A Boundary Perspective on Cosmological Correlators

Sadra Jazayeri

(Institut d'Astrophysique de Paris)

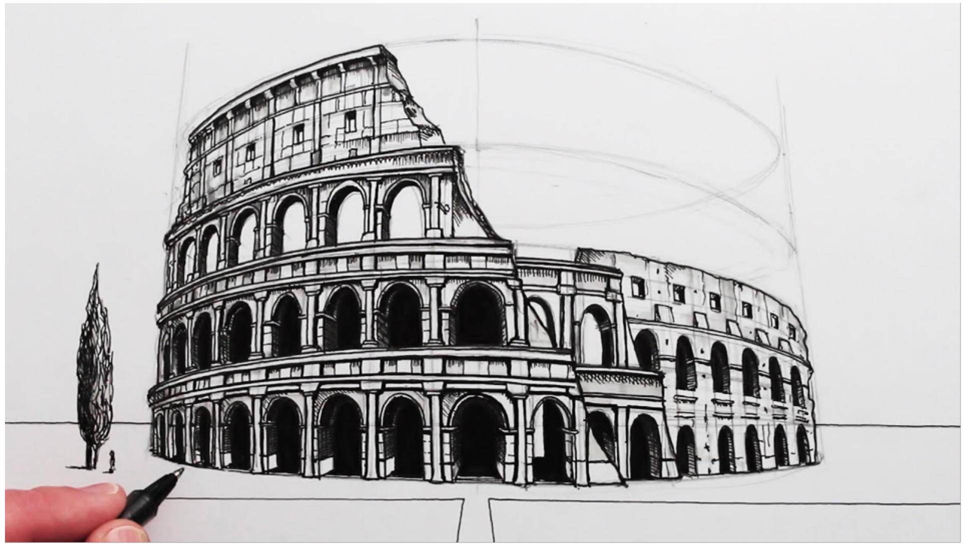
Based on

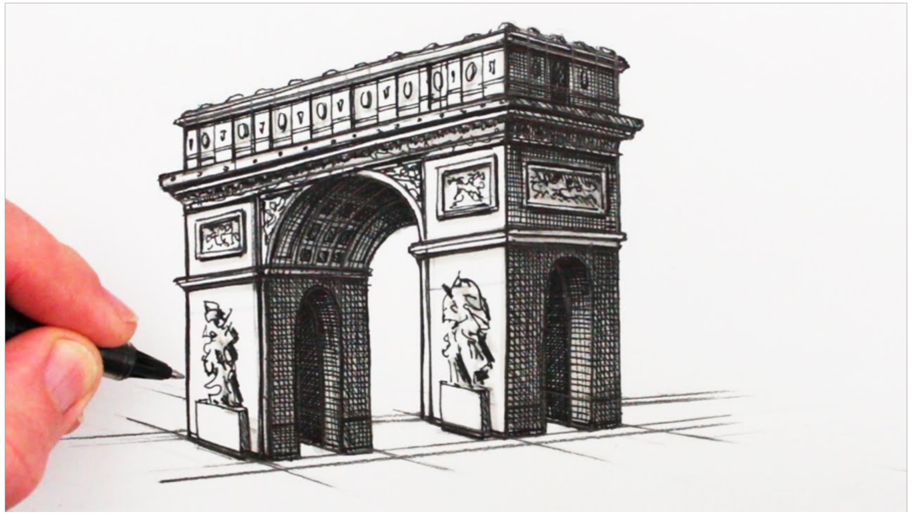
hep-th: 2103.05687
SJ, Enrico Pajer and David Stefanyszyn

hep-th: 2103.08640
Harry Goodhew, SJ, Mang Hei Gordon Lee, Enrico Pajer

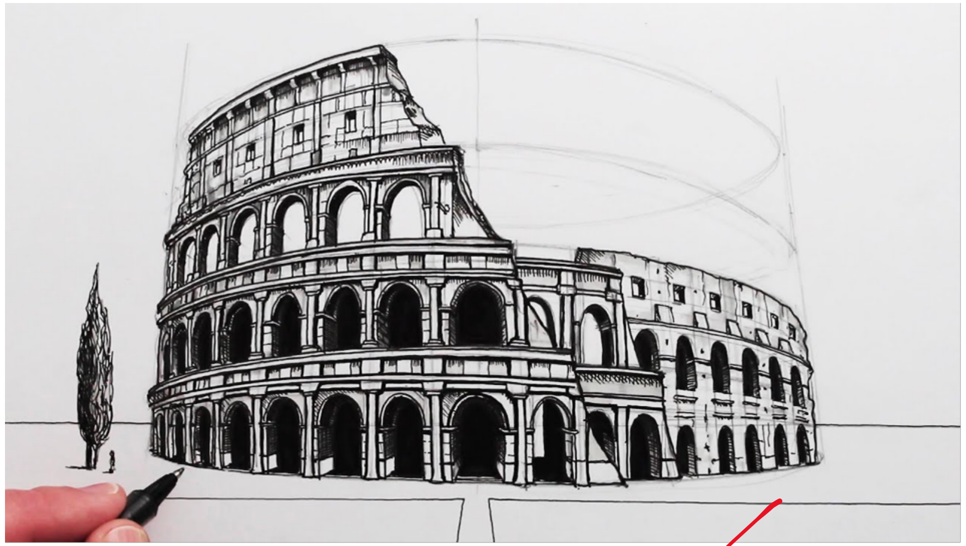
hep-th:2009.02898
Harry Goodhew, SJ and Enrico Pajer

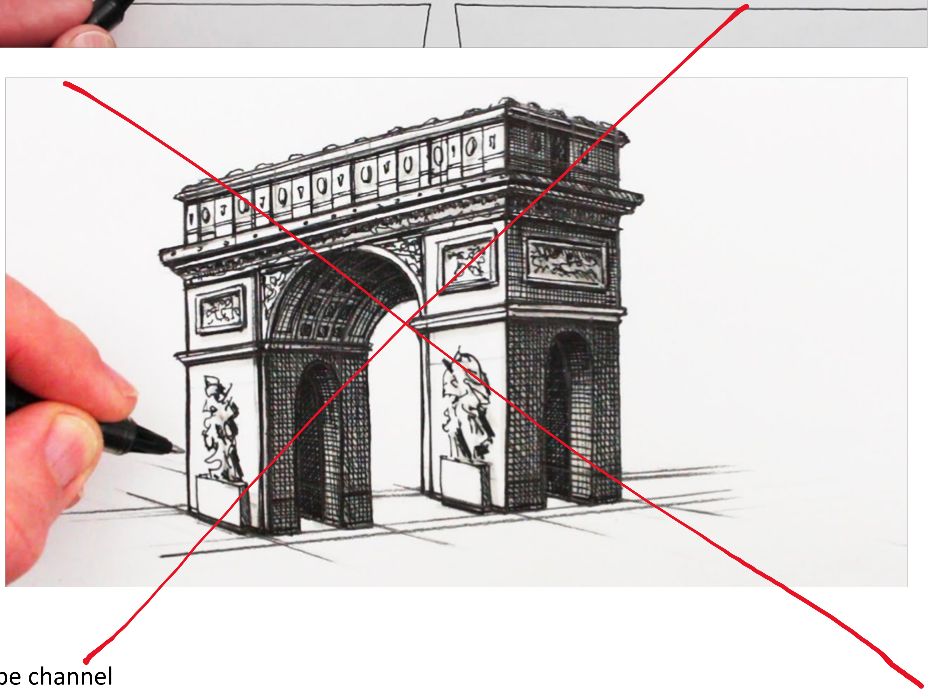
European Research Council
Established by the European Commission



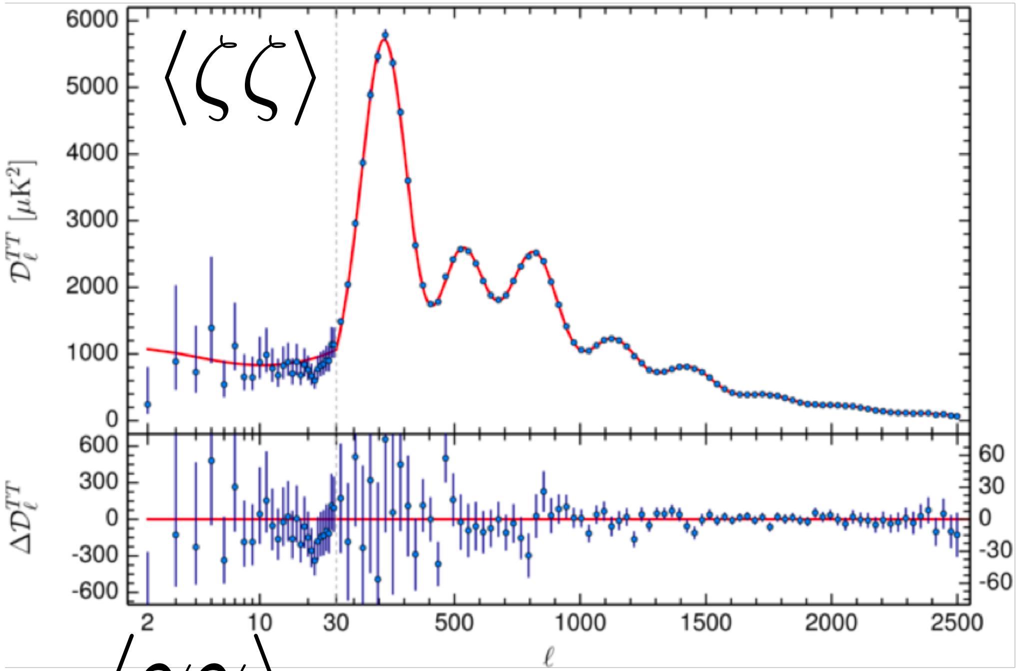


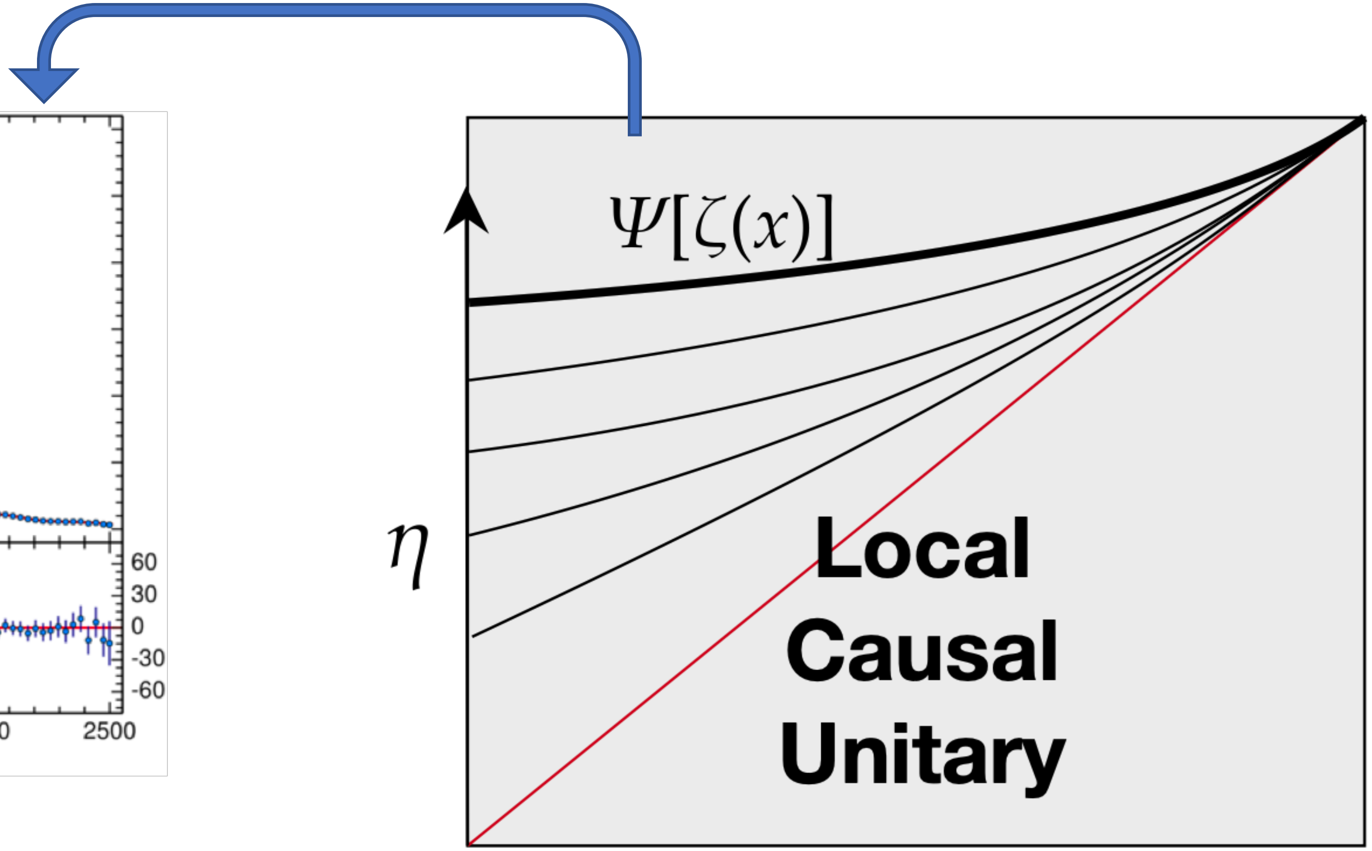
Credit:
Circle Line Art school YouTube channel
www.cometoparis.com
commons.wikimedia.org





