Black hole perturbations in modified gravity

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May 10, 2021

$\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$ seminar - Institut d'Astrophysique de Paris

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Introduction

- Modified gravity theories: predictions different from GR
- Important test: quasinormal modes of black holes
- Up to now, theoretical computations are rare
- Present a systematic algorithm to extract physical information and perform numerical analysis

Outline

1. Modified gravity: DHOST theories
    - Necessity for modified gravity
    - Importance of black holes
2. Quasinormal modes in GR
    - Perturbation setup
    - Schrödinger equations
3. Quasinormal modes in modified gravity
    - Similarities and differences
    - QNMs from the first order system
    - Numerical results

Modified gravity: DHOST theories

Motivation for beyond-GR theories

Testing deviations

- Design new tests of GR
- Know where to look in large amounts of data

Issues of GR

- Big Bang singularity
- Black hole interior singularity
- Dark energy

$\Rightarrow$ Important to look for extensions of GR

$\Rightarrow$ Recent and near-future experiments will give much insight

Various theories of modified gravity

Lovelock's theorem for gravity

- Fourth dimensional spacetime
- Only field is the metric
- Second order derivatives in equations

$\Rightarrow$ GR is the only possible theory

General procedure to construct a modified gravity theory:

Break one of Lovelock's hypotheses → Make sure the theory is not pathological → Take experimental constraints into account

Degenerate Higher-Order Scalar-Tensor theories

DHOST: add a scalar field and higher derivatives[1]

DHOST action

Ingredients: metric $g_{\mu\nu}$, scalar field ϕ of kinetic energy $X = \phi_\mu \phi^\mu$ with $\phi_\mu = \nabla_\mu \phi$.

$$S[g_{\mu\nu}, \phi] = \int \mathrm{d}^4x \sqrt{-g} \Big(F(X)R + P(X) + Q(X)\Box X + A_1(X)\phi_{\mu\nu}\phi^{\mu\nu} + A_2(X)(\Box\phi)^2 \\ + A_3(X)\phi^\mu\phi_{\mu\nu}\phi^\nu\Box\phi + A_4(X)\phi^\mu\phi_{\mu\nu}\phi^{\nu\rho}\phi_\rho + A_5(X)(\phi^\mu\phi_{\mu\nu}\phi_\nu)^2 \Big)$$

Degeneracy and stability: A_2, A_4 and A_5 are not free functions

$\Rightarrow$ *Most general* scalar-tensor theory

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[1] *Langlois, D. and Noui, K. arXiv: 1510.06930.*

Horndeski theory

Simplify the theory:

DHOST theory

- Higher derivatives
- 5 free functions

$$\xleftrightarrow{\tilde{g}_{\mu\nu} = A(X)g_{\mu\nu} + B(X)\phi_\mu\phi_\nu}$$

Horndeski theory

- Second-order derivatives only
- 3 free functions

$$S[g_{\mu\nu}, \phi] = \int \mathrm{d}^4x \left(F(X)R + P(X) + Q(X)\Box X + 2F'(X)\left(\phi_{\mu\nu}\phi^{\mu\nu} - (\Box\phi)^2\right)\right)$$

⇒ In the following, consider Horndeski

In vaccuum both theories are equivalent but the solutions may differ[2].

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[2] *Achour, J. B., Langlois, D., and Noui, K. arXiv: 1602.08398.*

Tests of modified gravity

Where to look for traces of modified gravity?

Black holes

- New solutions
- Different dynamics

Large scale structures

- Different growth rate
- Screenings

Cosmology

- Primordial GWs
- CMB

smaller → larger

- Each theory is tuned for a specific energy scale
- We focus on modifications of gravity in the black hole regime

Quasinormal modes and the ringdown

Ringdown of a merger: excited BH emits GW at precise frequencies, the **quasinormal modes**

