Inflationary correlators from the stochastic spectral expansion

arXiv:1904.11917, 1811.02586, 1910.?????

"*Scalar correlation functions in de Sitter space from the stochastic spectral expansion*",

"*Spectator Dark Matter*",

"*Scalar correlation functions for a double-well potential in de Sitter space*"

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Outline

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QFT in de Sitter (dS) space ($a(t) = e^{Ht}$)

- Field theory on curved backgrounds a well-known framework

  Textbook: N. Birrell & P. Davies (82)

- Often very difficult analytically, even for free scalars

- No vacuum state for free massless scalars in dS

  B. Allen (85); B. Allen & A Folacci (87)

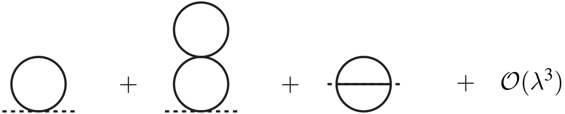
- Infrared (IR) divergences, $\hat{\phi} = \sum_{\mathbf{k}} \left[\hat{a}_{\mathbf{k}} f_k + \mathrm{h.c.}\right]$

$$\int_0^{\Lambda_{\mathrm{IR}}} dk\, k^2 |f_k|^2 \xrightarrow{t\to\infty} \infty$$

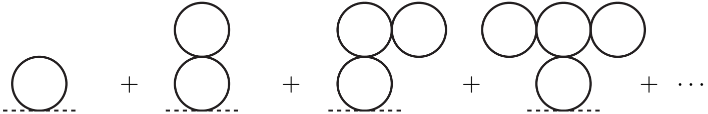
- Can be cured via *resumming*

Resummations, $V(\phi) = \frac{\lambda}{4}\phi^4$

- The perturbative expansion



- An example of a resummation



- A resummed propagator contains an infinite number of diagrams (i.e. is non-perturbative)
  - $\Rightarrow$ Interactions can cure the IR issues of the free theory

Stochastic approach for de Sitter

- Ingenious approach: IR as stochastic variables, noise from UV

  Starobinsky (86); Starobinsky & Yokoyama (94)

$\Rightarrow$ Classical statistics with the probability density $P(t, \phi)$

$$\dot{P}(t, \phi) = \frac{1}{3H} \frac{\partial}{\partial \phi} \left[P(t, \phi) V'(\phi) \right] + \frac{H^3}{8\pi^2} \frac{\partial^2}{\partial \phi^2} P(t, \phi)$$

(Fokker-Planck)

The 1-point equilibrium distribution

$$P(\phi) = N \exp\left[-\frac{8\pi^2}{3H^4} V(\phi) \right]$$

- Light fields $V''(\phi) \ll H^2$ excited by the expansion

What about correlators?

- The scalar "condensate" for $V(\phi) = \frac{\lambda}{4}\phi^4$

$$\langle\phi^2\rangle = \left(\frac{3}{2\pi^2}\right)^{1/2} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \frac{H^2}{\lambda^{1/2}}$$

- However, no information of the correlation length in $P(\phi)$
  - **Is there a condensate after all?!**

The crucial quantity

$$\langle\phi(0)\phi(\mathbf{x})\rangle = ?$$

- Also, in cosmology a very important quantity to know is

$$\langle\delta\rho(0)\delta\rho(\mathbf{x})\rangle = ? \, ; \qquad \delta\rho(\mathbf{x}) = \langle\rho\rangle - \rho(\mathbf{x})$$

Spectral expansion

Correlators for scalar ϕ from the stochastic approach

Three important assumptions:

1. ϕ is a spectator
2. Close to dS
3. Equilibrium

- Starobinsky & Yokoyama (94):
  solution via eigenvalues Λ_n and -functions $\psi_n(\phi)$:

$$\left[\frac{\partial^2}{\partial\phi^2} + \underbrace{v''(\phi) - v'(\phi)^2}_{\equiv W(\phi)}\right]\psi_n(\phi) = -\frac{8\pi^2\Lambda_n}{H^3}\psi_n(\phi)\,; \quad v(\phi) = \frac{4\pi^2}{3H^4}V(\phi)$$

$$\langle f[\phi(0)]f[\phi(t)]\rangle = \sum_n f_n^2 e^{-\Lambda_n t}\,; \qquad f_n = \int d\phi\,\psi_0(\phi)f(\phi)\psi_n(\phi)$$