Credit:
Circle Line Art school YouTube channel
www.cometoparis.com
commons.wikimedia.org





$$\langle\gamma\gamma\rangle$$

$$\langle\zeta\zeta\zeta\rangle$$

$$\langle\zeta\zeta\gamma\rangle$$

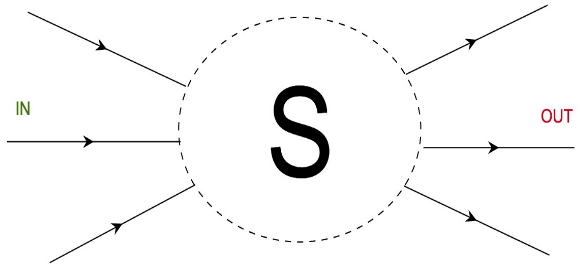
$$\langle\zeta\zeta\zeta\zeta....\rangle$$

Overview

- Shifting the Perspective: From The Bulk to the Boundary
- State-of-the-art Cosmological Bootstrap
- The Cosmological Optical Theorem (COT) and Cutting Rules
- The Manifest Locality Test (MLT)
- Bootstraping sample correlators in the EFT of single field inflation
- Concluding Remarks

Shifting The Perspective: From the Bulk to the Boundary

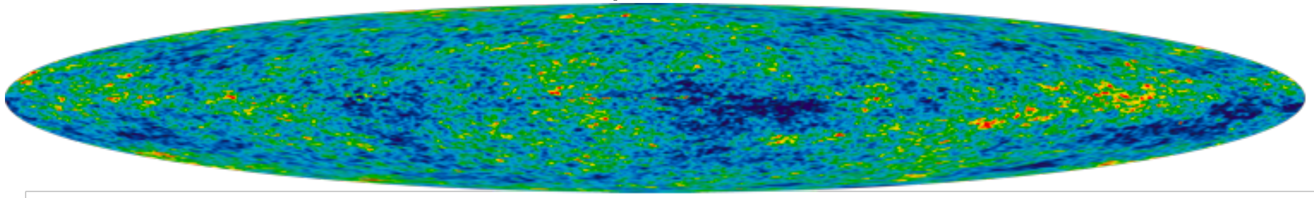
- In the past couple of decades, we have seen huge progress in the development of on-shell methods in the Scattering Amplitude Program. Cheung 2017, Benincasa 2013
- The idea is to reconstruct the actual observable, namely the S-matrix, from the principles of Lorentz invariance, Locality and Unitarity.
- These methods are especially useful in dealing with massless spinning particles, as the redundancies of gauge symmetries and diffeomorphisms can be abandoned altogether.

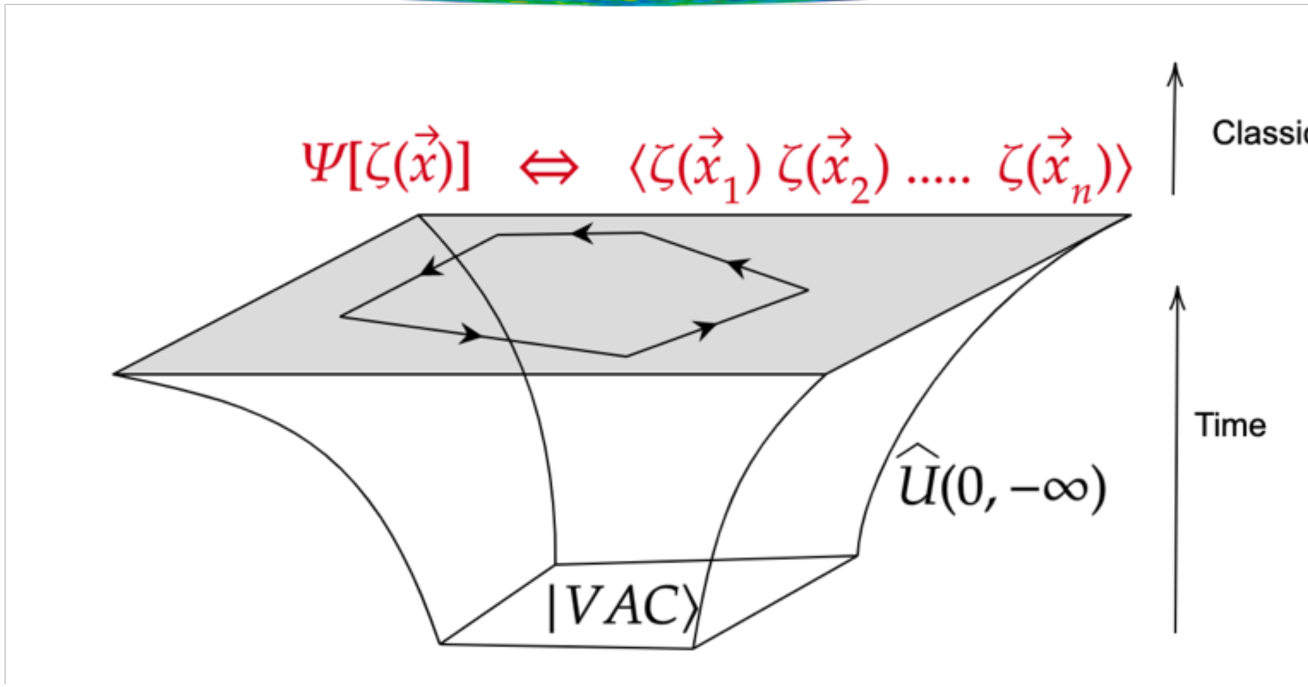


- some amazing upshots Benincasa-Cachazo 2007:

- *YM*=unique (low energy) interacting theory of massless spin-1 particles
- *GR*= unique (low energy) interacting theory of a massless spin-2 particles
- *SUGRA*= Unique (low every) interacting theory of a massless spin-3/2 and a massless spin-2 particle McGady-Rodina 2014

- The Chief observable in Cosmology: late time correlation functions (equivalently the *Wave Function of The Universe*)





$$ds^2 = a^2(\eta)(-d\eta^2 + d\mathbf{x}^2)$$

$$a(\eta) \sim -\frac{1}{\eta H}$$

$$\Psi[\phi(\mathbf{k})] = \exp\left(-\frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\psi_2(k)\phi(\mathbf{k})\phi(-\mathbf{k}) - \sum_{n>2}\frac{1}{n!}\prod_{m=1}^{n}\frac{d^3\mathbf{k}_1}{(2\pi)^3}\psi_n(\mathbf{k}_1,...,\mathbf{k}_n)\phi(\mathbf{k}_1)...\phi(\mathbf{k}_n)\right)$$

Leading Gaussian Piece *Perturbative Contributions from Interactions*

- Correlators (for physical momenta) can be obtained by integration over the fields space weighted by the probability $|\Psi[\phi(\mathbf{x})]|^2$

$$\langle \phi_{\mathbf{p}}(\eta_0)\phi_{-\mathbf{p}}(\eta_0)\rangle' = \frac{1}{2\,\mathrm{Re}\,\psi_2'(p)},$$

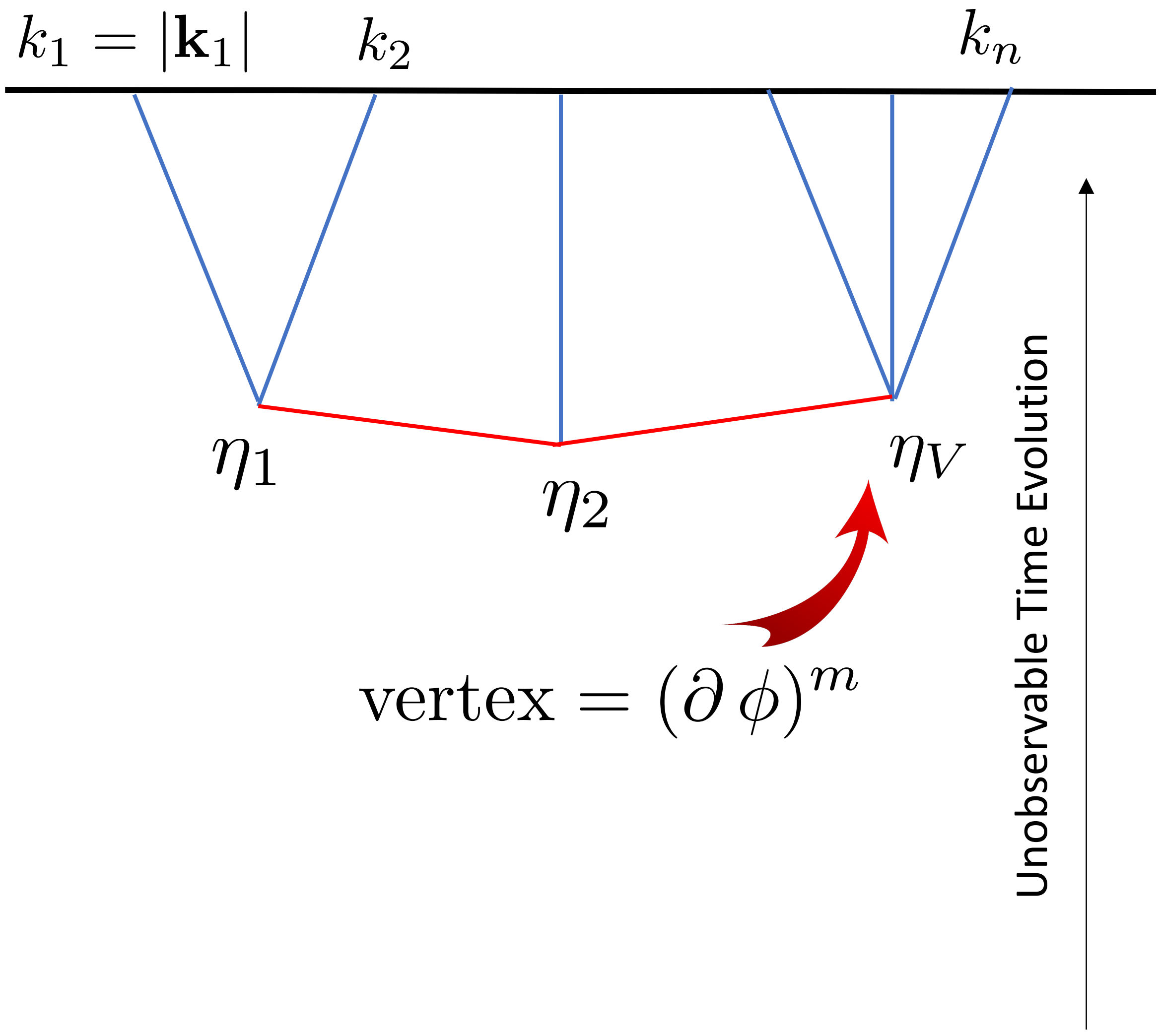
$$\left\langle \prod_{a=1}^{3} \phi_{\mathbf{p}_a}(\eta_0) \right\rangle' = -2\prod_{a=1}^{3}\frac{1}{2\,\mathrm{Re}\,\psi_2'(p_a)}\,\mathrm{Re}\left\{\psi_3'(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\right\},$$

$$\left\langle \prod_{a=1}^{4} \phi_{\mathbf{p}_a}(\eta_0) \right\rangle' = -2\prod_{a=1}^{4}\frac{1}{2\,\mathrm{Re}\,\psi_2'(p_a)}\left[\mathrm{Re}\left\{\psi_4'(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\right\} - \frac{\mathrm{Re}\left\{\psi_3'(\mathbf{p}_1,\mathbf{p}_2,-\mathbf{s})\right\}\mathrm{Re}\left\{\psi_3'(\mathbf{p}_3,\mathbf{p}_4,\mathbf{s})\right\}}{\mathrm{Re}\,\psi_2'(s)} - t - u\right]$$

