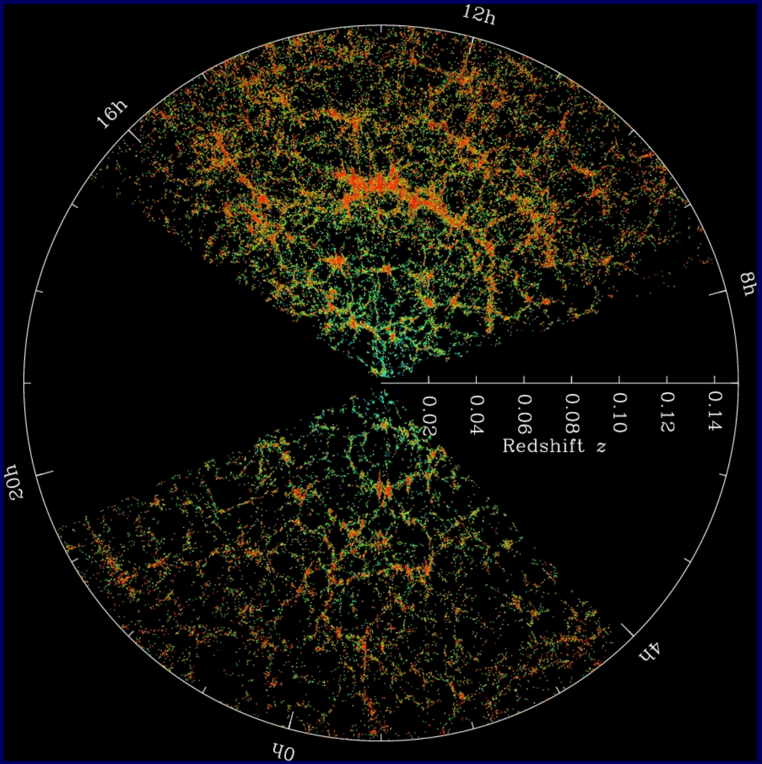
III – Averaging: intrinsic approach and proper-time foliations

- Intrinsic averaging operator
- Intrinsic-average commutation rule
- Intrinsic averaged Einstein equations
- Integrability condition and energy conservation equation
- Effective Friedmannian form
- Time parameter interpretation and application to proper-time foliations
- Manifestly covariant formulation

Structures from small to large scales : inhomogeneous distribution of matter



projection from the Millenium simulation at z=0
Credit: G. Lemson and the Virgo consortium

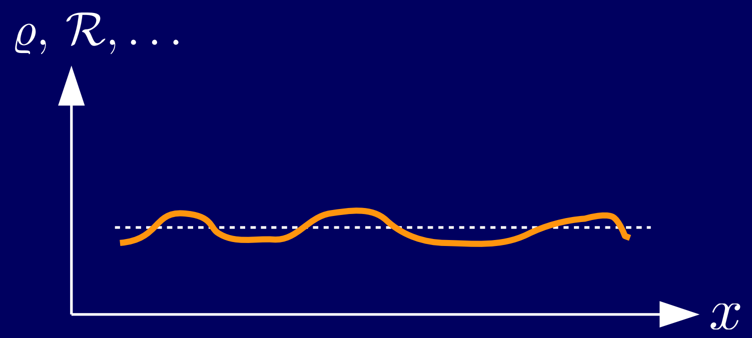


Credit: M. Blanton and the Sloan Digital Sky Survey

Statistical homogeneity ≠ strict local homogeneity:
nonlinear local deviations from homogeneous-isotropic models.

Impact (backreaction) on the large-scale dynamics ?
→ background-free coarse-graining or averaging procedures.

Nonlinear time evolution and averaging do not commute.



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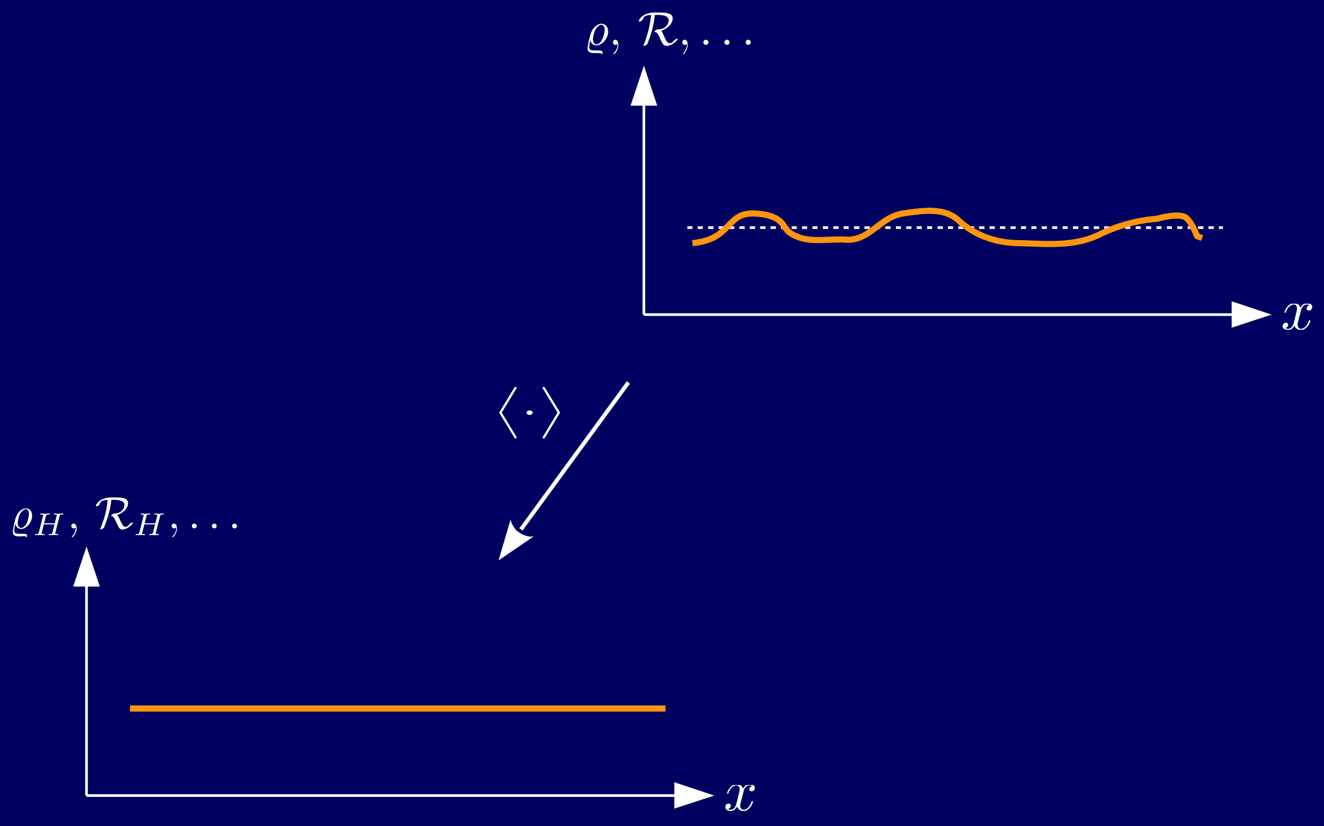

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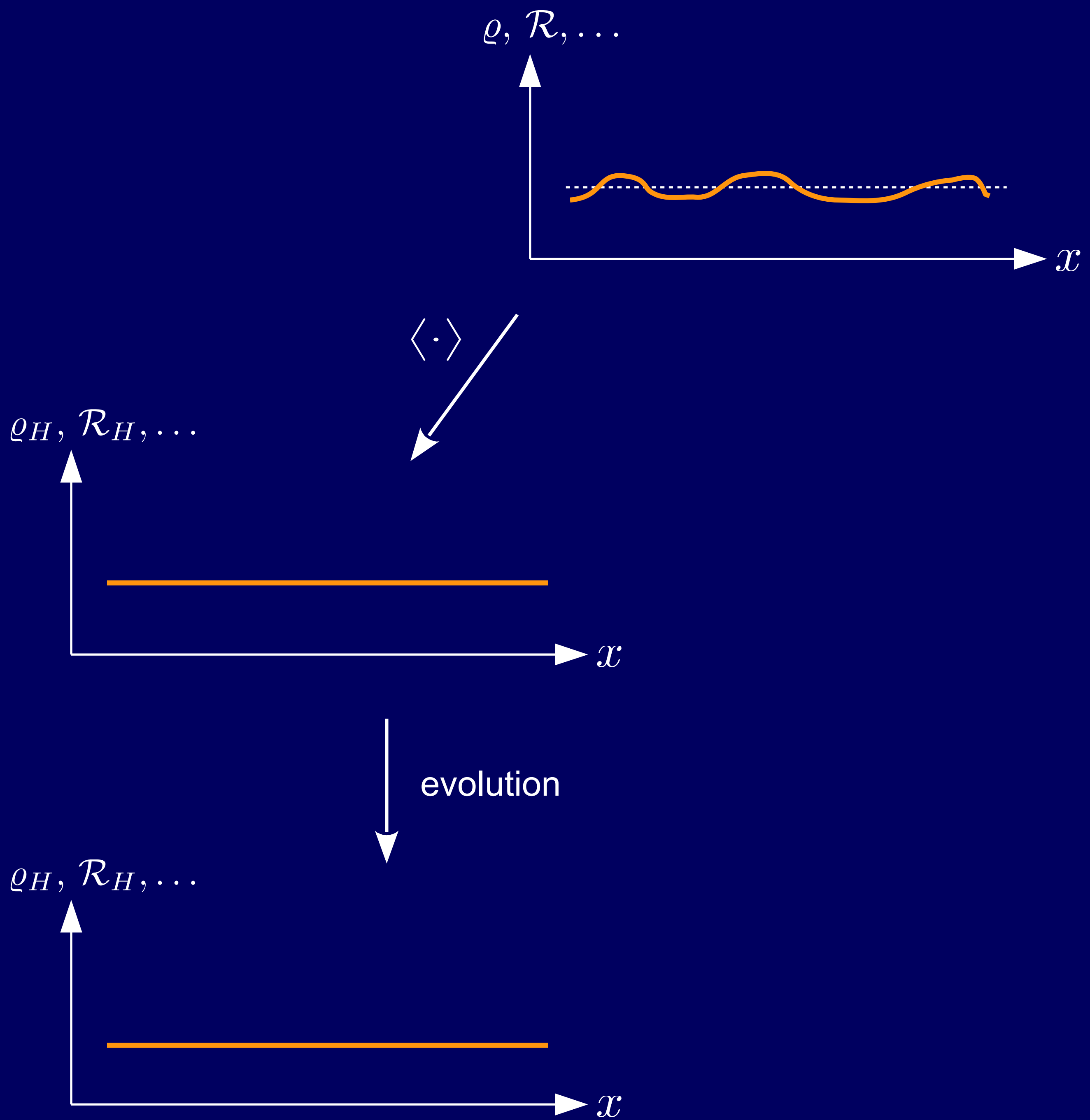