- Spatial correlators via dS invariance

Example: $\langle\delta\phi^2(0)\delta\phi^2(\mathbf{x})\rangle$

- **No** VEV, $\langle\phi\rangle = 0 \quad \Rightarrow \quad \delta\phi^2 = \phi^2 - \langle\phi^2\rangle$

$$\begin{aligned}\langle\delta\phi^2(0)\delta\phi^2(t)\rangle &= \langle[\phi^2(0) - \langle\phi^2(0)\rangle][\phi^2(t) - \langle\phi^2(t)\rangle]\rangle \\ &= \langle\phi^2(0)\phi^2(t)\rangle - \langle\phi^2\rangle^2 \\ &= \sum_{n=1}^{\infty}(\phi_n^2)^2 e^{-\Lambda_n t}\,; \qquad \phi_n^2 = \int d\phi\,\psi_0(\phi)\phi^2\psi_n(\phi)\end{aligned}$$

- dS invariant line-element:

$$y = \cosh H(t_1 - t_2) - \frac{H^2}{2}e^{H(t_1+t_2)}|\mathbf{x}_1 - \mathbf{x}_2|^2$$

- Finally, for $|\mathbf{x}|H \gtrsim 1$

$$\langle\delta\phi^2(0)\delta\phi^2(\mathbf{x})\rangle = \sum_{n=1}^{\infty}(\phi_n^2)^2(|\mathbf{x}|H)^{-2\Lambda_n/H}$$

Power spectrum, spectral tilt and the correlation length

- At cosmological distances $|\mathbf{x}H| \gg 1$

$$\langle f[\phi(0)]f[\phi(\mathbf{x})]\rangle \sim \frac{A_f}{(|\mathbf{x}|H)^{n_f-1}}\,; \quad A_f = f_n^2$$

$$n_f - 1 \sim 2\frac{\Lambda_n}{H}$$

$$\mathcal{P}_f(k) \sim A_f\,(n_f - 1)\left(\frac{k}{H}\right)^{n_f-1}$$

The correlation length

$$R_c \sim H^{-1} 2^{\frac{1}{n_f-1}}$$

- $n_f - 1$ is NOT the same for ϕ and $\delta\rho$:
  - Always blue

Example: $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4$

- Equation to solve

$$\left[\frac{\partial^2}{\partial\phi^2} - \left(\frac{4\pi^2}{3H^4}\right)^2 \left(m^4\phi^2 + 2\lambda m^2\phi^4 + \lambda^2\phi^6\right) + \frac{4\pi^2}{3H^4}\left(m^2 + 3\lambda\phi^2\right)\right]\psi_n(\phi) = -\frac{8\pi^2}{H^3}\Lambda_n\psi_n(\phi)$$

- Essentially, results are functions of a single parameter

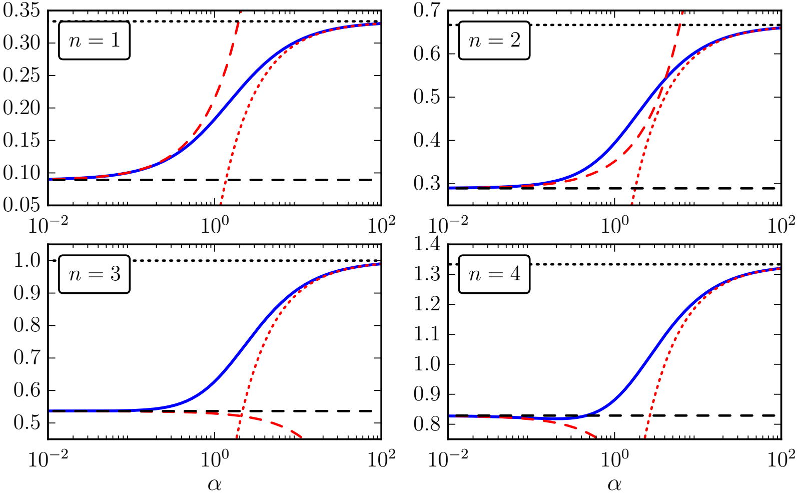
$$\alpha \equiv \frac{m^2}{H^2\sqrt{\lambda}}$$