Bulk Formalism Disadvantages:

- *Redundancies: Field Redef, Gauge Symmetries*
- Nested time integrals(Complicated even at tree level in contrast with flat space)

$$\psi_n(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n) \supset$$

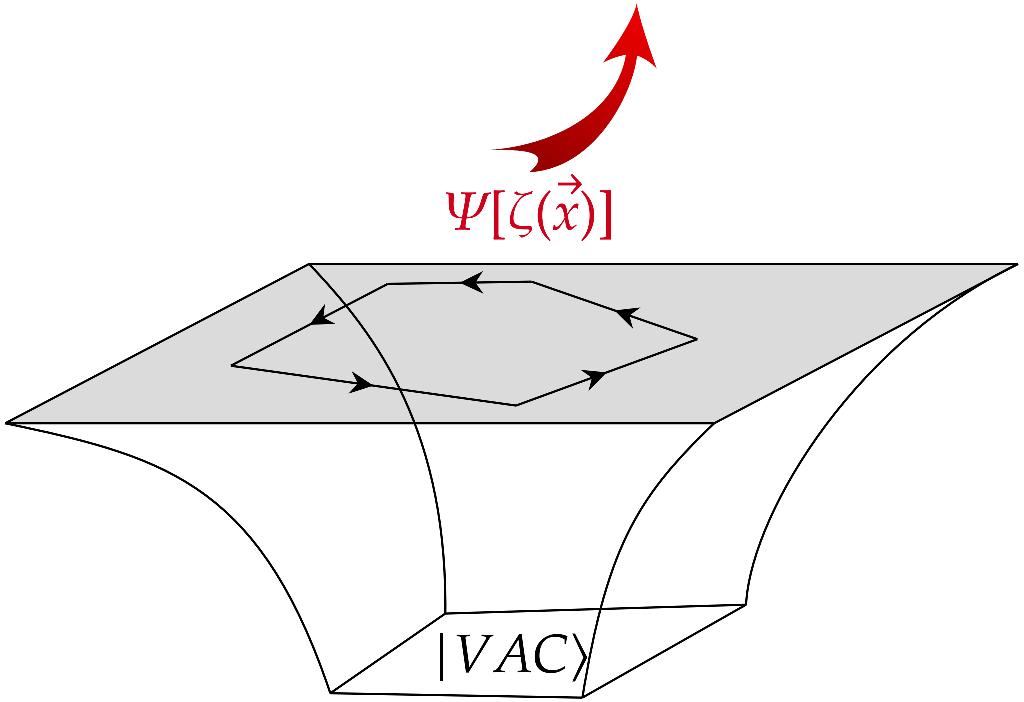


$$K(k,\eta) = \frac{\phi^+(k,\eta)}{\phi^+(k,\eta_0)}$$ Bulk-to-Boundary

$$G(\mathbf{k},\eta,\eta') =$$ Bulk-to-Bulk

$$i\left(\theta(\eta-\eta')\phi^+(\mathbf{k},\eta')\phi^-(\mathbf{k},\eta) + \theta(\eta'-\eta)\phi^+(\mathbf{k},\eta)\phi^-(\mathbf{k},\eta') - \frac{\phi^-(\mathbf{k},\eta_0)}{\phi^+(\mathbf{k},\eta_0)}\phi^+(\mathbf{k},\eta')\phi^+(\mathbf{k},\eta)\right).$$

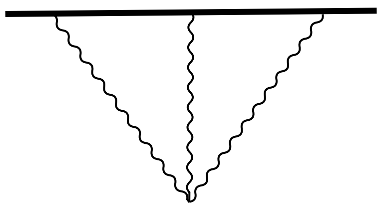
Boundary Principles?



Has that ever been useful?

Simplicity vs Complexity

non-perturbative derivation of tensor non-gaussianity from dS conformal symmetries Maldacena-Pimentel 2011



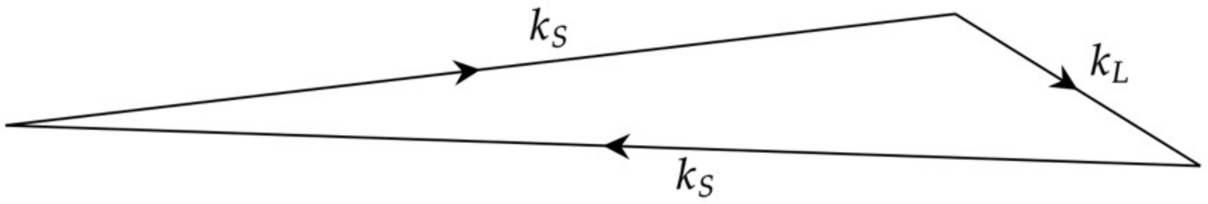
Boundary	Bulk
Conformal symmetry of dS on the boundary restricts the 3pt into only two possible shapes, $$R\,, W^3 \rightarrow \langle\gamma\gamma\gamma\rangle$$	Infinitely many operators contribute to the graviton cubic couplings $$R^2\,, R^2_{\mu\nu}\,, R\Box R, \ldots$$

non-perturbative derivation of soft theorems in single field inflation (Model Independent)

Maldacena 2002, Creminelli-Zaldarriaga 2004, Creminelli-Norena-Simonovic 2012, Hinterbichler-Hui-Khoury 2013

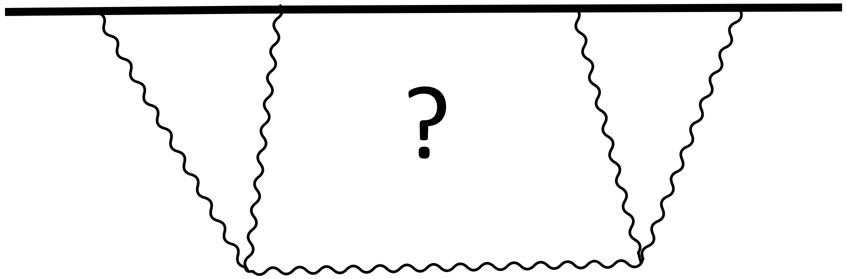
Boundary	Bulk
Symmetries generated by adiabatic modes non-linearly realized by ζ $\zeta \to \zeta(\mathbf{x}) + \lambda + \lambda \mathbf{x}.\nabla\zeta$	For each single field model one has to repeat Maldacena's computation for the Bispectra

$$\langle \zeta(\vec{k}_L)\,\zeta(\vec{k}_S)\,\zeta(\vec{k}_S)\rangle' = P(k_L)\frac{1}{k_S^3}\frac{\partial}{\partial k_S^3}\left(k_S^3\,P(k_S)\right)$$



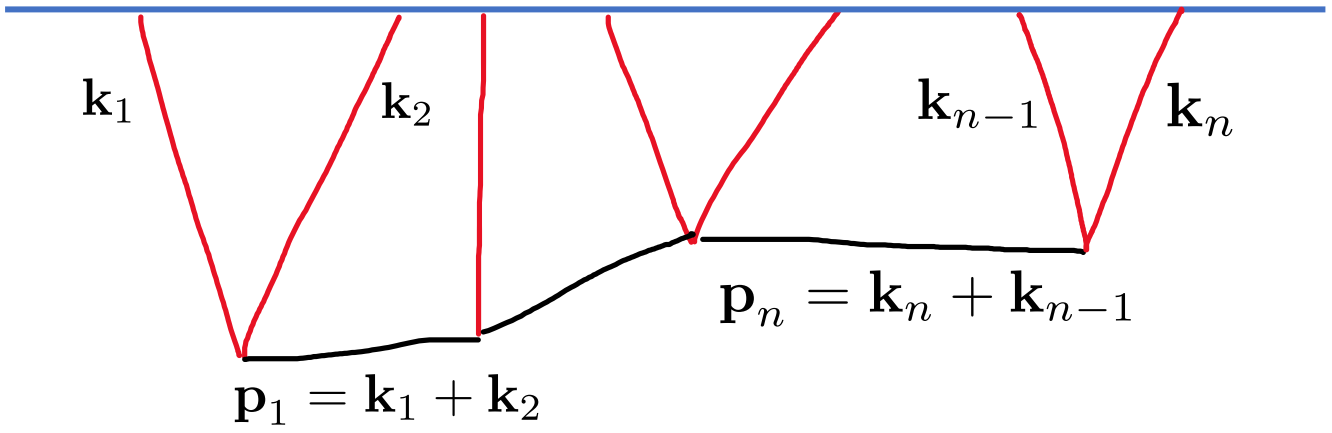
Tree level four-point function of gravitons

Boundary	Bulk
?	No Explicit computation to this date



The Cosmological Bootstrap

- The back bones of the bootstrap approach:
  - Analytical Properties of the Correlators (after analytical continuations)
  - Polology of the analytically continued *n*-point functions



$$\psi_n(|\mathbf{k}_1|, |\mathbf{k}_2|, \dots, |\mathbf{p}_1|, \dots, \mathbf{k}_a.\mathbf{k}_b, \dots)$$

$$\psi_n(k_1, k_2, \dots; p_1, p_2, \dots; \mathbf{k}_a.\mathbf{k}_b)$$

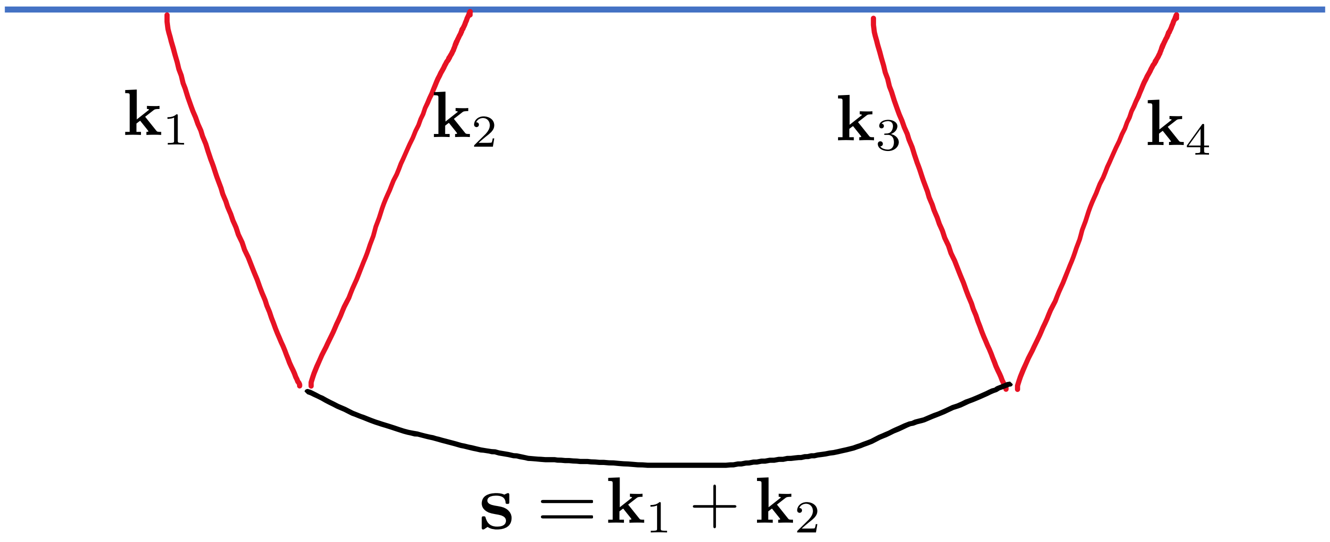
External and Internal Energies

$$\mathrm{Im}(k_a) < 0\,, \mathrm{Im}(p_a) < 0$$

Special cases of three and four point functions

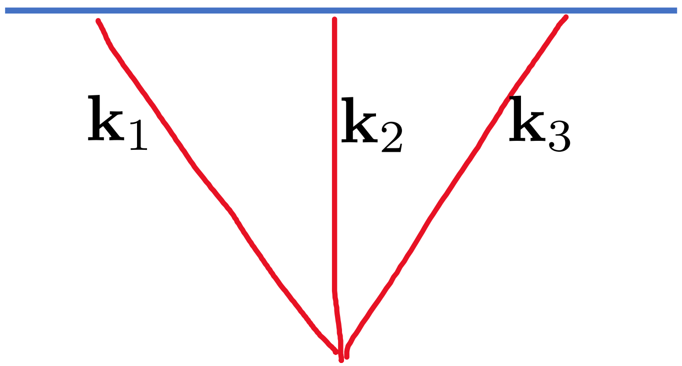
(*for the most part, I will focus on tree level diagrams of (massless or cc) scalars on a fixed dS background with Bunch-Davis vacuum*)

