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GW PHASE OF COMPACT BINARIES TO 4.5PN ORDER BEYOND THE EINSTEIN QUADRUPOLE FORMULA

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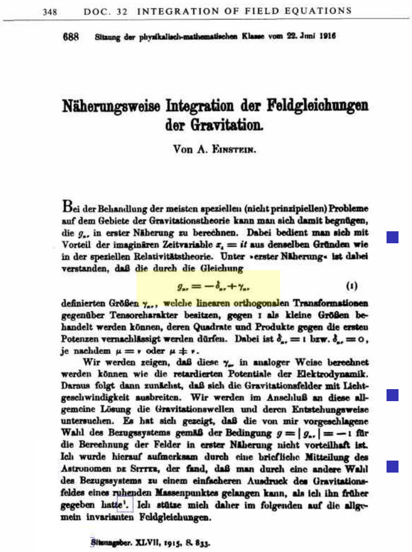
Einstein's first paper on gravitational radiation [Einstein 1916]



348 DOC. 32 INTEGRATION OF FIELD EQUATIONS

688 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. Einstein.

Bei der Behandlung der meisten speziellen (nicht prinzipiellen) Probleme auf dem Gebiete der Gravitationstheorie kann man sich damit begnügen, die $g_{\mu\nu}$ in erster Näherung zu berechnen. Dabei bedient man sich mit Vorteil der imaginären Zeitvariable $x_4 = it$ aus denselben Gründen wie in der speziellen Relativitätstheorie. Unter »erster Näherung« ist dabei verstanden, daß die durch die Gleichung

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \qquad (1)$$

definierten Größen $\gamma_{\mu\nu}$, welche linearen orthogonalen Transformationen gegenüber Tensorcharakter besitzen, gegen 1 als kleine Größen behandelt werden können, deren Quadrate und Produkte gegen die ersten Potenzen vernachlässigt werden dürfen. Dabei ist $\delta_{\mu\nu} = 1$ bzw. $\delta_{\mu\nu} = 0$, je nachdem $\mu = \nu$ oder $\mu \neq \nu$.

Wir werden zeigen, daß diese $\gamma_{\mu\nu}$ in analoger Weise berechnet werden können wie die retardierten Potentiale der Elektrodynamik. Daraus folgt dann zunächst, daß sich die Gravitationsfelder mit Lichtgeschwindigkeit ausbreiten. Wir werden im Anschluß an diese allgemeine Lösung die Gravitationswellen und deren Entstehungsweise untersuchen. Es hat sich gezeigt, daß die von mir vorgeschlagene Wahl des Bezugssystems gemäß der Bedingung $g = |g_{\mu\nu}| = -1$ für die Berechnung der Felder in erster Näherung nicht vorteilhaft ist. Ich wurde hierauf aufmerksam durch eine briefliche Mitteilung des Astronomen de Sitter, der fand, daß man durch eine andere Wahl des Bezugssystems zu einem einfacheren Ausdruck des Gravitationsfeldes eines ruhenden Massenpunktes gelangen kann, als ich ihn früher gegeben hatte[1]. Ich stütze mich daher im folgenden auf die allgemein invarianten Feldgleichungen.

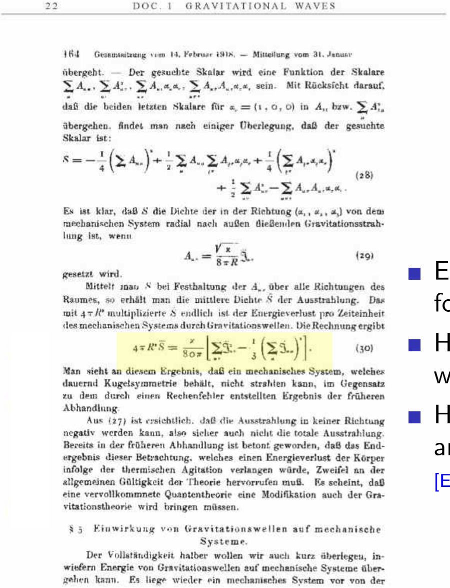
[1] Sitzungsber. XLVII, 1915, S. 833.

- Einstein considers a small perturbation of the Minkowski metric of special relativity

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}}$$

- Using harmonic coordinates he obtains a wave equation for the linear perturbation
- He makes an error in evaluating the energy pseudo-tensor of the gravitational wave

Einstein's second paper on gravitational radiation [Einstein 1918]


22 DOC. 1 GRAVITATIONAL WAVES

164 Gesamtsitzung vom 14. Februar 1918. — Mitteilung vom 31. Januar

übergeht. — Der gesuchte Skalar wird eine Funktion der Skalare $\sum_\mu A_{\mu\mu}$, $\sum_{\mu\nu} A^2_{\mu\nu}$, $\sum_{\mu\nu} A_{\mu\nu}\alpha_\mu\alpha_\nu$, $\sum_{\mu\nu\sigma} A_{\mu\sigma}A_{\mu\nu}\alpha_\sigma\alpha_\nu$ sein. Mit Rücksicht darauf, daß die beiden letzten Skalare für $\alpha_\nu = (1, 0, 0)$ in A_{11} bzw. $\sum_\mu A^2_{1\mu}$ übergehen, findet man nach einiger Überlegung, daß der gesuchte Skalar ist:

$$S = -\frac{1}{4}\left(\sum A_{\mu\mu}\right)^2 + \frac{1}{2}\sum_\mu A_{\mu\mu}\sum_{\rho\sigma} A_{\rho\sigma}\alpha_\rho\alpha_\sigma + \frac{1}{4}\left(\sum_{\rho\sigma} A_{\rho\sigma}\alpha_\rho\alpha_\sigma\right)^2 + \frac{1}{2}\sum_{\mu\nu} A^2_{\mu\nu} - \sum_{\mu\nu\sigma} A_{\mu\sigma}A_{\mu\nu}\alpha_\sigma\alpha_\nu . \qquad (28)$$

Es ist klar, daß S die Dichte der in der Richtung $(\alpha_1, \alpha_2, \alpha_3)$ von dem mechanischen System radial nach außen fließenden Gravitationsstrahlung ist, wenn

$$A_{\mu\nu} = \frac{\sqrt{\varkappa}}{8\pi R}\dddot{\mathfrak{J}}_{\mu\nu} \qquad (29)$$

gesetzt wird.

Mittelt man S bei Festhaltung der $A_{\mu\nu}$ über alle Richtungen des Raumes, so erhält man die mittlere Dichte $\bar{S}$ der Ausstrahlung. Das mit $4\pi R^2$ multiplizierte $\bar{S}$ endlich ist der Energieverlust pro Zeiteinheit des mechanischen Systems durch Gravitationswellen. Die Rechnung ergibt

$$4\pi R^2\bar{S} = \frac{\varkappa}{80\pi}\left[\sum_{\mu\nu}\dddot{\mathfrak{J}}^2_{\mu\nu} - \frac{1}{3}\left(\sum_\mu \dddot{\mathfrak{J}}_{\mu\mu}\right)^2\right]. \qquad (30)$$

Man sieht an diesem Ergebnis, daß ein mechanisches System, welches dauernd Kugelsymmetrie behält, nicht strahlen kann, im Gegensatz zu dem durch einen Rechenfehler entstellten Ergebnis der früheren Abhandlung.

Aus (27) ist ersichtlich, daß die Ausstrahlung in keiner Richtung negativ werden kann, also sicher auch nicht die totale Ausstrahlung. Bereits in der früheren Abhandlung ist betont geworden, daß das Endergebnis dieser Betrachtung, welches einen Energieverlust der Körper infolge der thermischen Agitation verlangen würde, Zweifel an der allgemeinen Gültigkeit der Theorie hervorrufen muß. Es scheint, daß eine vervollkommnete Quantentheorie eine Modifikation auch der Gravitationstheorie wird bringen müssen.

§ 5 Einwirkung von Gravitationswellen auf mechanische Systeme.