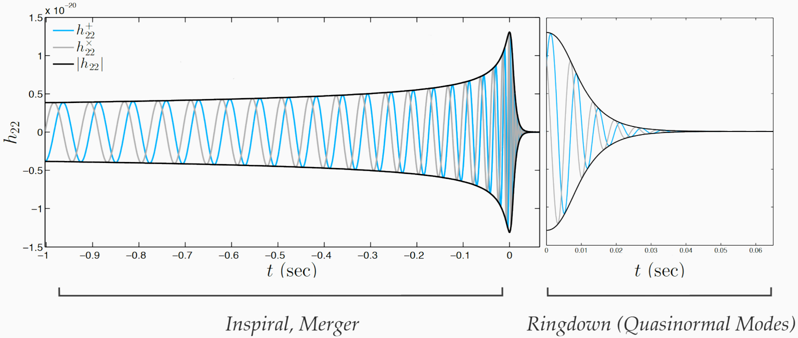

Figure 1: Ringdown phase of a binary black hole merger (L. London 2017)

Measuring quasinormal modes

- Discrete set (similar to plucked string)
- Complex frequencies: energy loss due to emission towards infinity
- Depend a lot of the theory → very good test

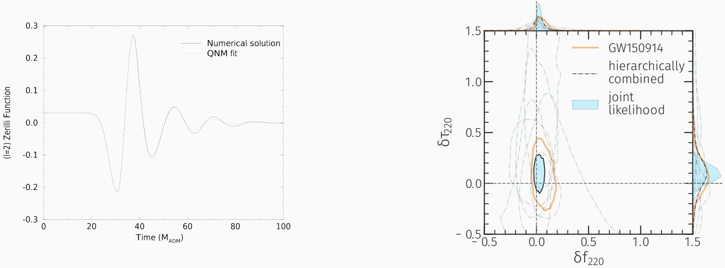

Figure 2: Principle of ringdown fit[3] and application to GW150914[4].

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[3] *Kokkotas, K. D. and Schmidt, B. G. 1999.*

[4] *Ghosh, A., Brito, R., and Buonanno, A. arXiv: 2104.01906.*

New black holes in DHOST: stealth solution

Metric sector: mimic GR

$$\mathrm{d}s^2 = -(1-\mu/r)\,\mathrm{d}t^2 + (1-\mu/r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

Scalar sector

$$\phi = qt + \psi(r)$$

$$X = -q^2 \;\Rightarrow\; \psi'(r) = q\frac{\sqrt{r\mu}}{r-\mu}$$

- Metric sector: similar to Schwarzschild ⇒ existing background tests still valid
- Scalar sector: time-dependant field and constant kinetic term
- Parametrization on F, P and Q for existence:

$$F(X) = 1\,,\quad F'(X) = \alpha\,,\quad F''(X) = \beta$$

$$P(X) = 0\,,\quad P'(X) = 0\,,\quad P''(X) = \gamma$$

$$Q(X) = 0\,,\quad Q'(X) = 0\,,\quad Q''(X) = \delta$$

New black holes in DHOST: BCL solution[5]

Parameters of Horndeski:

$$F(X) = f_0 + f_1\sqrt{X} \quad P(X) = -p_1 X\,, \quad Q(X) = 0$$

Metric sector: RN with imaginary charge

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \frac{1}{A(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

$$A(r) = 1 - \frac{r_m}{r} - \xi\frac{r_m^2}{r^2}\,, \quad \xi = \frac{f_1^2}{2f_0 p_1 r_m^2}$$

Scalar sector

$$\phi = \psi(r)\,, \quad \psi'(r) = \pm\frac{f_1}{p_1 r^2\sqrt{A(r)}}$$

$$X(r) = \frac{f_1^2}{p_1^2 r^4}$$

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[5] *Babichev, E., Charmousis, C., and Lehébel, A. arXiv: 1702.01938.*

Quasinormal modes in GR

Separating the degrees of freedom

1. Start with the Einstein-Hilbert action

$$S[g_{\mu\nu}] = \int \mathrm{d}^4x \, \sqrt{-g} \, R$$

2. Static spherically symmetric background

$$\bar{g}_{\mu\nu} = \mathrm{diag}(-A(r), 1/A(r), r^2, r^2 \sin^2\theta) \,, \quad A(r) = 1 - r_s/r$$

3. Perturb the metric: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and linearise Einstein's equations
   ⇒ obtain 10 equations
4. Decompose the components of $h_{\mu\nu}$ over spherical harmonics
5. Separate by parity: polar (even) and axial (odd) modes
6. Gauge fixing via $h_{\mu\nu} \longrightarrow h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$:
   - Polar modes: 7 equations for K, H_0, H_1, H_2
   - Axial modes: 3 equations for h_0, h_1
7. Fourier transform: $f(t,r) = \exp(-i\omega t) f(r)$

Reducing the number of equations

Two systems with more equations than variables → overconstrained?