Exchange Diagram



$$\psi_4(k_1, k_2, k_3, k_4, s)$$
$$+t \text{ and } u \text{ channels}$$

Contact Diagram

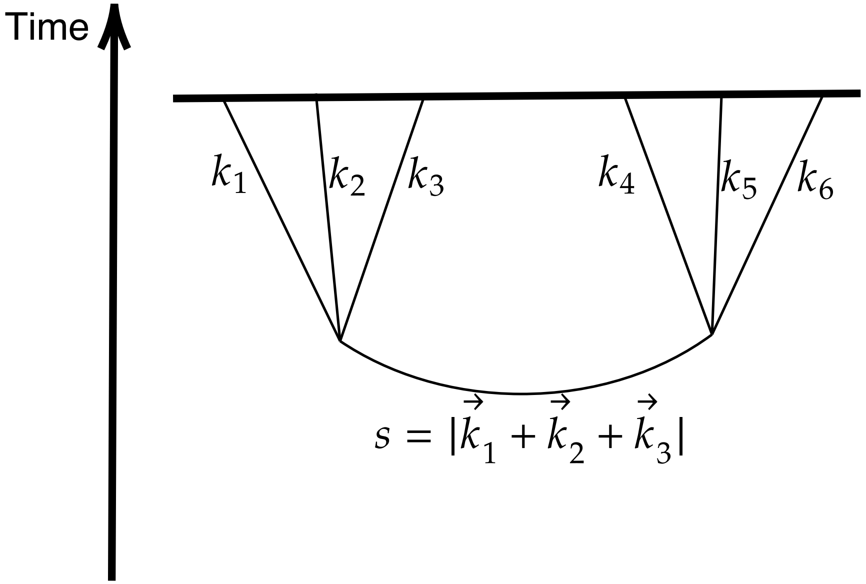


$$\psi_3(k_1, k_2, k_3)$$

- Example: phi4 theory of a conformally coupled field in dS

$$S = \int \sqrt{-g}\left(-\frac{1}{2}(\partial_\mu \varphi)^2 - H^2\varphi^2 - \lambda\,\varphi^4\right)$$

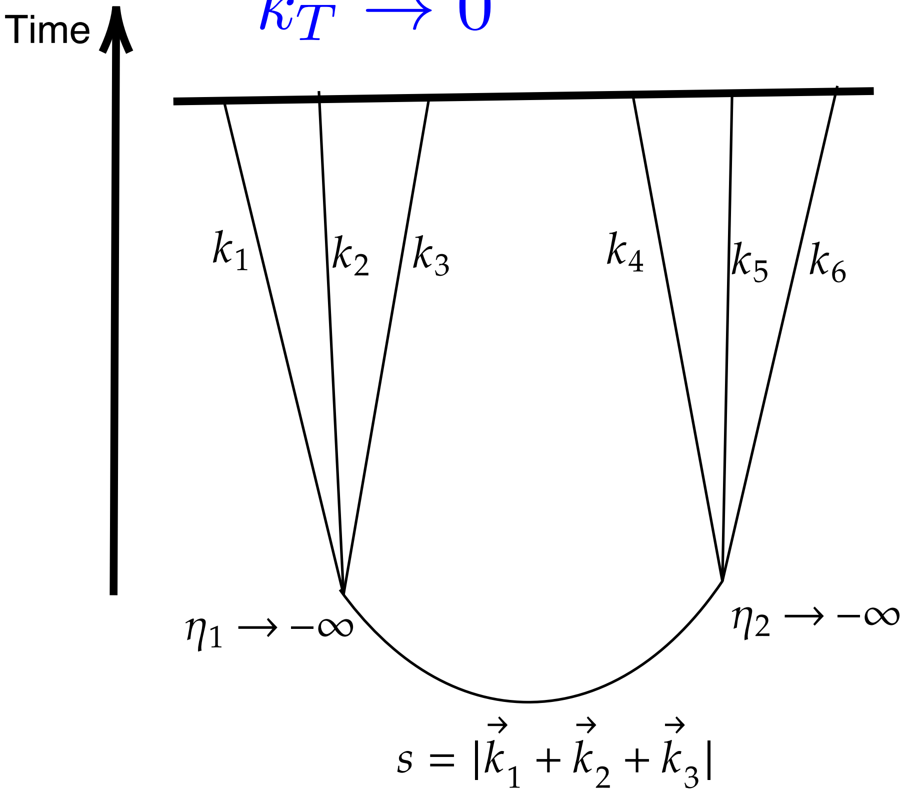
$$\psi_6 = \frac{g^2}{\eta_0^6\,(\sum_{a=1}^{6}\,k_a)(k_1+k_2+k_3+s)(k_4+k_5+k_6+s)}$$

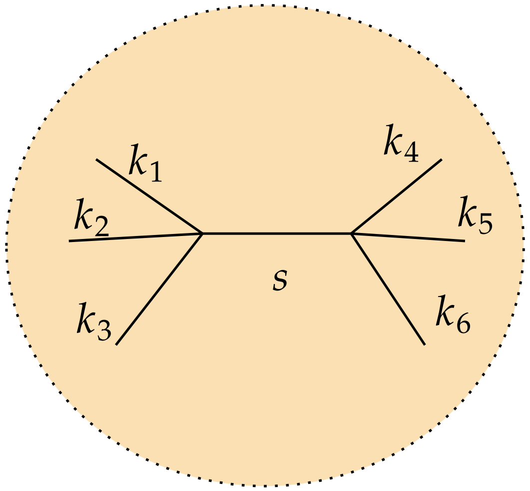


- Example: phi4 theory of a conformally coupled field in dS

$$S = \int \sqrt{-g}\left(-\frac{1}{2}(\partial_\mu \varphi)^2 - H^2\varphi^2 - \lambda\,\varphi^4\right)$$

$$\psi_6 = \frac{g^2}{\eta_0^6\,\underbrace{(\sum_{a=1}^{6}\ k_a)}_{k_T \to 0}(k_1+k_2+k_3+s)(k_4+k_5+k_6+s)} \longrightarrow \quad A_6 = \frac{g^2}{(k_1+k_2+k_3)^2 - s^2}$$



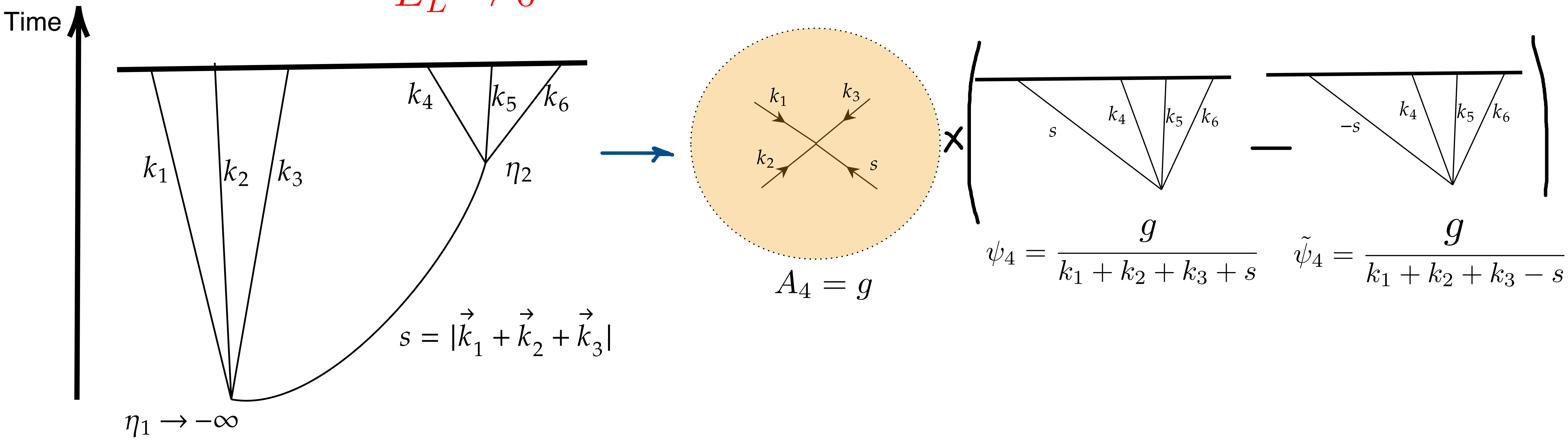


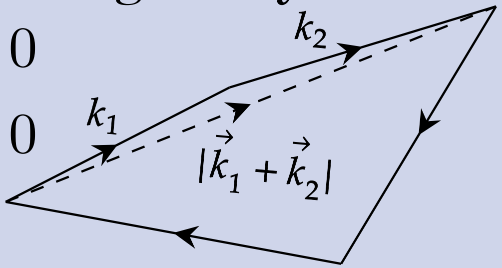
- Example: phi4 theory of a conformally coupled field in dS

$$S = \int \sqrt{-g}\left(-\frac{1}{2}(\partial_\mu \varphi)^2 - H^2\varphi^2 - \lambda\,\varphi^4\right)$$

$$\psi_6 = \frac{g^2}{\eta_0^6\,(\sum_{a=1}^{6}\ k_a)(k_1+k_2+k_3+s)(k_4+k_5+k_6+s)}$$

$E_L \to 0$



Type of the Pole/Singularity	Diagram Type	Behavior around the Pole
Total energy Pole $k_T = k_1 + k_2 + ... + k_n \to 0$	Contact Exchange (interaction inserted at infinite past)	$\lim_{k_T \to 0} \psi_n(\mathbf{k}_1, .., \mathbf{k}_n) = (k_1 ... k_n) \frac{A_n(1, ..., n)}{k_T^p}$ $p = 1 + \sum_\alpha (D_\alpha - 4)$ *Maldacena-Pimentel 2011, Raju 2012, Pajer 2020, Goodhew-SJ-Pajer 2020*
Partial Energy Poles $E_L = k_1 + k_2 + s \to 0$ $E_R = k_3 + k_4 + s \to 0$ (s-channel)	Exchange (one interaction inserted at infinite past)	$\lim_{E_{L,R} \to 0} \psi_4(k_1, .., k_4, s) =$ $\frac{1}{E_{L,R}^p} A_3(k_1, k_2, s) \times \tilde{\psi}_3(k_3, k_4, s)$ *Baumann-Duaso Pueyo,-Joyce-Lee-Pimentel 2020, Pajer 2020, Goodhew-SJ-Pajer 2020*
Collinear Singularity $k_1 + k_2 - s \to 0$ $k_3 + k_4 - s \to 0$ 	✗ Non-Bunch Davis Vacuum	✗

There are almost always subleading total and partial energy poles

$$\psi_{(\partial_\mu \phi)^4} = \frac{A_5(k_a)}{k_T^5} + \frac{A_4(k_a)}{k_T^4} + \ldots + A_0(k_a)\,.$$

We need more inputs!

de Sitter Symmetries SO(4,1) P_i, J_i, D, K_i

Arkani-Hamed, Maldacena 2014
Arkani-Hamed, Baumann, Lee, Pimentel 2018
Baumann, Duaso Puyeo, Joyce, Lee, Pimentel 2019
Baumann-Duaso Pueyo,-Joyce-Lee-Pimentel 2020

AdS to dS/Mellin Space

Sleight, Taronna 2021, 2020, 2019, 2018

$P_i, J_i, D, \cancel{K}_i$

Boostless Bootstrap

Pajer, 2020
SJ, Pajer, Stefanyszyn 2021

Other Boundary Inputs?

