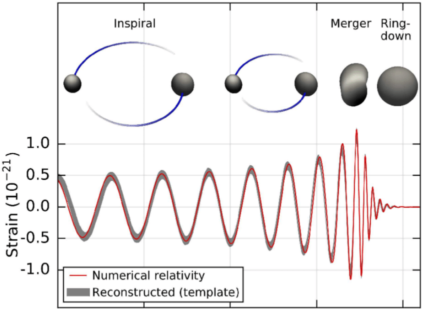
Predicting gravitational waveforms: tails-of-memory and the 4PN waveform

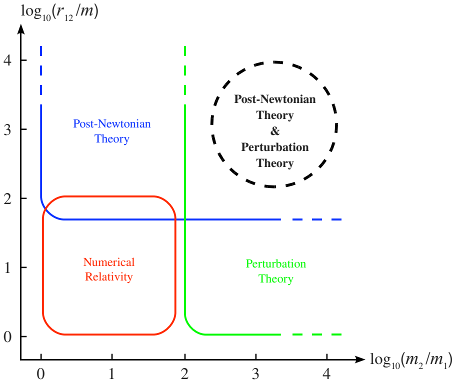
David Trestini
in collaboration with L. Blanchet, G. Faye, Q. Henry and F. Larrouturou

May 30, 2023

Setting and motivation

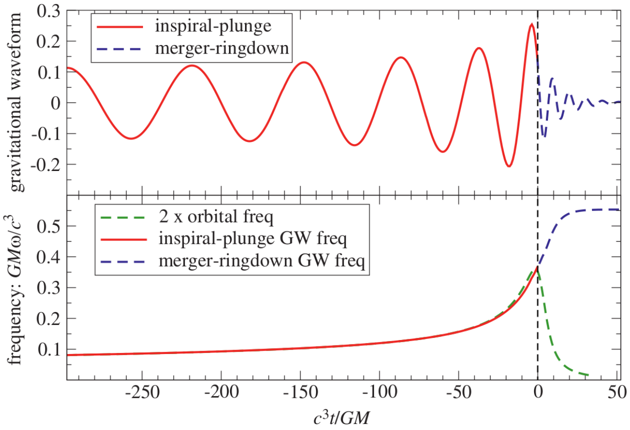
We want to predict the waveform of the inspiral phase using analytical methods. Main observable: the phase of the wave !





Setting and motivation

We want to predict the waveform of the inspiral phase using analytical methods. Main observable: the phase of the wave !



What accuracy do we need?

C. Cutler et al., "The Last Three Minutes" Phys. Rev. Lett. **70** (1993) 20:

> *The PN modulations are far less important than PN contributions to the secular growth of the waves' phase* $\phi = 2\pi \int f \mathrm{d}t$

and

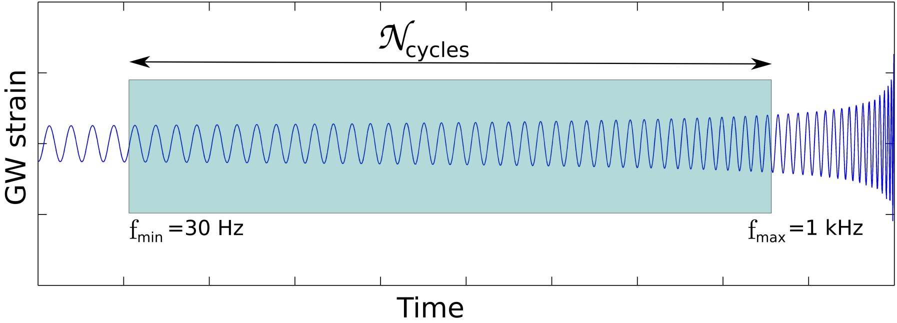
> *Although highly accurate wave form templates will* **not** *be needed when searching for waves, they* **will** *be needed when extracting the waves' information. Making optimal use of the interferometers' data will require general-relativity-based wave form templates whose phasing is correct to within a half cycle or so during the entire frequency sweep from* $\sim$ *10 Hz to* $\sim$ *1000 Hz.*

and

> *It is not at all clear how far beyond 2.5PN the template must be carried to keep its total phase error below a half cycle over the entire range from* $\sim$ *10 Hz to* $\sim$ *1000 Hz.*

Counting cycles

We count the number of cycles $\mathcal{N}_{\text{cycles}}$ in the detector's bandwidth (e.g. $[f_{\text{min}}, f_{\text{max}}] = [30\,\text{Hz}, 1\,\text{kHz}]$)



To use maximal capability of GW detector, we need $\boxed{\Delta\mathcal{N}_{\text{cycles}} < 1/2}$.

Behavior of the PN series

Cumulative contribution to the number of cycles

$$\mathcal{N}_{\text{cycles}} = \mathcal{N}^{\text{N}}_{\text{cycles}} + \mathcal{N}^{\text{1PN}}_{\text{cycles}} + \mathcal{N}^{\text{1.5PN}}_{\text{cycles}} + \dots$$

Approximate frequency bands by step functions $\Longrightarrow$ very crude, do better!

$\mathcal{N}_{\text{cycles}}$	LIGO/Virgo		ET		LISA	
f-band	$[30, 10^3]$ Hz		$[1, 10^4]$ Hz		$[10^{-4}, 10^{-1}]$ Hz	
$M_\odot$	1.4×1.4	10×10	1.4×1.4	500×500	$10^5 \times 10^5$	$10^7 \times 10^7$
N	$2\,562.599$	95.502	$744\,401.36$	37.90	$28\,095.39$	9.534
1PN	143.453	17.879	$4\,433.85$	9.60	618.31	3.386
1.5PN	-94.817	-20.797	$-1\,005.78$	-12.63	-265.70	-5.181
2PN	5.811	2.124	23.94	1.44	11.35	0.677
2.5PN	-8.105	-4.604	-17.01	-3.42	-12.47	-1.821
3PN	1.858	1.731	2.69	1.43	2.59	0.876
3.5PN	-0.627	-0.689	-0.93	-0.59	-0.91	-0.383

$\Longrightarrow$ can 4PN and 4.5PN contribute to more than a half cycle ?

Action and field equations in GR

Define the quantity: $h^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu} - \eta^{\mu\nu}$

The Einstein field equations (without cosmological constant) can be rewritten equivalently in the Landau-Lifschitz formulation:

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}(-g)T^{\mu\nu} + \Lambda^{\mu\nu}[h]$$
$$\partial_\nu h^{\mu\nu} = 0$$

The first equation is often written $\Box h^{\mu\nu} = (16\pi G/c^4)\tau^{\mu\nu}$, where we introduced the Landau-Lifschitz pseudo-tensor

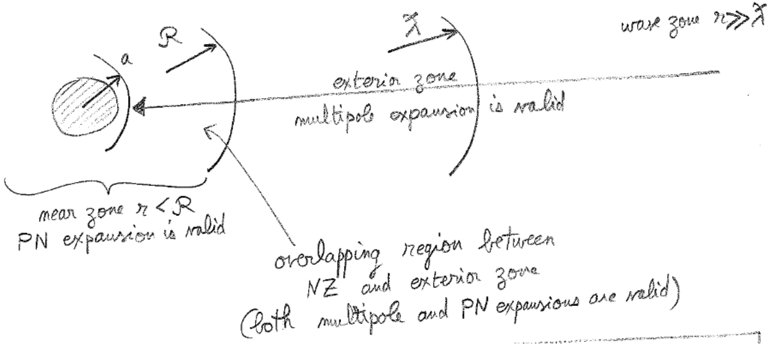
$$\tau^{\mu\nu} = (-g)T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}[h]$$

N.B.: $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$

The two expansions

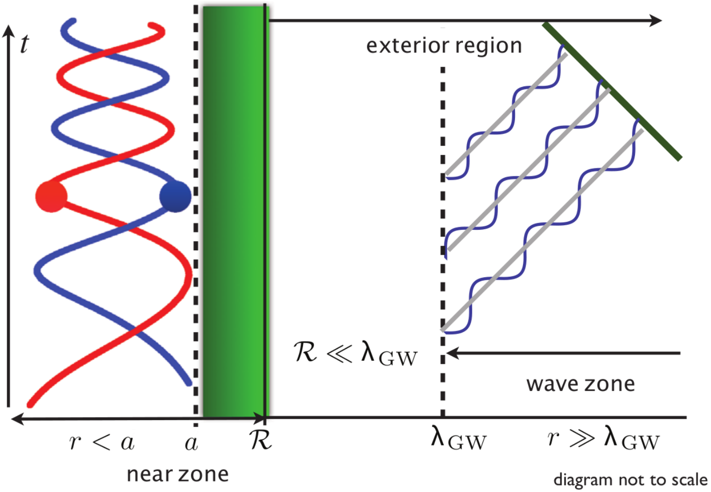
$\Box h^{\mu\nu} = (16\pi G/c^4)\tau^{\mu\nu}$ valid everywhere, but no exact solution for the wave generation problem. Two different approximations:

- near the source, assume small velocities ($v/c \ll 1 \Rightarrow$ **PN expansion**), valid only for small retardations ($r \ll \lambda_{\mathsf{GW}}$)
- in the exterior vacuum ($T^{\mu\nu} = 0$), not valid inside the matter source, assume $h \ll 1 \Rightarrow$ **post-Minkowskian (PM) expansion**



Near zone vs. exterior vacuum zone

Institut d'Astrophysique de Paris (SU) - Laboratoire Univers et Théories (Observatoire de Paris)



Interior PN solution

In the near zone, i.e. for $r < \mathcal{R} \ll \lambda_{\text{GW}}$, we want to solve

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}(-g)T^{\mu\nu} + \Lambda^{\mu\nu}[h] \qquad \text{and} \qquad \partial_\nu h^{\mu\nu} = 0$$

Assume matter is a fluid with $v \ll c$.