Der Vollständigkeit halber wollen wir auch kurz überlegen, inwiefern Energie von Gravitationswellen auf mechanische Systeme übergehen kann. Es liege wieder ein mechanisches System vor von der

- Einstein obtains the quadrupole formula for the energy flux of gravitational waves
- His calculation is valid only for systems with negligible internal gravity
- He makes a computational error and his result is wrong by a factor 2 ! [Eddington 1922]

Einstein's quadrupole formula [Einstein 1918]

$$4\pi R^2 \bar{\mathfrak{S}} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{\mathfrak{J}}_{\mu\nu}^2 - \frac{1}{3}\left(\sum_{\mu} \dddot{\mathfrak{J}}_{\mu\mu}\right)^2\right].$$

[Courtesy J. Mouette]

1 Quadrupole formula for the energy flux

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\mathrm{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 M_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 M_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}$$

2 Quadrupole formula for the GW amplitude

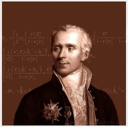
$$h_{ij}^{\mathrm{TT}} = \frac{2G}{c^4 r}\left\{\frac{\mathrm{d}^2 M_{ij}}{\mathrm{d}t^2}\left(t-\frac{r}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\mathrm{TT}} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

3 The quadrupole moment reduces to the usual Newtonian quadrupole

$$M_{ij}(t) = \int_{\mathrm{source}} \mathrm{d}^3\mathbf{x}\,\rho(\mathbf{x},t)\left(x_i x_j - \frac{1}{3}\delta_{ij}\,\mathbf{x}^2\right) + \mathcal{O}\left(\frac{v}{c}\right)^2$$

Gravitational radiation reaction

- Laplace [1776]: a finite speed of propagation of gravity would result in a damping of planetary orbits
- Poincaré [1907]: concept of "ondes gravifiques" and re-analysis of the Laplace effect

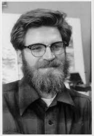


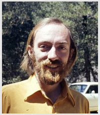
- Chandrasekhar & Esposito [1970]: radiation reaction is of order

$$\mathcal{O}\left(\frac{v}{c}\right)^5 \sim 2.5\text{PN}$$

- Burke & Thorne [1970]: simple expression of the radiation reaction

$$F_i^{\text{reac}} = \underbrace{-\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 M_{ij}}{\mathrm{d}t^5}}_{\text{small 2.5PN effect}} + \mathcal{O}\left(\frac{v}{c}\right)^7$$





Why quadrupole ? Einstein's equivalence principle

- For all test bodies $m_i = m_g$ [a]

$$\mathbf{F} = m_i \, \boldsymbol{a} \qquad (m_i = \text{inertial mass})$$
$$\mathbf{F}_g = m_g \, \boldsymbol{g} \qquad (m_g = \text{gravitational mass})$$

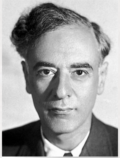
- Conservation of mass (like conservation of charge) $\Longrightarrow$ no monopole radiation
- Conservation of center-of-mass and angular momentum $\Longrightarrow$ no dipole radiation [Abraham 1914]

$$\text{mass dipole:} \quad \mathbf{I} = \sum m_g \mathbf{x} = \overbrace{\sum m_i \mathbf{x}}^{\text{CM integral}}$$

$$\text{current dipole:} \quad \mathbf{D} = \sum m_g \mathbf{x} \times \mathbf{v} = \overbrace{\sum m_i \mathbf{x} \times \mathbf{v}}^{\text{angular momentum}}$$

[a] Checked to level 10^{-15} by the MICROSCOPE satellite

Landau & Lifshitz [1941] derivation of the quadrupole formula

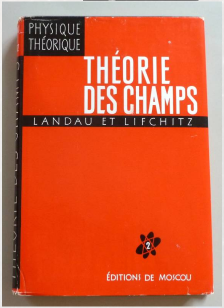


- The Einstein field equations can be written in terms of the "gothic" metric $\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$

$$\partial_{\rho\sigma}\Big[\mathfrak{g}^{\mu\nu}\mathfrak{g}^{\rho\sigma} - \mathfrak{g}^{\mu\rho}\mathfrak{g}^{\nu\sigma}\Big] = \frac{16\pi G}{c^4}|g|\big(T^{\mu\nu} + t^{\mu\nu}\big)$$

- The Landau-Lifshitz pseudo-tensor is

$$\boxed{t^{\mu\nu} = \frac{c^4}{32\pi G}\Big\{\mathfrak{g}^{\mu\nu}\mathfrak{g}_{\rho\sigma}\partial_\tau\mathfrak{g}^{\rho\lambda}\partial_\lambda\mathfrak{g}^{\sigma\tau} + \cdots\Big\}}$$



- The quadrupole formula follows directly from the conservation law of the pseudo-tensor

$$\partial_\nu\Big[|g|\big(T^{\mu\nu} + t^{\mu\nu}\big)\Big] = 0$$

- The derivation is valid for a self-gravitating source

Flux-balance equations

Balance equations are associated with the ten symmetries of the Poincaré group

1. Energy

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{G}{5c^5}\frac{\mathrm{d}^3 M_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 M_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{1}{c^7}\right)$$

2. Angular momentum [Papapetrou 1971; Thorne 1980]

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = -\frac{2G}{5c^5}\varepsilon_{ijk}\frac{\mathrm{d}^2 M_{jl}}{\mathrm{d}t^2}\frac{\mathrm{d}^3 M_{kl}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{1}{c^7}\right)$$

3. Linear momentum [Bonnor & Rotenberg 1961; Peres 1962; Bekenstein 1973; Thorne 1980]

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = -\frac{G}{c^7}\left[\frac{2}{63}\frac{\mathrm{d}^4 M_{ijk}}{\mathrm{d}t^4}\frac{\mathrm{d}^3 M_{jk}}{\mathrm{d}t^3} + \frac{16}{45}\varepsilon_{ijk}\frac{\mathrm{d}^3 M_{jl}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 S_{kl}}{\mathrm{d}t^3}\right] + \mathcal{O}\left(\frac{1}{c^9}\right)$$

4. Center-of-mass position [Kozameh, Nieva & Quirega 2018; Blanchet & Faye 2019]

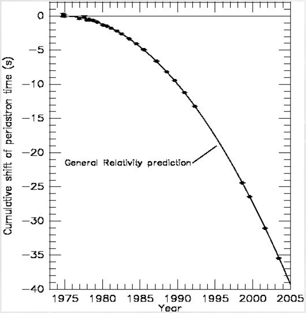
$$\frac{\mathrm{d}G_i}{\mathrm{d}t} = P_i - \frac{2G}{21c^7}\frac{\mathrm{d}^3 M_{ijk}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 M_{jk}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

The quadrupole formula works for the binary pulsar

$$4\pi R^2 \bar{\mathfrak{J}} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{\mathfrak{J}}_{\mu\nu}^2 - \frac{1}{3}\left(\sum_{\mu} \dddot{\mathfrak{J}}_{\mu\mu}\right)^2\right].$$

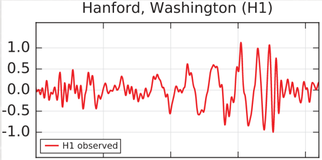
$$\dot{P} = -\frac{192\pi}{5c^5}\frac{m_1 m_2}{M^2}\left(\frac{2\pi G M}{P}\right)^{5/3} \underbrace{\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}}}_{\text{eccentricity enhancement factor}}$$

[Peters & Mathews 1963]



- Derivation based on flux-balance equation [Dyson 1969; Esposito & Harrison 1975; Wagoner 1975]
- Derivation based on EoM including the radiation reaction term at 2.5PN [Damour & Deruelle 1981; Damour 1982]
- Resolution of the radiation reaction controversy [Ehlers, Rosenblum, Goldberg & Havas 1976; Will & Walker 1980]

The quadrupole formula works for GW150914



$$4\pi R^2 \mathfrak{S} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{J}_{\mu\nu}^2 - \frac{1}{3}\left(\sum_{\mu} \dddot{J}_{\mu\mu}\right)^2\right].$$

- The GW frequency is given in terms of the chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ by

$$f = \frac{1}{\pi}\left[\frac{256}{5}\frac{G\mathcal{M}^{5/3}}{c^5}(t_c - t)\right]^{-3/8}$$

- The chirp mass is directly measured as

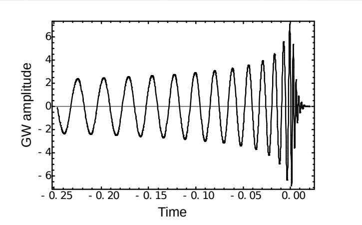
$$\boxed{\mathcal{M} = \left[\frac{5}{96}\frac{c^5}{G\pi^{8/3}} f^{-11/3}\dot{f}\right]^{3/5}}$$

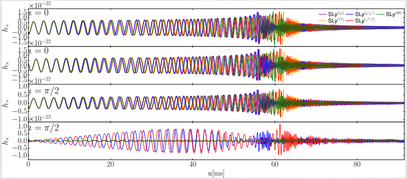
- The GW amplitude is predicted to be

$$h \sim 4.1 \times 10^{-22}\left(\frac{\mathcal{M}}{M_\odot}\right)^{5/6}\left(\frac{100\,\text{Mpc}}{R}\right)\left(\frac{100\,\text{Hz}}{f_{\text{merger}}}\right)^{-1/6} \sim 1.6 \times 10^{-21}$$

- The distance $R = 400\,\text{Mpc}$ is measured from the signal itself [Schutz 1986]

The gravitational chirp of compact binaries





- **Inspiralling phase**
  - Post-Newtonian theory
  - Point-particle approximation
  - Dependence on spin precession
  - Universality of the signal in GR
  - Effacing of the internal structure

    [Brillouin 1922; Damour 1982]
- **Late inspiral**
  - Post-Newtonian + Effective theory
  - Effects due to tidal interactions
  - Dependence on the internal structure (EoS)
- **Merger and post-merger**
  - Numerical relativity
  - Strong dependence on internal structure
  - Phenomenological models (EOB, IMR)

    [Buonanno & Damour 1999; Ajith *et al.* 2008]

Inspiralling binaries require high-order PN modelling

[Caltech "3mn paper" 1992; Blanchet & Schäfer 1993]

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The Last Three Minutes: Issues in Gravitational-Wave Measurements of Coalescing Compact Binaries

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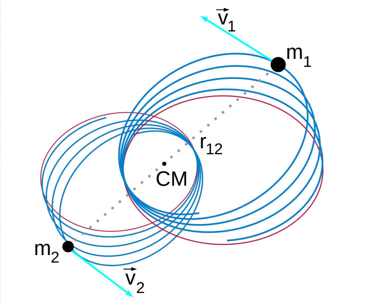
Gravitational-wave interferometers are expected to monitor the last three minutes of inspiral and final coalescence of neutron star and black hole binaries at distances approaching cosmological, where the event rate may be many per year. Because the binary's accumulated orbital phase can be measured to a fractional accuracy $\ll 10^{-3}$ and relativistic effects are large, the wave forms will be far more complex and carry more information than has been expected. Improved wave form modeling is needed as a foundation for extracting the waves' information, but is not necessary for wave detection.

