A new integrable parametrization for deformations to the Kerr photon ring

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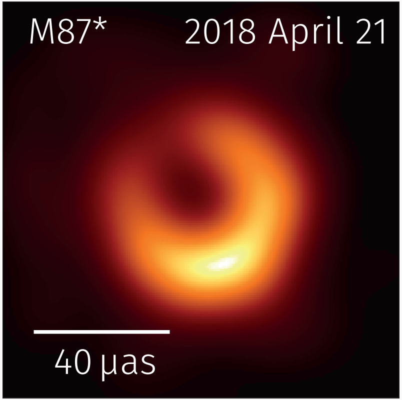
Introduction

- **First direct observation** of a black hole in 2019 [Akiyama et al. ’19]
- New opportunity to **test General Relativity** in its strong field regime
  - Extract observables from the observed image: *critical curve*
  - Parametrize deviations from the GR prediction
- This work: **analytical computation** of the critical curve beyond GR
- Exciting opportunities to apply this framework to **future observations**

Black hole critical curve

Results from the EHT collaboration

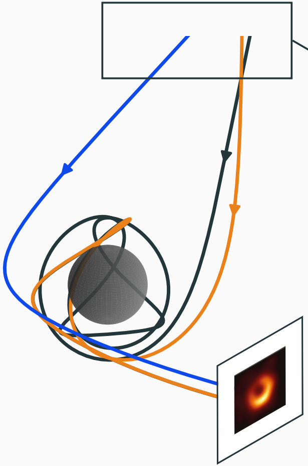


[Akiyama et al. '24]

First direct observation of the vicinity of a black hole in 2017

- Months of data using very-long-baseline interferometry
- First-of-its-kind observation followed by several others
- Black holes: strong field prediction of General Relativity $\longrightarrow$ perfect laboratories for testing the theory

Different trajectories around the black hole



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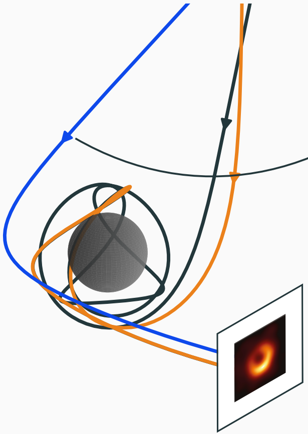

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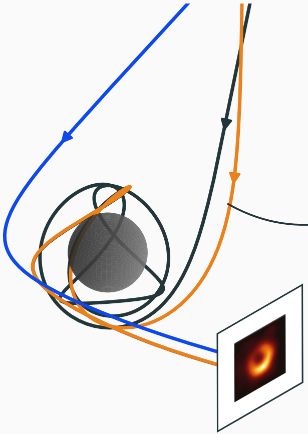

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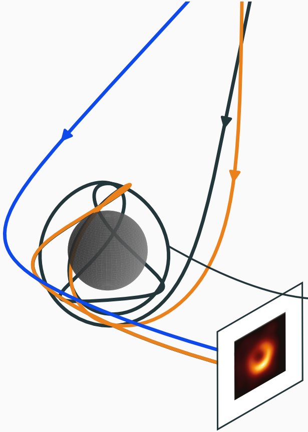

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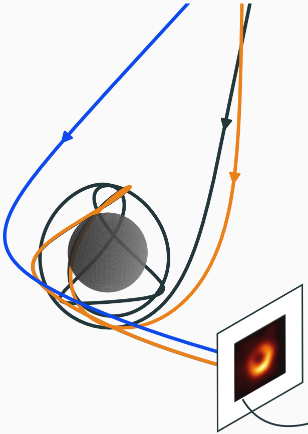
Different trajectories around the black hole



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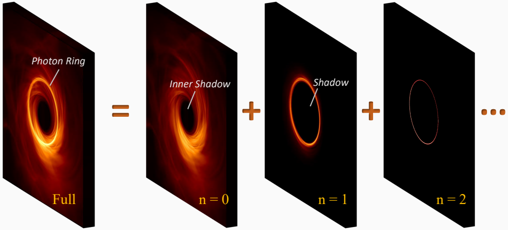
- Some light is directly deflected by the black hole, doing $n = 0$ revolutions around it
- Some light does $n = 1$ revolution around the black hole then escapes
- Some light is trapped for $n > 1$ revolutions but ultimately also escapes
- We observe a superposition of all possible trajectories: those with $n > 1$ are very close together

What do we see?