Cosmological Polytopes

Benincasa, L. McLeod, Vergu 2020
Benincasa 2019
Arkani-Hamed, Benincasa 2017, 2018

dS Correlators from Flat Space Correlators

Baumann, Chen, Joyce, Lee, Pimentel 2021

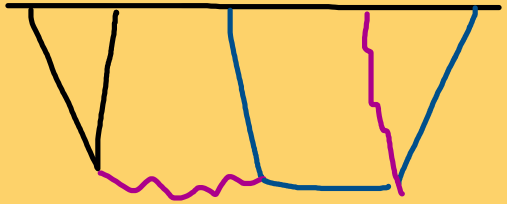
Recursive Bootstrap

Massless external Fields:
Rational Ansatz for contact terms

$$\psi_3(k_1,k_2,k_3) = \frac{Poly_{3+p}(k_1,k_2,k_3)}{k_T^p}$$

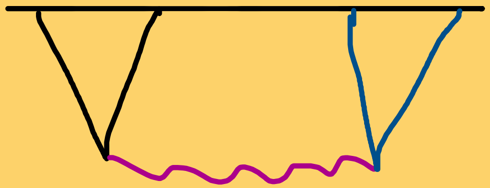
Manifest Locality

More complicated exchange Diagrams

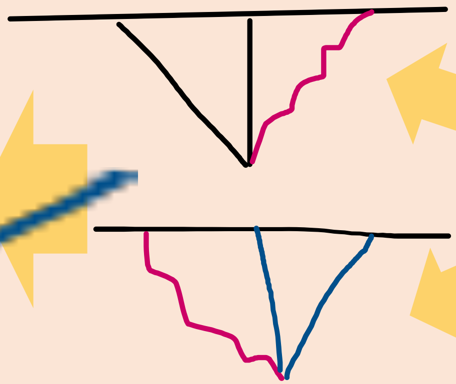


⋮

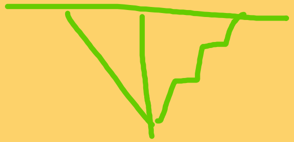
Single Exchange Diagrams Up to Contact Terms



Unitarity +Manifest Locality



All Contact Terms For Manifestly Local Theories



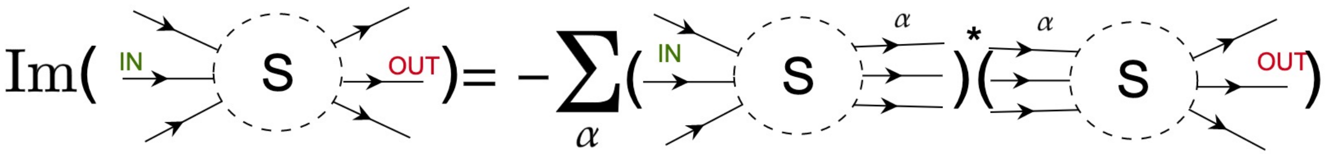
- Our Setup: probe scalar field in de Sitter space. The setup is similar to the one in the EFT of single field inflation in the decoupling limit Cheung et al 2008 (except that here we do not impose non-linearly realized boost symmetries while we assume exact scale invariance).

$$\int d\eta\, d^3\mathbf{x}\, a^2(\eta)\, \left[\frac{1}{2}\phi'^2 - \frac{c_s^2}{2}(\partial_i\phi)^2 + \frac{1}{3!}g_1\,\phi'^3 + \frac{1}{2}g_2\,\phi'(\partial_i\phi)^2 + \frac{1}{2}g_3\phi'(\partial^2\phi)^2 + \dots\right]$$

$$\phi(\eta, \mathbf{x}) \to \phi(\lambda\eta, \lambda\mathbf{x})$$

The Cosmological Optical Theorem (COT)

- In flat space, perturbative unitarity is encoded in the S-matrix optical theorem which formally



$$\mathrm{Im}\left(\frac{1}{p^2 - m^2 + i\epsilon}\right) = \delta(p^2 - m^2)\,.$$

- In Cosmology, a similar optical theorem follows from,
  - Reality of the couplings

$$g^* = g$$

  - The Hermitian Analyticity of the Bulk-to-Boundary Propagator

$$K^*_\sigma(z,\eta) = K_\sigma(-z^*,\eta)\,, \qquad \mathrm{Im}(z) < 0\,.$$

$$K_\phi(k,\eta) = \frac{\phi_+(k,\eta)}{\phi_+(k,\eta_0)} = (1 - ik\eta)\exp(+ik\eta)$$

  - The Factorization Property of the Bulk-to-Bulk Propagator

$$\mathrm{Im}\, G(s,\eta) = 2\, P(s,\eta_0)\, \mathrm{Im} K(s,\eta) \times \mathrm{Im} K(s,\eta')\,.$$

- For contact diagrams COT takes the following form,

$$\psi'_n(k_a, \hat{k}_a \cdot \hat{k}_b) + \left[\psi'_n(-k_a^*, \hat{k}_a \cdot \hat{k}_b)\right]^* = 0\,, \qquad k_a \in \mathbb{C}^{n-}\,.$$

For scale invariant correlators of massless fields this is trivially satisfied

(when there are IR-infinities e.g. with log branch cuts, it relates the coefficient of the logarithm to the imaginary part of the wfc)

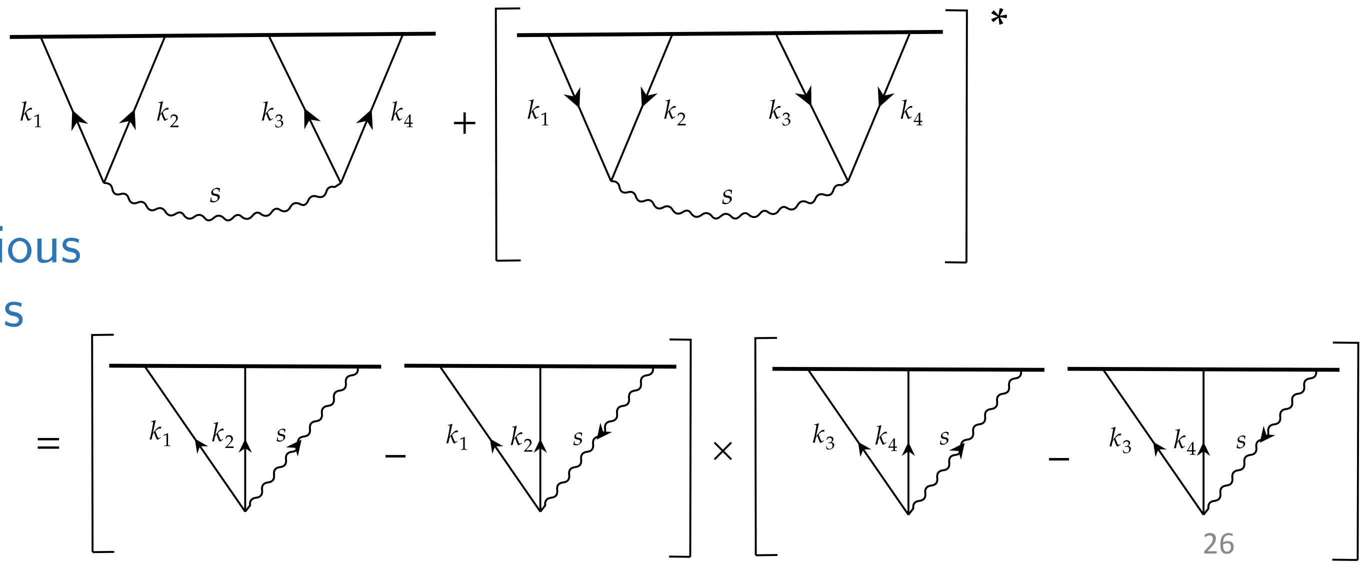
$$\psi_n(k_a) \sim i\,g \int d\eta\, F(\mathbf{k}_a.\mathbf{k}_b, \eta)\, K(k_1, \eta) \ldots K(k_n, \eta)\,.$$

$$\psi_n^*(-k_a^*) \sim -i\,g \int d\eta\, F(\mathbf{k}_a.\mathbf{k}_b, \eta)\, K^*(-k_1^*, \eta) \ldots K(-k_n^*, \eta)\,.$$

- For single exchange diagrams COT appears as cutting rules for wfc's,

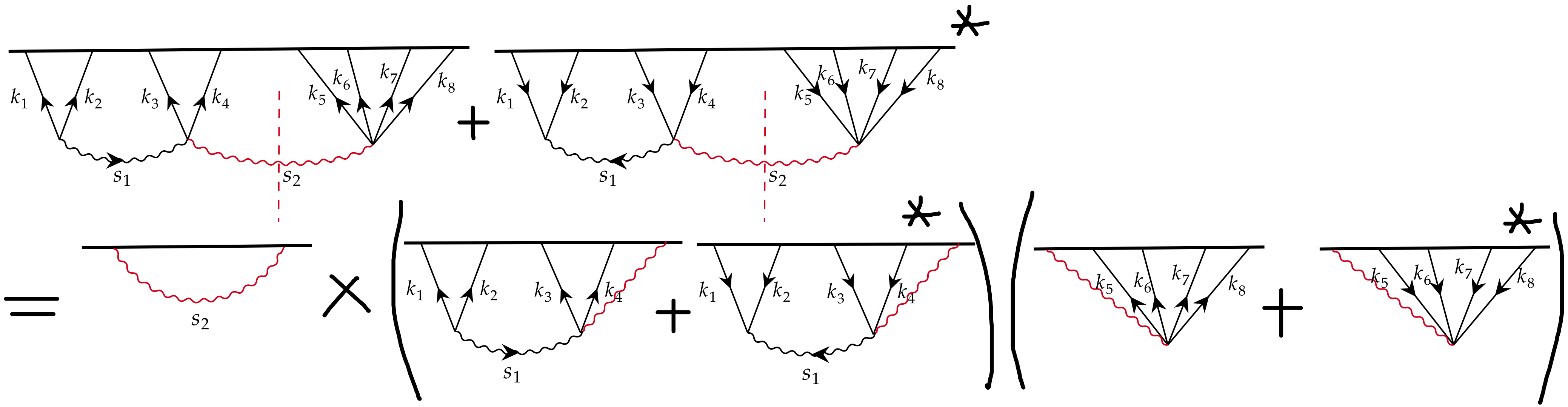
$$\psi_4(k_1,k_2,k_3,k_4,s)+\psi_4^*(-k_1^*,-k_2^*,-k_3^*,-k_4^*,s)=$$
$$P(s)\left(\psi_3(k_1,k_2,s)+\psi_3(-k_1^*,-k_2^*,s)\right)\left(\psi_3(k_3,k_4,s)+\psi_3(-k_3^*,-k_4^*,s)\right)$$

$$\psi_4(k_a,s)=g^2\int d\eta d\eta'\,K(k_1,\eta)K(k_2,\eta)G(s,\eta,\eta')K(k_3,\eta')K(k_4,\eta')$$



See H Goodhew-SJ-G Lee-E Pajer 2021 for various extensions to (i) other accelerating backgrounds
(ii) Higher order diagrams
(iii)external spinning fields

- Similar results hold for arbitrary tree diagrams (dubbed “cutting rules”)



- The COT extends to loop diagrams and it involves diagrams with multiple cuts

  E Pajer, S Melville 2021

- It applies to ***arbitrary accelerating backgrounds*** as long as there is a Bunch-Davis initial condition Goohew-Lee-SJ-Pajer 2021

Leading and Subleading Partial Energy Poles from the COT

- Around $E_L = 0$, the singular behaviour of ψ_4 is dictated by the RHS of the COT as the second terms is analytic there,

$$\psi_4 = \sum_{0<n\leq m} \frac{R_n(E_R, k_1\, k_2, k_3\, k_4, s)}{E_L^n} + \mathcal{O}(E_L^0)\,,$$

- The analytic part above is not totally arbitrary: it should cancel the collinear singularities in R_n
- The Laurent expansion around $E_L = 0$ should be consistent with the one around $E_R = 0$
- Terms that are regular in partial energies cannot be constrained by COT

	$\psi_4(k_a, s)$	$\psi_4(-k_a, s)$	$\psi_3(k_1, k_2, s)$	$\psi_3(k_1, k_2, -s)$
(partial energy pole) $E_L = 0$	✓	✗	✓	✗
(collinear pole) $E_L = 2s$	✗	✓	✗	✓
(total energy pole) $E_L + E_R = 2s$	✓	✓	✗	✗

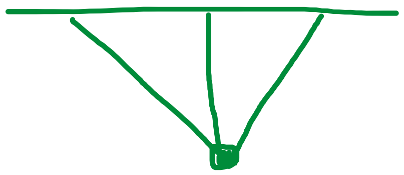
The Manifest Locality Test (MLT)

- We have already implemented a weak version of Locality within our polology: a diagram should have no pole other than the total energy and subdiagram poles. e.g. negative powers of external energies cannot appear in ψ_n
- But this is not enough. Consider the following non-local interaction,

$$\phi'^2 \frac{1}{\nabla^2} \phi'$$

Nevertheless, the resulting cubic contact term will still be regular at $k_a = 0$,

$$\psi_3 = \frac{2\left(k_2^2 k_3^2 + k_1^2 k_2^2 + k_1^2 k_3^2\right)}{k_1 + k_2 + k_3}$$



- Such secretly non-local contact terms can be excluded by looking at the singularities of the exchange diagram formed by gluing two copies of them,

$$\psi_4(k_a, s) + \psi_4(-k_a, s) = P(s)\left(\psi_3(k_1, k_2, s) - \psi_3(k_1, k_2, -s)\right) \times ((1,2) \leftrightarrow (3,4))$$