Post-Newtonian (PN) expansion $\equiv$ expansion in powers of $(v/c)^n$.
$\longrightarrow$ nPN order $\Longleftrightarrow$ correction in $(v/c)^{2n}$ to leading order

Since $r \ll \lambda_{\text{GW}}$, expand in retardations: $f(t - r/c) = f(t) - (r/c)f'(t) + \ldots$
(note that this blows up when $r \to \infty$)

This can be solved iteratively, order by order (though as some point one must include radiation effects)

Point particles

Change view: fluid becomes 2 point-particles with trajectories $y_1(t)$ and $y_2(t)$, and velocities $v_A(t) = dy_A(t)/dt$ (and define $v_A^\mu \equiv (c, v_A^i)$).

$$T^{\mu\nu}(t,\vec{x}) = \frac{1}{\sqrt{-g}} \sum_{A=1,2} \frac{m_A v_A^\mu(t) v_A^\nu(t)}{\sqrt{-\left. g_{\alpha\beta}\right|_{\vec{y}_A(t)} \frac{v_A^\alpha(t) v_A^\beta(t)}{c^2}}} \delta(\vec{x} - \vec{y}_A(t))$$

PN expansion simplifies it greatly, namely at lowest order

$$T^{\mu\nu}(t,\vec{x}) = m_1 v_1^\mu(t) v_1^\nu(t) \delta(\vec{x} - \vec{y}_1(t)) + (1 \leftrightarrow 2) + \mathcal{O}\left(1/c^2\right)$$

Denote total mass m, relative velocity v_{12} and separation r_{12}

Virial theorem $\Longrightarrow$ $\boxed{(v_{12}/c)^2 \approx (Gm)/(r_{12}c^2)}$

At higher order, evaluate divergent metric at location of particle

$\Longrightarrow$ need regularization scheme to remove infinite self-interaction

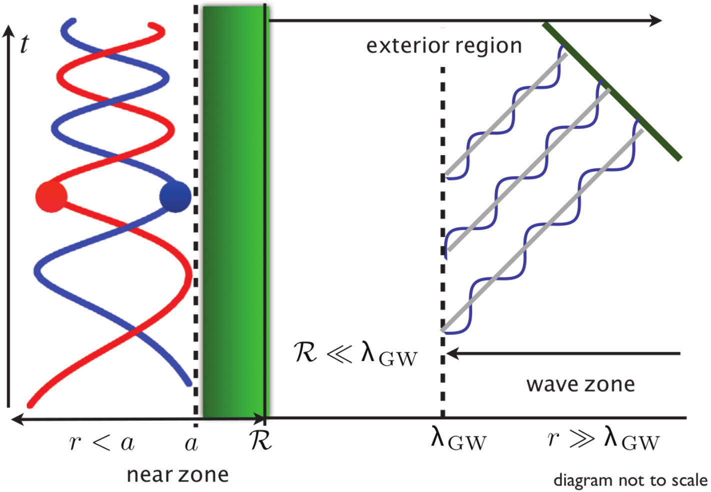
Equations of motion

In PN expansions, time-derivatives lead to the appearance of generalized accelerations $\mathrm{d}^k v_A^i/\mathrm{d}t^k$.

On shell, use "equations of motion" (EOM), e.g. at 1PN:

$$
\begin{aligned}
\vec{a}_A = & \sum_{B\neq A} \frac{G m_B \vec{n}_{BA}}{r_{AB}^2} \\
& + \frac{1}{c^2} \sum_{B\neq A} \frac{G m_B \vec{n}_{BA}}{r_{AB}^2} \bigg[v_A^2 + 2 v_B^2 - 4(\vec{v}_A \cdot \vec{v}_B) - \frac{3}{2} (\vec{n}_{AB} \cdot \vec{v}_B)^2 \\
& \quad - 4 \sum_{C\neq A} \frac{G m_C}{r_{AC}} - \sum_{C\neq B} \frac{G m_C}{r_{BC}} + \frac{1}{2} ((\vec{x}_B - \vec{x}_A) \cdot \vec{a}_B) \bigg] \\
& + \frac{1}{c^2} \sum_{B\neq A} \frac{G m_B}{r_{AB}^2} \left[\vec{n}_{AB} \cdot (4\vec{v}_A - 3\vec{v}_B) \right] (\vec{v}_A - \vec{v}_B) \\
& + \frac{7}{2c^2} \sum_{B\neq A} \frac{G m_B \vec{a}_B}{r_{AB}} + O(c^{-4})
\end{aligned}
$$

Near zone vs. exterior vacuum zone



Linearized exterior vacuum solution

Einstein equations in vacuum $\Box h^{\mu\nu} = \Lambda^{\mu\nu}[h]$ and $\partial_\nu h^{\mu\nu} = 0$

First step: solve the linear vacuum equations

$$\boxed{\begin{aligned} \Box h_1^{\mu\nu} &= 0 \\ \partial_\nu h_1^{\mu\nu} &= 0 \end{aligned}}$$

Physical conditions: stationary in the past, (i.e. no incoming radiation)
asymptotic flatness ...

Most general solution: canonical solution + residual gauge freedom

$$\boxed{h_1^{\mu\nu} = k_1^{\mu\nu}[\mathrm{I}_L, \mathrm{J}_L] + \partial\varphi_1^{\mu\nu}[\mathrm{W}_L, \mathrm{X}_L, \mathrm{Y}_L, \mathrm{Z}_L]}$$

where used shorthand $\partial\theta^{\mu\nu} \equiv \partial^\mu\theta^\nu + \partial^\mu\theta^\nu - \eta^{\mu\nu}\partial_\rho\theta^\rho$ and rescaled by G, i.e. $h = Gh_1 +$ (corrections).

Multipolar expansion notations

3+1 decomposition: $x^\mu = (ct, x^i)$ where $x^i = r\, n^i(\theta, \phi)$
Roman spatial indices: upstairs = downstairs (background 3-metric δ_{ij})
Multipolar expansion:

$$k_1^{00} = -\frac{4}{c^2} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \partial_L \left[\frac{1}{r} \mathrm{I}_L(t - r/c) \right]$$

- $L \equiv i_1 ... i_\ell$: a multi-index composed of ℓ spacelike indices
- $\partial_i \equiv \partial / \partial x^i$ is a spatial derivative. Then $\partial_L \equiv \partial_{i_1} ... \partial_{i_\ell}$
- $\mathrm{I}_L(u)$: mass-type source ℓ-polar moment (STF in all its indices)
- $u = t - r/c$: retarded time variable, and $f^{(n)}(u) \equiv \partial_t^n f(t - r/c)$

Note that

- monopole = constant ADM mass of the spacetime, i.e. $\mathrm{I}(u) = \mathrm{M}$
- dipole = constant, i.e. $\mathrm{I}_i^{(1)}(u) = 0$ (in COM frame, $\mathrm{I}_i(u) = 0$).

For pedagogy, truncate sum to $\ell \leq 2$ in COM frame

$$k_1^{00} = -\frac{4}{c^2}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\mathrm{I}_L(t-r/c)\right]$$

Since $\partial_i r = n_i$ and $\partial_i n_j = (\delta^{ij} - n^i n^j)/r$, it reads

$$k_1^{00} = -\frac{4}{c^2}\left\{\frac{M}{r} + \frac{(-1)^2}{2!}\partial_{ij}\left[\frac{\mathrm{I}_{ij}(t-r/c)}{r}\right]\right\}$$

$$= -\frac{4}{rc^2}\left\{M + \frac{n^i n^j}{2}\left(\frac{\mathrm{I}^{(2)}_{ij}(u)}{c^2} + \frac{3\mathrm{I}^{(1)}_{ij}(u)}{cr} + \frac{3\mathrm{I}^{(1)}_{ij}(u)}{r^2}\right)\right\}$$

where **mass-type source quadrupole** $\mathrm{I}_{ij}(u)$ is

- arbitrary function of u
- stationary for $u < -\mathcal{T}$
- STF, i.e. $\mathrm{I}_{ij} = \mathrm{I}_{ji}$ and $\mathrm{I}_{aa} = 0$

Full linearized vacuum metric

We recall our vacuum solution for $\Box h_1^{\mu\nu} = 0$ and $\partial_\nu h_1^{\mu\nu} = 0$ reads

$$\boxed{h_1^{\mu\nu} = k_1^{\mu\nu}[\mathrm{I}_L, \mathrm{J}_L] + \partial\varphi_1^{\mu\nu}[\mathrm{W}_L, \mathrm{X}_L, \mathrm{Y}_L, \mathrm{Z}_L]}$$

where

$$k_1^{00} = -\frac{4}{c^2}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_L\left[r^{-1}\mathrm{I}_L\right]$$

$$k_1^{0i} = \frac{4}{c^3}\sum_{\ell\geqslant 1}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left[r^{-1}\mathrm{I}^{(1)}_{iL-1}\right] + \frac{\ell}{\ell+1}\epsilon_{iab}\partial_{aL-1}\left[r^{-1}\mathrm{J}_{bL-1}\right]\right\}$$

$$k_1^{ij} = -\frac{4}{c^4}\sum_{\ell\geqslant 2}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left[r^{-1}\mathrm{I}^{(2)}_{ijL-2}\right] + \frac{2\ell}{\ell+1}\partial_{aL-2}\left[r^{-1}\epsilon_{ab(i}\mathrm{J}^{(1)}_{j)bL-2}\right]\right\}$$

$$\varphi_1^{0} = \frac{4}{c^3}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[r^{-1}\mathrm{W}_L\right]$$

$$\varphi_1^{i} = -\frac{4}{c^4}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\left\{\partial_{iL}\left[\frac{\mathrm{X}_L}{r}\right] + \partial_{L-1}\left[\frac{\mathrm{Y}_{iL-1}}{r}\right] + \frac{\ell}{\ell+1}\epsilon_{iab}\partial_{aL-1}\left[\frac{\mathrm{Z}_{bL-1}}{r}\right]\right\}$$