[Lupsasca et al. '24]



- Photons escape after n orbits so one region is singled out: the **critical curve**
- Convergence is exponential so $n \gtrsim 1$ is sufficient [Lupsasca et al. '24]

The critical curve contains information about the black hole spacetime while being **independent of the accretion disk**

Conception of a test of General Relativity

Observations prospects

- Peak in intensity producing a **clear interferometric pattern** allowing for extraction [Lupsasca et al. ’24]
- Future array of telescopes : the Black Hole EXplorer (BHEX) mission [Johnson et al. ’24]

Critical curve as a test of GR

- Can get information about the spacetime without modelling the accretion disk
- Analytical computation possible in General Relativity

Need for **theoretical predictions** for alternative theories beyond GR and exotic objects (wormholes...)

Principle of computation

Main focus: circular photon orbits around a Kerr black hole

- Such orbits are unstable
- Studied using the null geodesics

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0$$

1. Solve the equations for all positions near the BH
2. Find the impact of each geodesic on the observer's screen
3. Extract the shape of the critical curve

$\longrightarrow$ made much easier due to the integrability of geodesics on a Kerr black hole

Kerr black hole

Kerr metric: stationary axisymmetric black hole

$$\mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta\right)\sin^2\theta\,\mathrm{d}\phi^2 - \frac{4Mra\sin^2\theta}{\Sigma}\mathrm{d}t\,\mathrm{d}\phi$$

$$\Sigma = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 - 2Mr + a^2$$

Regions

- Outer horizon $r_+ = M + \sqrt{M^2 - a^2}$
- Inner horizon $r_- = M - \sqrt{M^2 - a^2}$
- Ergoregion $r_e^\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta}$

Tetrad

$$l^\mu\partial_\mu = \frac{r^2 + a^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi$$

$$n^\mu\partial_\mu = \frac{r^2 + a^2}{2\Sigma}\partial_t - \frac{\Delta}{2\Sigma}\partial_r + \frac{a}{2\Sigma}\partial_\phi$$

Integrability of the system

4D space: a geodesic $\mathcal{L}$ has **8 degrees of freedom** x^μ, p^μ

Symmetries of spacetime

- Stationarity: $\xi^\mu \partial_\mu = \partial_t$
- Axisymmetry: $\chi^\mu \partial_\mu = \partial_\phi$
- Rank-2 Killing tensor

$$K_{\mu\nu} = r^2 g_{\mu\nu} + 2\Sigma l_{(\mu} n_{\nu)}$$
$$\nabla_{(\mu} K_{\nu\rho)} = 0$$

Constants of motion on $\mathcal{L}$

- Energy $E = -\xi^\mu p_\mu$
- Angular momentum $L = \chi^\mu p_\mu$
- Hamiltonian $H = p_\mu p^\mu / 2$
- Carter constant [Carter '68]

$$K = K_{\mu\nu} p^\mu p^\nu$$

The system is integrable thanks to this additional symmetry

Petrov classification

Main idea: classify spacetimes by multiplicity of the principal null directions of the Weyl tensor. Higher multiplicity $\longrightarrow$ higher symmetry

$$\begin{tikzcd}
\text{Most general spacetime} & \mathrm{I} \arrow[l, dotted] \arrow[d] \arrow[dr] & & \text{Kerr spacetime} \\
& \mathrm{II} \arrow[r] \arrow[d] \arrow[dr] & \mathrm{D} \arrow[d] \arrow[dr] \arrow[ur, dotted] & \\
& \mathrm{III} \arrow[r] & \mathrm{N} \arrow[r] & \mathrm{O}
\end{tikzcd}$$

algebraically special spacetimes

$$\boxed{\frac{1}{2} C_{\mu\nu\rho\sigma} X^{\rho\sigma} = \lambda X^{\mu\nu}}$$

Algebraically special spacetimes have various interesting properties: analytic expressions, separability...