PACS numbers: 04.30.+x, 04.80.+z, 97.60.Jd, 97.60.Lf

$$\phi(t) = \phi_0 - \frac{M}{\mu}\left(\frac{GM\omega}{c^3}\right)^{-5/3}\Big\{1 + \underbrace{\frac{1\text{PN}}{c^2} + \frac{1.5\text{PN}}{c^3} + \cdots + \frac{3\text{PN}}{c^6} + \cdots}_{\text{to be computed with 3PN precision at least}}\Big\}$$

$$4\pi R^2 \bar{\mathcal{S}} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{J}_{\mu\nu}^2 - \frac{1}{3}\left(\sum_{\mu} \dddot{J}_{\mu\mu}\right)^2\right].$$

Post-Newtonian equations of motion



$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}\boldsymbol{n}_{12} + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \cdots\right]\boldsymbol{n}_{12} + \cdots\right\}}^{\text{1PN [Lorentz \& Droste 1917; EIH 1938]}}$$

$$+ \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\ \text{conservative \& dissipative (tail)}}} + \mathcal{O}\left[\left(\frac{v}{c}\right)^9\right]$$

Methods to compute PN equations of motion

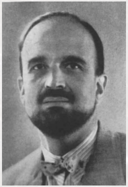
1. Traditional methods in classical GR
    - ADM Hamiltonian canonical formalism in GR
    - Fokker EH action in harmonic coordinates
    - Surface-integral approach *à la* EIH
    - Extended fluids in the compact body limit
2. QFT inspired methods
    - Effective-field theory
    - Scattering amplitude approach
3. Dimensional regularization is the common tool
   ['t Hooft & Veltman 1972; Bollini & Giambiagi 1972]
    - UV divergences: point particles modelling compact objects
    - IR divergences: integration over all space of formal PN expansion

4PN: state-of-the-art on equations of motion

3PN	[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
	[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002] [Blanchet, Damour & Esposito-Farèse 2004]	Harmonic EoM
	[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
	[Foffa & Sturani 2011]	Effective field theory
4PN	[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014, 2016]	ADM Hamiltonian
	[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017ab]	Fokker Lagrangian
	[Foffa & Sturani 2013, 2019; Foffa, Porto, Rothstein & Sturani 2019]	Effective field theory
	[Blümlein, Maier, Marquard & Schäfer 2020]	EFT Hamiltonian

- ADM Hamiltonian: One regularization ambiguity left at 4PN order and fixed by comparison with GSF calculations
- Fokker Lagrangian: First complete derivation of the EoM at 4PN order without regularization ambiguities

Fokker action of N particles [Fokker 1929]



1. Einstein-Hilbert action for a system of point particles

$$S_{\text{g.f.}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\Big[R \underbrace{-\frac{1}{2}g_{\mu\nu}\Gamma^\mu\Gamma^\nu}_{\text{Gauge-fixing term}}\Big] \underbrace{- \sum_A m_A c^2 \int \mathrm{d}t\,\sqrt{-(g_{\mu\nu})_A\, v_A^\mu v_A^\nu/c^2}}_{N \text{ point particles}}$$

2. The Fokker action is obtained by inserting an explicit PN solution of the Einstein field equations

$$g_{\mu\nu}(\mathbf{x},t) \longrightarrow \overline{g}_{\mu\nu}(\mathbf{x};\boldsymbol{y}_B(t),\boldsymbol{v}_B(t),\cdots)$$

3. The PN equations of motion of the N particles (self-gravitating system) are

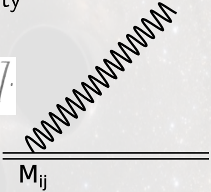
$$\boxed{\frac{\delta S_\text{F}}{\delta \boldsymbol{y}_A} \equiv \frac{\partial L_\text{F}}{\partial \boldsymbol{y}_A} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_\text{F}}{\partial \boldsymbol{v}_A}\right) + \cdots = 0}$$

Effective field theory approach

[Bertotti & Plebański 1960; Hari Dass & Soni 1982; Goldberger & Rothstein 2006]

- Two particles' world lines form a quadrupolar GW source
- The source emits radiation, i.e. a graviton (shown as a wiggly propagator line) propagates to infinity

$$4\pi R^2 \mathfrak{J} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{J}_{\mu\nu}^{2} - \frac{1}{3}\left(\sum_{\mu} \dddot{J}_{\mu\mu}\right)^2\right].$$

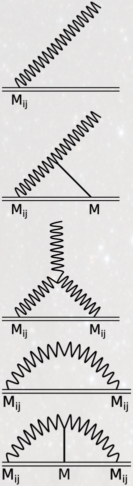


- The GW emission reacts back to the source, i.e. a graviton is emitted and then re-absorbed by the source

$$\mathbf{F}^{\text{reac}} = \mathcal{O}\left(\frac{v}{c}\right)^5$$

Diagrammatic expansion in EFT vs Post-Newtonian

Effective Field Theory



$\Longleftrightarrow$

Post-Newtonian

- emission from a quadrupole source
- tail effect in radiation field (1.5PN)
- non-linear memory effect (2.5PN)
- radiation reaction (2.5PN)
- tail in radiation reaction (4PN)

The EFT is equivalent to the traditional PN at the level of tree diagrams

Thorne's [1980] multipolar linearized vacuum solution

Newtonian potential, by reading the source's multipole moments off that potential, and by then inserting those moments into the gravitational-wave formulas of Part IV.

Soon thereafter, while writing the first draft of Chap. 36 of MTW, I found what I thought was a simple proof of Ipser's conjecture. That proof appears in the preliminary versions of MTW [Misner *et al*. (1970, 1971)] and is referred to in my review article with Bill Press on gravitational-wave astronomy [Press and Thorne (1972)]. However, much to my horror, in March 1973 while checking page proofs of the final version of MTW, I found a subtle but fatal flaw in my proof of Ipser's conjecture. After much agony I managed to rewrite the relevant material [Secs. 36.7 and 36.10 of Misner *et al*. (1973)] with a restriction to sources that have weak internal gravity—and without changing by even one the total number of lines of text.

In Part Two of this article I shall try to redeem myself by presenting a correct formulation and proof of Ipser's conjecture. This formulation will avoid the concept of the asymptotic Newtonian potential of a source; in its place will appear a prescription for reading the multipole moments of a source off its near-zone general relativistic metric. However, in all other respects the formalism will conform to Ipser's original ideas.

Part Two of this paper consists of five sections. The first four (Secs. VIII–XI) develop foundations for the strong-field, slow-motion wave-generation formalism. The last (Sec. XII) presents the formalism itself and describes a few applications.

Each of the four foundations is a derivation of the vacuum exterior gravitational field of a general isolated system. Section VIII derives that field for time-dependent systems in linearized theory. Section IX derives it in the near zone of slow-dependent systems in full general relativity using de Donder coordinates, and also matches that near-zone solution onto outgoing waves in the radiation zone. Section X specializes to time-independent general relativistic systems in de Donder coordinates; and Sec. XI extends the time-independent general relativistic case to any "asymptotically Cartesian and mass-centered" (ACMC) coordinate system.

For a more detailed overview see Sec. I.B, Box 2, and the table of contents—all in Part One of this article.

VIII. LINEARIZED THEORY

Here we express, in terms of time-dependent multipole moments, the linearized external gravitational field of an arbitrary isolated system. Similar expressions, but in different notation, have been given by Sachs and Bergmann (1958), Sachs (1961), Pirani

$$g_{\alpha\beta} = \eta_{\alpha\beta} + g^1_{\alpha\beta}\,. \qquad (8.1)$$

We shall denote by $\gamma^1_{\alpha\beta}$ the trace-reversed perturbation

$$\gamma^1_{\alpha\beta} \equiv g^1_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu}g^1_{\mu\nu}\,. \qquad (8.2)$$

(The reason for our "superscript 1" notation will become clear in Sec. IX. The nature of our coordinate system and basis vectors, and the rules for raising and lowering indices, are discussed in Sec. I.C.)

We introduce Lorentz gauge $\gamma^{1\,\alpha\beta}{}_{,\beta}=0$ for our gravitational field. Then, expressed in terms of covariant components, the gauge conditions and linearized vacuum field equations (Eqs. 18.8 of MTW) read

$$\gamma^1_{\alpha 0,0} = \gamma^1_{\alpha j,j}\,, \qquad (8.3a)$$

$$\Box\gamma^1_{\alpha\beta} \equiv -\gamma^1_{\alpha\beta,00} + \gamma^1_{\alpha\beta,jj} = 0\,. \qquad (8.3b)$$

We seek the most general symmetric gravitational field $\gamma^1_{\alpha\beta} = \gamma^1_{\beta\alpha}$ which satisfies these equations, and which has only outgoing waves (no incoming waves) at infinity; and we write that field as a sum over its multipole components.