$\mathcal{O}(s^0) + \ldots$ $\qquad \frac{1}{s^3} \qquad$ Naive scaling : $\mathcal{O}(s)$

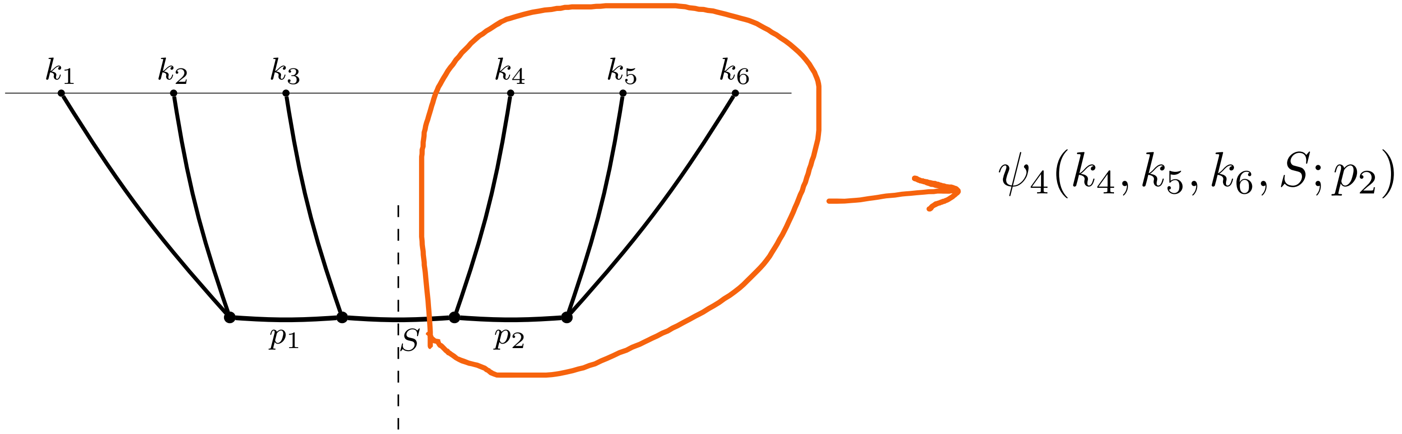
- The absence of spurious pole in $s = 0$ for the 4pt function demands the following,

$$\text{MLT for 3pt :} \quad \frac{\partial}{\partial s}\psi_3(k_1, k_2, s)\Big|_{s=0} = 0$$

- This is not satisfied by the previous example

- A few remarks about the ***MLT***:
  1) It can be straightforwardly generalized to any diagram,

$$\frac{\partial}{\partial k_c}\psi_n(k_1, ..., k_n; \{p\}; \{\mathbf{k}\})\Big|_{k_c=0} = 0$$



2) Same result can be obtained without looking at COT by directly looking at the bulk expression for ψ_n and that,

$$\lim_{S\to 0} K_\phi(\eta, S) = 1 + \frac{1}{2}(c_\phi S\eta)^2 + \frac{i}{3}(c_\phi S\eta)^3 + \mathcal{O}(S^5)$$

so when the vertices are manifestly local, the MLT will follow.

3) It was crucial to keep the external and internal energies fixed when sending $k_a \to 0$, therefore MLT is defined only for the analytically continued ψ_n

Bootstraping Contact Diagrams using the MLT

- Unitarity (COT) is trivially satisfied thanks to scale invariance.
- **Step I**: 3pt should be a rational function of three external energies and be symmetric under the permutation of the external legs (aka Bose symmetry). There should be no pole other than the total energy (Bunch Davis vacuum). p is an integer that characterizes the highest degree of the pole.

$$\psi_3^{(p)}(k_1, k_2, k_3) = \frac{1}{k_T^p} \sum_{n=0}^{\lfloor \frac{p+3}{3} \rfloor} \sum_{m=0}^{\lfloor \frac{p+3-3n}{2} \rfloor} C_{mn} k_T^{3+p-2m-3n} e_2^m e_3^n \qquad p = D - 3$$

$$k_T^{(3)} = e_1 = k_1 + k_2 + k_3, \qquad e_2 = k_1 k_2 + k_1 k_3 + k_2 k_3, \qquad e_3 = k_1 k_2 k_3.$$

- Step II. Apply the MLT: $$\frac{\partial}{\partial k_1}\psi_3(k_1,k_2,k_3)\Big|_{k_1=0}=0$$

- For p=3 (Ntotal=7-4=3) $\phi\phi'^2, \phi(\partial_i\phi)^2$

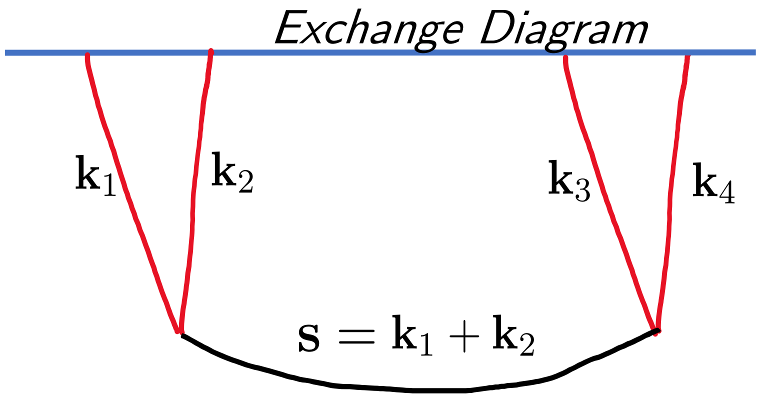
$\phi \to \phi + \phi^2$	$\psi_3^{\text{local}} = k_T^3 - 3k_T e_2 + 3e_3$
ϕ'^3	$\psi_3^{\text{EFT1}} = \frac{e_3^2}{k_T^3}$
$\phi'(\nabla\phi)^2$	$\psi_3^{\text{EFT2}} = \frac{1}{k_T^3}(k_T^6 - 3k_T^4 e_2 + 11k_T^3 e_3 - 4k_T^2 e_2^2 - 4k_T e_2 e_3 + 12e_3^2)$

- One can prove that to arbitrary order in derivatives (arbitrary p) the solutions to MLT can be attributed to a cubic contact term (proof for arbitrary contact terms?)

Bootstraping Exchange Diagrams via Partial Energy Recursion Relations

Here is the corresponding well posed mathematical question:

For a given cubic vertex what four-point functions satisfy both the COT+MLT, and the same time have the right pole structure?



$$\psi_4(k_1, k_2, k_3, k_4, s) + \psi_4^*(-k_1^*, -k_2^*, -k_3^*, -k_4^*, s) =$$
$$P(s)\left(\psi_3(k_1, k_2, s) + \psi_3(-k_1^*, -k_2^*, s)\right)\left(\psi_3(k_3, k_4, s) + \psi_3(-k_3^*, -k_4^*, s)\right)$$

$$\left.\frac{\partial \psi_4(k_a, s)}{\partial k_a}\right|_{k_a} = 0\,.$$

the best one can do is to solve COT+MLT up to an arbitrary quartic contact term (very sensible from an EFT standpoint)

- **One-Variable Shift of energies**

We choose the energy shift such that the residues of those poles are dictated by unitarity.

One convenient choice is the partial energy shift

$$\psi_4(E_L, E_R, k_1 k_2, k_3 k_4, s) \to \tilde{\psi}_4(z) = \psi_4(E_L + z, E_R - z, k_1 k_2, k_3 k_4, s)$$

$$\text{singularities of } \tilde{\psi}_4(z): \qquad z = -E_L \qquad \text{and} \qquad z = E_R\,.$$

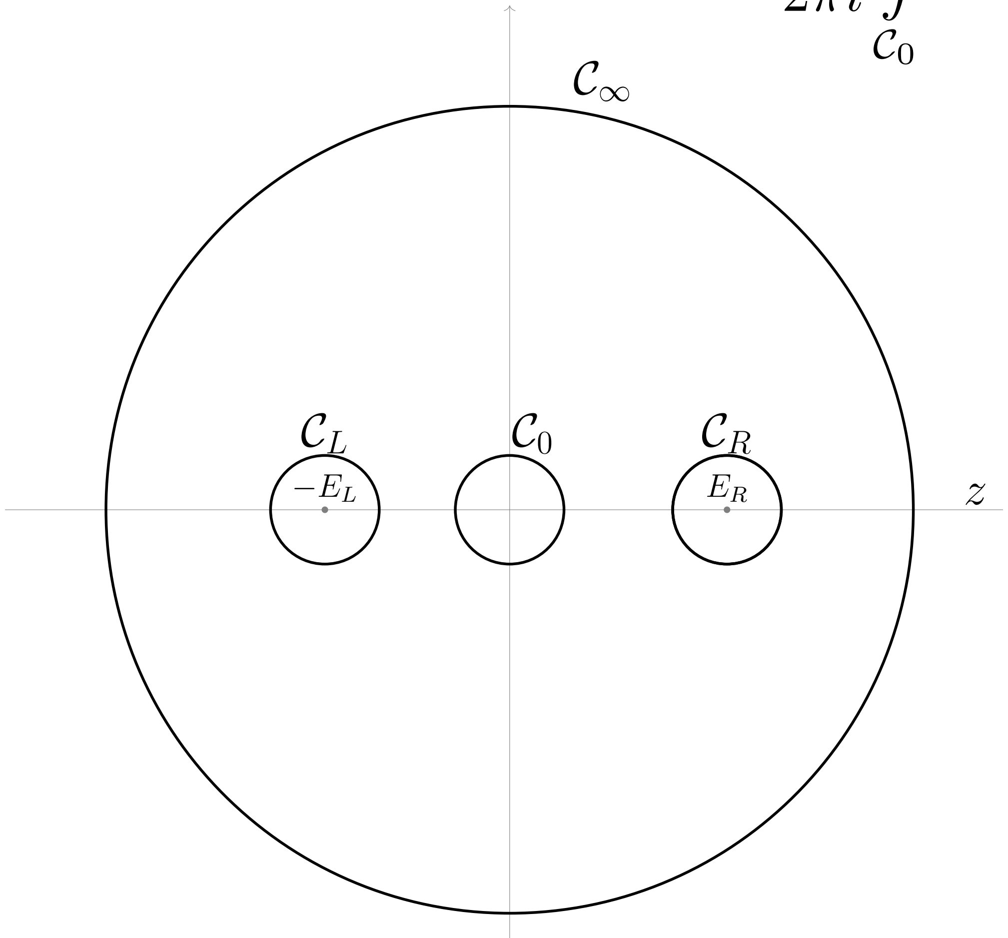
$$k_T(z) = (E_L - z) + (E_R + z) - 2s = k_T\,.$$

- **(partial energy recursion relations).**

We use the Cauchy theorem to relate $\psi_4(z)$ at origin $(=\psi_4(E_L,\ldots))$ to the residues of its associated poles and a boundary term at infinity

$$\psi_4(E_L, E_R, k_1k_2, k_3k_4, s) = \frac{1}{2\pi i}\oint_{\mathcal{C}_0} dz\, \frac{\tilde{\psi}_4(z)}{z} = -\mathrm{Res}\left[\frac{\tilde{\psi}_4(z)}{z}\right]_{z=-E_L} - \mathrm{Res}\left[\frac{\tilde{\psi}_4(z)}{z}\right]_{z=E_R} + B$$

$$= \psi_{\mathrm{Res}} + B\,,$$



ψ_{Res}: Entirely fixed by The COT

B: MLT+COT Up to a quartic contact term

Concluding Remarks and Future Directions

- We introduced a set of tools for bootstrapping the correlators of massless fields in Cosmology, namely the *COT*, the *MLT.*
- The MLT decides whether a correlators can originate from a manifestly local operator in the Lagrangian, while the COT is satisfied by correlators that arise from unitary theories.
- Using the MLT contact diagrams can be bootstrapped, while the combination of the COT and the MLT gives us the power to bootstrap more complicated Tree-diagrams.

- Some fruitful generalizations ahead of us,

    - For spinning fields, it is essential to replace the MLT with a more relaxed version of locality that can accommodate obviously legitimate theories such as gauge theories and GR in dS. Also our partial energy shifts should be modified beyond its current single channel format. By doing so, hopefully we can answer questions like what are the consistent 4pt of gravitons in dS.

    - It would be nice to extend our methods to situations where branch cuts are present. In particular correlators with massive fields or Logarithmic IR-singularities.