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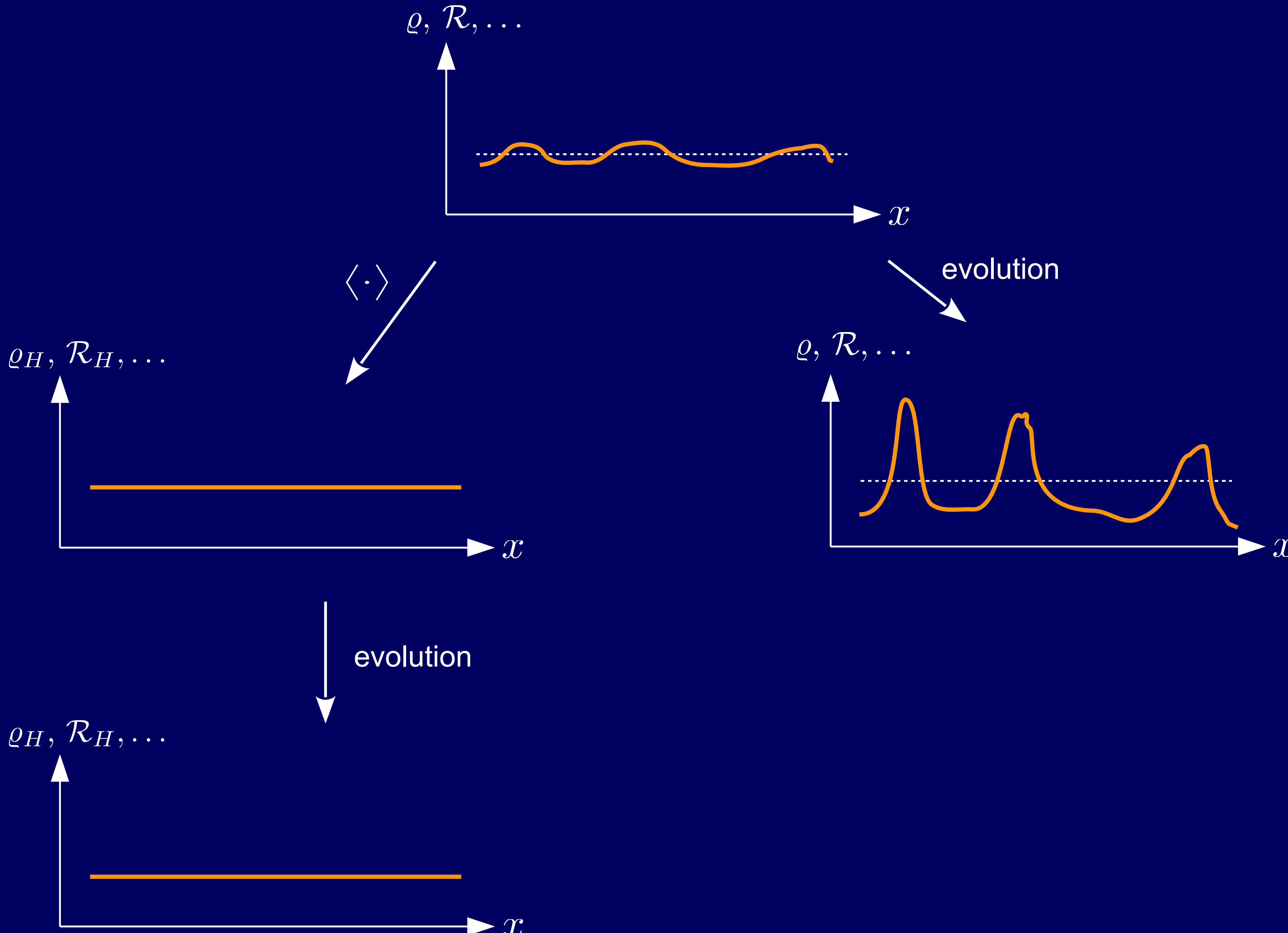

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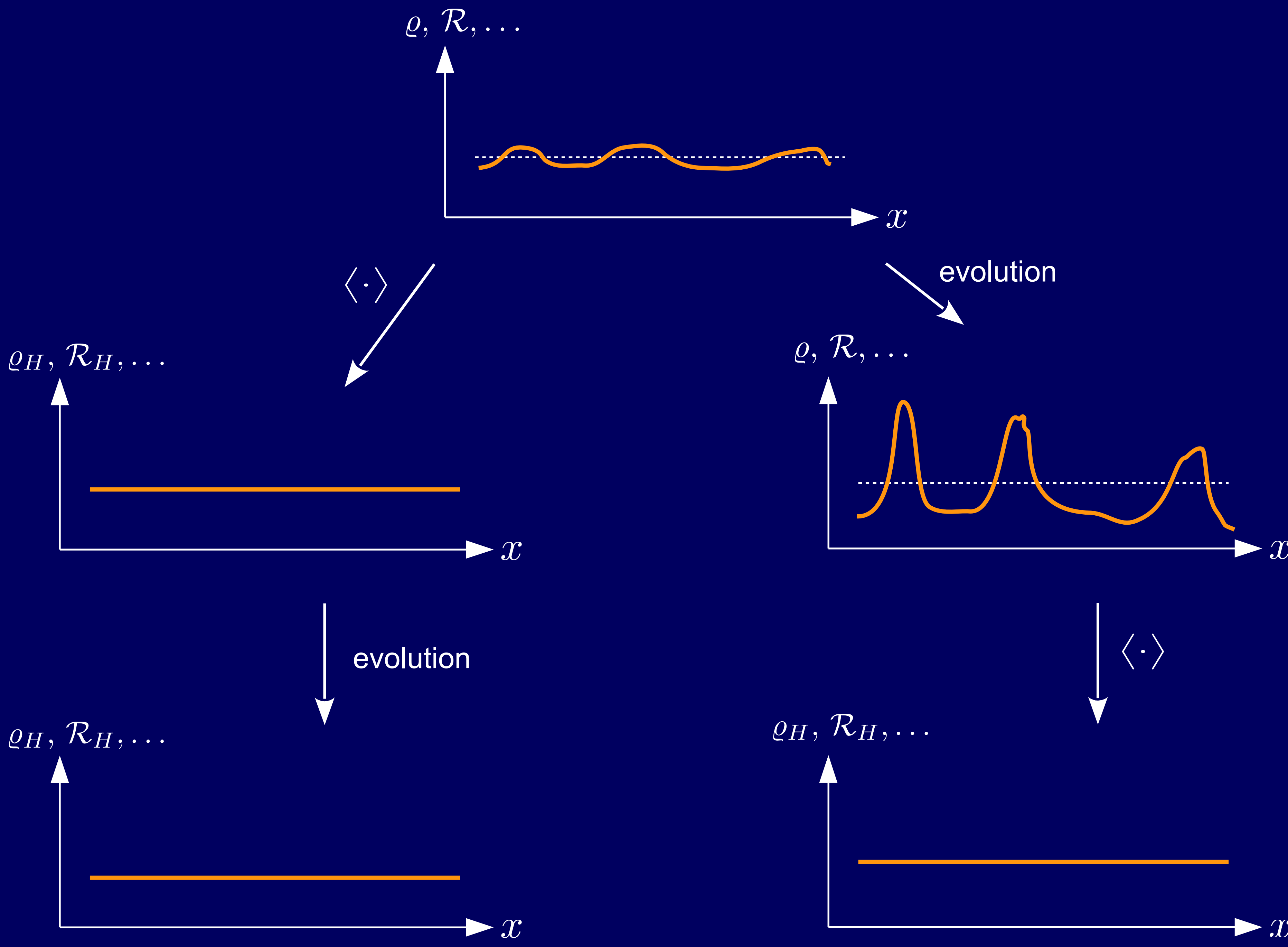
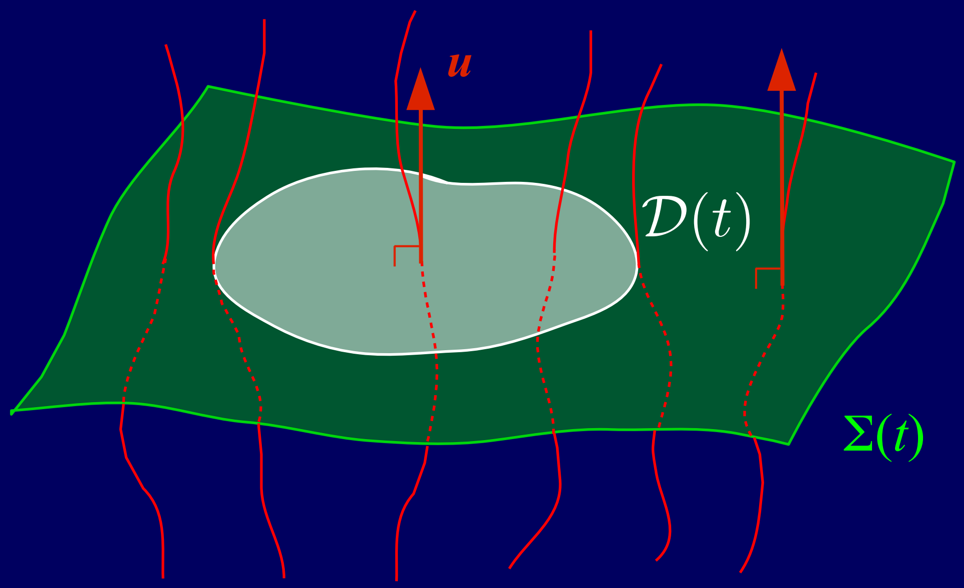