Killing tower

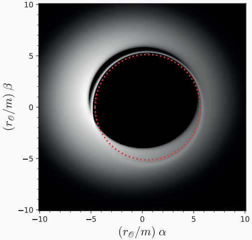
- $K_{\mu\nu}$ can be built from a Killing-Yano 2-form:

$$K_{\mu\nu} = -Y_{\mu\alpha} Y^{\alpha}{}_{\nu} \,, \qquad \nabla_\alpha Y_{\mu\nu} + \nabla_\nu Y_{\mu\alpha} = 0$$

- This Killing-Yano 2-form exists because the Kerr metric is of Petrov type D and has no acceleration [Kubiznak, Frolov '07]
- One can introduce a principal tensor $\mathbf{h} = \star\mathbf{Y}$
- From this principal tensor, one can construct an abundant structure of tensors with specific algebraic relations [Frolov et al. '17]
- In particular, one can recover
  - the complete integrability of the geodesic motion
  - integrability of the motion of spinning particles [Ramond '22]
  - separability of the perturbation equations for spin s [Teukolsky '72]

Summary

[Gourgoulhon '25]



- The image of a black hole contains a critical curve which does not depend on the accretion disk
- Comparing this curve to the theoretical prediction yields a new test of GR in its strong regime
- Theoretical shape can be obtained analytically due to the integrability of the geodesic motion

$\longrightarrow$ How can one obtain the critical curve in beyond-GR theories?

Kerr off-shell family

Roadmap for generalizing Kerr

No-hair theorem

Kerr is the unique asymptotically flat stationary and axisymmetric vacuum solution of GR

Modified gravity

- Extensions of General Relativity [Deffayet et al. '11; Langlois, Noui '16]
- Disformal transformations [Anson et al. '21; Ben Achour et al. '20]
- *ad hoc* parametrizations of the metric functions [Johannsen, Psaltis '10]
- **This work**: focus on parametrizing without requiring a specific action

Preserving symmetries

- Plebanski-Demianski family that maintains Petrov type D [Plebanski, Demianski '76]
- Preserving separability of Hamilton-Jacobi [Carter '68]
- **This work**: unique solution preserving the Killing tower [Krtous et al. '07]

Existing parametrizations

$$\boxed{\mathrm{d}s^2 = g_{tt}\,\mathrm{d}t^2 + 2g_{t\phi}\,\mathrm{d}t\,\mathrm{d}\phi + g_{rr}\,\mathrm{d}r^2 + g_{\theta\theta}\,\mathrm{d}\theta^2 + g_{\phi\phi}\,\mathrm{d}\phi^2}$$ [Yagi et al. '24]

$$g_{tt} = -\frac{\tilde{\Sigma}(A_5 - a^2 A_2^2 \sin^2\theta)}{\rho^4}\,, \quad g_{t\phi} = \frac{aA_5(A_5 - A_0)\tilde{\Sigma}\sin^2\theta}{\rho^4}\,, \quad g_{\phi\phi} = \frac{\tilde{\Sigma}\sin^2\theta A_5(A_1^2 - a^2 A_5 \sin^2\theta)}{\rho^4}$$

$$\tilde{\Sigma} = r^2 + a^2\cos^2\theta + f(r) + g(\theta)\,, \quad g_{\theta\theta} = \tilde{\Sigma}\,, \quad g_{rr} = \frac{\tilde{\Sigma}}{A_5}$$

$$\rho^4 = a^4 A_2^2 A_5 \sin^4\theta + a^2\sin^2\theta(A_0^2 - 2A_0A_5 - A_1^2A_2^2) + A_1^2 A_5\,.$$

- Provides both radial and polar deformations
- General case: Petrov type I
- Radial deformations can keep a Killing-Yano 2-form
- No such 2-form for polar deformations