Full vacuum PM solution

The linearized metric $h_1^{\mu\nu}$ is the **seed** to construct the full metric. Assuming $h \ll 1$, write the PM expansion of the metric:

$$h = Gh_1 + G^2 h_2 + G^3 h_3 + \ldots$$

which should solve the **non-linear vacuum equation** $\Box h^{\mu\nu} = \Lambda^{\mu\nu}$. Thus at each order $n \geq 2$, we find

$$\Box h_n^{\mu\nu} = \Lambda_n^{\mu\nu}[h_1, \ldots, h_{n-1}]$$

For example, at quadratic order, we have $\Box h_2^{\mu\nu} = N^{\mu\nu}[h_1, h_1]$ where

$$\begin{aligned} N^{\alpha\beta}[h,h] = & - h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \frac{1}{2}\partial^\alpha h_{\mu\nu}\partial^\beta h^{\mu\nu} - \frac{1}{4}\partial^\alpha h \partial^\beta h \\ & + \partial_\nu h^{\alpha\mu}\left(\partial^\nu h^\beta_\mu + \partial_\mu h^{\beta\nu}\right) - 2\partial^{(\alpha} h_{\mu\nu}\partial^\mu h^{\beta)\nu} \\ & + \eta^{\alpha\beta}\left[-\frac{1}{4}\partial_\tau h_{\mu\nu}\partial^\tau h^{\mu\nu} + \frac{1}{8}\partial_\mu h \partial^\mu h + \frac{1}{2}\partial_\mu h_{\nu\tau}\partial^\nu h^{\mu\tau}\right] \end{aligned}$$

Full vacuum PM solution

Iteratively solve a wave equation with harmonic gauge condition

$$\Box h_n^{\mu\nu} = \Lambda_n^{\mu\nu}[h_1, \ldots, h_{n-1}] \qquad \text{and} \qquad \partial_\mu h_n^{\mu\nu} = 0$$

but **with a non-compact source** (recall $h_1 \underset{r\to\infty}{\sim} r^{-1}$) **that can diverge at** $r = 0$. Schematically:

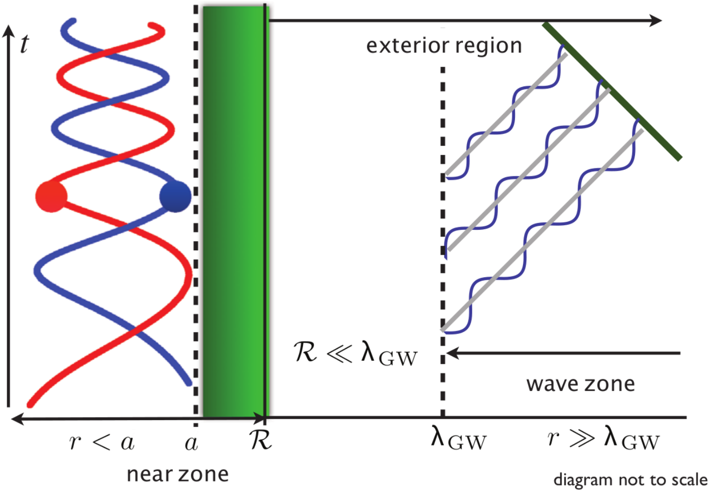
$$\boxed{h_n = u_n + v_n} \qquad \text{where} \qquad \begin{cases} u_n = \underset{B=0}{\mathrm{FP}}\,\Box^{-1}\left[\left(\frac{r}{r_0}\right)^B \Lambda_n\right] \\ \Box v_n = 0 \qquad \text{and} \qquad \partial_\nu v_n^{\mu\nu} = -\partial_\nu u^{\mu\nu} \end{cases}$$

with the retarded inverse d'Alembert operator defined as

$$\Box^{-1} f(\mathbf{x}, t) \equiv -\frac{1}{4\pi}\int \frac{\mathrm{d}\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} f\left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)$$

and where $\underset{B=0}{\mathrm{FP}}\left[\sum_{k=-n}^{\infty} \alpha_k(\mathbf{x}, t) B^k\right] \equiv \alpha_0(\mathbf{x}, t)$

Near zone vs. exterior vacuum zone

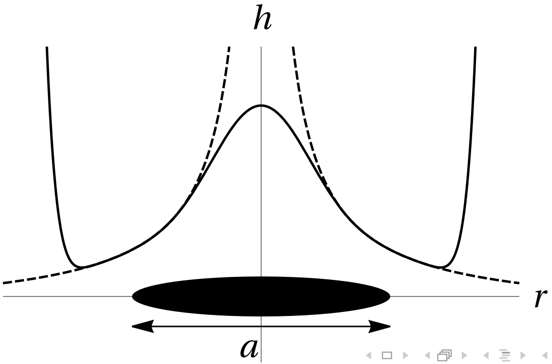


Matching equation

To link the two approximations, we promote a numerical equality in the overlapping zone to a formal matching of two asymptotic series

$$\overline{\mathcal{M}(h)} \equiv \sum \hat{n}_L r^p (\ln r)^q F(t) \equiv \mathcal{M}(\overline{h})$$

$$\mathsf{NZ}_{r\to 0}\left[\mathsf{Multipolar}_{a/r\to 0}\right] \equiv \mathsf{FZ}_{r\to\infty}\left[\mathsf{PN}_{c\to\infty}\right]$$



Result of matching

Implementing matching yields **explicit** expressions of source and gauge moments in terms of PN expansion of $\tau^{\mu\nu}$ (denoted $\overline{\tau}^{\mu\nu}$). For example:

$$\mathrm{I}_L(u) = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \sum_{k=0}^{\infty} \left(\frac{r}{c}\right)^{2k} \left\{ \frac{a_{\ell,k}}{c^2} \hat{x}_L \frac{\mathrm{d}^{2k}\left(\overline{\tau}^{00} + \overline{\tau}^{aa}\right)}{\mathrm{d}u^{2k}} + \frac{b_{\ell,k}}{c^3} \hat{x}_{iL} \frac{\mathrm{d}^{2k+1}\overline{\tau}^{0i}}{\mathrm{d}u^{2k+1}} + \frac{c_{\ell,k}}{c^4} \hat{x}_{ijL} \frac{\mathrm{d}^{2k+2}\overline{\tau}_{ij}}{\mathrm{d}t^{2k+2}} \right\}$$

At lowest PN order, we recover the usual Newtonian formula

$$\boxed{\mathrm{I}_{ij} = \int \mathrm{d}^3\mathbf{x}\, \rho(x) \hat{x}_{ij} + \mathcal{O}\left(\frac{1}{c^2}\right)}$$

where $\hat{x}_{ij} = x^i x^j - \delta^{ij} r^2/3$ and $\rho = (T^{00}/c^2)$.

Radiative moments

In theory, we have now constructed the full metric

$$h = Gh_1 + G^2 h_2 + G^3 h_3 + \dots$$

as a complicated functional of the source and gauge moments (I_L etc.)

What's the observable ? We are observers **very far from the source**
$\Longrightarrow$ we can **relinearize** in the very small parameter $Gm/(rc^2)$ where r is the source-observer distance. $[Gm/(rc^2) \ll Gm/(r_{12}c^2) \ll 1]$

Modulo a **coordinate transformation** and a transverse-traceless (TT) projection, we recover (asymptotically) a multipolar structure, with **radiative moments**:

$$\boxed{h_{\mathsf{TT}} \underset{r\to+\infty}{\sim} \frac{1}{r}\sum_{\ell=2}^{\infty} \hat{n}_L \Big(\mathcal{U}_L(u) + \epsilon \mathcal{V}_L(u)\Big)}$$

Canonical linearized metric

Residual gauge freedom $\varphi_1^{\mu\nu}$: **necessary for matching** procedure. But for MPM construction, easier to choose $\varphi_1^{\mu} = 0$.
$\Longrightarrow$ metric written in terms of only two *canonical* moments M_L and S_L

$$h_1^{00} = -\frac{4G}{c^2} \sum_{\ell \geqslant 0} \frac{(-)^\ell}{\ell!} \partial_L \left[r^{-1} \mathrm{M}_L\right]$$

$$h_1^{0i} = \frac{4G}{c^3} \sum_{\ell \geqslant 1} \frac{(-)^\ell}{\ell!} \left\{ \partial_{L-1} \left[r^{-1} \mathrm{M}_{iL-1}^{(1)}\right] + \frac{\ell}{\ell+1} \epsilon_{iab} \partial_{aL-1} \left[r^{-1} \mathrm{S}_{bL-1}\right] \right\}$$

$$h_1^{ij} = -\frac{4G}{c^4} \sum_{\ell \geqslant 2} \frac{(-)^\ell}{\ell!} \left\{ \partial_{L-2} \left[r^{-1} \mathrm{M}_{ijL-2}^{(2)}\right] + \frac{2\ell}{\ell+1} \partial_{aL-2} \left[r^{-1} \epsilon_{ab(i} \mathrm{S}_{j)bL-2}^{(1)}\right] \right\}$$

We will see see that $\mathrm{M}_L \neq \mathrm{I}_L$ and $\mathrm{S}_L \neq \mathrm{J}_L$, but differ by non-linear corrections. For example :

$$\mathrm{M}_{ij} = \mathrm{I}_{ij} + \frac{4G}{c^5} \left[\mathrm{W}^{(2)} \mathrm{I}_{ij} - \mathrm{W}^{(1)} \mathrm{I}_{ij}^{(1)}\right] + \mathcal{O}\left(\frac{1}{c^7}\right)$$