Application to a simple framework (T. Buchert, GRG **32**, 105 (2000))

Spatial volume-averaging scheme for scalars
for a model universe filled with irrotational dust (pressureless matter):

Consider a compact domain $\mathcal{D}$ within a spatial slice Σ, defined as a global rest frame for the dust fluid.



Adapted coordinate system:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + h_{ij}\,\mathrm{d}x^i \mathrm{d}x^j$$

Volume: $\mathcal{V}_{\mathcal{D}}(t) \equiv \int_{\mathcal{D}} \sqrt{h}\,\mathrm{d}^3x$

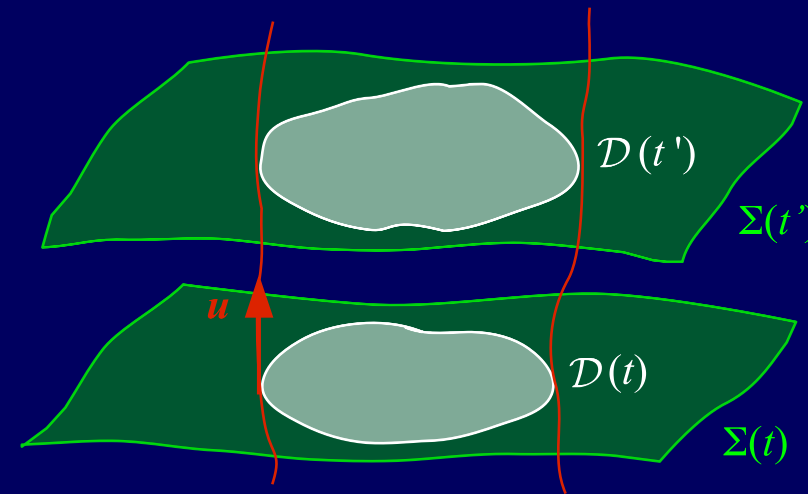
$$\big(\, h = \det(h_{ij}) \,\big)$$

$\rightarrow$ Average of a scalar $\psi(t, x^k)$: $\left\langle \psi \right\rangle_{\mathcal{D}}(t) \equiv \dfrac{1}{\mathcal{V}_{\mathcal{D}}} \displaystyle\int_{\mathcal{D}} \psi \sqrt{h}\,\mathrm{d}^3x$

Conventions:
- $c = 1$
- metric signature $(-,+,+,+)$
- Greek letters for space-time indices (0 to 3), Latin letters for spatial indices (1 to 3)

Application to a simple framework (T. Buchert, GRG **32**, 105 (2000))

Assume a **comoving domain**, following the fluid propagation.



→ **Commutation rule** for averaging and (Lagrangian) time derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi\rangle_{\mathcal{D}} = \left\langle\frac{\mathrm{d}\psi}{\mathrm{d}t}\right\rangle_{\mathcal{D}} + \langle\Theta\,\psi\rangle_{\mathcal{D}} - \langle\Theta\rangle_{\mathcal{D}}\langle\psi\rangle_{\mathcal{D}}$$

expansion scalar (Θ)

Regional evolution equations: define an effective scale factor for the domain:

$$a_{\mathcal{D}} \equiv \left(\frac{\mathcal{V}_{\mathcal{D}}}{\mathcal{V}_{\mathcal{D}_\mathrm{i}}}\right)^{1/3} \longrightarrow \frac{1}{a_{\mathcal{D}}}\frac{\mathrm{d}a_{\mathcal{D}}}{\mathrm{d}t} = \frac{1}{3}\langle\Theta\rangle_{\mathcal{D}}$$

Application to a simple framework (T. Buchert, GRG **32**, 105 (2000))

Averaging scalar projections
of the Einstein equations:

$$3\,\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = -4\pi G \langle \varrho \rangle_{\mathcal{D}} + \Lambda + \mathcal{Q}_{\mathcal{D}} \;;$$

$$3\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 = 8\pi G \langle \varrho \rangle_{\mathcal{D}} + \Lambda - \frac{1}{2}\langle \mathcal{R} \rangle_{\mathcal{D}} - \frac{1}{2}\mathcal{Q}_{\mathcal{D}}$$

$$\langle \varrho \rangle_{\mathcal{D}}^{\cdot} + 3\,\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\langle \varrho \rangle_{\mathcal{D}} = 0 \;\rightarrow\; \langle \varrho \rangle_{\mathcal{D}} = \frac{\langle \varrho \rangle_{\mathcal{D}}(t_{\mathbf{i}})}{a_{\mathcal{D}}^3}$$

Kinematical backreaction:

$$\mathcal{Q}_{\mathcal{D}} \equiv \frac{2}{3}\left\langle \left(\Theta - \langle \Theta \rangle_{\mathcal{D}}\right)^2 \right\rangle_{\mathcal{D}} - 2\,\langle \sigma^2 \rangle_{\mathcal{D}}$$

Integrability condition:

$$\dot{\mathcal{Q}}_{\mathcal{D}} + 6\,\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\mathcal{Q}_{\mathcal{D}} + \langle \mathcal{R} \rangle_{\mathcal{D}}^{\cdot} + 2\,\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\langle \mathcal{R} \rangle_{\mathcal{D}} = 0$$

Application to a simple framework (T. Buchert, GRG **32**, 105 (2000))

Averaging scalar projections of the Einstein equations:

$$3\,\frac{\ddot a_{\mathcal D}}{a_{\mathcal D}} = -4\pi G \left\langle \varrho \right\rangle_{\mathcal D} + \Lambda + \mathcal{Q}_{\mathcal D}\;;$$

$$3\left(\frac{\dot a_{\mathcal D}}{a_{\mathcal D}}\right)^2 = 8\pi G \left\langle \varrho \right\rangle_{\mathcal D} + \Lambda - \frac{1}{2}\left\langle \mathcal{R} \right\rangle_{\mathcal D} - \frac{1}{2}\mathcal{Q}_{\mathcal D}$$

$$\left\langle \varrho \right\rangle_{\mathcal D}^{\cdot} + 3\,\frac{\dot a_{\mathcal D}}{a_{\mathcal D}}\left\langle \varrho \right\rangle_{\mathcal D} = 0 \;\rightarrow\; \left\langle \varrho \right\rangle_{\mathcal D} = \frac{\left\langle \varrho \right\rangle_{\mathcal D}(t_{\mathbf i})}{a_{\mathcal D}^3}$$

Compare with Friedmann:

$$3\,\frac{\ddot a}{a} = -4\pi G \varrho + \Lambda\;;$$

$$3\,\frac{\dot a^2}{a^2} = 8\pi G \varrho + \Lambda - \frac{3k}{a^2}$$

$$\dot\varrho + 3\,\frac{\dot a}{a}\varrho = 0 \;\rightarrow\; \varrho = \frac{\varrho(t_{\mathbf i})}{a^3}$$

Kinematical backreaction:

$$\mathcal{Q}_{\mathcal D} \equiv \frac{2}{3}\left\langle \left(\Theta - \left\langle \Theta \right\rangle_{\mathcal D}\right)^2 \right\rangle_{\mathcal D} - 2\left\langle \sigma^2 \right\rangle_{\mathcal D}$$

$$\sigma^2 = 0$$

Integrability condition:

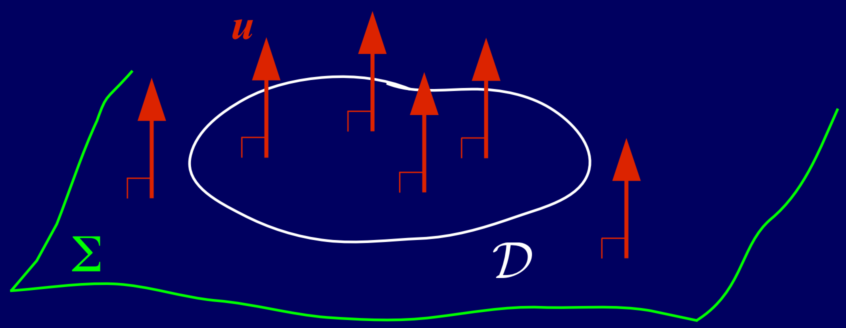
$$\dot{\mathcal{Q}}_{\mathcal D} + 6\,\frac{\dot a_{\mathcal D}}{a_{\mathcal D}}\mathcal{Q}_{\mathcal D} + \left\langle \mathcal{R} \right\rangle_{\mathcal D}^{\cdot} + 2\,\frac{\dot a_{\mathcal D}}{a_{\mathcal D}}\left\langle \mathcal{R} \right\rangle_{\mathcal D} = 0$$

$$\mathcal{R} = \frac{6k}{a^2}$$

Later generalized to irrotational perfect fluids (T. Buchert, GRG **33**, 1381 (2001))

General spatial foliations

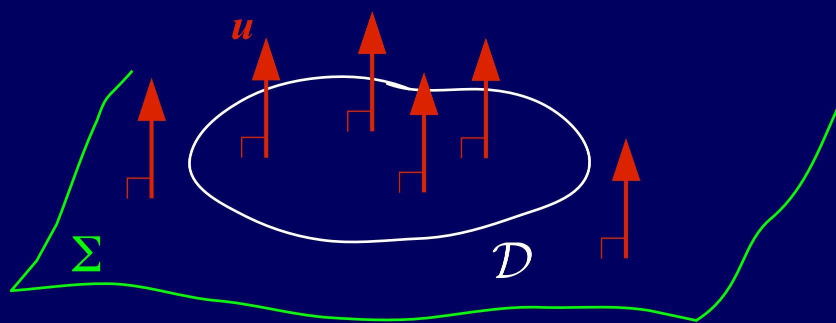
Towards tilted hypersurfaces



Fluid-orthogonal foliation
(irrotational fluid)

General spatial foliations

Towards tilted hypersurfaces



Fluid-orthogonal foliation
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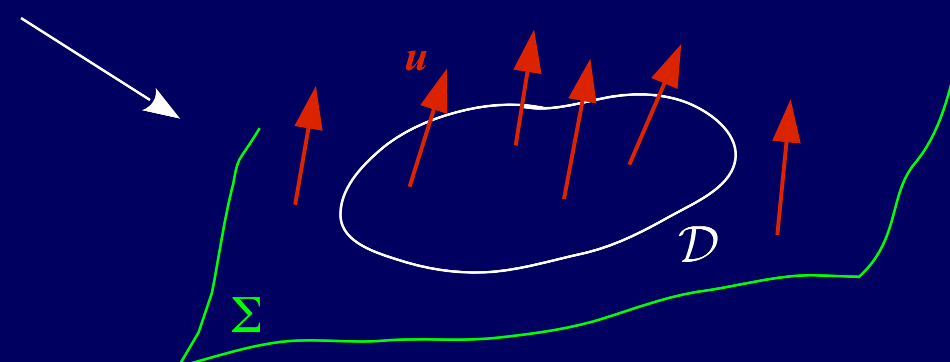


Arbitrary, tilted foliation

→ describe any fluid flow; obtain averaged equations in any foliation (*e.g.* for numerical simulations); study the consequences of a change of foliation…

Existing literature on generalizations of the fluid-orthogonal scalar averaging procedure of Buchert 2000, 2001 to arbitrary foliations:

Kasai, M., Asada, H., Futamase, T.: Toward a no-go theorem for an accelerating universe through a nonlinear backreaction, Progr. Theor. Phys. **115**, 827 (2006); Tanaka, H., Futamase, T.: A phantom does not result from a backreaction, Progr. Theor. Phys. **117**, 183 (2007)

Larena, J.: Spatially averaged cosmology in an arbitrary coordinate system, Phys. Rev. D **79**, 084006 (2009)

Brown, I.A., Behrend, J., Malik, K.A.: Gauges and cosmological backreaction, J. Cosmol. Astropart. Phys., JCAP0911:027 (2009)

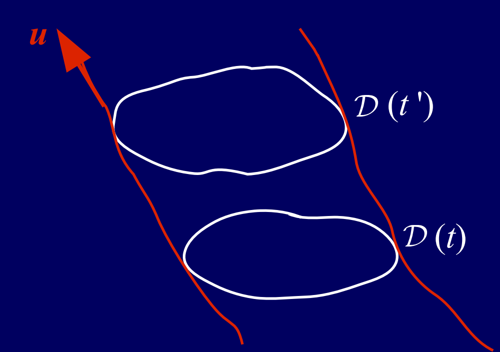
Gasperini, M., Marozzi, G., Veneziano, G.: Gauge invariant averages for the cosmological backreaction, J. Cosmol. Astropart. Phys., JCAP0903:011 (2009); Gasperini, M., Marozzi, G., Veneziano, G.: A covariant and gauge invariant formulation of the cosmological “backreaction”, J. Cosmol. Astropart. Phys., JCAP1002:009 (2010)

Räsänen, S.: Light propagation in statistically homogeneous and isotropic universes with general matter content, J. Cosmol. Astropart. Phys., JCAP1003:018 (2010)

Beltrán Jiménez, J., de la Cruz-Dombriz, Á., Dunsby, P.K.S., Sáez-Gómez, D.: Backreaction mechanism in multifluid and extended cosmologies, J. Cosmol. Astropart. Phys., JCAP1405:031 (2014)

Smirnov, J.: Gauge-invariant average of Einstein equations for finite volumes, [*arXiv:1410.6480*] (2014)

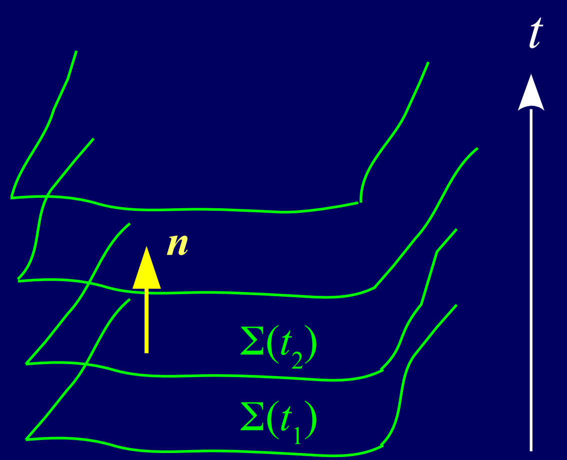
Some of these works apply to global averaging domains. Otherwise, the domain propagation matters.



In the above works, it is never comoving
→ different physical system considered for each foliation...

Building a foliation — Lapse and Shift

Choice of a spatial foliation (slicing)
$\longleftrightarrow$ Choice of the unit time-like normal vector field $\boldsymbol{n}$, irrotational (Frobenius theorem)



Adapted coordinates set (t, x^i):
time is constant on each hypersurface
and used as a label: $\Sigma(t)$;
arbitrary spatial coordinates x^i

$\rightarrow$ in these coordinates:

$$n^\mu = \frac{1}{N}(1, -N^i) \ , \quad n_\mu = -N(1,0)$$

lapse (set by foliation choice and time normalization)

shift (can be set through the propagation of the spatial coordinates)

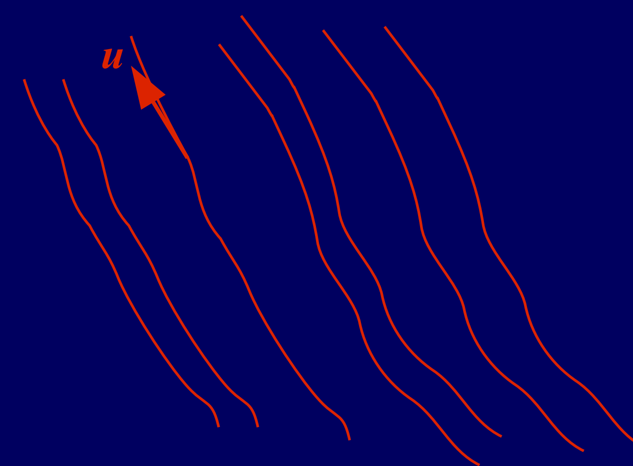
The fluid flow

Universe filled with a **single fluid**, characterized by its 4-velocity field $\boldsymbol{u}$, rest-mass density ϱ and general energy-momentum tensor

$$T_{\mu\nu} = \epsilon\, u_\mu u_\nu + 2\, q_{(\mu} u_{\nu)} + p\, b_{\mu\nu} + \pi_{\mu\nu}$$

energy density (ϵ); heat vector (q); isotropic pressure (p); (traceless) anisotropic pressure ($\pi_{\mu\nu}$)

$$(b_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu)$$



$\boldsymbol{u}$ defines a 1D threading congruence (flow lines)