Axial modes

- 2 first-order equations
- 1 second-order equation

Polar modes

- 4 first-order equations
- 2 second-order equations
- 1 algebraic equation

Interestingly, each system is equivalent to a 2-dimensional system of the form[6]

$$\frac{\mathrm{d}X}{\mathrm{d}r} = M(r)X$$

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[6] *Regge, T. and Wheeler, J. A. 1957; Zerilli, F. J. 1970.*

Final system of equations

Axial modes

$$X_{\text{axial}} = {}^{t}\begin{pmatrix} h_0 & h_1/\omega \end{pmatrix}$$

$$M_{\text{axial}} = \begin{pmatrix} \frac{2}{r} & 2i\lambda\frac{r-r_s}{r^3} - i\omega^2 \\ -\frac{r^2}{(r-r_s)^2} & -\frac{r_s}{r(r-r_s)} \end{pmatrix}$$

(set $2(\lambda + 1) = \ell(\ell + 1)$)

Polar modes

$$X_{\text{polar}} = {}^{t}\begin{pmatrix} K & H_1/\omega \end{pmatrix}$$

$$M_{\text{polar}} = \frac{1}{3r_s + 2\lambda r}\begin{pmatrix} \frac{a_{11}(r)+b_{11}(r)\omega^2}{r(r-r_s)} & \frac{a_{12}(r)+b_{12}(r)\omega^2}{r^2} \\ \frac{a_{21}(r)+b_{21}(r)\omega^2}{2(r-r_s)^2} & \frac{a_{22}(r)+b_{22}(r)\omega^2}{r(r-r_s)} \end{pmatrix}$$

⇒ goal to achieve: simplify these complicated differential systems

Effect of a change of variables

Changing the functions in X is not a change of basis for M!

Change of variables

$$\frac{\mathrm{d}X}{\mathrm{d}r} = M(r)X\,, \quad X = P(r)\tilde{X}$$

$$\frac{\mathrm{d}\tilde{X}}{\mathrm{d}r} = \tilde{M}(r)\tilde{X}\,, \quad \tilde{M} = P^{-1}MP - P^{-1}\frac{\mathrm{d}P}{\mathrm{d}r}$$

Main idea: find a change of variables that will put the equation into a better form

Usual change of variables: propagation equation

Canonical form for $\tilde{M}$:

$$\tilde{M} = \begin{pmatrix} 0 & 1 \\ V(r) - \frac{\omega^2}{c^2} & 0 \end{pmatrix}$$

Physical interpretation

$$\begin{cases} \tilde{X}_0' = \tilde{X}_1 \,, \\ \tilde{X}_1' = (V(r) - \omega^2/c^2)\tilde{X}_0 \end{cases} \Rightarrow \quad \frac{\mathrm{d}^2\tilde{X}_0}{\mathrm{d}r_*^2} + \left(\frac{\omega^2}{c^2} - V(r)\right)\tilde{X}_0 = 0 \,, \quad \frac{\mathrm{d}r}{\mathrm{d}r_*} = A(r)$$

Schrödinger equation with potential V

r_*: “tortoise coordinate”, $r = r_s \longrightarrow r_* = -\infty$ and $r = +\infty \longrightarrow r_* = +\infty$

Interpretation of the equations

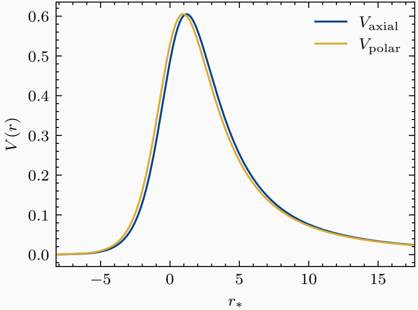
Axial case:

$$P_{\text{axial}} = \begin{pmatrix} 1 - r_s/r & r \\ ir^2/(r - r_s) & 0 \end{pmatrix}, \quad c = 1$$