Can use familiar techniques from QM

$\Rightarrow$ Solutions via the simple ”overshoot/undershoot” method

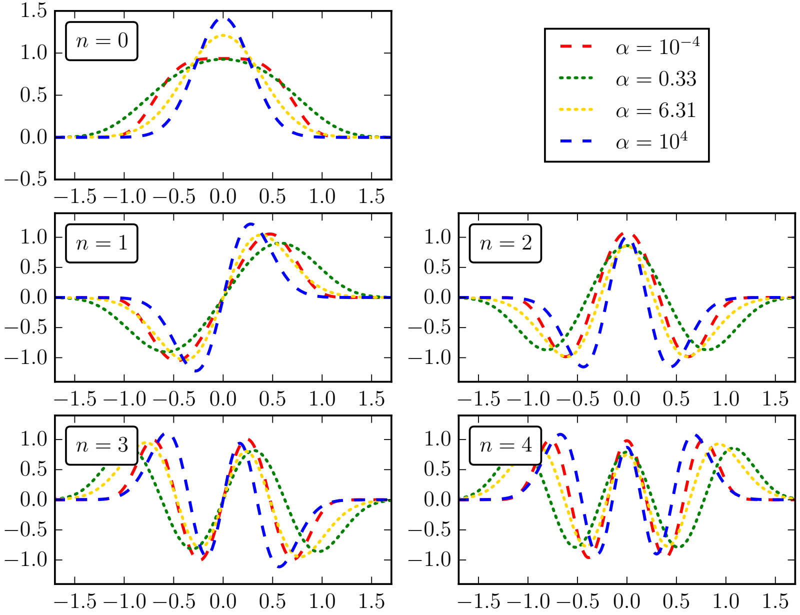
- $\alpha \to \infty \quad \Rightarrow$ quadratic
- $\alpha \to 0 \quad \Rightarrow$ quartic

Eigenvalues for $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4$



$$\tilde{\Lambda}_n(\alpha) = \frac{\Lambda_n}{\lambda^{1/2}H + m^2/H}\,; \qquad \alpha = \frac{m^2}{H^2\sqrt{\lambda}}$$

Eigenfunctions for $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4$



Approximation schemes, $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4$

- Consider the density contrast in the stochastic formalism

$$\delta \equiv \frac{\delta\rho}{\langle\rho\rangle} \approx \frac{V(\phi) - \langle V(\phi)\rangle}{\langle V(\phi)\rangle}$$

$$\Rightarrow \quad \langle\delta(0)\delta(\mathbf{x})\rangle \overset{|\mathbf{x}|H \gg 1}{\approx} \frac{A_\delta}{(|\mathbf{x}|H)^{n_\delta - 1}}\,; \quad n_\delta - 1 = 2\frac{\Lambda_2}{H}\,,\ A_\delta = (\delta_2)^2$$

Common approximations:

Linear EOM for ϕ

$$\left(\Box - m^2\right)\phi = 0$$

Gaussian for δ

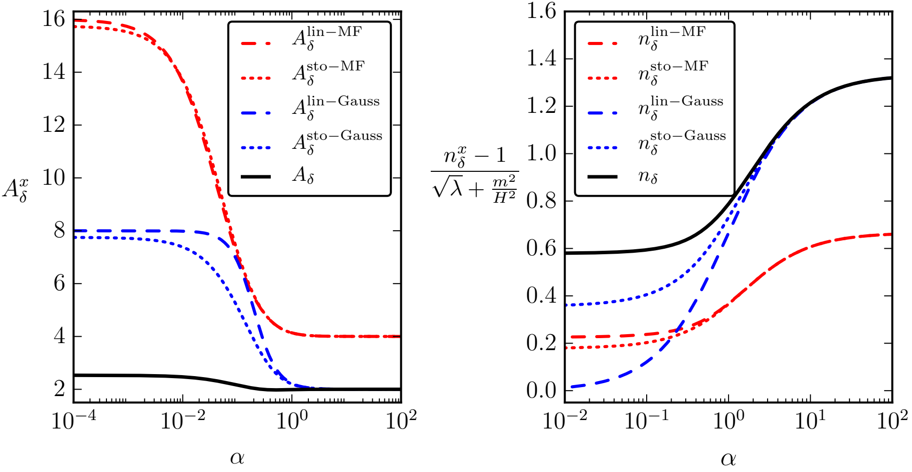
- Assume Wick's theorem

Mean field for δ

$$\varphi \equiv \sqrt{\langle\phi^2\rangle}$$

$$\langle\delta^{\mathrm{MF}}(0)\delta^{\mathrm{MF}}(\mathbf{x})\rangle = \left(\frac{V'(\varphi)}{V(\varphi)}\right)^2 \langle\phi(0)\phi(\mathbf{x})\rangle$$

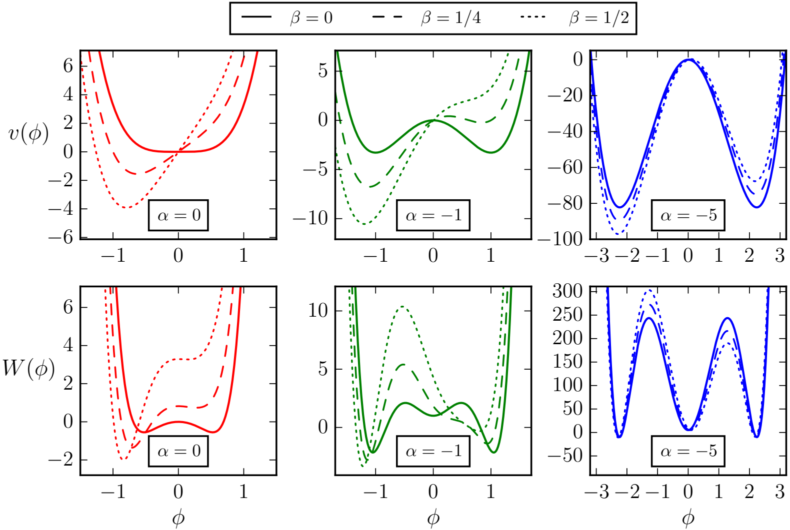
Comparison between approximations



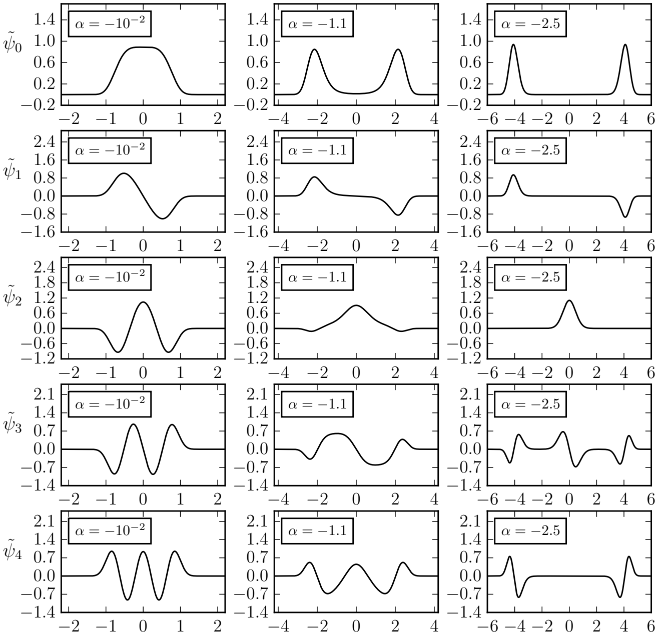
All approximations overestimate power on large scales!