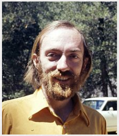
The general outgoing-wave solution to the field equation $\Box\gamma^1_{\alpha\beta}=0$ in multipole notation has the following form [see Eq. (2.51), where we must set $\epsilon=+1$ (outgoing waves) and we must make the identifications $\gamma^1_{00}=F$, $\gamma^1_{0j}=V_j$, $\gamma^1_{jk}=U_{jk}$]:

$$\gamma^1_{00} = \sum_{l=0}^{\infty}\left[r^{-1}\mathcal{A}_{A_l}(t-r)\right]_{,A_l}\,, \qquad (8.4a)$$

$$\gamma^1_{0j} = \sum_{l=1}^{\infty}\left\{\left[r^{-1}\mathcal{B}_{jA_{l-1}}(t-r)\right]_{,A_{l-1}} + \left[r^{-1}\epsilon_{jpq}\mathcal{C}_{qA_{l-1}}(t-r)\right]_{,pA_{l-1}}\right\} + \sum_{l=0}^{\infty}\left[r^{-1}\mathcal{D}_{A_l}(t-r)\right]_{,jA_l}\,, \qquad (8.4b)$$

$$\begin{aligned}\gamma^1_{jk} = {} & \sum_{l=0}^{\infty}\delta_{jk}\left[r^{-1}\mathcal{E}_{A_l}(t-r)\right]_{,A_l} + \sum_{l=2}^{\infty}\left\{\left[r^{-1}\mathcal{F}_{jkA_{l-2}}(t-r)\right]_{,A_{l-2}} + \left[r^{-1}\epsilon_{pq(j}\mathcal{G}_{k)qA_{l-2}}(t-r)\right]_{,pA_{l-2}}\right\} \\ & + \sum_{l=1}^{\infty}\left[\left[r^{-1}\mathcal{H}_{jA_{l-1}}(t-r)\right]_{,kA_{l-1}} + \left[r^{-1}\epsilon_{jpq}\mathcal{I}_{qA_{l-1}}(t-r)\right]_{,kpA_{l-1}}\right]^S + \sum_{l=0}^{\infty}\left[r^{-1}\mathcal{K}_{A_l}(t-r)\right]_{,jkA_l}\,. \end{aligned} \qquad (8.4c)$$

Here ϵ_{ijk} is the completely antisymmetric Levi-Civita tensor; the capital script quantities are the multipole moments, which are arbitrary functions of retarded time $t-r$ and are symmetric and trace-free (STF) on all their tensor indices; and all other details of notation are explained in Sec. I.C. The gauge conditions

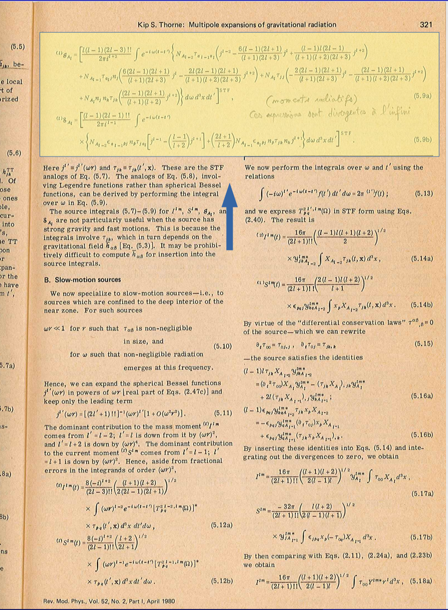


- Most general solution of the Einstein vacuum linearized field equations in harmonic coordinates

$$G h_1^{\mu\nu}\Big[\underbrace{M_L(u), S_L(u)}_{\text{multipole moments}}\Big]$$

[see also Sachs & Bergmann 1958; Pirani 1964]

Sad situation of the field in the 1980's



$$ {}^{(l)}\mathcal{I}_{A_l} = \Bigg[\frac{l(l-1)(2l-3)!!}{2\pi i^{l+2}} \int e^{-i\omega(t-t')}\Big\{ N_{A_{l-2}}\tau_{a_{l-1}a_l}\Big(j^{l-2} - \frac{6(l-1)(2l+1)}{(l+1)(2l+3)} j^l + \frac{(l-1)l(2l-1)}{(l+1)(l+2)(2l+3)} j^{l+2}\Big) + N_{A_{l-1}}\tau_{a_l j} n_j\Big(\frac{6(2l-1)(2l+1)}{(l+1)(2l+3)} j^l - \frac{2l(2l-1)(2l+1)}{(l+1)(l+2)(2l+3)} j^{l+2}\Big) + N_{A_l}\tau_{jj}\Big(-\frac{2(2l-1)(2l+1)}{(l+1)(2l+3)} j^l - \frac{(2l-1)(2l+1)}{(l+1)(l+2)(2l+3)} j^{l+2}\Big) + N_{A_l} n_j n_k \tau_{jk}\Big(\frac{(2l-1)(2l+1)}{(l+1)(l+2)} j^{l+2}\Big)\Big\} d\omega\, d^3x\, dt'\Bigg]^{\mathrm{STF}}, \quad (5.9a) $$

(moments radiatifs)

Ces expressions sont divergentes à l'infini

$$ {}^{(l)}\mathcal{S}_{A_l} = \Bigg[\frac{(l-1)(2l-1)!!}{2\pi i^{l-1}} \int e^{-i\omega(t-t')} \times \Big\{ N_{A_{l-2}}\epsilon_{a_{l-1}pj} n_p \tau_{ja_l}\Big[j^{l-1} - \Big(\frac{l-1}{l+2}\Big) j^{l+1}\Big] + \Big(\frac{2l+1}{l+2}\Big) N_{A_{l-1}}\epsilon_{a_l pj} n_p \tau_{jk} n_k j^{l+1}\Big\} d\omega\, d^3x\, dt'\Bigg]^{\mathrm{STF}}. \quad (5.9b) $$

Here $j^{l'} \equiv j^{l'}(\omega r)$ and $\tau_{jk} \equiv \tau_{jk}(t', \mathbf{x})$. These are the STF analogs of Eq. (5.7). The analogs of Eq. (5.8), involving Legendre functions rather than spherical Bessel functions, can be derived by performing the integral over ω in Eq. (5.9).

The source integrals (5.7)–(5.9) for I^{lm}, S^{lm}, $\mathcal{I}_{A_l}$, and $\mathcal{S}_{A_l}$ are not particularly useful when the source has strong gravity and fast motions. This is because the integrals involve τ_{jk}, which in turn depends on the gravitational field $\bar{h}_{\alpha\beta}$ [Eq. (5.3)]. It may be prohibitively difficult to compute $\bar{h}_{\alpha\beta}$ for insertion into the source integrals.

B. Slow-motion sources

We now specialize to slow-motion sources—i.e., to sources which are confined to the deep interior of the near zone. For such sources

$\omega r \ll 1$ for r such that $\tau_{\alpha\beta}$ is non-negligible in size, and for ω such that non-negligible radiation emerges at this frequency. (5.10)

Hence, we can expand the spherical Bessel functions $j^{l'}(\omega r)$ in powers of ωr [real part of Eqs. (2.47c)] and keep only the leading term

$$ j^{l'}(\omega r) = [(2l'+1)!!]^{-1}(\omega r)^{l'}[1 + O(\omega^2 r^2)]. \quad (5.11) $$

The dominant contribution to the mass moment ${}^{(l)}I^{lm}$ comes from $l' = l-2$; $l' = l$ is down from it by $(\omega r)^2$, and $l' = l+2$ is down by $(\omega r)^4$. The dominant contribution to the current moment ${}^{(l)}S^{lm}$ comes from $l' = l-1$; $l' = l+1$ is down by $(\omega r)^2$. Hence, aside from fractional errors in the integrands of order $(\omega r)^2$,

$$ {}^{(l)}I^{lm}(t) = \frac{8(-i)^{l+2}}{(2l-3)!!}\Big(\frac{(l+1)(l+2)}{2(2l-1)(2l+1)}\Big)^{1/2} \times \int (\omega r)^{l-2} e^{-i\omega(t-t')}[T^{2\,l-2,lm}_{pq}(\Omega)]^* \times \tau_{pq}(t', \mathbf{x})\, d^3x\, dt'\, d\omega, \quad (5.12a) $$

$$ {}^{(l)}S^{lm}(t) = \frac{8(-i)^{l+2}}{(2l-1)!!}\Big(\frac{l+2}{2l+1}\Big)^{1/2} \times \int (\omega r)^{l-1} e^{-i\omega(t-t')}[T^{2\,l-1,lm}_{pq}(\Omega)]^* \times \tau_{pq}(t', \mathbf{x})\, d^3x\, dt'\, d\omega. \quad (5.12b) $$