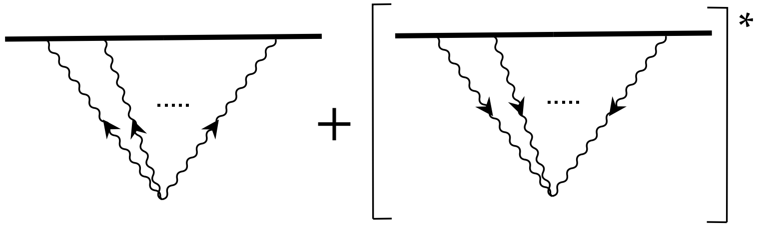
    - COT remains a perturbative statement unlike the optical theorem in flat space. What is the non-perturbative version of it probably stated for the full wavefunction of the universe? $\Psi\{\phi(\mathbf{x})\}$

Quanta magazine

Backup

- As a result,

$$\psi_n'(k_a, \hat{k}_a \cdot \hat{k}_b) + \left[\psi_n'(-k_a^*, \hat{k}_a \cdot \hat{k}_b)\right]^* = 0\,, \qquad k_a \in \mathbb{C}^{n-}.$$



- For IR-finite contact terms among massless particles, COT implies

$$\psi_n^\phi = k_T^3\, f(\frac{k_n}{k_T}, \hat{k}_a.\hat{k}_b) \Rightarrow \qquad \mathrm{Im}(\psi_n'^\phi) = 0 \qquad \text{(massless field)}$$

- Very non-trivial implications for IR-divergent contact terms (with branch cuts). E.g. ϕ^3 theory in dS

$$\psi_3'^\phi = \frac{g}{H^4}\Big[(-2+2\gamma+i\pi)e_1^3 + (4-6\gamma-3i\pi)e_2e_1 + (2+6\gamma+3i\pi)e_3 + 2\left(e_1^3-3e_2e_1+3e_3\right)\log\left(-e_1\eta_0\right) + \frac{2i}{\eta_0}(e_1^2-2e_2) + \frac{2i}{\eta_0^3}\Big]$$

$$\ln(-k_T - i\epsilon) = -i\pi + \ln(k_T)$$

$$e_1 = k_1+k_2+k_3\,, \qquad e_2 = k_1k_2+k_1k_3+k_2k_3\,, \qquad e_3 = k_1k_2k_3\,.$$

- Application: Factorization on the pole from COT

$$E_L = k_1 + k_2 + s \to 0$$

Analytic around $E_L = 0$

$$\psi_4'^s(k_1,k_2,k_3,k_4,s) + [\psi_4'^s(-k_1,-k_2,-k_3,-k_4,s)]^* =$$

$$P_\sigma(s)\left[\psi_3'^{\phi\phi\sigma}(k_1,k_2,s) - \psi_3'^{\phi\phi\sigma}(k_1,k_2,-s)\right]\left[\psi_3'^{\phi\phi\sigma}(k_3,k_4,s) - \psi_3'^{\phi\phi\sigma}(k_3,k_4,-s)\right]$$

Three particle Amplitude

$$\lim_{E_{L,R}\to 0} \psi_4(k_1,..,k_4,s) = \frac{1}{E_{L,R}^p} A_3(k_1,k_2,s) \times \tilde{\psi}_3(k_3,k_4,s)$$

- No matter how much effort we put in, the best we can do is to solve these equations up to,

$$\psi_4(k_a, s) + \psi_4(-k_a, s) = 0\,, \qquad \frac{\partial \psi_4}{\partial k_a}\bigg|_{k_a=0} = 0$$

- The first equation tells us that
  - ψ_4 can only have a total energy pole (no partial energy is allowed)
  - it can depend only on even powers of s , therefore, $\psi_4 = \psi_4(k_a, s^2) = \psi_4(k_a, \mathbf{k}_1.\mathbf{k}_2)$

- As such, the corresponding MLT turns out to be the *MLT for contact terms* (not exchange diagrams),

$$\frac{\partial}{\partial k_a}\psi_4(k_a, \mathbf{k}_a.\mathbf{k}_b)\bigg|_{k_a=0} = 0 \;\Rightarrow \psi_4 = \text{quartic contact term}$$

- In other words, the best one can do is to solve COT+MLT up to an arbitrary quartic contact term (very sensible from an EFT standpoint).

- The residue sector ψ_{Res} is entirely fixed by the RHS of the COT,

$$\tilde{\psi}_4(z) = \sum_{0<n\leq m} \frac{A_n(E_R, E_L, k_1k_2, k_3k_4, s)}{(z+E_L)^n} + \mathcal{O}(z+E_L)\,,$$

$$A_n = \frac{1}{(m-n)!}\left[\partial_z^{m-n}(z+E_L)^m\, \Xi(E_L+z, E_R-z, k_1k_2, k_3k_4, s)\right]_{z=-E_L}\,.$$

- ψ_{Res} constituents the partial energy singularities of ψ_4 , while the boundary term can only have total energy poles simply because

$$B = \frac{1}{2\pi i}\int_{C_\infty} \frac{\psi_4(E_L+z, E_R-z)}{z} \quad \Rightarrow \quad B(E_R, E_L) = B(E_R+c, E_L-c)$$

To illustrate the last two steps, consider the trispectrum with two copies of $\frac{1}{2} g_2\, \phi'(\partial_i \phi)^2$ vertices

$$
\begin{aligned}
\psi_{\text{Res}}^{\text{EFT2}} = &-\frac{g_2^2}{8k_T^5 E_L^3 E_R^3}\Big[\tfrac{1}{2} E_L^2 E_R^2 k_T^{10} - E_L^3 E_R^2 k_T^9 + 2E_L E_R^2 k_1 k_2 k_T^9 - 5E_L^3 E_R^3 k_T^8 - 5E_L^4 E_R^2 k_T^8 - 12E_R^2 k_1^2 k_2^2 k_T^8 - 12E_L^2 E_R^2 k_1 k_2 k_T^8 + 2E_L E_R k_1 k_2 k_3 k_4 k_T^8 + \\
&30E_L^4 E_R^3 k_T^7 + 10E_L^5 E_R^2 k_T^7 - 12E_R^3 k_1^2 k_2^2 k_T^7 + 28E_L E_R^2 k_1^2 k_2^2 k_T^7 - 10E_L E_R^4 k_1 k_2 k_T^7 - 4E_L^2 E_R^3 k_1 k_2 k_T^7 + 30E_L^3 E_R^2 k_1 k_2 k_T^7 - 24E_R k_1^2 k_2^2 k_3 k_4 k_T^7 - \\
&20E_L^2 E_R k_1 k_2 k_3 k_4 k_T^7 - 15E_L^4 E_R^4 k_T^6 - 20E_L^5 E_R^3 k_T^6 - 5E_L^6 E_R^2 k_T^6 + 48E_R^4 k_1^2 k_2^2 k_T^6 + 8E_L E_R^3 k_1^2 k_2^2 k_T^6 - 4E_L^2 E_R^2 k_1^2 k_2^2 k_T^6 + 72k_1^2 k_2^2 k_3^2 k_4^2 k_T^6 + \\
&10E_L E_R^5 k_1 k_2 k_T^6 + 36E_L^2 E_R^4 k_1 k_2 k_T^6 - 14E_L^3 E_R^3 k_1 k_2 k_T^6 - 40E_L^4 E_R^2 k_1 k_2 k_T^6 + 96E_R^2 k_1^2 k_2^2 k_3 k_4 k_T^6 + 32E_L E_R k_1^2 k_2^2 k_3 k_4 k_T^6 + 36E_L^2 E_R^2 k_1 k_2 k_3 k_4 k_T^6 + \\
&40E_L^3 E_R k_1 k_2 k_3 k_4 k_T^6 - 10E_L^5 E_R^4 k_T^5 - 5E_L^6 E_R^3 k_T^5 - E_L^7 E_R^2 k_T^5 - 12E_R^5 k_1^2 k_2^2 k_T^5 - 32E_L E_R^4 k_1^2 k_2^2 k_T^5 + 52E_L^2 E_R^3 k_1^2 k_2^2 k_T^5 - 24E_L^3 E_R^2 k_1^2 k_2^2 k_T^5 - \\
&48E_L k_1^2 k_2^2 k_3^2 k_4^2 k_T^5 - 14E_L^2 E_R^5 k_1 k_2 k_T^5 + 2E_L^3 E_R^4 k_1 k_2 k_T^5 + 46E_L^4 E_R^3 k_1 k_2 k_T^5 + 30E_L^5 E_R^2 k_1 k_2 k_T^5 - 144E_R^3 k_1^2 k_2^2 k_3 k_4 k_T^5 - 48E_L E_R^2 k_1^2 k_2^2 k_3 k_4 k_T^5 + \\
&24E_L^2 E_R k_1^2 k_2^2 k_3 k_4 k_T^5 - 88E_L^3 E_R^2 k_1 k_2 k_3 k_4 k_T^5 - 40E_L^4 E_R k_1 k_2 k_3 k_4 k_T^5 + 10E_L^5 E_R^5 k_T^4 + 15E_L^6 E_R^4 k_T^4 + 6E_L^7 E_R^3 k_T^4 + E_L^8 E_R^2 k_T^4 - 12E_R^6 k_1^2 k_2^2 k_T^4 + \\
&8E_L E_R^5 k_1^2 k_2^2 k_T^4 + 16E_L^2 E_R^4 k_1^2 k_2^2 k_T^4 - 8E_L^3 E_R^3 k_1^2 k_2^2 k_T^4 - 4E_L^4 E_R^2 k_1^2 k_2^2 k_T^4 - 192E_L^2 k_1^2 k_2^2 k_3^2 k_4^2 k_T^4 + 32E_L E_R k_1^2 k_2^2 k_3^2 k_4^2 k_T^4 - 2E_L E_R^7 k_1 k_2 k_T^4 - \\
&4E_L^2 E_R^6 k_1 k_2 k_T^4 - 12E_L^3 E_R^5 k_1 k_2 k_T^4 - 32E_L^4 E_R^4 k_1 k_2 k_T^4 - 34E_L^5 E_R^3 k_1 k_2 k_T^4 - 12E_L^6 E_R^2 k_1 k_2 k_T^4 + 96E_R^4 k_1^2 k_2^2 k_3 k_4 k_T^4 - 48E_L E_R^3 k_1^2 k_2^2 k_3 k_4 k_T^4 - \\
&64E_L^2 E_R^2 k_1^2 k_2^2 k_3 k_4 k_T^4 - 24E_L^3 E_R k_1^2 k_2^2 k_3 k_4 k_T^4 + 12E_L^3 E_R^3 k_1 k_2 k_3 k_4 k_T^4 + 32E_L^4 E_R^2 k_1 k_2 k_3 k_4 k_T^4 + 20E_L^5 E_R k_1 k_2 k_3 k_4 k_T^4 - 12E_L E_R^6 k_1^2 k_2^2 k_T^3 + \\
&8E_L^2 E_R^5 k_1^2 k_2^2 k_T^3 + 80E_L^3 E_R^4 k_1^2 k_2^2 k_T^3 + 88E_L^4 E_R^3 k_1^2 k_2^2 k_T^3 + 28E_L^5 E_R^2 k_1^2 k_2^2 k_T^3 - 48E_L^3 k_1^2 k_2^2 k_3^2 k_4^2 k_T^3 - 256E_L^2 E_R k_1^2 k_2^2 k_3^2 k_4^2 k_T^3 - 2E_L^2 E_R^7 k_1 k_2 k_T^3 - \\
&6E_L^3 E_R^6 k_1 k_2 k_T^3 - 4E_L^4 E_R^5 k_1 k_2 k_T^3 + 4E_L^5 E_R^4 k_1 k_2 k_T^3 + 6E_L^6 E_R^3 k_1 k_2 k_T^3 + 2E_L^7 E_R^2 k_1 k_2 k_T^3 - 24E_R^5 k_1^2 k_2^2 k_3 k_4 k_T^3 + 112E_L E_R^4 k_1^2 k_2^2 k_3 k_4 k_T^3 - \\
&16E_L^2 E_R^3 k_1^2 k_2^2 k_3 k_4 k_T^3 - 40E_L^3 E_R^2 k_1^2 k_2^2 k_3 k_4 k_T^3 - 32E_L^4 E_R k_1^2 k_2^2 k_3 k_4 k_T^3 + 56E_L^4 E_R^3 k_1 k_2 k_3 k_4 k_T^3 + 12E_L^5 E_R^2 k_1 k_2 k_3 k_4 k_T^3 - 4E_L^6 E_R k_1 k_2 k_3 k_4 k_T^3 - \\
&12E_L^2 E_R^6 k_1^2 k_2^2 k_T^2 - 48E_L^3 E_R^5 k_1^2 k_2^2 k_T^2 - 72E_L^4 E_R^4 k_1^2 k_2^2 k_T^2 - 48E_L^5 E_R^3 k_1^2 k_2^2 k_T^2 - 12E_L^6 E_R^2 k_1^2 k_2^2 k_T^2 + 144E_L^4 k_1^2 k_2^2 k_3^2 k_4^2 k_T^2 - 256E_L^2 E_R^2 k_1^2 k_2^2 k_3^2 k_4^2 k_T^2 - \\
&192E_L^3 E_R k_1^2 k_2^2 k_3^2 k_4^2 k_T^2 - 48E_L E_R^5 k_1^2 k_2^2 k_3 k_4 k_T^2 + 128E_L^2 E_R^4 k_1^2 k_2^2 k_3 k_4 k_T^2 + 136E_L^3 E_R^3 k_1^2 k_2^2 k_3 k_4 k_T^2 - 16E_L^4 E_R^2 k_1^2 k_2^2 k_3 k_4 k_T^2 + 24E_L^5 E_R k_1^2 k_2^2 k_3 k_4 k_T^2 - \\
&24E_L^4 E_R^4 k_1 k_2 k_3 k_4 k_T^2 - 32E_L^5 E_R^3 k_1 k_2 k_3 k_4 k_T^2 - 8E_L^6 E_R^2 k_1 k_2 k_3 k_4 k_T^2 - 432E_L^3 E_R^2 k_1^2 k_2^2 k_3^2 k_4^2 k_T + 432E_L^4 E_R k_1^2 k_2^2 k_3^2 k_4^2 k_T - 72E_L^4 E_R^3 k_1 k_2 k_3^2 k_4^2 k_T - \\
&72E_L^5 E_R^2 k_1 k_2 k_3^2 k_4^2 k_T + 72E_L^4 E_R^3 k_1^2 k_2^2 k_3 k_4 k_T + 72E_L^5 E_R^2 k_1^2 k_2^2 k_3 k_4 k_T + 864E_L^3 E_R^3 k_1^2 k_2^2 k_3^2 k_4^2 + 864E_L^4 E_R^2 k_1^2 k_2^2 k_3^2 k_4^2 + (E_L \to E_R, k_1 k_2 \to \\
&k_3 k_4)\Big].
\end{aligned}
$$