Kerr-off-shell family

Kerr off-shell: most general spacetime beyond Kerr
with **Killing tower preserved** [Krtous et al. '07]

$$\mathrm{d}s^2 = -\frac{\Delta_r(r)}{\Sigma}\left(\mathrm{d}\tau + y^2\mathrm{d}\varphi\right)^2 + \frac{\Delta_y(y)}{\Sigma}\left(\mathrm{d}\tau - r^2\mathrm{d}\varphi\right)^2 + \frac{\Sigma}{\Delta_r(r)}\mathrm{d}r^2 + \frac{\Sigma}{\Delta_y(y)}\mathrm{d}y^2\,, \quad \Sigma = r^2 + y^2$$

Observations

- Does not rely on a specific action
- Provides both radial and polar deviations
- Integrability makes the critical curve computation **analytical**

Theory

- Preserves the Killing tower of Kerr: expect the same properties (Teukolsky equation, etc.)
- Preserves the Petrov type D of Kerr
- Recover Kerr with

$$\Delta_r = r^2 - 2Mr + a^2\,, \quad \Delta_y = a^2 - y^2\,,$$
$$y = a\cos\theta\,, \quad \varphi = \phi/a\,, \quad \tau = t - a\phi\,.$$

Concrete examples

Radial deformations

- $\Delta_r(r) = r^2 - 2Mr + a^2 + \alpha M^2$: Kerr-MOG [Moffat '15]
- $\Delta_r(r) = r^2 + a^2 - 2rMe^{-\ell M/r}$: Simpson-Visser regular model [Simpson, Visser '22]
- $\Delta_r(r) = r^2 - 2Mr + a^2 + qM^2 \log\left(\frac{r}{M}\right)$: logarithmic corrections

Polar deformations

- $\Delta_y(y) = a^2 - y^2 + py^4$: corrections that maintain the y-parity

Hamiltonian dynamics

1. Geodesic parametrized by $x^\mu(\lambda)$ and $p^\mu(\lambda)$ in Hamiltonian formulation:

$$\{\tau, p_\tau\} = 1\,, \qquad \{r, p_r\} = 1\,, \qquad \{y, p_y\} = 1\,, \qquad \{\varphi, p_\varphi\} = 1\,. \tag{1}$$

2. Conserved charges: E and L from Killing vectors, K from the Killing-Yano and H
3. Reformulate the dynamical system:

$$\Sigma \frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = \frac{r^2(r^2E - L)}{\Delta_r} - \frac{y^2(y^2E + L)}{\Delta_y}\,, \qquad \Sigma \frac{\mathrm{d}r}{\mathrm{d}\lambda} = \pm\sqrt{V_r(r)}\,,$$

$$\Sigma \frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} = \frac{r^2E - L}{\Delta_r} + \frac{y^2E + L}{\Delta_y}\,, \qquad \Sigma \frac{\mathrm{d}y}{\mathrm{d}\lambda} = \pm\sqrt{V_y(y)}\,.$$

Full geodesic motion

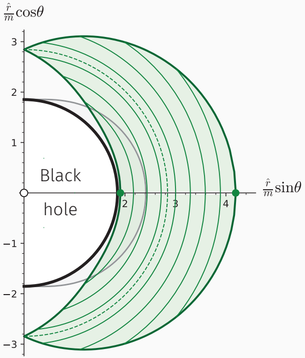
4. Label each null geodesic by $(\ell, k) = (L/E, K/E^2)$
5. Obtain the final form for the geodesic motion:

$$\frac{\mathrm{d}\tau}{\mathrm{d}\lambda'} = \frac{r^2(r^2-\ell)}{\Delta_r} - \frac{y^2(y^2+\ell)}{\Delta_y}\,, \qquad \frac{\mathrm{d}r}{\mathrm{d}\lambda'} = \pm\sqrt{\mathcal{V}_r(r)}\,, \quad \mathcal{V}_r(r) = (r^2-\ell)^2 - \Delta_r k\,,$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda'} = \frac{r^2-\ell}{\Delta_r} + \frac{y^2+\ell}{\Delta_y}\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}\lambda'} = \pm\sqrt{\mathcal{V}_y(y)}\,, \quad \mathcal{V}_y(y) = \Delta_y k - (y^2+\ell)^2\,.$$