At the horizon and infinity:

$$X_0(t, r) \propto e^{-i\omega(t \pm r_*)}$$

$\Rightarrow$ Propagation of waves



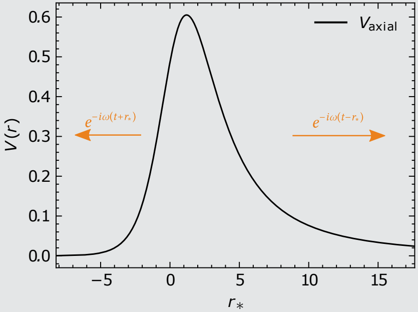
Physical interpretation

- Free propagation at $c = 1$ near the horizon and infinity
- Scattering by the potential V
- At infinity: recover gravitational waves in Minkowski

Computation of the modes

Quasinormal modes

- Waves *ingoing* at the horizon: $e^{-i\omega(t+r_*)}$
- Waves *outgoing* at infinity: $e^{-i\omega(t-r_*)}$



- 2 boundary conditions + 2nd order system ⟶ conditions on ω
- “Eigenvalue problem”: find values of parameter such that solutions exist
- Very different from plucked string: wave propagation at each boundary!

Quasinormal modes in modified gravity

Summary: computation of QNMs in GR

① Linearized Einstein's eqs → ② Gauge choice → ③ Background → ④ First-order system → ⑤ Schrödinger equations → ⑥ Numerical computation

Major difficulties:

① Many different theories

③ Many different backgrounds

⑤ Highly non-trivial change of variables!

New challenges in modified gravity

New theories

Scalar-tensor: new scalar degree of freedom that couples to the polar mode

New backgrounds

Stealth solution: time-dependant scalar field, *lose staticity*

Schrödinger equation

In general, very hard to solve:

$$\begin{pmatrix} 0 & 1 \\ V(r) - \frac{\omega^2}{c^2} & 0 \end{pmatrix} = P^{-1}MP - P^{-1}\frac{\mathrm{d}P}{\mathrm{d}r}$$

⇒ need for a systematic approach that does not rely on specific simplifications

Example: polar BCL perturbations

$$A(r) = 1 - \frac{r_m}{r} - \xi \frac{r_m^2}{r^2}\,, \quad \xi = \frac{f_1^2}{2f_0 p_1 r_m^2}\,, \quad \phi'(r) = \pm \frac{f_1}{p_1 r^2 \sqrt{A(r)}}$$

$$M(r) = \begin{pmatrix} -\frac{1}{r} + \frac{U}{2r^3A} & \frac{U}{r^4} & \frac{i(1+\lambda)}{\omega r^2} & \frac{V}{r^3} \\ \frac{\omega^2 r^2}{A^2} - \frac{\lambda}{A} - \frac{r_m}{2rA} + \frac{r_m^2 S}{4r^4A^2} & -\frac{2}{r} - \frac{UV}{2r^5A} & -\frac{i\omega r}{A} + \frac{i(1+\lambda)U}{2r^3\omega A} & -\frac{\lambda}{A} - \frac{3U}{2r^3A} - \frac{\xi^2 r_m^4}{2r^4A} \\ -\frac{i\omega V}{r^2A} & \frac{2i\omega}{r} - \frac{i\omega U}{r^3A} & -\frac{U}{r^3A} & -\frac{i\omega V}{r^2A} \\ -\frac{1}{r} + \frac{U}{2r^3A} & \frac{2}{r^2} - \frac{U^2}{2r^6A} & -\frac{i\omega}{A} + \frac{i(1+\lambda)}{\omega r^2} & \frac{1}{r} - \frac{U}{2r^3A} - \frac{UV}{2r^5A} \end{pmatrix}$$

$$U(r) = r_m(r + \xi r_m)\,, \qquad V(r) = r^2 + \xi r_m^2\,, \qquad S(r) = r^2 + 2\xi r(2r_m - r) + 2\xi^2 r_m^2\,.$$