The 2.5PN relation between canonical and radiative moments reads

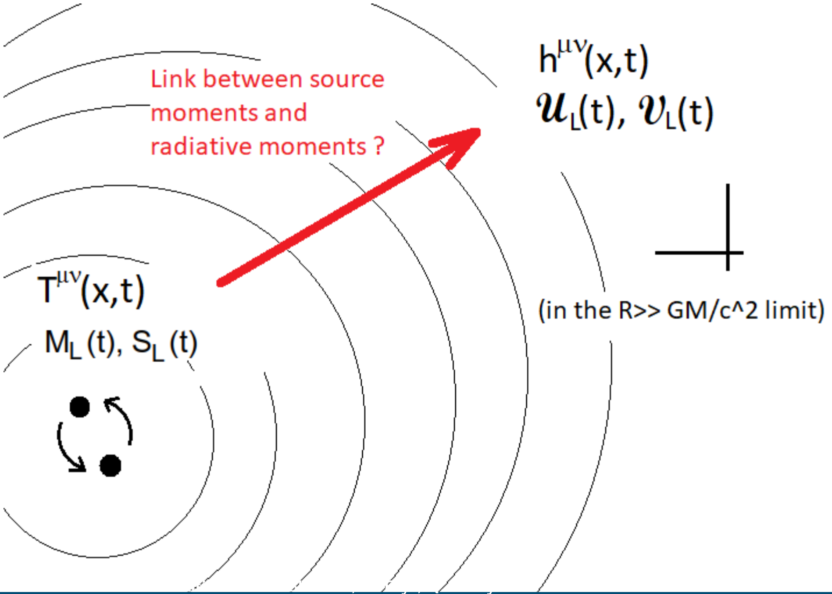
$$\mathcal{U}_{ij} = \mathrm{M}^{(2)}_{ij}(u) + \frac{2G\mathrm{M}}{c^3}\int_0^{\infty} \mathrm{d}\tau \left[\ln\left(\frac{\tau}{2b_0}\right) + \frac{11}{12}\right] \mathrm{M}^{(4)}_{ij}(u-\tau) - \frac{2G}{7c^5}\int_0^{\infty} \mathrm{d}\tau\, \mathrm{M}^{(3)}_{a\langle i}\mathrm{M}^{(3)}_{j\rangle a}(u-\tau) + (\text{instantaneous terms}) + \mathcal{O}\left(\frac{1}{c^6}\right)$$

there are two **non local** integrals in play. The waveform depends on the entire **history** of the binary. These integrals are:

- **tails**: linear waves back-scatter against the curvature of spacetime
- **memory**: linear GWs reradiate linear GWs

If we have a model for the evolution of the binary (in this case: **quasi-circular orbits**), we can compute these integrals explicitly.

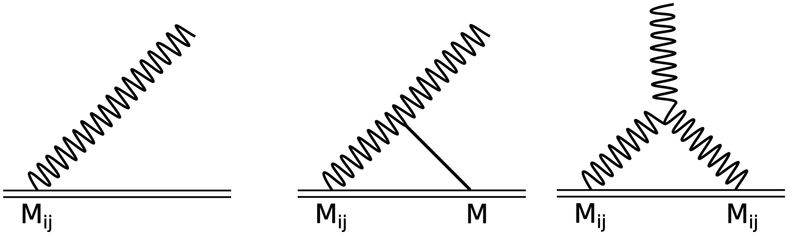
Source and radiative moments



Tails and memory

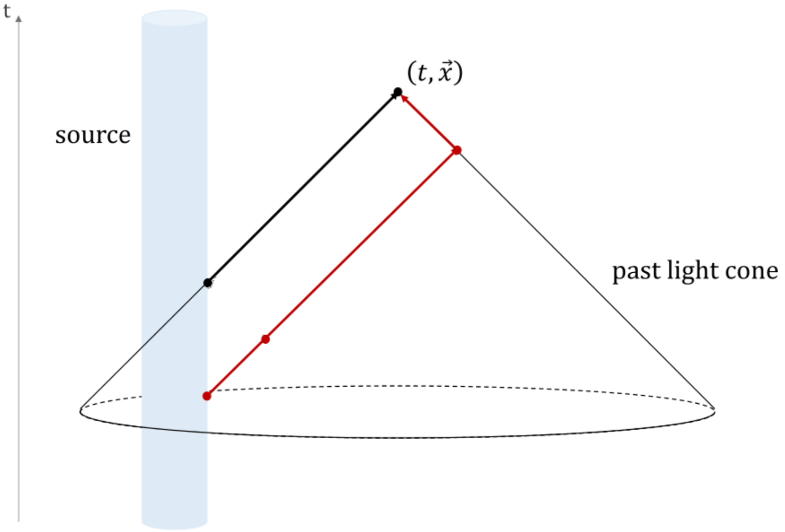
The 2.5PN relation between canonical and radiative moments reads

$$\mathcal{U}_{ij} = \mathrm{M}^{(2)}_{ij}(u) + \frac{2G\mathrm{M}}{c^3}\int_0^{\infty} \mathrm{d}\tau \left[\ln\left(\frac{\tau}{2b_0}\right) + \frac{11}{12}\right] \mathrm{M}^{(4)}_{ij}(u-\tau)$$
$$- \frac{2G}{7c^5}\int_0^{\infty} \mathrm{d}\tau\, \mathrm{M}_{a\langle i}\mathrm{M}_{j\rangle a}(u-\tau) + (\text{instantaneous terms}) + \mathcal{O}\left(\frac{1}{c^6}\right)$$



Linear quadrupolar wave — Tail — Memory

Illustration: tails



Flux for quasicircular orbits

Energy flux carried by GWs:

$$\mathcal{F} = \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left\{ \alpha_\ell \, \mathcal{U}_L^{(1)} \mathcal{U}_L^{(1)} + \frac{\beta_\ell}{c^2} \, \mathcal{V}_L^{(1)} \mathcal{V}_L^{(1)} \right\}$$

Lowest order: Einstein quadrupole formula $\boxed{\mathcal{F} = G/(5c^5)\, \mathcal{U}_{ij}^{(1)} \mathcal{U}_{ij}^{(1)}}$

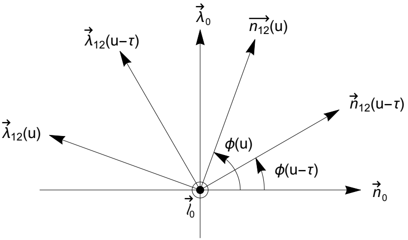
For **quasi-circular orbits** define:

$$\boxed{y \equiv \left(\frac{G m \omega}{c^3} \right)^{2/3} = \mathcal{O}\left(\frac{1}{c^2} \right)}$$

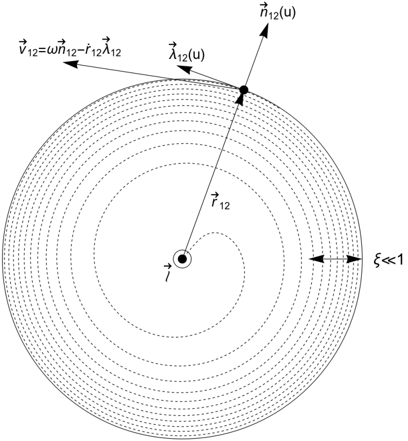
where ω is the orbital frequency. The 1.5PN flux (including tails) reads

$$\mathcal{F} = \frac{32 c^5}{5G} \nu^2 y^5 \left\{ 1 + \left(-\frac{1247}{336} - \frac{35}{12} \right) y + 4\pi y^{3/2} + \ldots \right\}$$

Quasicircular orbits



$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} \equiv \omega(t)$$



Energy-flux balance equation

We postulate energy conservation:

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}}$$

where E is the conservative energy of the bound system and $\mathcal{F}$ the energy flux of GWs carried at infinity.

If we have explicit expression in terms of the variable y, i.e. $\mathcal{F}(y)$ and $E(y)$, we recast the balance equation as

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{\mathcal{F}}{(\mathrm{d}E/\mathrm{d}y)}} \implies y(t) = \dots \qquad \text{(the frequency chirp)}$$

$$\boxed{\frac{\mathrm{d}\phi}{\mathrm{d}y} = -\frac{c^3 x^{3/2}}{Gm}\frac{\mathrm{d}E/\mathrm{d}y}{\mathcal{F}(y)}} \implies \phi(y) = \dots \quad \text{where} \quad \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} \equiv \omega(t)$$

The three types of moments

Moment type	Source	Gauge	Canonical	Radiative
Notation	I_L, J_L	W_L,X_L,Y_L,Z_L	M_L, S_L	$\mathcal{U}_L$, $\mathcal{V}_L$
What does it parametrize	Linearized metric, general gauge		Linearized metric, canonical gauge	Full metric, asymptotically
How to compute	Directly from stress-energy tensor		$\mathrm{M}_L = \mathrm{I}_L + \dots$	$\mathcal{U}_L = \mathrm{M}_L^{(\ell)} + \dots$

$$h_{\mathsf{TT}} \underset{r\to+\infty}{\sim} \frac{1}{r}\sum_{\ell=2}^{\infty} \hat{n}_L \Big(\mathcal{U}_L(u) + \epsilon \mathcal{V}_L(u)\Big)$$

$$\mathcal{F} = \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left\{ \alpha_\ell \mathcal{U}_L^{(1)} \mathcal{U}_L^{(1)} + \frac{\beta_\ell}{c^2} \mathcal{V}_L^{(1)} \mathcal{V}_L^{(1)} \right\}$$

Question ?

Questions?