Photon ring

- **Spherical photon orbits**: set of positions where a null geodesic with $r =$ cst can exist
- **Critical null geodesic**: geodesic with the same (ℓ, k) as a spherical photon orbits but with $r \neq$ cst
  $\longrightarrow$ geodesics infinitesimally close to these evolve towards the observer and build the critical curve



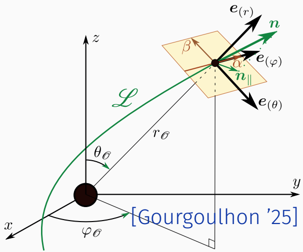
(adapted from [Gourgoulhon '25])

Observation of the curve

- Stationary observer at $(r_{\mathcal{O}}, y_{\mathcal{O}})$ with no angular momentum and $r_{\mathcal{O}} \gg r_+$
- 2D coordinates (α, β) on the screen [Bardeen '73]
- Orthogonal frame

$$\boldsymbol{e}_{(\tau)} = \boldsymbol{\partial}_\tau\,, \qquad \boldsymbol{e}_{(y)} = \frac{\sqrt{\Delta_y(y_{\mathcal{O}})}}{r_{\mathcal{O}}}\boldsymbol{\partial}_y\,,$$

$$\boldsymbol{e}_{(r)} = \boldsymbol{\partial}_r\,, \qquad \boldsymbol{e}_{(\varphi)} = \frac{1}{r_{\mathcal{O}}\sqrt{\Delta_y(y_{\mathcal{O}})}}\boldsymbol{\partial}_\varphi$$



[Gourgoulhon '25]

- **Spherical photon orbits** at $r = r_0$ have $\mathcal{V}_r(r_0) = 0$ and $\mathcal{V}_r'(r_0) = 0$, yielding

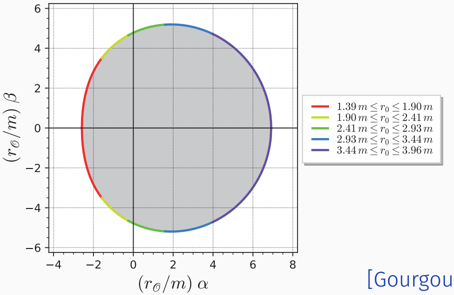
$$\boxed{\ell(r_0) = r_0\left(r_0 - \frac{4\Delta_r(r_0)}{\Delta_r'(r_0)}\right)\,, \qquad k(r_0) = \frac{16 r_0^2\Delta_r(r_0)}{\Delta_r'(r_0)}}$$

Projection on the screen

Impact of the critical geodesics on the screen at large $r_{\mathcal{O}}$:

$$\alpha(r_0) = -\frac{\sqrt{\Delta_y(y_{\mathcal{O}})}}{r_{\mathcal{O}}}\left(1 + \frac{y_{\mathcal{O}}^2 + \ell(r_0)}{\Delta_y}\right), \quad \beta(r_0) = \pm\frac{1}{r_{\mathcal{O}}}\left(k(r_0) - \frac{(y_{\mathcal{O}}^2 + \ell)^2}{\Delta_y}\right)^{1/2}.$$

$\longrightarrow$ parametric analytical expression describing the critical curve



[Gourgoulhon '25]

Summary

- Consider the most general spacetime that preserves the Killing tower of symmetries of Kerr
- Find the parameters $\ell(r_0)$ and $k(r_0)$ that describe all spherical photon orbits
- The critical curve is built from geodesics infinitesimally close to these
- Obtain an analytical parametric shape $(\alpha(r_0), \beta(r_0))$ for the critical curve
- Result true for any value of Δ_r and Δ_y

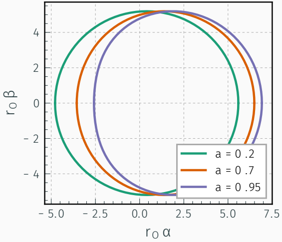
$\longrightarrow$ How can one constrain GR using this result?