First-order system and boundary conditions

Main idea

Skip step ⑤: get boundary conditions and perform numerical computations **from the first-order system**

Steps to perform

- Find asymptotic behaviour at the horizon and infinity
- Identify ingoing and outgoing modes
- Use a numerical method that does not require Schrödinger equations

Naively:

$$\frac{\mathrm{d}X}{\mathrm{d}r} = MX\,, \quad M(r) = M_p r^p + \mathcal{O}(r^{p-1}) \quad \Rightarrow \quad X \sim \exp\left(M_p \frac{r^{p+1}}{p+1}\right) X_c$$

Failure of naive approach

Axial Schwarzschild

$$M(r) = \begin{pmatrix} 0 & -i\omega^2 \\ -i & 0 \end{pmatrix} + \mathcal{O}\left(\frac{1}{r}\right)$$

$$X \sim \begin{pmatrix} e^{i\omega r} & 0 \\ 0 & e^{-i\omega r} \end{pmatrix} X_c$$

Polar Schwarzschild

$$M(r) = \begin{pmatrix} 0 & 0 \\ \frac{i\omega^2}{\lambda} & 0 \end{pmatrix} r^2 + \mathcal{O}(r)$$

$$X \sim \begin{pmatrix} 1 & 0 \\ \frac{i\omega^2}{\lambda}\frac{r^3}{3} & 1 \end{pmatrix} X_c$$

Problem

- We do not recover the $e^{\pm i\omega r_*}$ behaviour all the time!
- This is because of a *nilpotent* leading order in the polar case
- A more advanced mathematical treatment is needed

Mathematical results

Solution: behaviour studied in[7], mathematical algorithm from[8]

Mathematical algorithm

Main idea: *diagonalize M order by order* using

$$\tilde{M} = P^{-1}MP - P^{-1}\frac{\mathrm{d}P}{\mathrm{d}r}$$

⇒ important result: diagonalization is always possible!

General result:

$$M = M_p r^p + M_{p-1} r^{p-1} + \dots$$

$$\tilde{M} = D_q r^q + D_{q-1} r^{q-1} + \dots$$

$$X \sim e^{D(r)} r^{D_{-1}} F(r) X_c$$

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[7] *Wasow, W. 1965.*

[8] *Balser, W. 1999.*

Example for the BCL solution: polar perturbations

Horizon

$$\tilde{M} \sim \begin{pmatrix} -i\omega/c_0 & & & \\ & i\omega/c_0 & & \\ & & 1/2 & 1 \\ & & & 1/2 \end{pmatrix}$$

Infinity

$$\tilde{M} \sim \begin{pmatrix} -i\omega & & & \\ & i\omega & & \\ & & -\sqrt{2}\omega & \\ & & & \sqrt{2}\omega \end{pmatrix}$$

$$\underbrace{\hphantom{-i\omega \quad i\omega}}_{\text{Gravitational part}} \quad \underbrace{\hphantom{-\sqrt{2}\omega \quad \sqrt{2}\omega}}_{\text{Scalar part}}$$

What can we deduce from this?

- We decoupled both modes but only *locally*
- The gravitational mode propagates at $c = 1$ at infinity and c_0 at the horizon
- Always one ingoing and one outgoing gravitational mode
- The scalar mode does not propagate

“Recipe” for the computation of quasinormal modes

① Linearized Einstein’s eqs → ② Gauge choice → ③ Background → ④ First-order system → ⑤ Asymptotical behaviour → ⑥ Numerical computation

- Generic algorithm that should work for any modified gravity theory
- Go around the technical difficulties of steps ① and ③
- Caveat: we do not get the full decoupled equations for the modes ⇒ impossible to get a potential
- Asymptotical behaviour is enough to obtain boundary conditions for numerical resolution