The 4PN program

Before 4PN program, flux/phase/waveform a 3.5PN precision ($\sim (v/c)^{7/2}$). Steps for 4PN (comments on 4.5PN later):

1. 4PN EOM and conservative energy (Bernard et al. 2017, 2018)
   $\longrightarrow$ dimensional regulation for UV & IR divergences.
2. I_{ij} at 4PN (Marchand et al. 2020, Larrouturou et al. 2021ab)
   $\longrightarrow$ dimensional regulation for UV & IR divergences.
3. Relation between source and canonical moments (Blanchet et al. 2022, 2023)
   $\longrightarrow$ in 3D then in d dimension
4. Relation between canonical and radiative moment (Trestini et al. 2022, 2023, Larrouturou et al. 2021b).
   $\longrightarrow$ in 3D + correction from dimensional regularization
5. Flux and $(2, 2)$ mode at 4PN (Blanchet et al. 2023ab)
6. Energy & flux $\Rightarrow$ phase at 4PN (by energy-flux balance equation)

Before 4PN program, flux/phase/waveform a 3.5PN precision ($\sim (v/c)^{7/2}$). Steps for 4PN (comments on 4.5PN later):

1. 4PN EOM and conservative energy (Bernard et al. 2017, 2018)
   $\longrightarrow$ dimensional regulation for UV & IR divergences.
2. I_{ij} at 4PN (Marchand et al. 2020, Larrouturou et al. 2021ab)
   $\longrightarrow$ dimensional regulation for UV & IR divergences.
3. Relation between source and canonical moments (Blanchet et al. 2022, 2023)
   $\longrightarrow$ in 3D then in d dimension
4. **Relation between canonical and radiative moment (Trestini et al. 2022, 2023, Larrouturou et al. 2021b).**
   $\longrightarrow$ **in 3D + correction from dimensional regularization**
5. **Flux and $(2,2)$ mode at 4PN (Blanchet et al. 2023ab)**
6. **Energy & flux $\Rightarrow$ phase at 4PN (by energy-flux balance equation)**

Asymptotic structure of metric

In harmonic coordinates, when $r \to +\infty$ (u kept fixed), metric behaves as

$$h \sim \frac{1}{r} \sum_{\ell=0}^{\infty} \sum_{p=0}^{\infty} \log^p(r) \hat{n}_L F_L(u) + o\left(\frac{1}{r}\right)$$

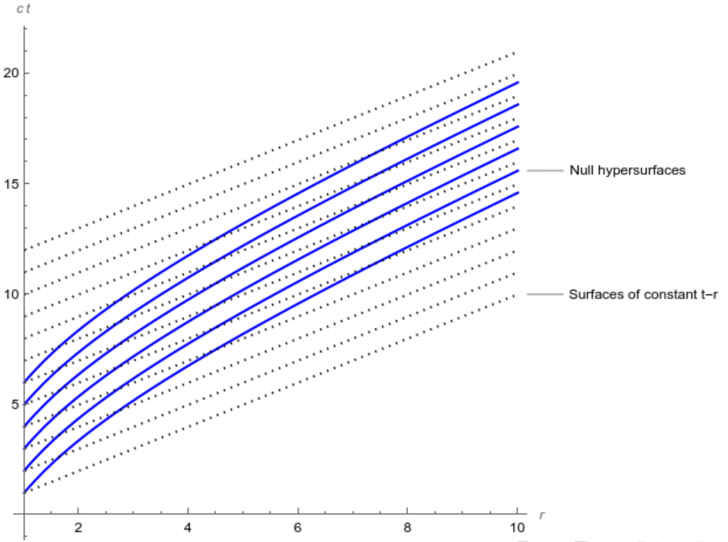
$\longrightarrow$ logarithms spoil multipolar structure, cannot extract "$1/r$" piece

Pure coordinate effect! Perform $T = t - (2\mathrm{M}/c^2)\ln(r/b_0)$ and $X_i = x_i$.
Removes logarithms from all terms stemming from $\mathrm{M}^n \times \mathrm{M}_{ij}$ interactions
$\longrightarrow$ these were the only problematic terms up to 3.5PN

This did not work for $\mathrm{M} \times \mathrm{M}_{ij} \times \mathrm{M}_{ij}$

Solution: **modify MPM construction** such as to remove logarithms automatically (we leave harmonic coordinates)

$u = t - r/c$ not a null coordinate!



Radiative MPM construction

Not in harmonic coordinates: equation in exterior vacuum becomes:

$$\boxed{\Box h^{\mu\nu} - \partial H^{\mu\nu} = \Lambda^{\mu\nu}[h]} \qquad \text{where} \qquad H^{\mu} \equiv \partial_{\rho} h^{\rho\mu}$$

Most general solution parametrized by new moments $\overline{\mathrm{M}}_L$ and $\overline{\mathrm{S}}_L$:

$$\boxed{h_1^{\mu\nu} = k_1^{\mu\nu}[\overline{\mathrm{M}}_L, \overline{\mathrm{S}}_L] + \partial \xi_1^{\mu\nu}}$$

where

$$k_1[\overline{\mathrm{M}}_L, \overline{\mathrm{S}}_L] \sim \sum_{\ell \geqslant 2} \left\{ \partial_L \left[r^{-1} \overline{\mathrm{M}}_L^{(2)} \right] + \epsilon\, \partial_L \left[r^{-1} \overline{\mathrm{S}}_{\mathrm{L}} \right] \right\}$$

$$\boxed{\xi_1^{\mu} = \frac{2\mathrm{M}}{c^2} \eta^{0\mu} \ln(r/b_0)}$$

with this coordinate transformation, $u = t - r/c$ is a null coordinate at linearized order, i.e. $g(\partial_u, \partial_u) = \mathcal{O}(G^2)$

Radiative construction

PM expansion

$$h = Gh_1 + G^2 h_2 + G^3 h_3 + \ldots$$

For $n \geq 2$, we need to solve

$$\boxed{\Box h_n^{\mu\nu} - \partial H_n^{\mu\nu} = \Lambda_n^{\mu\nu}[h_1, \ldots h_{n-1}]} \qquad \text{where} \qquad H_n^{\mu} \equiv \partial_\rho h_n^{\rho\mu}$$

Crucial: when $r \to +\infty$, $\Lambda_n^{\mu\nu} = r^{-2} k^\mu k^\nu \sigma_n(u, \mathbf{n}) + \mathcal{O}(1/r^3)$

Solved for $n \geq 2$ by:

$$\boxed{h_n^{\mu\nu} = u_n^{\mu\nu} + v_n^{\mu\nu} + \partial \xi_n^{\mu\nu}}$$

where

$$u_n^{\mu\nu} = \underset{B=0}{\mathrm{FP}} \Box^{-1} \left[(r/r_0)^B \, \Lambda_n^{\mu\nu} \right]$$

$$\Box v_n^{\mu\nu} = 0 \qquad \partial_\nu v_n^{\mu\nu} = -\partial_\nu u^{\mu\nu}$$

$$\xi_n^{\mu\nu} \equiv \underset{B=0}{\mathrm{FP}} \Box^{-1} \left[\left(\frac{r}{r_0} \right)^B \frac{c k^\mu}{2r^2} \int_0^\infty \mathrm{d}\tau \, \sigma_n(u - \tau, \mathbf{n}) \right]$$

Moment redefintion

We can now construct a radiative coordinate system such that $g(\partial_u, \partial_u) = \mathcal{O}(G^{n+1})$, for n arbitrarily large.

In radiative coordinates, we find that our asymptotic structure is free of far-zone logarithms.

$$h \sim \frac{1}{r} \sum_{\ell=0}^{\infty} \hat{n}_L F_L(u) + \mathcal{O}\left(\frac{1}{r^2}\right)$$

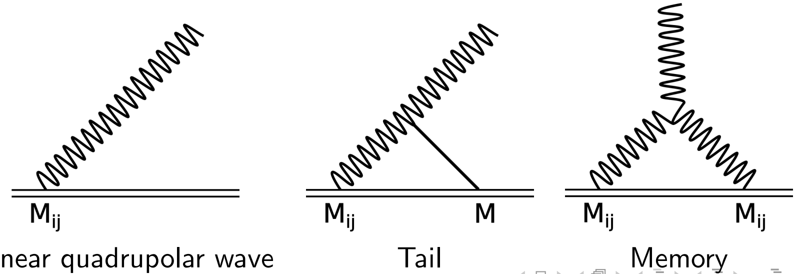
The price to pay is that we do not have the same canonical moments: we find (DT., F. Larrouturou and L. Blanchet 2023)

$$\begin{aligned}\overline{\mathrm{M}}_{ij} &= \mathrm{M}_{ij} - \frac{26}{15}\frac{G\,\mathrm{M}}{c^3}\,\mathrm{M}^{(1)}_{ij} + \frac{124}{45}\frac{G^2\mathrm{M}^2}{c^6}\,\mathrm{M}^{(2)}_{ij} \\ &+ \frac{G^2\mathrm{M}}{c^8}\left[-\frac{8}{21}\,\mathrm{M}_{a\langle i}\mathrm{M}^{(4)}_{j\rangle a} - \frac{8}{7}\,\mathrm{M}^{(1)}_{a\langle i}\mathrm{M}^{(3)}_{j\rangle a} - \frac{8}{9}\,\epsilon_{ab\langle i}\mathrm{M}^{(3)}_{j\rangle a}\mathrm{S}_b\right],\end{aligned}$$

Nonlinear effects

In general relativity at 4PN, the radiative quadrupole is related to the source quadrupole by the following interactions:

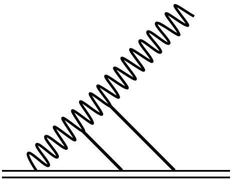
$$\mathcal{U}_{ij} = \mathrm{I}_{ij}^{(2)} + G\Big\{\mathrm{Tail}[\mathrm{M}\times\mathrm{I}_{ij}] + \mathrm{Memory}[\mathrm{I}_{ab}\times I_{ij}]\Big\}$$
$$+ G^2\Big\{\text{Tail-of-tail}[\mathrm{M}\times\mathrm{M}\times\mathrm{I}_{ij}] + \text{Tail-of-memory}[\mathrm{M}\times\mathrm{I}_{ij}\times\mathrm{I}_{pq}]$$
$$+ \text{Spin-quadrupole tails}[\mathrm{M}\times\mathrm{S}_k\times\mathrm{I}_{ij}]\Big\}\Big\}$$



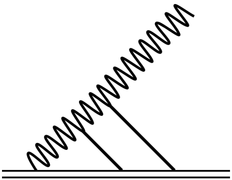
Nonlinear effects

In general relativity at 4PN, the radiative quadrupole is related to the source quadrupole by :

$$\mathcal{U}_{ij} = \mathrm{I}^{(2)}_{ij} + G\Big\{\mathrm{Tail}[\mathrm{M}\times\mathrm{I}_{ij}] + \mathrm{Memory}[\mathrm{I}_{ab}\times I_{ij}]\Big\}$$
$$+ G^2\Big\{\text{Tail-of-tail}[\mathrm{M}\times\mathrm{M}\times\mathrm{I}_{ij}] + \text{Tail-of-memory}[\mathrm{M}\times\mathrm{I}_{ij}\times\mathrm{I}_{pq}]$$
$$+ \text{Spin-quadrupole tails}[\mathrm{M}\times\mathrm{S}_k\times\mathrm{I}_{ij}]\Big\}\Big\}$$