For this example, COT demands the following piece in the boundary term,

$$B_{\text{COT}} = \frac{25}{2}s^3$$

Also ψ_{Res} does not satisfy the MLT, and we need a second contribution to B,

$$B_{\text{MLT}}^{\text{EFT2}} = -\frac{12g_2^2(k_1^2k_2^2 + k_3^2k_4^2)s^2}{k_T^3} + \frac{4g_2^2 s^4}{k_T} - 5g_2^2 k_T s^2$$

The remaining boundary term is equivalent to the following contact term (by direct bulk computation)

$$\frac{1}{g_2^2}\Delta\mathcal{L}_{\text{int}}^{\text{EFT2}} = -\frac{5}{2}\phi'^4 + 2\phi'^2(\nabla\phi)^2 + 9a(\eta)\phi\phi'^3 - 17a^2(\eta)\phi^2\phi'^2 + \frac{17}{2}a^2(\eta)\phi^2[\phi'^2 - (\nabla\phi)^2]$$

A Four-Step Receipe

- **Step 0: Change of Kinematics**

$$
\begin{aligned}
\psi_4 : &\qquad (k_1, k_2, k_3, k_4, s) \to (E_L, E_R, k_1 k_2, k_3 k_4, s)\,, \\
\psi_3^L : &\qquad (k_1, k_2, s) \to (E_L, k_1 k_2, s)\,, \\
\psi_3^R : &\qquad (k_3, k_4, s) \to (E_R, k_3 k_4, s)\,.
\end{aligned}
$$

$$\psi_4(E_L, E_R, k_1 k_2, k_3 k_4, s) + \psi_4(-E_L + 2s, -E_R + 2s, k_1 k_2, k_3 k_4, s) = \Xi$$

where

$$
\begin{aligned}
\Xi = P(s)\,&(\psi_3(E_L, k_1 k_2, s) - \psi_3(E_L - 2s, k_1 k_2, -s)) \\
&\times (\psi_3(E_R, k_3 k_4, s) - \psi_3(E_R - 2s, k_3 k_4, -s))\,.
\end{aligned}
$$

- **<u>Step 1: One-Variable Shift of energies</u>** (inspired from BCFW but with crucial differences).

We choose the energy shift such that the residues of those poles are dictated by unitarity.

One convenient choice is the partial energy shift

$$\psi_4(E_L, E_R, k_1k_2, k_3k_4, s) \to \tilde{\psi}_4(z) = \psi_4(E_L + z, E_R - z, k_1k_2, k_3k_4, s)$$

$$\text{singularities of } \tilde{\psi}_4(z): \qquad z = -E_L \qquad \text{and} \qquad z = E_R\,.$$

Notica that the total energy pole is not shifted (which is not fixed by the RHS of the COT)

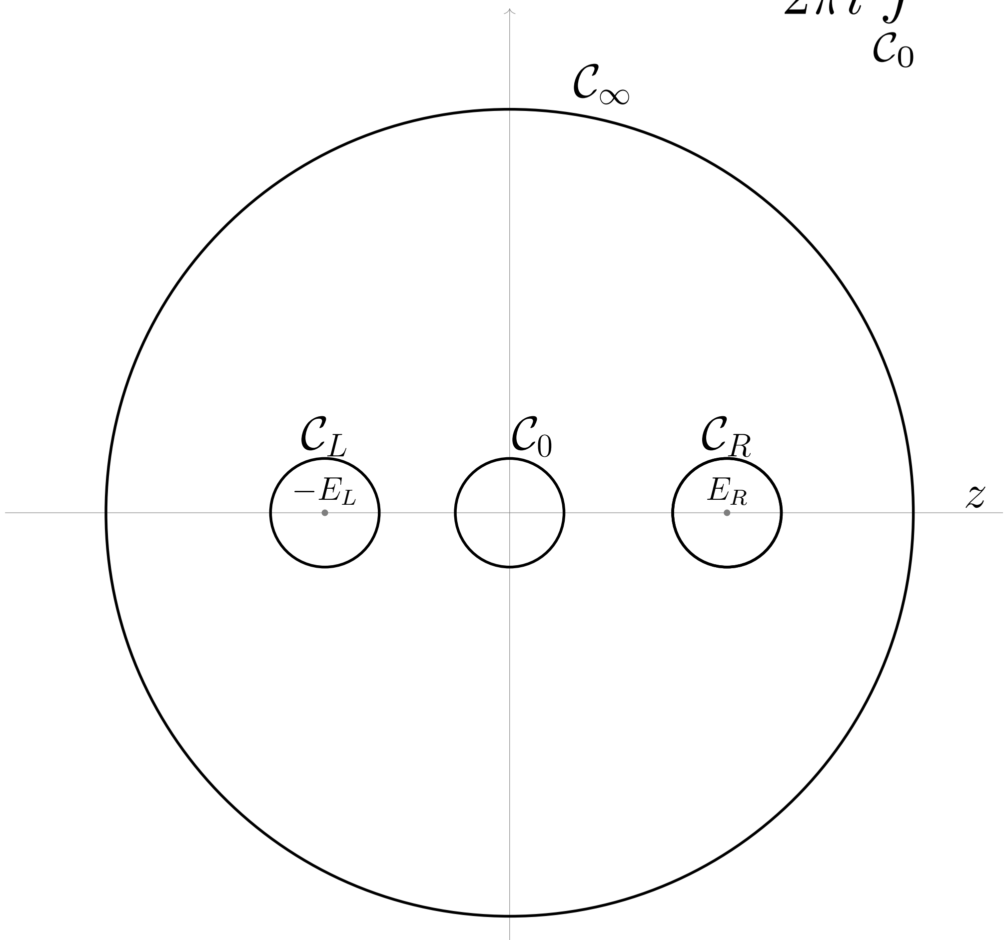
$$k_T(z) = (E_L - z) + (E_R + z) - 2s = k_T\,.$$

- <u>Step 2 (partial energy recursion relations).</u>

We use the Cauchy theorem to relate $\psi_4(z)$ at origin ($=\psi_4(E_L,\dots)$) to the residues of its associated poles and a boundary term at infinity

$$\psi_4(E_L,E_R,k_1k_2,k_3k_4,s)=\frac{1}{2\pi i}\oint_{\mathcal{C}_0} dz\,\frac{\tilde{\psi}_4(z)}{z}=-\mathrm{Res}\left[\frac{\tilde{\psi}_4(z)}{z}\right]_{z=-E_L}-\mathrm{Res}\left[\frac{\tilde{\psi}_4(z)}{z}\right]_{z=E_R}+B$$

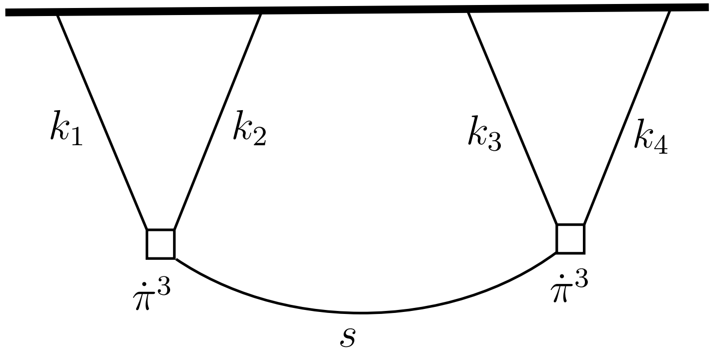
$$=\psi_{\mathrm{Res}}+B\,,$$



ψ_{Res}: Entirely fixed by The COT

B: MLT+COT Up to a quartic contact term

Let us consider a simple example to illustrate Step I.



$$A_3 = -\frac{4g_1^2(k_1k_2k_3k_4s)^2}{k_T^3(E_L+E_R)^3}[3(E_L+E_R)^2 - 6(E_L+E_R)s + 4s^2],$$

$$A_2 = -\frac{48g_1^2(k_1k_2k_3k_4s)^2}{k_T^4(E_L+E_R)^4}[(E_L+E_R)^3 - 3(E_L+E_R)^2s + 4(E_L+E_R)s^2 - 2s^3],$$

$$A_1 = -\frac{24g_1^2(k_1k_2k_3k_4s)^2}{k_T^5(E_L+E_R)^5}[5(E_L+E_R)^4 - 20(E_L+E_R)^3s + 40(E_L+E_R)^2s^2 - 40(E_L+E_R)s^3 + 16s^4],$$

The apparently spurious singularities in the Laurent expansion miraculously cancel in the actual residue,

$$\psi_{\text{Res}} = \sum_{0<n\leq m}\frac{A_n(E_L,E_R,k_1k_2,k_3k_4,s)}{E_L^n} + \sum_{0<n\leq m}\frac{A_n(E_R,E_L,k_3k_4,k_1k_2,s)}{E_R^n}.$$

$$= -4g_1^2(k_1k_2k_3k_4s)^2\left[\frac{6}{k_T^5E_LE_R} + \frac{3}{k_T^4E_LE_R}\left(\frac{1}{E_L}+\frac{1}{E_R}\right) + \frac{1}{k_T^3E_LE_R}\left(\frac{1}{E_L}+\frac{1}{E_R}\right)^2 + \frac{1}{k_T^2E_L^2E_R^2}\left(\frac{1}{E_L}+\frac{1}{E_R}\right) + \frac{1}{k_TE_L^3E_R^3}\right]$$

Compared to the Bulk result,

$$\Delta\mathcal{L}_{\text{int}}^{\text{EFT1}} = -\frac{g_1^2}{4!}\phi'^4$$

- Step 3 (back to the COT). ψ_{Res} is not gauranteed to satisfy the COT by itself. Therefore, COT might already necessitates a boundary term. However, because the RHS of the COT is of O(s^3) or higher and that B cannot have partial energy poles, the only possibily for it is,

$$B_{\rm COT}(k_a, s) = \frac{1}{2}\alpha_0\, s^3$$

  where α_0 is supposedly fixed by the RHS of the COT.

- Step 4 (boundary term from MLT) $\psi_{Res} + B_{COT}$ is not gauranteed to satisfy the MLT either. In such cases a second conribution in the boundary term must be present such that,

$$\frac{\partial}{\partial k_a} B_{\rm MLT}\Big|_{k_a=0} = -\frac{\partial}{\partial k_a}\psi_{\rm Res}\Big|_{k_a=0}$$

Notice that on the right hand side above, a lot of cancellation should happen such that the result is free of any partial energy pole.