Numerical method

Decomposition onto Chebyshev polynomials T_n: $f = \sum_{i=0}^{N} f_i T_i$

ODE

$$X = {}^t\begin{pmatrix} X_0 & \dots & X_n \end{pmatrix}$$

$$\boxed{\frac{\mathrm{d}X}{\mathrm{d}r} = M(r,\omega)X}$$

+ boundary conditions

Numerical system

$$X = {}^t\begin{pmatrix} X_{0i} & \dots & X_{ni} \end{pmatrix}$$

$$\boxed{D_{ij}X_j = M_{ij}(\omega)X_j}$$

+ boundary conditions

- Linear algebra problem: generalized eigenvalue problem
- Procedure: find ω for $N = N_0$, then $N = N_1 > N_0$, keep the common values

BCL axial modes

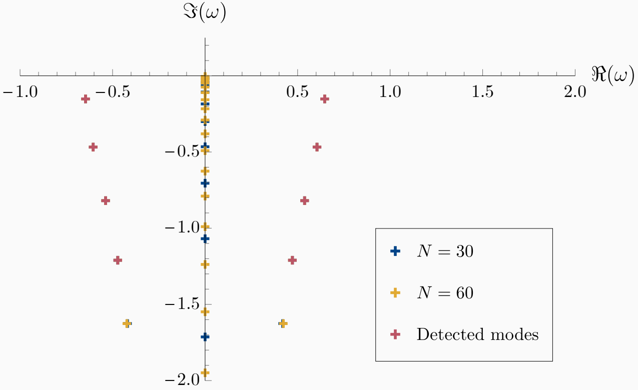

Figure 3: Axial QNMs found for the BCL solution with $\xi = 0.5$, $r_m = 1$, $\lambda = 2$.

BCL polar modes

- Impose gravitational mode b.c. at horizon and infinity
- Obtain modes even though the full system is not decoupled!

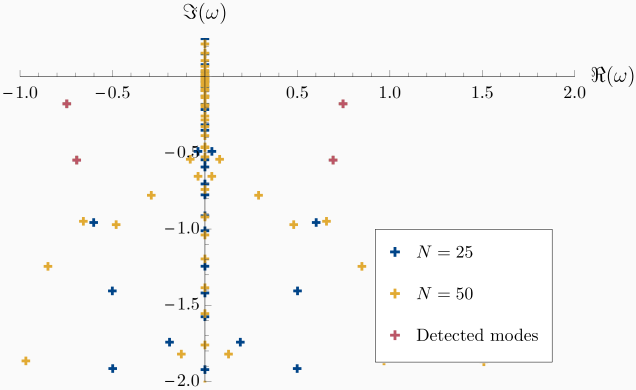

Figure 4: Polar gravitational QNMs found for the BCL solution with $\xi = 10^{-4}$, $r_m = 1, \lambda = 2$.

Isospectrality

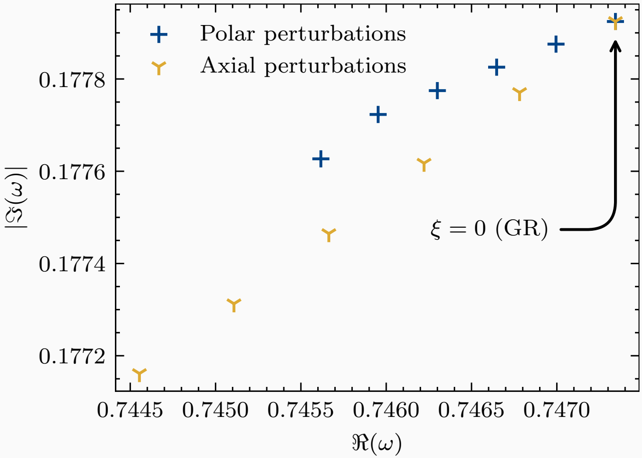

Figure 5: Tracking of the fundamental mode for axial and polar gravitational modes as ξ varies.

Conclusion

- Computing quasinormal modes can be very difficult in modified theories of gravity
- We propose a new technique: use the **first-order system** instead of looking for Schrödinger-like equations
- A mathematical algorithm enables us to decouple the modes asymptotically, which allows us to find their physical behaviour and obtain boundary conditions
- We can use these boundary conditions to numerically compute the quasinormal modes frequencies
- The method is theory-agnostic: it can be applied to any theory of gravity and any background

Thank you for your attention!