Kinematic variables of the fluid:

$$\nabla_\mu u_\nu = -u_\mu\, a_\nu + \frac{1}{3}\Theta\, b_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}$$

acceleration (a_ν); expansion scalar (Θ); shear ($\sigma_{\mu\nu}$); vorticity ($\omega_{\mu\nu}$)

$$\sigma^2 \equiv \frac{1}{2}\sigma^{\mu\nu}\sigma_{\mu\nu}\ ; \qquad \omega^2 \equiv \frac{1}{2}\omega^{\mu\nu}\omega_{\mu\nu}$$

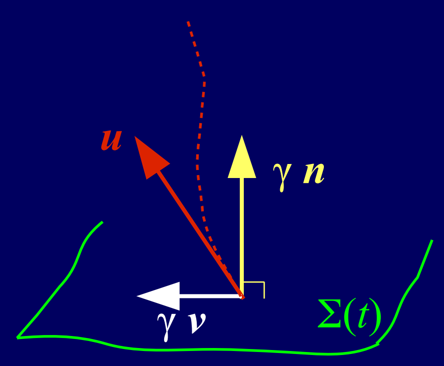
Foliation and fluid — Tilt

Decomposition of $\boldsymbol{u}$ with respect to the foliation:

$$\boldsymbol{u} = \gamma(\boldsymbol{n} + \boldsymbol{v})$$

with $\boldsymbol{v}$ such that $n^\mu v_\mu = 0$ (**tilt vector**)

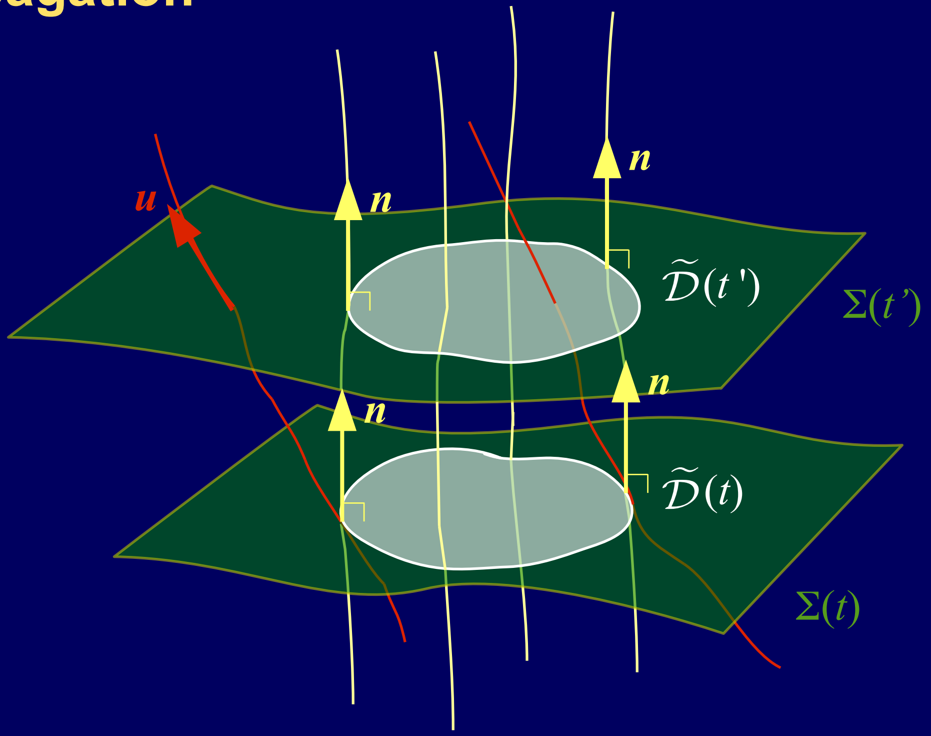
and $\gamma = -n^\mu u_\mu = \dfrac{1}{\sqrt{1 - v^\mu v_\mu}}$ (**tilt factor** or Lorentz factor)



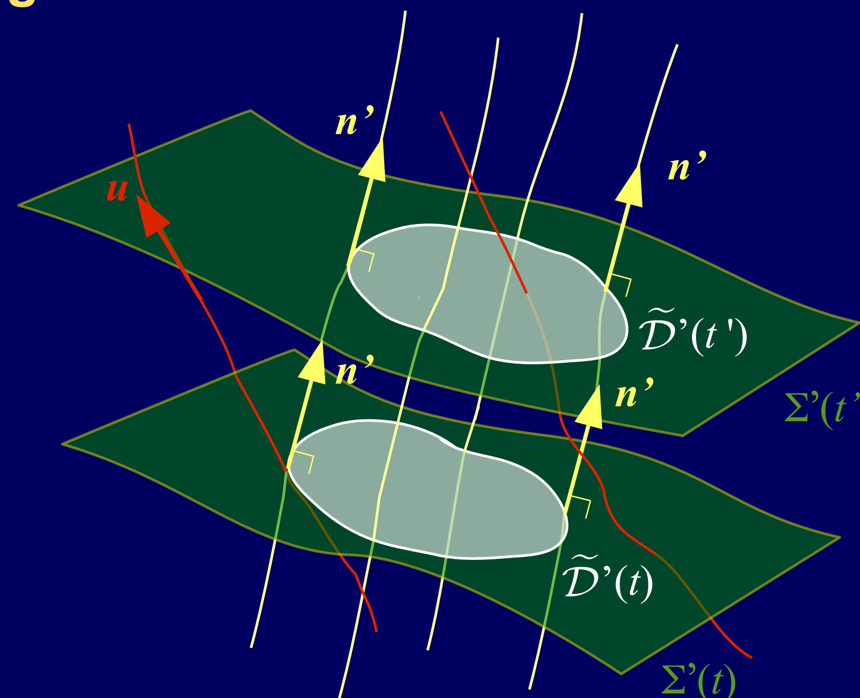
Domain propagation

In the literature: non-global averaging domains $\widetilde{\mathcal{D}}$ propagate along $\boldsymbol{\partial_t}$ or along $\boldsymbol{n}$.

→non-conservation of the collection of fluid elements over the evolution; dependence of the studied system in the foliation (or even coordinates)



Domain propagation



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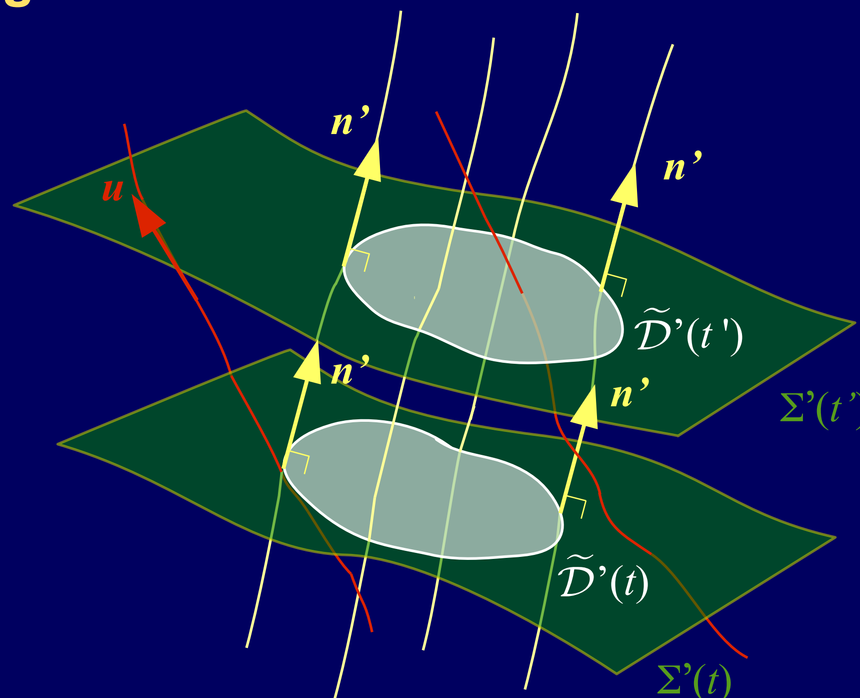
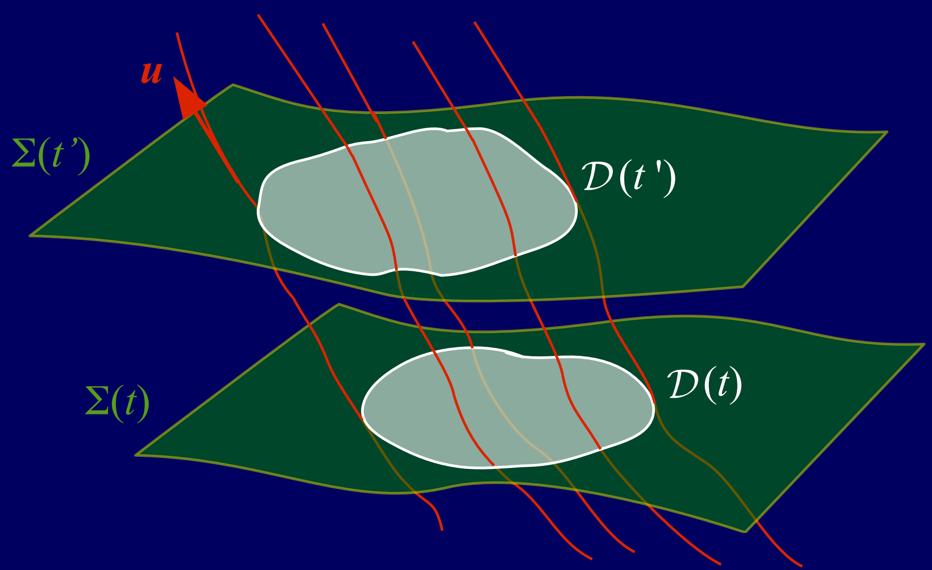