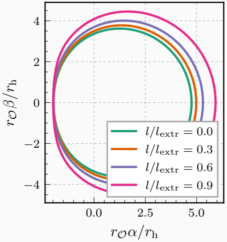
Beyond-GR results

Predicted photon rings

Kerr

- Recover the circle shape of Schwarzschild when $a \to 0$
- Asymmetric deformation when the spin increases

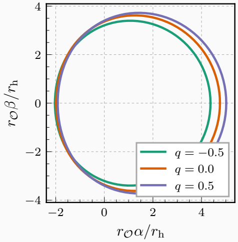




Simpson-Visser

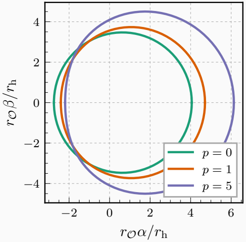
- New parameter ℓ affects the global scale of the curve
- No deformation of the Kerr shape

More examples



Log deviations

- Global modification of both the scale and the shape
- Effect barely visible at low values of q



Polar deformation

- Deformation similar to the one induces by the log term
- High values of p required to see differences

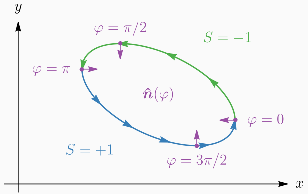
Building an observable

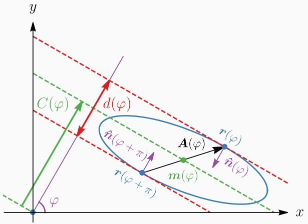
Quantitative description?

- Position $(\alpha = 0, \beta = 0)$ not available on a screen
- Interferometric observation does not yield directly $(\alpha(r_0), \beta(r_0))$
- Parametrization of closed convex curves [Gralla, Lupsasca '20]

$\longrightarrow$ describe the curve by

$$f(\varphi) = \frac{d(\varphi)}{2} + C(\varphi)$$

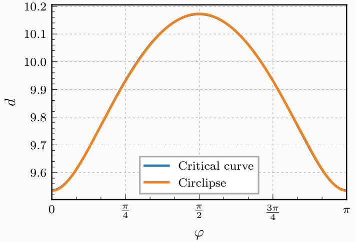




The circlipse parametrization

Parametrize the convex hull d by a circlipse shape [Gralla, Lupsasca '20]

$$f(\varphi) = \underbrace{R_0 + \sqrt{R_1^2 \sin^2\varphi + R_2^2 \cos^2\varphi}}_{d(\varphi)/2} + \underbrace{(X-\chi)\cos\varphi + \arcsin(\chi\cos\varphi)}_{C(\varphi)}$$



a	y_O	R_0	R_1	R_2	Residuals
0.95	0.6	9.23	0.94	0.30	10^{-4}
0.5	0	9.51	0.82	0.72	10^{-4}

$\longrightarrow$ the shape is sufficient to describe all GR parameter space

Degeneracy beyond GR

Beyond-GR modifications are degenerate with GR parameters variation:

Parameters	M	a	ℓ	$y_{\mathcal{O}}$	R_0	R_1	R_2	Residuals
Kerr	1	0.2		0	10.0	0.394	0.370	1.88×10^{-5}
SV	1.32	0.1	0.82	0	10.0	0.402	0.380	1.53×10^{-5}

$\longrightarrow$ cannot discriminate up to 10^{-2} between Kerr and Simpson-Visser!

$\longrightarrow$ Independent measurements of mass and spin required to test General Relativity

Summary

- We can obtain the critical curve for a broad class of radial and polar deformations of Kerr
- The resulting image can be fitted by a circlipse shape for all examples we considered, as in General Relativity
- This implies a degeneracy in the measurement of the critical curve
- It will require independent measurements of black hole mass and spin to be lifted

Conclusion

- Observing the critical curve is a new way to test GR in its strong regime
- This curve can be computed analytically due to the integrability of the geodesic motion in Kerr and does not depend on the accretion disk
- Our work: consider the unique generalization of Kerr that preserves its Killing tower symmetry
- We obtain analytically the critical curve for this class
- Constraining deviations from GR will be possible by critical curve measurements provided independent measurements of black hole parameters are realised

Thank you for your attention!