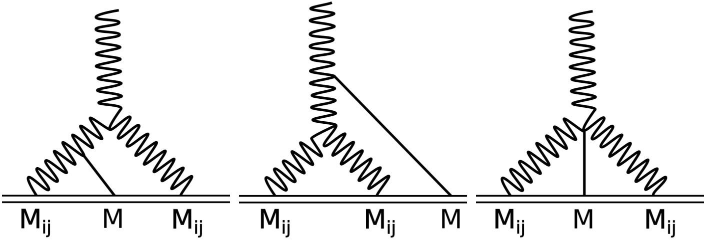
Tail-of-tail



Spin-quadrupole tail

In general relativity at 4PN, the radiative quadrupole is related to the source quadrupole by :

$$\begin{aligned}\mathcal{U}_{ij} = \mathrm{I}^{(2)}_{ij} &+ G\Big\{\mathrm{Tail}[\mathrm{M}\times\mathrm{I}_{ij}] + \mathrm{Memory}[\mathrm{I}_{ab}\times I_{ij}]\Big\}\\ &+ G^2\Big\{\mathrm{Tail\text{-}of\text{-}tail}[\mathrm{M}\times\mathrm{M}\times\mathrm{I}_{ij}] + \mathrm{Tail\text{-}of\text{-}memory}[\mathrm{M}\times\mathrm{I}_{ij}\times\mathrm{I}_{pq}]\\ &\qquad + \text{Spin-quadrupole tails}[\mathrm{M}\times\mathrm{S}_k\times\mathrm{I}_{ij}]\Big\}\Big\}\end{aligned}$$



Tails of memory

General integration technique

We want to solve explicitly

$$\Box\Psi_L = \hat{n}_L \, S(r,u)\,, \tag{1}$$

where $S(r,u) = \mathcal{O}(r^{\ell+5})$ when $r \to 0$ with u kept fixed. We define

$$R_\alpha(\rho, s) \equiv \rho^\ell \int_\alpha^\rho \mathrm{d}\lambda \, \frac{(\rho-\lambda)^\ell}{\ell!} \left(\frac{2}{\lambda}\right)^{\ell-1} S(\lambda, s)$$

where α is an arbitrary constant. Then the solution reads

$$\Psi_L = \int_{-\infty}^{t-r} \mathrm{d}s \, \hat{\partial}_L \left[\frac{R_\alpha\left(\frac{t-r-s}{2}, s\right) - R_\alpha\left(\frac{t+r-s}{2}, s\right)}{r} \right]$$

If $S(r,u)$ does not converge fast enough, consider $(r/r_0)^B S(r,u)$ instead, and take the finite part when $B \to 0$.

Cubic integration formula

Most difficult integral to compute for the $\mathrm{M} \times \mathrm{M}_{ij} \times \mathrm{M}_{ij}$ interaction:

$$\underset{k,m}{\Psi_L} = \underset{B=0}{\mathrm{FP}} \Box^{-1} \left[\left(\frac{r}{r_0}\right)^B r^{-k} \, G(t-r) \int_1^{+\infty} \mathrm{d}x \, Q_m(x) F(t-rx) \right]$$

where F and G identically vanish for $t < -\mathcal{T}$ and Q_m is the Legendre function of second kind.

For $k \geq 3$, we can recursively bring ourselves to the case where $k=1$ and $k=2$. In the latter case, we don't need the finite parts:

$$\underset{k,m}{\Psi_L} = -\frac{\hat{n}_L}{2r} \int_0^{+\infty} \mathrm{d}\rho \, G(u-\rho) \int_0^{+\infty} \mathrm{d}\tau \, F(u-\rho-\tau) \underset{k,m}{K_\ell}(\rho,\tau,r)\,,$$

where the kernel reads

$$\underset{k,m}{K_\ell}(\rho,\tau,r) = \tau^{1-k} \int_{\frac{2\tau}{\rho+2r}}^{\frac{2\tau}{\rho}} \mathrm{d}y \, y^{k-2} Q_m(y+1) \, \Pi_\ell \left(1 - \frac{\rho y}{\tau}, 1 + \frac{\rho}{r}\right)$$

Seperating the logarithms

We now know how to integrate all terms, but we want to make sure that the far-zone logarithms explicitly vanish in the radiative construction. To do this, we explicitly extract the logarithmic dependency of the kernels, e.g.

$$\underset{1,m}{K_\ell}(\rho,\tau,r) = \frac{1}{4}\ln^2\left(\frac{r}{r_0}\right) - \frac{1}{2}\ln\left(\frac{r}{r_0}\right)\left[\ln\left(\frac{\tau}{2r_0}\right) + 2H_m\right] + \underset{1,m}{\overline{K}_\ell}(\rho,\tau)$$

This leads to defining elementary functionals

$$\underset{k,m}{\overline{\Psi}_\ell}[F,G] \equiv \int_0^{+\infty} \mathrm{d}\rho\, G(u-\rho) \int_0^{+\infty} \mathrm{d}\tau\, F(u-\rho-\tau)\, \underset{k,m}{\overline{K}_\ell}(\rho,\tau)$$

Tails-of-memory: raw result

With the two types of elementary functionals and kernels in hand, namely

$$\underset{k,m}{\overline{\Psi}_\ell}[F,G] \equiv \int_0^{+\infty} \mathrm{d}\rho\, G(u-\rho) \int_0^{+\infty} \mathrm{d}\tau\, F(u-\rho-\tau)\, \underset{k,m}{\overline{K}_\ell}(\rho,\tau)$$

$$\underset{k,m}{\overline{\chi}_\ell}[F,G] \equiv \int_0^{+\infty} \mathrm{d}\rho\, G(u-\rho) \int_0^{+\infty} \mathrm{d}\tau\, F(u-\rho-\tau)\, \underset{k,m}{\overline{L}_\ell}(\rho,\tau)$$

we find the explicit but untractable result

$$\mathcal{U}_{ij}^{\mathrm{M}\times\overline{\mathrm{M}}_{ij}\times\overline{\mathrm{M}}_{ij}}\Big| = \frac{G^2\mathrm{M}}{c^8}\sum_{m,\ell,n}\Big\{\mathcal{A}^n_{m,\ell}\,\underset{1,m}{\overline{\Psi}_\ell}\Big[\overline{\mathrm{M}}^{(n)}_{a\langle i},\overline{\mathrm{M}}^{(8-n)}_{j\rangle a}\Big] + \mathcal{B}^n_{m,\ell}\,\underset{2,m}{\overline{\Psi}_\ell}\Big[\overline{\mathrm{M}}^{(n)}_{a\langle i},\overline{\mathrm{M}}^{(7-n)}_{j\rangle a}\Big]$$
$$+\,\mathcal{C}^n_{n,\ell}\,\underset{1,m}{\overline{\chi}_\ell}\Big[\overline{\mathrm{M}}^{(n)}_{a\langle i},\overline{\mathrm{M}}^{(8-n)}_{j\rangle a}\Big] + \mathcal{D}^n_{m,\ell}\,\underset{2,m}{\overline{\chi}_\ell}\Big[\overline{\mathrm{M}}^{(n)}_{a\langle i},\overline{\mathrm{M}}^{(7-n)}_{j\rangle a}\Big]\Big\}$$
$$+\,(\text{terms that have a more standard and tractable form})$$

Impossible to get a simpler integration formula, but one can hope for a simpler end result. **Idea: integrate by parts to have only one derivative combination**

Simplification method

Introducing a regularizing lower bound ϵ which we be taken to 0 at the end, we can integrate by parts. We find

$$\mathcal{U}_{ij}^{\mathrm{M}\times\overline{\mathrm{M}}_{ij}\times\overline{\mathrm{M}}_{ij}} = \mathrm{M}\int_{\epsilon}^{+\infty}\mathrm{d}\rho\,\overline{\mathrm{M}}_{a\langle i}(u-\rho)\int_0^{+\infty}\mathrm{d}\tau\,\overline{\mathrm{M}}^{(8)}_{j\rangle a}(u-\rho-\tau)\,\Omega(\rho,\tau) + (\text{surface term})_\epsilon + (\text{standard terms})$$

One would expect $\Omega(\rho,\tau)$ to be insanely complicated, with polylogarithms, etc. But actually, all the complicated terms neatly cancel out:

$$\Omega(\rho,\tau) = \frac{7613764}{165375} - \frac{1024076}{18375}\frac{\tau}{\rho} - \frac{2074}{63}\left(\frac{\tau}{\rho}\right)^2 - \frac{104}{15}\left(\frac{\tau}{\rho}\right)^3 + \frac{634076}{55125}\ln\left(\frac{\rho}{2r_0}\right) + \frac{384}{175}\frac{\tau}{\rho}\ln\left(\frac{\rho}{2r_0}\right) - \frac{144}{175}\ln\left(\frac{\rho}{2r_0}\right)^2 + \frac{8}{7}\ln\left(\frac{\tau}{2r_0}\right)$$

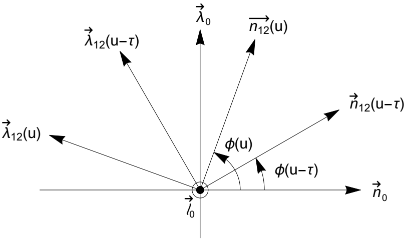
This can then be massaged into a tractable form (and finite when $\epsilon \to 0$)