We now perform the integrals over ω and t' using the relations

$$ \int (-i\omega)^{l'} e^{-i\omega(t-t')} f(t')\, dt'\, d\omega = 2\pi\, {}^{(l')}f(t); \quad (5.13) $$

and we express $T^{2\,l',lm}_{pq}(\Omega)$ in STF form using Eqs. (2.40). The result is

$$ {}^{(2)}I^{lm}(t) = \frac{16\pi}{(2l+1)!!}\Big(\frac{(l-1)l(l+1)(l+2)}{2}\Big)^{1/2} \times \mathcal{Y}^{lm*}_{jkA_{l-2}} \int X_{A_{l-2}}\tau_{jk}(t, \mathbf{x})\, d^3x, \quad (5.14a) $$

$$ {}^{(1)}S^{lm}(t) = \frac{16\pi}{(2l+1)!!}\Big(\frac{2(l-1)l(l+2)}{l+1}\Big)^{1/2} \times \epsilon_{pqj}\mathcal{Y}^{lm*}_{kqA_{l-2}} \int x_p X_{A_{l-2}}\tau_{jk}(t, \mathbf{x})\, d^3x. \quad (5.14b) $$

By virtue of the "differential conservation laws" $\tau^{\alpha\beta}{}_{,\beta} = 0$ of the source—which we can rewrite

$$ \partial_t \tau_{00} = \tau_{0j,j}, \quad \partial_t \tau_{0j} = \tau_{jk,k} \quad (5.15) $$

—the source satisfies the identities

$$ (l-1)l\,\tau_{jk}X_{A_{l-2}}\mathcal{Y}^{lm*}_{jkA_{l-2}} = (\partial_t{}^2\tau_{00})X_{A_l}\mathcal{Y}^{lm*}_{A_l} - (\tau_{jk}X_{A_l})_{,jk}\mathcal{Y}^{lm*}_{A_l} + 2l(\tau_{jk}X_{A_{l-1}})_{,j}\mathcal{Y}^{lm*}_{kA_{l-1}}; \quad (5.16a) $$

$$ (l-1)\epsilon_{pqj}\mathcal{Y}^{lm*}_{kqA_{l-2}}\tau_{jk}x_p X_{A_{l-2}} = -\epsilon_{pqj}\mathcal{Y}^{lm*}_{qA_{l-1}}(\partial_t\tau_{0j})x_p X_{A_{l-1}} + \epsilon_{pqj}\mathcal{Y}^{lm*}_{qA_{l-1}}(\tau_{jk}x_p X_{A_{l-1}})_{,k}. \quad (5.16b) $$

By inserting these identities into Eqs. (5.14) and integrating out the divergences to zero, we obtain

$$ I^{lm} = \frac{16\pi}{(2l+1)!!}\Big(\frac{(l+1)(l+2)}{2(l-1)l}\Big)^{1/2}\mathcal{Y}^{lm*}_{A_l}\int \tau_{00}X_{A_l}\, d^3x, \quad (5.17a) $$

$$ S^{lm} = \frac{-32\pi}{(2l+1)!!}\Big(\frac{l(l+2)}{2(l-1)(l+1)}\Big)^{1/2} \times \mathcal{Y}^{lm*}_{jA_{l-1}}\int \epsilon_{jpq}x_p(-\tau_{0q})X_{A_{l-1}}\, d^3x. \quad (5.17b) $$

By then comparing with Eqs. (2.11), (2.24a), and (2.23b) we obtain

$$ I^{lm} = \frac{16\pi}{(2l+1)!!}\Big(\frac{(l+1)(l+2)}{2(l-1)l}\Big)^{1/2}\int \tau_{00}Y^{lm*}r^l\, d^3x, \quad (5.18a) $$

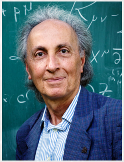
- Multipole moments given by divergent integrals [Epstein & Wagoner 1975; Thorne 1980]

$$ \sim \int \mathrm{d}^3\mathbf{x}\, r^\ell\, \hat{n}^L(\theta,\varphi)\, \underbrace{\tau^{\mu\nu}(\mathbf{x},u)}_{\text{pseudo-tensor}} $$

- PN iteration yields divergent Poisson-like integrals from 3PN [Anderson & DeCanio 1975; Kerlick 1980]
- Treatment of point-particles in non-linear GR poorly understood [Infeld & Plebański 1960]
- Tails, memory, tails-of-tails, ... completely ignored

Multipolar-post-Minkowskian expansion (BDI)

[Blanchet & Damour 1986, 1988, 1989, 1992; Damour & Iyer 1991; Blanchet 1987, 1995, 1996, 1998abc]



1. Start from Thorne's linearized solution $h_1^{\mu\nu}[M_L, S_L]$
2. Look for the general multipolar expansion outside the source in the form of a post-Minkowskian expansion

$$h^{\mu\nu}[M_L, S_L] = G\, h_1^{\mu\nu} + G^2\, h_2^{\mu\nu} + \cdots$$

3. Iterate that multipole expansion using a regularization scheme based on analytic continuation in $B \in \mathbb{C}$

$$\underbrace{\mathop{\text{Finite Part}}_{B=0} \Box_{\text{ret}}^{-1}\left[(r/r_0)^B f\right]}_{\text{treats UV divergence when } r \to 0}$$

4. One obtains the most general solution of the field equations outside the source and the expansion at future null infinity ($\mathcal{J}^+$) is in agreement with the Bondi-Sachs [1962] formalism [Blanchet 1987]

A powerful integration formula

- To each post-Minkowskian order one has to solve

$$\Box\Psi_L = \underbrace{\hat{n}_L\, S(r, t - r/c)}_{\text{source with given multipolarity } \ell}$$

- To cure UV divergences one defines the regularized source

$$S^B(r, t - r/c) \equiv \left(\frac{r}{r_0}\right)^B S(r, t - r/c)$$

- The solution is obtained by analytic continuation in B as

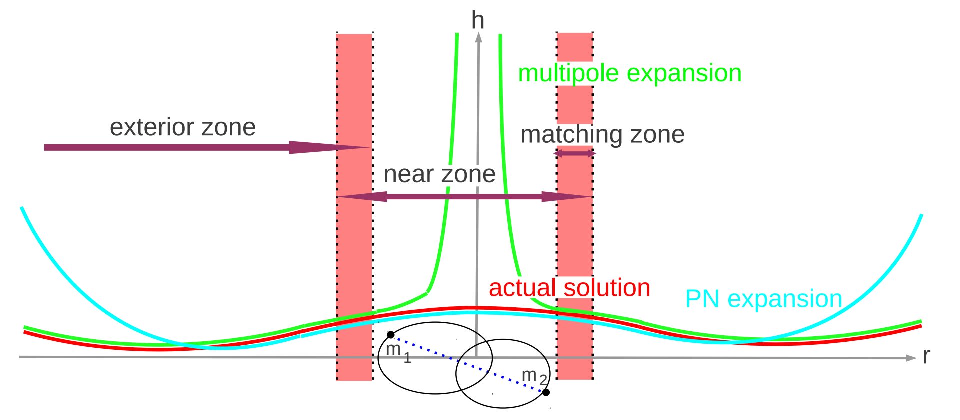
$$\Psi_L = \mathop{\mathrm{FP}}_{B=0} \int_{-\infty}^{t-r/c} \mathrm{d}u\, \hat{\partial}_L \left[\frac{R^B\left(\frac{t-u-r}{2}, u\right) - R^B\left(\frac{t-u+r}{2}, u\right)}{r}\right]$$

where $R^B(\rho, u) \equiv \rho^\ell \int_0^\rho \mathrm{d}r\, \frac{(\rho - r)^\ell}{\ell!} \left(\frac{2}{r}\right)^{\ell-1} S^B(r, u)$

The MPM-PN formalism

[Blanchet 1998; Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2005]

The MPM outer metric is matched to the PN inner field of the source

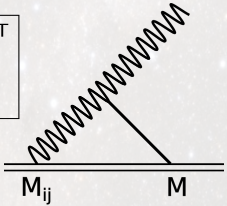


matching equation $\Longrightarrow$ $\overline{\mathcal{M}(h)} = \mathcal{M}(\bar{h})$

The gravitational wave tail effect [Blanchet & Damour 1988, 1992]

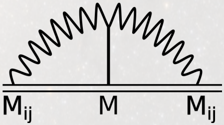
- In the far zone a 1.5PN effect beyond the quadrupole formula

$$h^{\text{tail}}_{ij} = \frac{2G}{c^4 r}\left[\frac{2GM}{c^3}\int_{-\infty}^{t-r/c} \mathrm{d}u\, M^{(4)}_{ij}(u)\ln\left(\frac{t-r/c-u}{P}\right)\right]^{\text{TT}}$$



- In the near zone a 4PN effect made of a radiation reaction part and a conservative part modifying the particle action [Foffa & Sturani 2012]

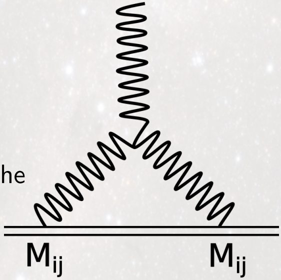
$$S^{\text{tail}} = \frac{G^2 M}{5c^8}\,\text{Pf}\iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|}\, M^{(3)}_{ij}(t)\, M^{(3)}_{ij}(t')$$