Instead: physical system as a flow tube: given set of fluid elements

Sections by the spatial slices: **comoving** domain

→ Preservation of rest mass and fluid elements collection, same system considered whatever the foliation

Averaging: geometric approach

Averaging operator and commutation rule

Using the hypersurface Riemannian volume measure, $n^{\mu}\mathrm{d}\sigma_{\mu} = \sqrt{h}\, d^3x$:
$\left(h_{\mu\nu} \equiv g_{\mu\nu} + n_{\mu}n_{\nu}\, ,\; h \equiv \det(h_{ij})\right)$

$$\mathcal{V}^h_{\mathcal{D}} \equiv \int_{\mathcal{D}} \sqrt{h(t,x^i)}\, \mathrm{d}^3x \quad ; \quad \langle \psi \rangle^h_{\mathcal{D}} \equiv \frac{1}{\mathcal{V}_{\mathcal{D}}} \int_{\mathcal{D}} \psi(t,x^i)\, \sqrt{h(t,x^i)}\, \mathrm{d}^3x$$

$$a^h_{\mathcal{D}} \equiv \left(\frac{\mathcal{V}^h_{\mathcal{D}}}{\mathcal{V}^h_{\mathcal{D}\mathbf{i}}} \right)^{1/3}$$; fluid-comoving domain

$$\longrightarrow \quad \frac{1}{a^h_{\mathcal{D}}} \frac{\mathrm{d}a^h_{\mathcal{D}}}{\mathrm{d}t} = \frac{1}{3} \left\langle -N\mathcal{K} + \left(Nv^i\right)_{||i} \right\rangle^h_{\mathcal{D}}$$

trace of the extrinsic curvature ($-N\mathcal{K}$); spatial covariant derivative ($_{||i}$)

commutation rule:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \psi \rangle^h_{\mathcal{D}} = \left\langle \frac{\mathrm{d}}{\mathrm{d}t} \psi \right\rangle^h_{\mathcal{D}} - \left\langle -N\mathcal{K} + \left(Nv^i\right)_{||i} \right\rangle^h_{\mathcal{D}} \langle \psi \rangle^h_{\mathcal{D}} + \left\langle \left(-N\mathcal{K} + \left(Nv^i\right)_{||i} \right) \psi \right\rangle^h_{\mathcal{D}}$$

Averaged Einstein equations and backreaction terms

Averaging the scalar 3+1 Einstein equations:

$$3\,\frac{1}{a_{\mathcal{D}}^h}\frac{\mathrm{d}^2 a_{\mathcal{D}}^h}{\mathrm{d}t^2} = -4\pi G\left\langle N^2(\epsilon+3p)\right\rangle_{\mathcal{D}}^h + \left\langle N^2\right\rangle_{\mathcal{D}}^h \Lambda + \mathcal{Q}_{\mathcal{D}}^h + \mathcal{P}_{\mathcal{D}}^h + \frac{1}{2}\mathcal{T}_{\mathcal{D}}^h\,;$$

$$3\left(\frac{1}{a_{\mathcal{D}}^h}\frac{\mathrm{d}a_{\mathcal{D}}^h}{\mathrm{d}t}\right)^2 = 8\pi G\left\langle N^2\epsilon\right\rangle_{\mathcal{D}}^h + \left\langle N^2\right\rangle_{\mathcal{D}}^h \Lambda - \frac{1}{2}\left\langle N^2\,\mathcal{R}\right\rangle_{\mathcal{D}}^h - \frac{1}{2}\mathcal{Q}_{\mathcal{D}}^h - \frac{1}{2}\mathcal{T}_{\mathcal{D}}^h$$

hypersurfaces intrinsic curvature scalar

with the backreaction terms:

kinematical: $\mathcal{Q}_{\mathcal{D}}^h \equiv \left\langle N^2\left(\mathcal{K}^2 - \mathcal{K}_{ij}\mathcal{K}^{ij}\right)\right\rangle_{\mathcal{D}}^h - \frac{2}{3}\left(\left\langle -N\mathcal{K} + \left(Nv^i\right)_{||i}\right\rangle_{\mathcal{D}}^h\right)^2$

dynamical: $\mathcal{P}_{\mathcal{D}}^h \equiv \left\langle NN^{||i}{}_{||i} - \mathcal{K}\,\frac{\mathrm{d}N}{\mathrm{d}t}\right\rangle_{\mathcal{D}}^h + \left\langle \left(\left(Nv^i\right)_{||i}\right)^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(Nv^i\right)_{||i}\right) - 2N\mathcal{K}\left(Nv^i\right)_{||i} - N^2 v^i \mathcal{K}_{||i}\right\rangle_{\mathcal{D}}^h$

stress-energy: $\mathcal{T}_{\mathcal{D}}^h \equiv -16\pi G\left\langle N^2\left((\gamma^2-1)(\epsilon+p) + 2\,\gamma v^\alpha q_\alpha + v^\alpha v^\beta \pi_{\alpha\beta}\right)\right\rangle_{\mathcal{D}}^h$

(+ integrability condition and averaged energy conservation equation)

T. Buchert, PM and X. Roy, *in prep.*

Manifestly covariant form — Window function

The averages, effective evolution equations and backreactions are covariant.

This can be made explicit by defining the same spatial averages from

$$I(\psi) = \int_{\mathcal{M}} \psi \, W \sqrt{|\det(g_{\mu\nu})|}\, \mathrm{d}^4 x$$

$$\longrightarrow \; \mathcal{V} = I(1) \quad ; \quad \langle \psi \rangle = \frac{I(\psi)}{I(1)}$$

with the spatial slice ($A=A_0$) and averaging domain ($B \leq B_0$) selected by

the window function $W = n^\mu \nabla_\mu \big(H(A - A_0) \big) \, H(B_0 - B)$.

(M. Gasperini, G. Marozzi, G. Veneziano, JCAP 02(2010)009 (2010))

Comoving domain choice: set by requiring $\boldsymbol{u} \cdot \boldsymbol{\nabla} B = 0$

→ get the same commutation rule, averaged equations, backreactions, under their manifestly covariant form.

Consequences of the geometric approach

→ An averaging scheme useful to analyse the behaviour of geometric quantities of the hypersurfaces such as $\langle \mathcal{R} \rangle^h_{\mathcal{D}}$, but implicit contributions of the fluid kinematic variables (vorticity ??)

Features curvatures of Σ, hence derivatives of $\boldsymbol{n}$
→high sensitivity to the foliation choice

The equations can be rewritten in terms of the kinematic variables, using local relations. But contributions from the tilt still appear…

E.g.: $$\frac{3}{a^h_{\mathcal{D}}}\frac{\mathrm{d}a^h_{\mathcal{D}}}{\mathrm{d}t} = \left\langle -N\mathcal{K} + \left(Nv^i\right)_{||i} \right\rangle^h_{\mathcal{D}} = \left\langle \frac{N}{\gamma}\Theta - \frac{1}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t} \right\rangle^h_{\mathcal{D}}$$

Averaging: intrinsic approach and proper-time foliations

Intrinsic averaging operator

Define from the fluid not only the domain propagation, but also the volume measure:

$$n^\mu \mathrm{d}\sigma_\mu = \sqrt{h}\, d^3x \;\longmapsto\; u^\mu \mathrm{d}\sigma_\mu = \sqrt{b}\, d^3x = \gamma\sqrt{h}\,\mathrm{d}^3x \quad \big(b \equiv \det(b_{ij})\big)$$

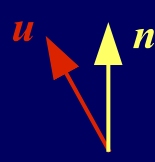
(or $\star\underline{\boldsymbol{n}} \;\longmapsto\; \star\underline{\boldsymbol{u}}$)

or: $W = n^\mu \nabla_\mu\big(H(A-A_0)\big)H(B_0-B) \mapsto W = u^\mu \nabla_\mu\big(H(A-A_0)\big)H(B_0-B)$

→ Define the **fluid proper volume** within $\mathcal{D}$, and associated averages:

$$\mathcal{V}_\mathcal{D}(t) \equiv \int_\mathcal{D} \sqrt{b(t,x^i)}\,\mathrm{d}^3x \;\longrightarrow\; \langle\psi\rangle_\mathcal{D} \equiv \frac{1}{\mathcal{V}_\mathcal{D}}\int_\mathcal{D} \psi(t,x^i)\,\sqrt{b(t,x^i)}\,\mathrm{d}^3x$$

Still a direct generalization of the fluid-orthogonal framework!



$$-u^\mu n_\mu = \gamma$$

Intrinsic-average commutation rule

$$a_{\mathcal{D}} \equiv \left(\frac{\mathcal{V}_{\mathcal{D}}}{\mathcal{V}_{\mathcal{D}_{\mathbf{i}}}} \right)^{1/3}$$

\+ fluid-comoving domain

$$\longrightarrow \quad \frac{1}{a_{\mathcal{D}}} \frac{\mathrm{d}a_{\mathcal{D}}}{\mathrm{d}t} = \frac{1}{3} \left\langle \frac{N}{\gamma} \Theta \right\rangle_{\mathcal{D}}$$