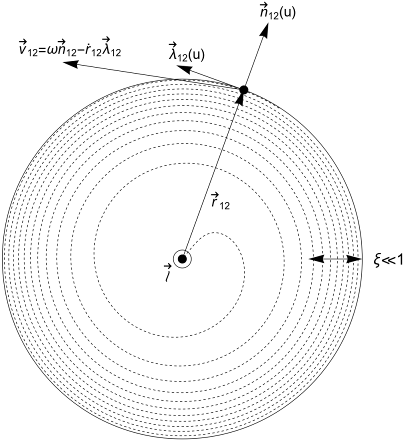
Tails-of-memory: result

After the $\overline{\mathrm{M}}_{ij} \to \mathrm{M}_{ij}$ conversion, we find (DT & L. Blanchet 2022)

$$\begin{aligned}
\mathcal{U}_{ij}^{\mathrm{M}\times\mathrm{M}_{ij}\times\mathrm{M}_{ij}} = \frac{8G^2\mathrm{M}}{7c^8}\Bigg\{ & \int_0^{+\infty}\mathrm{d}\rho\, \mathrm{M}^{(4)}_{a\langle i}(u-\rho)\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(4)}_{j\rangle a}(u-\rho-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)-\frac{1613}{270}\right] \\
& -\frac{5}{2}\int_0^{+\infty}\mathrm{d}\tau\,(\mathrm{M}^{(3)}_{a\langle i}\mathrm{M}^{(4)}_{j\rangle a})(u-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{3}{2}\ln\left(\frac{\tau}{2b_0}\right)\right] \\
& -3\int_0^{+\infty}\mathrm{d}\tau\,(\mathrm{M}^{(2)}_{a\langle i}\mathrm{M}^{(5)}_{j\rangle a})(u-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{11}{12}\ln\left(\frac{\tau}{2b_0}\right)\right] \\
& -\frac{5}{2}\int_0^{+\infty}\mathrm{d}\tau\,(\mathrm{M}^{(1)}_{a\langle i}\mathrm{M}^{(6)}_{j\rangle a})(u-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{3}{10}\ln\left(\frac{\tau}{2b_0}\right)\right] \\
& -\int_0^{+\infty}\mathrm{d}\tau\,(\mathrm{M}_{a\langle i}\mathrm{M}^{(7)}_{j\rangle a})(u-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)-\frac{1}{4}\ln\left(\frac{\tau}{2b_0}\right)\right] \\
& -2\mathrm{M}^{(2)}_{a\langle i}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(5)}_{j\rangle a}(u-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{27521}{5040}\right] \\
& -\frac{5}{2}\mathrm{M}^{(1)}_{a\langle i}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(6)}_{j\rangle a}(u-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{15511}{3150}\right] \\
& +\frac{1}{2}\mathrm{M}_{a\langle i}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(7)}_{j\rangle a}(u-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)-\frac{6113}{756}\right]\Bigg\}.
\end{aligned}$$

Quasicircular orbits





$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} \equiv \omega(t)$$

The phase: orbital vs. wave

It is necessary to distinguish:

- the orbit phase ϕ and the orbital frequency $\omega = \mathrm{d}\phi/\mathrm{d}t$ (as well as the associated $y = (Gm\omega c^{-3})^{2/3}$)
- the phase associated to the wave ψ and the wave frequency $\Omega = \mathrm{d}\psi/\mathrm{d}t$ (as well as the associated $x = (Gm\Omega c^{-3})^{2/3}$)

Due to tails, they differ by a 1.5PN term

$$\boxed{\psi = \phi - \frac{2G\mathrm{M}\omega}{c^3}\ln\left(\frac{\omega}{\omega_0}\right)} \quad \text{where} \quad \omega_0 \equiv \frac{c e^{11/12-\gamma_E}}{4b_0}$$

The frequency is then modified by 4PN term $[\nu = m_1 m_2/(m_1+m_2)^2]$

$$\Omega = \omega\left\{1 - \frac{192}{5}\nu\left(\frac{Gm\omega}{c^3}\right)^{8/3}\left[\ln\left(\frac{\omega}{\omega_0}\right)+1\right] + \mathcal{O}\left(\frac{1}{c^{10}}\right)\right\}$$

$$x = y\left\{1 - \frac{192}{5}\nu\, y^4\left[\ln\left(\frac{y}{y_0}\right)+\frac{2}{3}\right] + \mathcal{O}(y^5)\right\}$$

The gravitational waveform h_{TT} can be decomposed into the two usual polarization, h_+ and $h_\times$ as

$$h^{ij}_{\mathsf{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

This is practically summarized is the complex variable $\boxed{h \equiv h_+ - \mathrm{i} h_\times}$
Expanded onto basis of spin-weighted spherical harmonics ${}_sY_{\ell \mathrm{m}}(\theta, \phi)$:

$$h_+ - \mathrm{i} h_\times = \sum_{\ell \geq 2} \sum_{\mathrm{m}=-\ell}^{\ell} h_{\ell \mathrm{m}}(T, R) Y^{-2}_{\ell \mathrm{m}}(\theta, \phi)$$

where $h_{\ell \mathrm{m}}$ is normalized as:

$$h_{\ell \mathrm{m}} = \frac{8 G m \nu x}{c^2 R} \sqrt{\frac{\pi}{5}} H_{\ell \mathrm{m}}(T) e^{-\mathrm{i m} \psi(T)}$$

$(2,2)$ mode at 4PN

We recover the partial 4PN result in the extreme mass ratio limit of Tagoshi and Sasaki (1994).

$$
\begin{aligned}
H_{22} = 1 &+ \left(-\frac{107}{42} + \frac{55}{42}\nu\right)x + 2\pi x^{3/2} + \left(-\frac{2173}{1512} - \frac{1069}{216}\nu + \frac{2047}{1512}\nu^2\right)x^2 \\
&+ \left[-\frac{107\pi}{21} + \left(\frac{34\pi}{21} - 24\,\mathrm{i}\right)\nu\right]x^{5/2} \\
&+ \left[\frac{27027409}{646800} - \frac{856}{105}\,\gamma_\mathrm{E} + \frac{428\,\mathrm{i}\,\pi}{105} + \frac{2\pi^2}{3} + \left(-\frac{278185}{33264} + \frac{41\pi^2}{96}\right)\nu - \frac{20261}{2772}\nu^2 \right. \\
&\qquad \left. + \frac{114635}{99792}\nu^3 - \frac{428}{105}\ln(16x)\right]x^3 \\
&+ \left[-\frac{2173\pi}{756} + \left(-\frac{2495\pi}{378} + \frac{14333\,\mathrm{i}}{162}\right)\nu + \left(\frac{40\pi}{27} - \frac{4066\,\mathrm{i}}{945}\right)\nu^2\right]x^{7/2} \\
&+ \left[-\frac{846557506853}{12713500800} + \frac{45796}{2205}\gamma_\mathrm{E} - \frac{22898}{2205}\mathrm{i}\pi - \frac{107}{63}\pi^2 + \frac{22898}{2205}\ln(16x) \right. \\
&\qquad + \left(-\frac{336005827477}{4237833600} + \frac{15284}{441}\gamma_\mathrm{E} - \frac{219314}{2205}\mathrm{i}\pi - \frac{9755}{32256}\pi^2 + \frac{7642}{441}\ln(16x)\right)\nu \\
&\qquad \left. + \left(\frac{256450291}{7413120} - \frac{1025}{1008}\pi^2\right)\nu^2 - \frac{81579187}{15567552}\nu^3 + \frac{26251249}{31135104}\nu^4\right]x^4 + \mathcal{O}\left(x^{9/2}\right).
\end{aligned}
$$

Flux: the 4.5PN piece

It was shown in T. Marchand, L. Blanchet & G. Faye (2016) that in order to control the 4.5PN piece of the flux, only the hereditary pieces of the radiative moments were needed. The most difficult such piece is the "tails-of-tails-of-tails", due to a *quartic* $\mathrm{M}^3 \times \mathrm{M}_{ij}$ interaction. It reads

$$\begin{aligned}
\mathcal{U}_{ij}^{\mathrm{M}^3\times\mathrm{M}_{ij}} = \frac{G^3\mathrm{M}^3}{c^9}\int_0^{+\infty} \mathrm{d}\tau\, \mathrm{M}_{ij}^{(6)}(u-\tau)\Bigg[&\frac{4}{3}\ln^3\left(\frac{c\tau}{2b_0}\right) + \frac{11}{3}\ln^2\left(\frac{c\tau}{2b_0}\right)\\
&- \frac{428}{105}\ln\left(\frac{c\tau}{2b_0}\right)\ln\left(\frac{c\tau}{2r_0}\right) + \frac{124627}{11025}\ln\left(\frac{c\tau}{2b_0}\right)\\
&- \frac{1177}{315}\ln\left(\frac{c\tau}{2r_0}\right) + \frac{129268}{33075} + \frac{428}{315}\pi^2\Bigg]
\end{aligned}$$

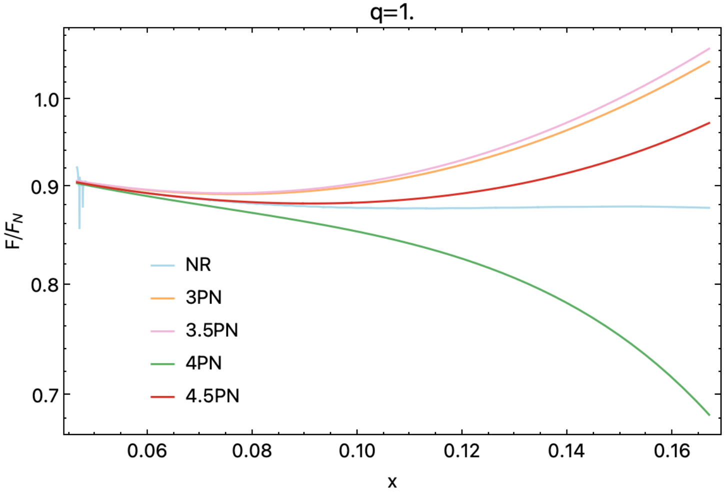
Flux at 4.5PN

We recover the partial 4PN result in the extreme mass ratio limit of Tagoshi and Sasaki (1994).