The non-linear memory effect

- Coupling between two quadrupole moments $M_{ij} \times M_{ij}$
[Blanchet, habilitation thesis 1991; Blanchet & Damour 1992]

$$h^{\text{mem}}_{ij} = \frac{2G}{c^4 r}\left[-\frac{2G}{7c^5}\int_{-\infty}^{t-r/c} \mathrm{d}u\, M^{(3)}_{k\langle i}(u)\, M^{(3)}_{j\rangle k}(u)\right]^{\text{TT}}$$



- Exact derivation based on the asymptotic behaviour of the field at future null and time like infinity [Christodoulou 1992]

- Physical interpretation: GW re-emission of gravitons
[Thorne 1992; Will & Wiseman 1992; Favata 2009, 2011; Nichols 2017]

$$h^{\text{mem}}_{ij} = -\frac{4G}{c^4 r}\left[\int \mathrm{d}\Omega'\, \frac{n'^i n'^j}{1 - \boldsymbol{n}\cdot\boldsymbol{n}'}\,\frac{\mathrm{d}E^{\text{GW}}}{\mathrm{d}\Omega'}(\boldsymbol{n}', t - r/c)\right]^{\text{TT}}$$

3.5PN: previously the state-of-the-art on GW field

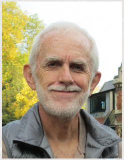
$$4\pi R^2 \mathfrak{S} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{J}_{\mu\nu}^{2} - \frac{1}{3}\left(\sum_{\mu} \dddot{J}_{\mu\mu}\right)^{2}\right].$$

1.5PN
- [Epstein & Wagoner 1975; Wagoner & Will 1976] — EW moments
- [Blanchet & Damour 1989; Blanchet & Schäfer 1989, 1993] — BD moments
- [Poisson 1993; Wiseman 1993]

2.5PN
- [Blanchet, Damour & Iyer 1995] — MPM-PN formalism
- [Will & Wiseman 1996] — DIRE formalism
- [BDIWW 1995; BIWW 1996]

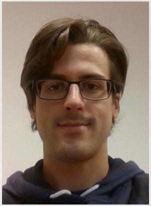
3.5PN
- [Blanchet 1998] — MPM-PN formalism
- [BIJ 2002; BFIJ 2002; BDEI 2005]

- The Direct Integration of the Relaxed Equations (DIRE) method [Will & Wiseman 1996] is equivalent to the MPM-PN formalism for general matter systems [Blanchet 2004]



The 4.5PN phasing of compact binaries

Based on recent collaborations with

Guillaume Faye, **Quentin Henry**, **François Larrouturou**,
Tanguy Marchand & **David Trestini**

Field equations and Green's function in d dimensions

- Einstein's field equations in harmonic (de Donder) coordinates

$$\partial_\nu h^{\mu\nu} = 0 \qquad \text{(harmonic gauge condition)}$$

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu} \qquad \text{(wave equation in } D = d+1 \text{ dimensions)}$$

$$\tau^{\mu\nu} = |g| T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu} \qquad \text{(matter + gravitation pseudo tensor)}$$

- The Green's function is implemented in the real space-time domain

$$G_{\text{ret}}(\mathbf{x}, t) = -\frac{\tilde{k}}{4\pi}\frac{\theta(t-r)}{r^{d-1}}\gamma_{\frac{1-d}{2}}\left(\frac{t}{r}\right)$$

$$\gamma_{\frac{1-d}{2}}(z) \equiv \frac{2\sqrt{\pi}}{\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2}-1)}\left(z^2-1\right)^{\frac{1-d}{2}}$$

The multipole expansion outside the matter source

- The multipole expansion $\mathcal{M}(h^{\mu\nu})$ is a retarded solution the *vacuum* field equations $\Box\mathcal{M}(h^{\mu\nu}) = \mathcal{M}(\Lambda^{\mu\nu})$ valid formally everywhere except at $r = 0$

$$\mathcal{M}(h^{\mu\nu}) = \overbrace{\mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\mathrm{ret}}\left[\left(\frac{r}{r_0}\right)^B}^{\text{regularization when } r\to 0} \mathcal{M}(\Lambda^{\mu\nu})\right] \underbrace{-\frac{4G}{c^4}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\,\hat{\partial}_L\widetilde{\mathcal{F}}^{\mu\nu}_L}_{\text{retarded homogeneous solution}}$$

$$\Box\widetilde{\mathcal{F}}^{\mu\nu}_L(r,t) = 0 \quad \text{in } d \text{ dimensions}$$

- The multipole moment functions $\mathcal{F}^{\mu\nu}_L(t)$ are symmetric-trace-free (STF) with respect to their ℓ indices $L \equiv i_1 \cdots i_\ell$

$$\widetilde{\mathcal{F}}^{\mu\nu}_L(r,t) = \frac{\tilde{k}}{r^{d-2}}\int_1^{+\infty}\mathrm{d}z\,\gamma_{\frac{1-d}{2}}(z)\,\mathcal{F}^{\mu\nu}_L(t-zr)$$

The multipole expansion matched to the PN source

- Explicit matching to a general extended PN isolated source gives

$$\mathcal{F}_L^{\mu\nu}(t) = \overbrace{\mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}^d\mathbf{x} \left(\frac{r}{r_0}\right)^B}^{\text{IR regularization}} \hat{x}_L \int_{-1}^{1} \mathrm{d}z\, \delta_\ell^{(d)}(z) \underbrace{\overline{\tau}^{\mu\nu}(\mathbf{x}, t + zr)}_{\text{PN expansion of the pseudo-tensor}}$$

$$\delta_\ell^{(d)}(z) \equiv \frac{\Gamma\left(\frac{d}{2} + \ell\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d-1}{2} + \ell\right)} \left(1 - z^2\right)^{\frac{d-3}{2} + \ell}$$

- The $B\varepsilon$ regularization
  - first apply the limit $B \to 0$ in generic dimensions $d = 3 + \varepsilon$
  - then the usual dimensional regularization when $\varepsilon \to 0$

Mass and current irreducible multipole moments

[Henry, Faye and Blanchet 2020]

- The irreducible decomposition of $\mathcal{F}_L^{\mu\nu}$ reads (with $\langle\cdots\rangle$ the STF projection)

$$
\begin{aligned}
\mathcal{F}_L^{00} &= R_L \\
\mathcal{F}_L^{0i} &= T_{iL}^{(+)} + T^{(0)}_{i|\langle i_\ell L-1\rangle} + \delta_{i\langle i_\ell} T^{(-)}_{L-1\rangle} \\
\mathcal{F}_L^{ij} &= U_{ijL}^{(+2)} + \underset{L}{\mathrm{STF}}\,\underset{ij}{\mathrm{STF}}\Big[U^{(+1)}_{i|i_\ell jL-1} + \delta_{ii_\ell} U^{(0)}_{jL-1} + \delta_{ii_\ell} U^{(-1)}_{j|i_{\ell-1}L-2} \\
&\qquad + \delta_{ii_\ell}\delta_{ji_{\ell-1}} U^{(-2)}_{L-2} + W_{ij|i_\ell i_{\ell-1}L-2}\Big] + \delta_{ij} V_L
\end{aligned}
$$

- The "mass-type" contributions R_L, $T^{(+)}_{L+1}$, $T^{(-)}_{L-1}$, $U^{(+2)}_{L+2}$, $U^{(0)}_L$, $U^{(-2)}_{L-2}$, V_L are STF in the ordinary sense
- The "current-type" contributions $T^{(0)}_{i|\langle i_\ell L-1\rangle}$, $U^{(+1)}_{i|i_{\ell+1}L}$, $U^{(-1)}_{i|i_{\ell-1}L-2}$ have more complicated symmetries

Mass and current irreducible multipole moments

[Henry, Faye and Blanchet 2020]

- The mass moment M_L is given by the usual STF moment, but the generalization of the current moment involves two tensors $S_{i|L}$ and $K_{ij|L}$ having the symmetries of mixed Young tableaux

$$M_L = \begin{array}{|c|c|c|}\hline i_\ell & \dots & i_1 \\ \hline \end{array}$$

$$S_{i|L} = \begin{array}{|c|c|c|c|}\hline i_\ell & i_{\ell-1} & \dots & i_1 \\ \hline i & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \qquad K_{ij|L} = \begin{array}{|c|c|c|c|c|}\hline i_\ell & i_{\ell-1} & i_{\ell-2} & \dots & i_1 \\ \hline j & i & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}$$

- The tensor $K_{ij|L}$ is absent in 3 dimensions

$$\sharp(\text{components}) = \frac{(d-3)d(d-1)_{\ell-2}(2\ell+d-2)(\ell+d-1)}{2\ell(\ell+1)(\ell-2)!}$$

and plays no role with dimensional regularization

The irreducible mass quadrupole moment

- Posing

$$\overline{\Sigma} \equiv \frac{2}{d-1}\frac{(d-2)\overline{\tau}^{00}+\overline{\tau}^{ii}}{c^2} \qquad \overline{\Sigma}^i \equiv \frac{\overline{\tau}^{i0}}{c} \qquad \overline{\Sigma}^{ij} \equiv \overline{\tau}^{ij}$$

$$\overline{\Sigma}_{[\ell]}(\mathbf{x},t) = \int_{-1}^{1} \mathrm{d}z\, \delta_\ell^{(d)}(z)\, \overline{\Sigma}(\mathbf{x},t+zr)$$

$$\boxed{\begin{aligned} M_{ij} = \frac{d-1}{2(d-2)} \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^d\mathbf{x} \left(\frac{r}{r_0}\right)^B \Bigg\{ & \hat{x}^{ij}\,\overline{\Sigma}_{[2]} - \frac{4(d+2)}{d(d+4)}\,\hat{x}^{ijk}\,\dot{\overline{\Sigma}}^k_{[3]} \\ & + \frac{2(d+2)}{d(d+1)(d+6)}\,\hat{x}^{ijkl}\,\ddot{\overline{\Sigma}}^{kl}_{[4]} \\ & - \frac{4(d-3)(d+2)}{d(d-1)(d+4)}\, B\, \hat{x}^{ijk}\,\frac{x^l}{r^2}\,\overline{\Sigma}^{kl}_{[3]} \Bigg\} \end{aligned}}$$