Intrinsic commutation rule:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \psi \rangle_{\mathcal{D}} = \left\langle \frac{\mathrm{d}}{\mathrm{d}t} \psi \right\rangle_{\mathcal{D}} + \left\langle \frac{N}{\gamma} \Theta \, \psi \right\rangle_{\mathcal{D}} - \left\langle \frac{N}{\gamma} \Theta \right\rangle_{\mathcal{D}} \langle \psi \rangle_{\mathcal{D}}$$

Intrinsic averaged Einstein equations

Averaging the Raychaudhuri equation and the Hamilton contraint:

$$3\,\frac{1}{a_{\mathcal{D}}}\frac{\mathrm{d}^2 a_{\mathcal{D}}}{\mathrm{d}t^2} = -\,4\pi G\left\langle\,\tilde{\epsilon}+3\tilde{p}\,\right\rangle_{\mathcal{D}} + \Lambda\left\langle\frac{N^2}{\gamma^2}\right\rangle_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}} + \mathcal{P}_{\mathcal{D}}\ ;$$

$$3\left(\frac{1}{a_{\mathcal{D}}}\frac{\mathrm{d}a_{\mathcal{D}}}{\mathrm{d}t}\right)^2 = 8\pi G\left\langle\,\tilde{\epsilon}\,\right\rangle_{\mathcal{D}} + \Lambda\left\langle\frac{N^2}{\gamma^2}\right\rangle_{\mathcal{D}} - \frac{1}{2}\left\langle\tilde{\mathscr{R}}\right\rangle_{\mathcal{D}} - \frac{1}{2}\mathcal{Q}_{\mathcal{D}}$$

introducing intrinsic kinematical and dynamical backreactions:

$$\mathcal{Q}_{\mathcal{D}} \equiv \frac{2}{3}\left\langle\left(\tilde{\Theta}-\left\langle\tilde{\Theta}\right\rangle_{\mathcal{D}}\right)^2\right\rangle_{\mathcal{D}} - 2\left\langle\tilde{\sigma}^2\right\rangle_{\mathcal{D}} + 2\left\langle\tilde{\omega}^2\right\rangle_{\mathcal{D}}$$

$$\mathcal{P}_{\mathcal{D}} \equiv \left\langle\tilde{\mathcal{A}}\right\rangle_{\mathcal{D}} + \left\langle\tilde{\Theta}\,\frac{\gamma}{N}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{N}{\gamma}\right)\right\rangle_{\mathcal{D}}$$

with the "curvature" $\mathscr{R} \equiv \nabla_\mu u^\nu\,\nabla_\nu u^\mu - \nabla_\mu u^\mu\,\nabla_\nu u^\nu + {}^{(4)}R + 2\,{}^{(4)}R_{\mu\nu}\,u^\mu u^\nu$

and the rescaled variables $\left(\frac{N}{\gamma} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\right)$: $\tilde{\Theta} \equiv \frac{N}{\gamma}\Theta \quad ; \quad \tilde{\sigma} \equiv \frac{N}{\gamma}\sigma \quad ; \quad \tilde{\omega} \equiv \frac{N}{\gamma}\omega$

$$\tilde{\mathscr{R}} \equiv \frac{N^2}{\gamma^2}\mathscr{R} \quad ; \quad \tilde{\epsilon} \equiv \frac{N^2}{\gamma^2}\epsilon \quad ; \quad \tilde{p} \equiv \frac{N^2}{\gamma^2}p \quad ; \quad \tilde{\mathcal{A}} \equiv \frac{N^2}{\gamma^2}\nabla_\mu a^\mu$$

Integrability condition and energy conservation law

Integrability condition:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}_{\mathcal{D}} + 6H_{\mathcal{D}}\mathcal{Q}_{\mathcal{D}} + \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \tilde{\mathscr{R}} \right\rangle_{\mathcal{D}} + 2H_{\mathcal{D}}\left\langle \tilde{\mathscr{R}} \right\rangle_{\mathcal{D}} + 4H_{\mathcal{D}}\mathcal{P}_{\mathcal{D}}$$
$$= 16\pi G\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle\tilde{\epsilon}\rangle_{\mathcal{D}} + 3H_{\mathcal{D}}\langle\tilde{\epsilon}+\tilde{p}\rangle_{\mathcal{D}}\right) + 2\,\Lambda\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \frac{N^2}{\gamma^2}\right\rangle_{\mathcal{D}}$$

Averaged energy conservation equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\tilde{\epsilon}\right\rangle_{\mathcal{D}} + 3H_{\mathcal{D}}\left\langle\tilde{\epsilon}+\tilde{p}\right\rangle_{\mathcal{D}} = \left\langle\tilde{\Theta}\right\rangle_{\mathcal{D}}\left\langle\tilde{p}\right\rangle_{\mathcal{D}} - \left\langle\tilde{\Theta}\,\tilde{p}\right\rangle_{\mathcal{D}}$$
$$-\left\langle\frac{N^3}{\gamma^3}(\nabla_\mu q^\mu + q^\mu a_\mu + \pi^{\mu\nu}\sigma_{\mu\nu})\right\rangle_{\mathcal{D}} + 2\left\langle\tilde{\epsilon}\,\frac{\gamma}{N}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{N}{\gamma}\right)\right\rangle_{\mathcal{D}}$$

with $H_{\mathcal{D}} \equiv \frac{1}{a_{\mathcal{D}}}\frac{\mathrm{d}a_{\mathcal{D}}}{\mathrm{d}t} = \frac{1}{3}\left\langle\tilde{\Theta}\right\rangle_{\mathcal{D}}$

Effective Friedmannian form

$$3\,\frac{1}{a_{\mathcal{D}}}\frac{\mathrm{d}^2 a_{\mathcal{D}}}{\mathrm{d}t^2} = -\,4\pi G\left(\epsilon_{\mathcal{D}}^{\mathrm{eff}} + 3p_{\mathcal{D}}^{\mathrm{eff}}\right) + \Lambda\;;$$

$$3\left(\frac{1}{a_{\mathcal{D}}}\frac{\mathrm{d}a_{\mathcal{D}}}{\mathrm{d}t}\right)^2 = 8\pi G\,\epsilon_{\mathcal{D}}^{\mathrm{eff}} + \Lambda - \frac{3k_{\mathcal{D}}}{{a_{\mathcal{D}}}^2}$$

+ Integrability condition: $$\frac{\mathrm{d}}{\mathrm{d}t}\epsilon_{\mathcal{D}}^{\mathrm{eff}} + \frac{3}{a_{\mathcal{D}}}\frac{\mathrm{d}a_{\mathcal{D}}}{\mathrm{d}t}\left(\epsilon_{\mathcal{D}}^{\mathrm{eff}} + p_{\mathcal{D}}^{\mathrm{eff}}\right) = 0$$

$$\epsilon_{\mathcal{D}}^{\mathrm{eff}} = \langle\tilde{\epsilon}\rangle_{\mathcal{D}} - \frac{1}{16\pi G}\mathcal{Q}_{\mathcal{D}} - \frac{1}{16\pi G}\mathcal{W}_{\mathcal{D}} + \frac{1}{8\pi G}\Lambda\left(\left\langle\frac{N^2}{\gamma^2}\right\rangle_{\mathcal{D}} - 1\right)\;;$$

$$p_{\mathcal{D}}^{\mathrm{eff}} = \langle\tilde{p}\rangle_{\mathcal{D}} - \frac{1}{16\pi G}\mathcal{Q}_{\mathcal{D}} + \frac{1}{48\pi G}\mathcal{W}_{\mathcal{D}} - \frac{1}{8\pi G}\Lambda\left(\left\langle\frac{N^2}{\gamma^2}\right\rangle_{\mathcal{D}} - 1\right) - \frac{1}{12\pi G}\mathcal{P}_{\mathcal{D}}$$

with $\mathcal{W}_{\mathcal{D}} = \left\langle\tilde{\mathscr{R}}\right\rangle_{\mathcal{D}} - \frac{6k_{\mathcal{D}}}{a_{\mathcal{D}}^2}$ and, e.g., $6k_{\mathcal{D}} = \left\langle\tilde{\mathscr{R}}\right\rangle_{\mathcal{D}}(t_{\mathbf{i}})\,.$

→ effective energy sources with evolving equation of state, or sum of coupled sources

Time parameter interpretation and application to proper-time foliations

$\mathrm{d}^2 a_{\mathcal{D}}/\mathrm{d}t^2$, $\mathrm{d}a_{\mathcal{D}}/\mathrm{d}t$ may not have an interpretation equivalent to the comoving proper-time terms $\ddot{a}/a$, $\dot{a}/a$ of the Friedmann equations.
It must be interpreted in relation to the meaning of *t*, once it is specified.
(The same holds for any similar general averaging formalism...)

Recovering a simple meaning: choice of the normalization of *t*, in any foliation.

Or: choose a foliation at **constant proper time** *τ* of the fluid (starting from a given hypersurface) and set *t* = *τ*, *i.e.*, *N* = *γ*.