$$\begin{aligned}
\mathcal{F} = &\frac{32c^5}{5G}\nu^2 x^5 \\
\times\Bigg\{ &1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\
&+ \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_\mathrm{E} - \frac{856}{105}\ln(16\,x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right]x^3 \\
&+ \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} \\
&+ \Bigg[-\frac{323105549467}{3178375200} + \frac{232597}{4410}\gamma_\mathrm{E} - \frac{1369}{126}\pi^2 + \frac{39931}{294}\ln 2 - \frac{47385}{1568}\ln 3 + \frac{232597}{8820}\ln x \\
&\quad + \left(-\frac{1452202403629}{1466942400} + \frac{41478}{245}\gamma_\mathrm{E} - \frac{267127}{4608}\pi^2 + \frac{479062}{2205}\ln 2 + \frac{47385}{392}\ln 3 + \frac{20739}{245}\ln x\right)\nu \\
&\quad + \left(\frac{1607125}{6804} - \frac{3157}{384}\pi^2\right)\nu^2 + \frac{6875}{504}\nu^3 + \frac{5}{6}\nu^4\Bigg]x^4 \\
&+ \Bigg[\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\mathrm{E} - \frac{3424}{105}\ln(16\,x) + \left(\frac{2062241}{22176} + \frac{41}{12}\pi^2\right)\nu \\
&\quad - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3\Bigg]\pi x^{9/2} + \mathcal{O}(x^5)\Bigg\}.
\end{aligned}$$

Comparison with NR (preliminary)

Credits: Héctor Estellés Estrella



Conservative energy at 4PN

Recall energy-flux balance equation

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}$$

Conservative energy for two point-particles on a circular orbit:

$$\begin{aligned} E = -\frac{m\nu c^2 x}{2}\Bigg\{ & 1 + \left(-\frac{3}{4} - \frac{\nu}{12}\right)x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}\right)x^2 \\ & + \left[-\frac{675}{64} + \left(\frac{34445}{576} - \frac{205}{96}\pi^2\right)\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right]x^3 \\ & + \left[-\frac{3969}{128} + \left(-\frac{123671}{5760} + \frac{9037}{1536}\pi^2 + \frac{896}{15}\gamma_\mathsf{E} + \frac{448}{15}\ln(16x)\right)\nu \right. \\ & \left. + \left(-\frac{498449}{3456} + \frac{3157}{576}\pi^2\right)\nu^2 + \frac{301}{1728}\nu^3 + \frac{77}{31104}\nu^4\right]x^4 + \mathcal{O}(x^5)\Bigg\} \end{aligned}$$

Frequency chirp at 4.5PN

Define a dimensionless time variable $\boxed{\tau = \nu c^3 (t_0 - t)/(5Gm)}$

$$
\begin{aligned}
x = \frac{\tau^{-1/4}}{4} \Bigg\{ & 1 + \left(\frac{743}{4032} + \frac{11}{48}\nu\right)\tau^{-1/4} - \frac{1}{5}\pi\,\tau^{-3/8} \\
& + \left(\frac{19583}{254016} + \frac{24401}{193536}\nu + \frac{31}{288}\nu^2\right)\tau^{-1/2} + \left(-\frac{11891}{53760} + \frac{109}{1920}\nu\right)\pi\,\tau^{-5/8} \\
& + \Bigg[-\frac{10052469856691}{6008596070400} + \frac{1}{6}\pi^2 + \frac{107}{420}\gamma_\mathrm{E} - \frac{107}{3360}\ln\left(\frac{\tau}{256}\right) \\
& \qquad + \left(\frac{3147553127}{780337152} - \frac{451}{3072}\pi^2\right)\nu - \frac{15211}{442368}\nu^2 + \frac{25565}{331776}\nu^3\Bigg]\tau^{-3/4} \\
& + \left(-\frac{113868647}{433520640} - \frac{31821}{143360}\nu + \frac{294941}{3870720}\nu^2\right)\pi\,\tau^{-7/8} \\
& + \Bigg[-\frac{2518977598355703073}{3779358859513036800} + \frac{9203}{215040}\gamma_\mathrm{E} + \frac{9049}{258048}\pi^2 + \frac{14873}{1128960}\ln 2 + \frac{47385}{1605632}\ln 3 - \frac{9203}{3440640}\ln\tau \\
& \qquad + \left(\frac{718143266031997}{576825222758400} + \frac{244493}{1128960}\gamma_\mathrm{E} - \frac{65577}{1835008}\pi^2 + \frac{15761}{47040}\ln 2 - \frac{47385}{401408}\ln 3 - \frac{244493}{18063360}\ln\tau\right)\nu \\
& \qquad + \left(-\frac{1502014727}{8323596288} + \frac{2255}{393216}\pi^2\right)\nu^2 - \frac{258479}{33030144}\nu^3 + \frac{1195}{262144}\nu^4\Bigg]\tau^{-1}\ln\tau \\
& + \Bigg[-\frac{9965202491753717}{5768252227584000} + \frac{107}{600}\gamma_\mathrm{E} + \frac{23}{600}\pi^2 - \frac{107}{4800}\ln\left(\frac{\tau}{256}\right) \\
& \qquad + \left(\frac{8248609881163}{2746786775040} - \frac{3157}{30720}\pi^2\right)\nu - \frac{3590973803}{20808990720}\nu^2 - \frac{520159}{1634992128}\nu^3\Bigg]\pi\,\tau^{-9/8} + \mathcal{O}\left(\tau^{-5/4}\right)\Bigg\}
\end{aligned}
$$

The phase at 4.5PN

$$
\begin{aligned}
\psi = -\frac{x^{-5/2}}{32\nu}\Bigg\{ & 1 + \left(\frac{3715}{1008} + \frac{55}{12}\nu\right)x - 10\pi x^{3/2} \\
& + \left(\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2\right)x^2 + \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi x^{5/2}\ln\left(\frac{x}{x_0}\right) \\
& + \Bigg[\frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_\mathsf{E} - \frac{856}{21}\ln(16\,x) \\
& \qquad + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3\Bigg]x^3 \\
& + \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi x^{7/2} \\
& + \Bigg[\frac{2550713843998885153}{2214468081745920} - \frac{9203}{126}\gamma_\mathsf{E} - \frac{45245}{756}\pi^2 - \frac{252755}{2646}\ln 2 - \frac{78975}{1568}\ln 3 - \frac{9203}{252}\ln x \\
& \qquad + \left(-\frac{680712846248317}{337983528960} - \frac{488986}{1323}\gamma_\mathsf{E} + \frac{109295}{1792}\pi^2 - \frac{1245514}{1323}\ln 2 + \frac{78975}{392}\ln 3 - \frac{244493}{1323}\ln x\right)\nu \\
& \qquad\quad + \left(\frac{7510073635}{24385536} - \frac{11275}{1152}\pi^2\right)\nu^2 + \frac{1292395}{96768}\nu^3 - \frac{5975}{768}\nu^4\Bigg]x^4 \\
& + \Bigg[-\frac{93098188434443}{150214901760} + \frac{1712}{21}\gamma_\mathsf{E} + \frac{80}{3}\pi^2 + \frac{856}{21}\ln(16x) \\
& \qquad + \left(\frac{1492917260735}{1072963584} - \frac{2255}{48}\pi^2\right)\nu - \frac{45293335}{1016064}\nu^2 - \frac{10323755}{1596672}\nu^3\Bigg]\pi x^{9/2} + \mathcal{O}(x^5)\Bigg\}
\end{aligned}
$$

C. Cutler et al., "The Last Three Minutes" Phys. Rev. Lett. **70** (1993) 20:

> *The PN modulations are far less important than PN contributions to the secular growth of the waves' phase* $\phi = 2\pi \int f \mathrm{d}t$

and

> *Although highly accurate wave form templates will* **not** *be needed when searching for waves, they* **will** *be needed when extracting the waves' information. Making optimal use of the interferometers' data will require general-relativity-based wave form templates whose phasing is correct to within a half cycle or so during the entire frequency sweep from* $\sim$ *10 Hz to* $\sim$ *1000 Hz.*

and

> *It is not at all clear how far beyond 2.5PN the template must be carried to keep its total phase error below a half cycle over the entire range from* $\sim$ *10 Hz to* $\sim$ *1000 Hz.*

Cumulative contribution to the number of cycles

$$\mathcal{N}_{\text{cycles}} = \mathcal{N}^{\text{N}}_{\text{cycles}} + \mathcal{N}^{\text{1PN}}_{\text{cycles}} + \mathcal{N}^{\text{1.5PN}}_{\text{cycles}} + \dots$$

$\mathcal{N}_{\text{cycles}}$	LIGO/Virgo		ET		LISA	
f-band	$[30, 10^3]$ Hz		$[1, 10^4]$ Hz		$[10^{-4}, 10^{-1}]$ Hz	
$M_\odot$	1.4×1.4	10×10	1.4×1.4	500×500	$10^5 \times 10^5$	$10^7 \times 10^7$
N	2 562.599	95.502	744 401.36	37.90	28 095.39	9.534
1PN	143.453	17.879	4 433.85	9.60	618.31	3.386
1.5PN	-94.817	-20.797	−1 005.78	-12.63	-265.70	-5.181
2PN	5.811	2.124	23.94	1.44	11.35	0.677
2.5PN	-8.105	-4.604	-17.01	-3.42	-12.47	-1.821
3PN	1.858	1.731	2.69	1.43	2.59	0.876
3.5PN	-0.627	-0.689	-0.93	-0.59	-0.91	-0.383
4PN	-0.107	-0.064	-0.12	-0.04	-0.12	-0.013
4.5PN	0.098	0.118	0.14	0.10	0.14	0.065

Conclusion

With the computation of the 4.5PN phase, we now seem to have an accurate-enough waveform for current and future detector !

Caveat: this is a simplistic model, more detailed study needed

References:

1. DT, F. Larrouturou and L. Blanchet, Class. Quant. Grav. **40** (2023) 5, 055006, arXiv:2209.02719
2. DT and L. Blanchet, Phys. Rev. D **107** (2023) 10, 104048, arXiv:2301.09395
3. L. Blanchet, G. Faye, Q. Henry, F. Larrouturou and DT, *submitted to* Phys. Rev. Lett., arXiv:2304.11185.
4. L. Blanchet, G. Faye, Q. Henry, F. Larrouturou and DT, *submitted to* Phys. Rev. D, arXiv:2304.11186.