- The $B\varepsilon$ regularization is systematically applied (the limit $B \to 0$ is finite)

Techniques to compute the 4PN mass quadrupole

- Method of super-potentials

$$\int \mathrm{d}^3\mathbf{x}\, r^B\, \hat{x}_L \overbrace{\phi}^{\text{linear potential}} \underbrace{P}_{\text{difficult potential}} = \int \mathrm{d}^3\mathbf{x}\, r^B \Big(\Psi_L^\phi\, \Delta P + \underbrace{\partial_i \Big[\partial_i \Psi_L^\phi P - \Psi_L^\phi \partial_i P \Big]}_{\text{yields a surface term}} \Big)$$

where Ψ_L^ϕ is obtained from the super-potentials ϕ_{2k} of $\phi = \phi_0$ as

$$\boxed{\Psi_L^\phi = \Delta^{-1}\big(\hat{x}_L\, \phi\big) = \sum_{k=0}^{\ell} \frac{(-2)^k \ell!}{(\ell-k)!}\, x_{\langle L-K} \partial_{K\rangle} \overbrace{\phi_{2k+2}}^{\Delta\phi_{2k+2} = \phi_{2k}}}$$

- Method of surface integrals

$$\boxed{\mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}^3\mathbf{x}\, r^B \hat{x}_L\, \Delta G = -(2\ell+1) \int \mathrm{d}\Omega\, \hat{n}_L X_\ell(\boldsymbol{n})}$$

where X_ℓ is the coefficient of $r^{-\ell-1}$ in the expansion of G when $r \to +\infty$

- Schwartz distributional derivatives in d dimensions systematically applied

Completion of the 4PN mass quadrupole moment

[Larrouturou, Blanchet, Henry & Faye 2021ab]

- All UV divergences treated by dimensional regularization and all UV poles shown to be renormalized by appropriate shifts of the particles' worldlines
- Presence at 4PN order of a non-local-in-time term associated with tail radiation mode and containing a crucial IR pole
- IR divergences (poles $\propto \frac{1}{d-3}$) appear already at 3PN order but are cancelled (as well as the finite part beyond) by poles coming from "tails-of-tails" propagating in the wave zone
- At 4PN order the IR poles are cancelled by more complicated "tails-of-memory" but there remains a crucial finite contribution specifically due to dimensional regularization
- Finally we have obtained the finite renormalized 4PN quadrupole moment of compact binaries ready to be used for 4PN/4.5PN templates

Non-linear interactions at 4.5PN order

$$U_{ij}(u) = M^{(2)}_{ij}(u) + \underbrace{\frac{G M}{c^3} \int_0^{+\infty} \mathrm{d}\tau\, M^{(4)}_{ij}(u-\tau) \left[2\ln\left(\frac{c\tau}{2b_0}\right) + \frac{11}{6}\right]}_{\text{1.5PN tail}}$$

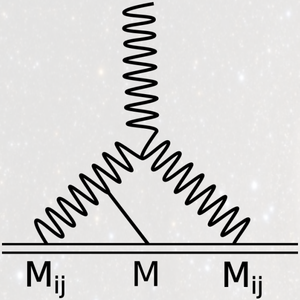
$$+ \frac{G}{c^5}\left\{\underbrace{\frac{2}{7}\int_0^{+\infty} \mathrm{d}\tau\, M^{(3)}_{a<i} M^{(3)}_{j>a}(u-\tau)}_{\text{2.5PN memory}} + \cdots\right\}$$

$$+ \underbrace{\frac{G^2 M^2}{c^6} \int_0^{+\infty} \mathrm{d}\tau\, M^{(5)}_{ij}(u-\tau) \left[2\ln^2\left(\frac{c\tau}{2r_0}\right) + \frac{57}{35}\ln\left(\frac{c\tau}{2r_0}\right) + \frac{124627}{22050}\right]}_{\text{3PN tails-of-tail [Blanchet 1998]}}$$

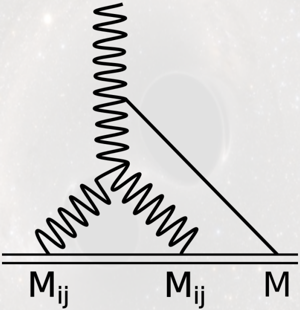
$$+ \frac{G^2}{c^8}\left\{ \quad \text{4PN tails-of-memory} \quad M \times M_{ij} \times M_{ij} \quad \right\}$$

$$+ \underbrace{\frac{G^3 M^3}{c^9} \int_0^{+\infty} \mathrm{d}\tau\, M^{(6)}_{ij}(u-\tau) \left[\frac{4}{3}\ln^3\left(\frac{c\tau}{2r_0}\right) + \cdots + \frac{129268}{33075}\right]}_{\text{4.5PN tail-of-tail-of-tail [Marchand, Blanchet \& Faye 2017; Messina \& Nagar 2017]}} + \mathcal{O}\left(\frac{1}{c^{10}}\right)$$

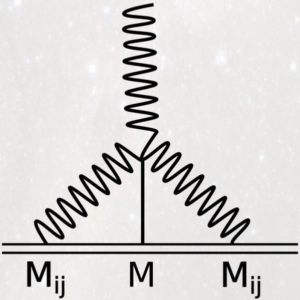
Gravitational-wave tails of memory



$(M_{ij} \times M) \times M_{ij}$



$(M_{ij} \times M_{ij}) \times M$



$M_{ij} \times M \times M_{ij}$

Gravitational-wave tails of memory [Trestini & Blanchet 2023]

- Computation performed using the MPM construction in radiative coordinates which avoids far zone logarithms which plague harmonic coordinates

$$U_{ij}^{M\times M_{ij}\times M_{ij}} = \frac{2G^2 M}{7c^8}\Bigg\{\underbrace{\int_0^{+\infty}\mathrm{d}\rho\, M^{(4)}_{a\langle i}(u-\rho)\int_0^{+\infty}\mathrm{d}\tau\, M^{(4)}_{j\rangle a}(u-\rho-\tau)\ln\left(\frac{\tau}{2r_0}\right)}_{\text{“genuine” tail-of-memory}}$$

$$+\underbrace{\int_0^{+\infty}\mathrm{d}\tau\,\left(M^{(3)}_{a\langle i}M^{(4)}_{j\rangle a}\right)(u-\tau)\left[-15\ln\left(\frac{\tau}{2b_0}\right)-10\ln\left(\frac{\tau}{2r_0}\right)\right]}_{\text{tail-like term}}$$

$$+\cdots\cdots\cdots$$

$$\underbrace{-\,8M^{(2)}_{a\langle i}\int_0^{+\infty}\mathrm{d}\tau\, M^{(5)}_{j\rangle a}(u-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{27521}{5040}\right]}_{\text{tail-like term}}\Bigg\}$$

- The 4PN “genuine” tail-of-memory (containing the memory effect) can be retrived from general expressions for the memory effect

Gravitational-wave tails of memory [Trestini & Blanchet 2023]

1 The quadrupole memory in the waveform of any source can be expressed as

$$U_{ij}^{\mathrm{mem}} = -\frac{2G}{7c^5}\int_0^{+\infty} \mathrm{d}\tau\, U^{(1)}_{k\langle i}(u-\tau)\, U^{(1)}_{j\rangle k}(u-\tau)$$

2 Computing the dominant $\mathrm{M}_{ij} \times \mathrm{M}_{ij}$ interaction followed by the subdominant one $\mathrm{M} \times \mathrm{M}_{ij} \times \mathrm{M}_{ij}$ we need the radiative quadrupole at 1.5PN order

$$U_{ij} = M^{(2)}_{ij} + \frac{2GM}{c^3}\int_0^{+\infty} \mathrm{d}\tau\, M^{(4)}_{ij}(u-\tau)\left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{11}{12}\right] + \mathcal{O}\left(\frac{1}{c^5}\right)$$

3 Injecting it into U_{ij}^{mem} we obtain at 4PN order

$$\boxed{\begin{aligned} U_{ij}^{\mathrm{mem}} &= -\frac{2G}{7c^5}\int_0^{+\infty} \mathrm{d}\tau\, M^{(3)}_{a\langle i}(u-\tau)\, M^{(3)}_{j\rangle a}(u-\tau) \\ &\quad - \frac{8G^2M}{7c^8}\int_0^{+\infty} \mathrm{d}\rho\, M^{(3)}_{a\langle i}(u-\rho)\int_0^{+\infty} \mathrm{d}\tau \ln\left(\frac{c\tau}{2b_0}\right) M^{(5)}_{j\rangle a}(u-\rho-\tau) \end{aligned}}$$