Spec. exp. for double-well: TM & A. Rajantie in prep.

$$V(\phi) = \mu^3\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4\,; \quad \alpha \equiv \frac{m^2}{\sqrt{\lambda}H^2}\,; \quad \beta \equiv \frac{\mu^3}{\lambda^{1/4}H^3}\,.$$

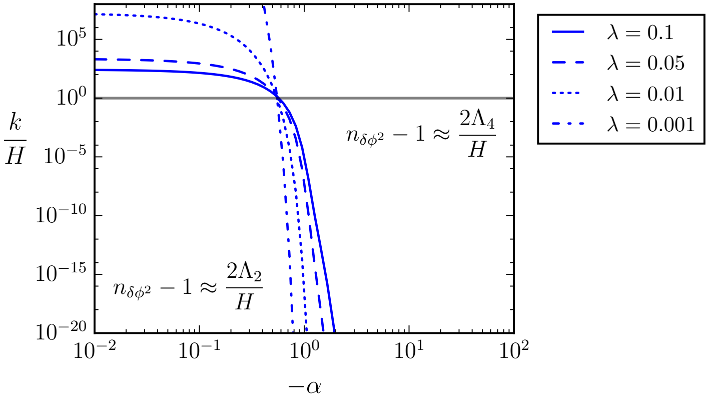


Symmetric case, $\beta = 0$



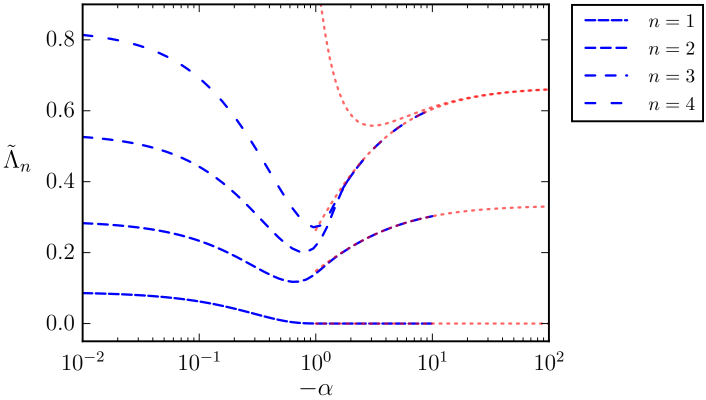
$\mathcal{P}_{\delta\phi^2}(k)$ for symmetric case, $(\delta\phi^2 \equiv \phi^2 - \langle\phi^2\rangle)$

$$\langle\delta\phi^2(0)\delta\phi^2(\mathbf{x})\rangle = \sum_{n=1}^{\infty}(\phi_n^2)^2(|\mathbf{x}|H)^{-2\Lambda_n/H}\,; \qquad \phi_n^2 = \int d\phi\,\psi_0(\phi)\phi^2\psi_n(\phi)$$



- Feature: the tilt jumps on large scales

Eigenvalues, symmetric case, $\beta = 0$



- At the limit of deep wells numerics becomes difficult, but analytics work
- Spectrum does NOT agree with Starobinsky & Yokoyama (94)

Asymmetric case, $\beta \neq 0$

- Numerically a straightforward generalization
- Very rich phenomenology

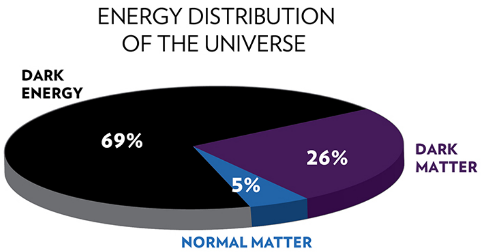
Interesting application:

- Early Universe signatures of different BSM theories

- Also interesting: periodic potentials (axion)
  - Continuous eigenvalue bands?
- Generalization beyond dS?

Dark Matter

The Dark Matter (DM) Paradigm



Known properties of DM

- Massive, Cold & Dark
- Suppressed interactions
- $\Omega_{\rm DM} \sim 26.8\%$
- **Adiabatic**

Open questions

- Fundamental nature
- Production mechanism

Many ways of generating the DM abundance!