For such a choice:

$$\left(\dot{} \equiv u^\mu \partial_\mu = \frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}t} \right)$$

$$3\,\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = -4\pi G \left\langle \epsilon + 3p \right\rangle_{\mathcal{D}} + \Lambda + \mathcal{Q}_{\mathcal{D}} + \mathcal{P}_{\mathcal{D}} \, ;$$

$$3\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \right)^2 = 8\pi G \left\langle \epsilon \right\rangle_{\mathcal{D}} + \Lambda - \frac{1}{2} \left\langle \mathscr{R} \right\rangle_{\mathcal{D}} - \frac{1}{2} \mathcal{Q}_{\mathcal{D}} \, ,$$

with

$$\mathcal{Q}_{\mathcal{D}} = \frac{2}{3} \left\langle \left(\Theta - \left\langle \Theta \right\rangle_{\mathcal{D}} \right)^2 \right\rangle_{\mathcal{D}} - 2 \left\langle \sigma^2 \right\rangle_{\mathcal{D}} + 2 \left\langle \omega^2 \right\rangle_{\mathcal{D}} \; ; \quad \mathcal{P}_{\mathcal{D}} = \left\langle \nabla_\mu a^\mu \right\rangle_{\mathcal{D}}$$

Time parameter interpretation and application to proper-time foliations

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T. Buchert, PM and X. Roy, arXiv:1805.10455, accepted by CQG (2018);
T. Buchert, PM and X. Roy, *in prep.*

Manifestly covariant formulation

We can similarly recover the intrinsic averages and the previous equations under a more explicitly covariant form, using

$$W = u^\mu \nabla_\mu \big(H(A - A_0) \big) H(B_0 - B) \,.$$

Or: use a more general window function to include any volume measure:

$$W = V^\mu \nabla_\mu \big(H(A - A_0) \big) H(B_0 - B)$$

→ **Commutation rule**:

$$\frac{\partial \langle \psi \rangle}{\partial A_0} = \left\langle \frac{Z^\mu \nabla_\mu \psi}{Z^\sigma \nabla_\sigma A} \right\rangle + \left\langle \frac{\big(\psi - \langle \psi \rangle\big) \nabla_\mu \left(Z^\mu \frac{V^\kappa \nabla_\kappa A}{Z^\sigma \nabla_\sigma A} \right)}{V^\nu \nabla_\nu A} \right\rangle - \left\langle \frac{\big(\psi - \langle \psi \rangle\big) Z^\mu \nabla_\mu B \, \delta(B_0 - B)}{Z^\sigma \nabla_\sigma A} \right\rangle$$

Comoving domain: $\boldsymbol{u} \cdot \boldsymbol{\nabla} B = 0$; take ***Z*** = ***u***, and ***V*** = ***u*** or ***n*** (or *ϱ* ***u***…); apply to the Hamilton and Raychaudhuri equations...

A. Heinesen, PM, T. Buchert, *in prep.*

Summary

• Scalar averaging schemes to describe inhomogeneous universes in any spatial foliation, for a general single-fluid model. Resulting averaged equations for a **comoving domain**; always feature **backreaction terms** of local structures on the large-scale evolution.

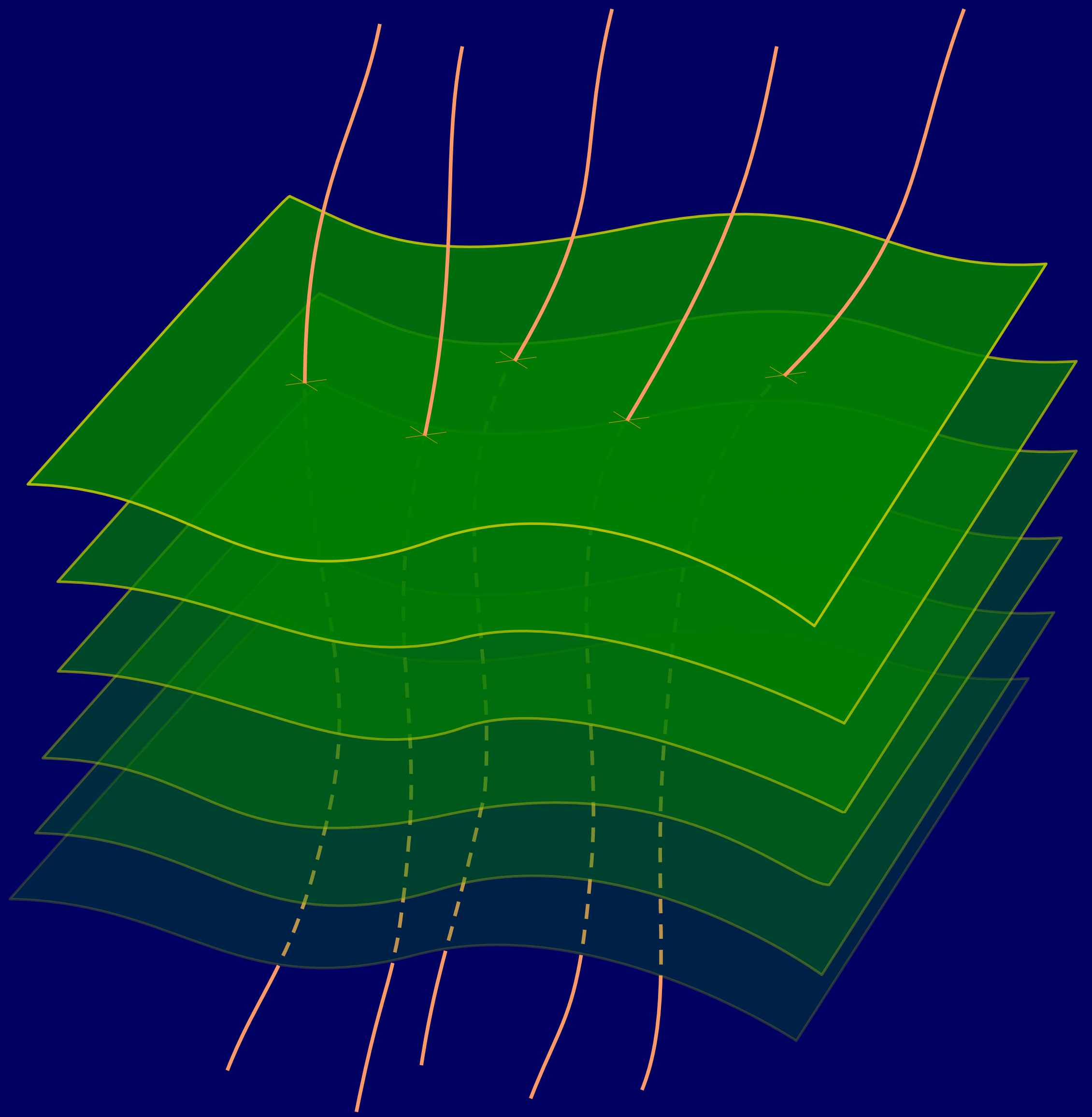
• Two averaging operators from "natural" volume measures. One using the hypersurface volume measure: sheds some light on the geometric properties of the slices. The other using the fluid proper volume measure: provides simpler average equations that directly show the contributions from the fluid rest-frame properties, with less dependence on the foliation choice.

• May be written under an explicitly covariant form; may formally encompass any volume measure.

• In specific applications: choose a suitable foliation and a meaningful t parameter, interpret time derivatives (and lapse N) accordingly.

• One choice of particular physical interest: the constant proper time foliation, built from the fluid flow, well-suited to the intrinsic approach. Provides simple equations and a natural interpretation of "time" and time derivatives.

• More explicit determination of dependence on the foliation? (with Asta Heinesen) Application to specific fluid models? Lagrangian approximation schemes?



THANK YOU FOR YOUR ATTENTION!

SUPPLEMENTARY MATERIAL

Intrinsic-average integrability condition and energy conservation law in a proper-time foliation

Integrability condition:

$$\dot{\mathcal{Q}}_{\mathcal{D}} + 6H_{\mathcal{D}}\mathcal{Q}_{\mathcal{D}} + \langle \mathscr{R} \rangle^{\cdot}_{\mathcal{D}} + 2H_{\mathcal{D}} \langle \mathscr{R} \rangle_{\mathcal{D}} + 4H_{\mathcal{D}}\mathcal{P}_{\mathcal{D}} = 16\pi G \left(\langle \epsilon \rangle^{\cdot}_{\mathcal{D}} + 3H_{\mathcal{D}} \langle \epsilon + p \rangle_{\mathcal{D}} \right)$$

Averaged energy conservation equation:

$$\langle \epsilon \rangle^{\cdot}_{\mathcal{D}} + 3H_{\mathcal{D}} \langle \epsilon + p \rangle_{\mathcal{D}} = \langle \Theta \rangle_{\mathcal{D}} \langle p \rangle_{\mathcal{D}} - \langle \Theta\, p \rangle_{\mathcal{D}} - \langle \nabla_{\mu} q^{\mu} + q^{\mu} a_{\mu} + \pi^{\mu\nu} \sigma_{\mu\nu} \rangle_{\mathcal{D}}$$

+ Rest-mass conservation equation:

$$\dot{M}_{\mathcal{D}} = 0 \, ; \quad \langle \varrho \rangle_{\mathcal{D}} = \frac{M_{\mathcal{D}}}{\mathcal{V}_{\mathcal{D}}} \; : \quad \langle \varrho \rangle^{\cdot}_{\mathcal{D}} + 3H_{\mathcal{D}} \langle \varrho \rangle_{\mathcal{D}} = 0$$

Here, $H_{\mathcal{D}} = \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = \frac{1}{3} \langle \Theta \rangle_{\mathcal{D}}$.