4 This is perfectly consistent with our direct 4PN calculation

Tail modulation of the GW phase at the 4PN order

[Wiseman 1993; Blanchet & Schäfer 1993; Blanchet, Iyer, Will & Wiseman 1996]

1. Because of GW tails the GW phase ψ differs from the orbital phase ϕ by a logarithmic, tail-induced phase modulation

$$\psi = \phi - \frac{2GM\,\omega}{c^3} \ln\left(\frac{\omega}{\omega_0}\right)$$

2. The GW frequency $\Omega = \dot{\psi}$ is shifted with respect to the orbital one $\omega = \dot{\phi}$

$$\Omega = \omega - \frac{2GM\,\dot{\omega}}{c^3} \left[\ln\left(\frac{\omega}{\omega_0}\right) + 1\right]$$

$$\boxed{\Omega = \omega \left\{ 1 \overbrace{- \frac{192}{5}\,\nu \left(\frac{Gm\omega}{c^3}\right)^{8/3} \left[\ln\left(\frac{\omega}{\omega_0}\right) + 1\right]}^{\text{4PN effect}} + \mathcal{O}\left(\frac{1}{c^{10}}\right) \right\}}$$

3. Expressing the flux and modes in terms of the directly observable GW phase ψ and frequency Ω we find that all arbitrary constants cancel out at 4PN order

Post-adiabatic calculation of the tail integral

1. The tail integral arises at the 1.5PN order

$$\propto \int_0^{+\infty} \mathrm{d}\tau \, \big[\omega(u-\tau)\big]^\alpha \, \overbrace{\mathrm{e}^{-\mathrm{i}n\,\phi(u-\tau)}}^{\text{oscillating phase}} \, \ln\left(\frac{\tau}{\tau_0}\right)$$

2. At 4PN order we must include a 2.5PN post-adiabatic correction

$$\boxed{\xi(u) \equiv \frac{\dot{\omega}(u)}{\omega^2(u)} = \mathcal{O}\left(\frac{1}{c^5}\right)}$$

3. Changing variable $\tau \longrightarrow v = \xi[\phi(u) - \phi(u-\tau)]$

$$\propto \frac{\mathrm{e}^{-\mathrm{i}n\phi(u)}}{\xi(u)} \int_0^{+\infty} \mathrm{d}v \, \underbrace{\big[\omega(u-\tau(v))\big]^{\alpha-1} \, \mathrm{e}^{\frac{\mathrm{i}n\,v}{\xi(u)}} \, \ln\left(\frac{\tau(v)}{\tau_0}\right)}_{\text{fast oscillating integrand in the limit } \xi(u)\to 0}$$

4. The integral can be computed by replacing the integrand by its expansion when $v \to 0$ which yields the asymptotic post-adiabatic expansion

The 4.5PN GW energy flux for circular orbits

[Blanchet, Faye, Henry, Larrouturou & Trestini 2023ab]

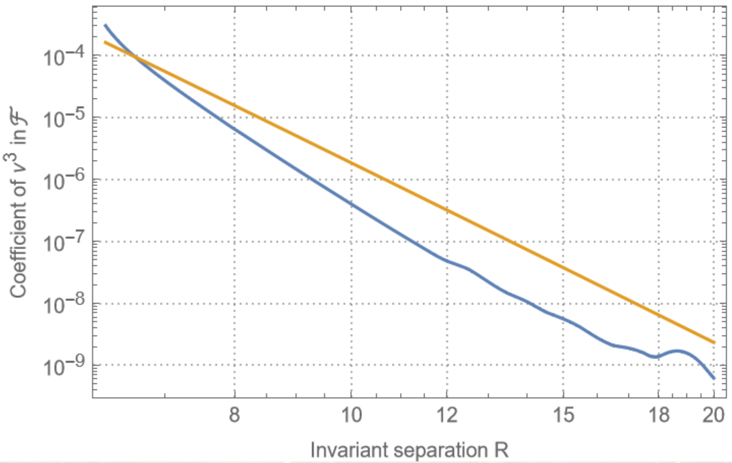
$$
\begin{aligned}
\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} \\
& + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\
& + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_\text{E} - \frac{856}{105}\ln(16\,x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right]x^3 \\
& + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} \\
& + \Bigg[-\frac{323105549467}{3178375200} + \frac{232597}{4410}\gamma_\text{E} - \frac{1369}{126}\pi^2 + \frac{39931}{294}\ln 2 - \frac{47385}{1568}\ln 3 + \frac{232597}{8820}\ln x \\
& \quad + \left(-\frac{1452202403629}{1466942400} + \frac{41478}{245}\gamma_\text{E} - \frac{267127}{4608}\pi^2 + \frac{479062}{2205}\ln 2 + \frac{47385}{392}\ln 3 + \frac{20739}{245}\ln x\right)\nu \\
& \quad + \left(\frac{1607125}{6804} - \frac{3157}{384}\pi^2\right)\nu^2 + \frac{6875}{504}\nu^3 + \frac{5}{6}\nu^4\Bigg]x^4 \\
& + \Bigg[\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\text{E} - \frac{3424}{105}\ln(16\,x) + \left(\frac{2062241}{22176} + \frac{41}{12}\pi^2\right)\nu \\
& \quad - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3\Bigg]\pi x^{9/2} \Bigg\}
\end{aligned}
$$

In the test-mass limit $\nu \to 0$, we exactly retrieve the result of linear black-hole perturbation theory [Tagoshi & Sasaki 1994; Tanaka, Tagoshi & Sasaki 1996]

Comparison with second-order GSF results

[Warburton, Pound, Wardell, Miller & Durkan 2021]

The 4.5PN flux agrees well with recent numerical second-order self-force results



[Courtesy to Adam Pound]

The 4.5PN phase evolution of compact binaries

[Blanchet, Faye, Henry, Larrouturou & Trestini 2023ab]

Apply the energy flux-balance equation $\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}$

$$
\begin{aligned}
\psi = \psi_0 - \frac{x^{-5/2}}{32\nu}\Bigg\{ & 1 + \left(\frac{3715}{1008} + \frac{55}{12}\nu\right)x - 10\pi x^{3/2} \\
& + \left(\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2\right)x^2 + \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi x^{5/2}\ln x \\
& + \Bigg[\frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_E - \frac{856}{21}\ln(16\,x) \\
& \quad + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3\Bigg]x^3 \\
& + \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi x^{7/2} \\
& + \Bigg[\frac{2550713843998885153}{2214468081745920} - \frac{9203}{126}\gamma_E - \frac{45245}{756}\pi^2 - \frac{252755}{2646}\ln 2 - \frac{78975}{1568}\ln 3 - \frac{9203}{252}\ln x \\
& \quad + \left(-\frac{680712846248317}{337983528960} - \frac{488986}{1323}\gamma_E + \frac{109295}{1792}\pi^2 - \frac{1245514}{1323}\ln 2 + \frac{78975}{392}\ln 3 - \frac{244493}{1323}\ln x\right)\nu \\
& \qquad + \left(\frac{7510073635}{24385536} - \frac{11275}{1152}\pi^2\right)\nu^2 + \frac{1292395}{96768}\nu^3 - \frac{5975}{768}\nu^4\Bigg]x^4 \\
& + \Bigg[-\frac{93098188434443}{150214901760} + \frac{1712}{21}\gamma_E + \frac{80}{3}\pi^2 + \frac{856}{21}\ln(16x) \\
& \quad + \left(\frac{1492917260735}{1072963584} - \frac{2255}{48}\pi^2\right)\nu - \frac{45293335}{1016064}\nu^2 - \frac{10323755}{1596672}\nu^3\Bigg]\pi x^{9/2}\Bigg\}
\end{aligned}
$$

Number of cycles contributed by each PN order

[Blanchet, Faye, Henry, Larrouturou & Trestini 2023ab]

Contribution of each PN order to the total number of accumulated cycles

Detector	LIGO/Virgo		ET		LISA	
Masses ($M_\odot$)	1.4×1.4	10×10	1.4×1.4	500×500	$10^5 \times 10^5$	$10^7 \times 10^7$
PN order	cumulative number of cycles					
Newtonian	2 562.599	95.502	744 401.36	37.90	28 095.39	9.534
1PN	143.453	17.879	4 433.85	9.60	618.31	3.386
1.5PN	−94.817	−20.797	−1 005.78	−12.63	−265.70	−5.181
2PN	5.811	2.124	23.94	1.44	11.35	0.677
2.5PN	−8.105	−4.604	−17.01	−3.42	−12.47	−1.821
3PN	1.858	1.731	2.69	1.43	2.59	0.876
3.5PN	−0.627	−0.689	−0.93	−0.59	−0.91	−0.383
4PN	−0.107	−0.064	−0.12	−0.04	−0.12	−0.013
4.5PN	0.098	0.118	0.14	0.10	0.14	0.065

- The PN approximation seems to converge well for comparable masses
- This suggests that systematic errors due to the PN modeling may be dominated by statistical errors and negligible for LISA
- However, this should be confirmed by detailed investigations along the lines [Owen, Haster, Perkins, Cornish & Yunes 2023]