Spectator Dark Matter; (Peebles & Vilenkin (99))

A decoupled singlet with the potential

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4$$

- Field excitations during inflation if $m \ll H$

$\Rightarrow$ Energy density at the end of inflation:

$$\rho_{\rm DM} \sim \langle V(\phi)\rangle = \langle \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4\rangle \approx \langle \frac{\lambda}{4}\phi^4\rangle \sim H_{\rm end}^4$$

- Abundance today

$$\frac{\Omega_\phi h^2}{0.12} \sim \left(\frac{H_{\rm end}}{M_{\rm P}}\right)^{3/2} \frac{m}{\rm GeV}.$$

Correct abundance with the appropriate choice for m for a given $H_{\rm end}$

Isocurvature

- Isocurvature/entropy perturbations

$$S \equiv \frac{3}{4}\frac{\delta\rho_\gamma}{\rho_\gamma} - \frac{\delta\rho_{\rm DM}}{\rho_{\rm DM}}$$

adiabatic $\Rightarrow$ $S = 0$ $\Leftrightarrow$ $\delta\left(\frac{n_\gamma}{n_{\rm DM}}\right) = 0\,,$

- A decoupled spectator is not adiabatic

$$S \sim \frac{\delta\rho_\phi}{\rho_\phi}$$

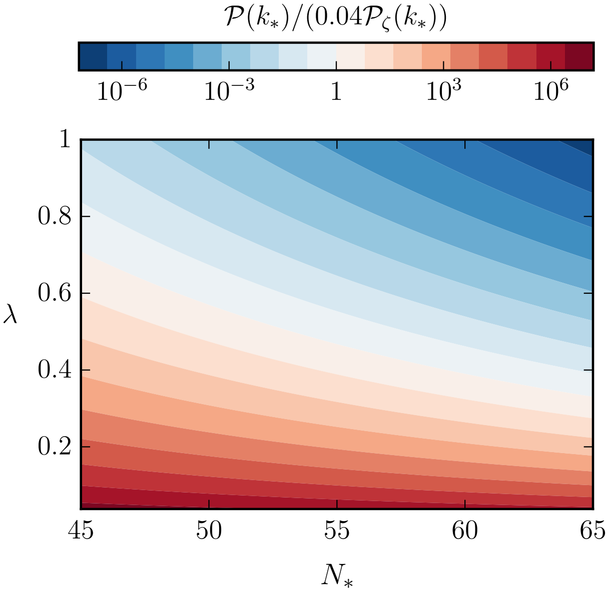
- Planck places stringent bounds on isocurvature

$$\mathcal{P}(k_*) \lesssim 0.040\mathcal{P}_\zeta(k_*)\,.$$

Isocurvature bound sensitive only to scales $\sim k_*$

A sufficiently blue spectrum can avoid the bound!

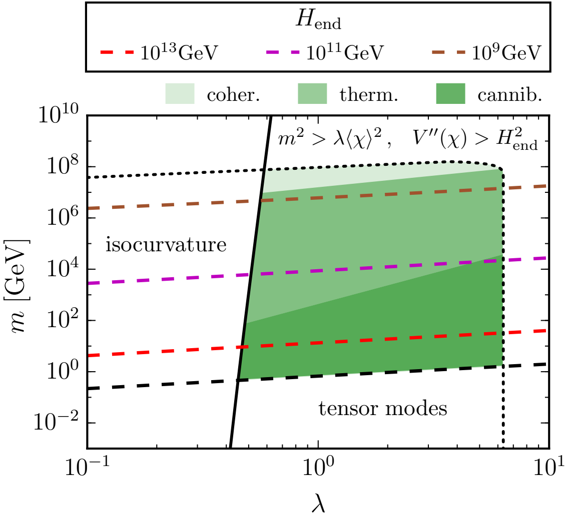
Isocurvature constraints



- Bound avoided for $\lambda \gtrsim 0.45$

Viable parameter range

- A large range of viable masses
- λ is perturbative, $\lambda \ll 4\pi$



- For $V(\phi) = \frac{1}{2}m^2\phi^2$ see: Kuzmin & Tkachev (98)
  - For $m \sim H$ stochastic approach questionable (and not needed)

Conclusions

- The stochastic spectral expansion is a powerful tool for calculating correlators
- Rich phenomenology for double-well potentials
- Decoupled stochastic spectators are a viable candidate of DM

Thank You!